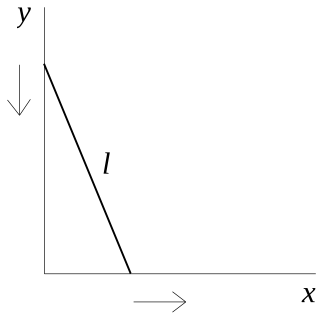
1. [Maximum mark: 30]

This question asks you to investigate a ladder of length l m sliding down a wall and across the floor.



(a) Find the equation of the path of the midpoint of the ladder, in terms of x, y and l. [2]

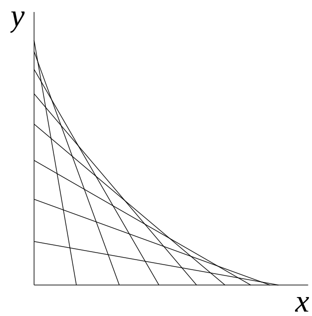
(b) The speed, in m s^{-1}, of the end of the ladder on the wall is equal to the distance, in m, of the other end from the wall. Find the time it takes the ladder to move from vertical to horizontal. [6]

(c) Now assume that the end of the ladder on the floor has constant speed 1 m s^{-1}.

(i) Find the speed of the other end just before the ladder hits the floor.

(ii) Is your answer to (c)(i) realistic? Explain. [7]

As the ladder slides, an envelope curve appears. The ladder is always tangent to the envelope curve.



(d) (i) On the same set of axes, in the first quadrant, sketch and label $x^n + y^n = l^n$ for $n = \frac{1}{2}, 1, 2, 3$.

(ii) Show that the equation of the tangent to $x^n + y^n = l^n$, $n \neq 0$, at (a, b) is $y = -a^{n-1}b^{1-n}x + b^{1-n}l^n$.

(iii) Find the equation of the envelope curve, in terms of x, y and l. [15]

2a. *[2 marks]*

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$.

2**b.** *[3 marks]*

For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

local maximum points;

2**c.** *[1 mark]*

local minimum points;

2**d.** *[2 marks]*

On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$.

2**e.** *[3 marks]*

For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of

local maximum points;

2**f.** *[1 mark]*

local minimum points.

2**g.** *[4 marks]*

Solve the equation $f_n{}'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$.

2h. *[2 marks]*

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$.

2**i.** *[2 marks]*

Consider $f_{n+1}(x) = \cos\big((n+1)\arccos x\big)$.

Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$.

2j. *[3 marks]*

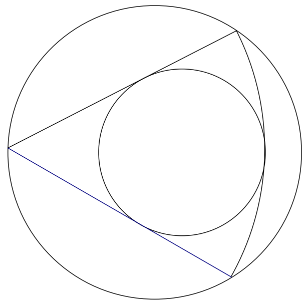
Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x), n \in \mathbb{Z}^+$.

2**k.** *[2 marks]*

Hence express $f_3(x)$ as a cubic polynomial.

3.. [Maximum mark: 24]

This question asks you to investigate a circle inscribed in a sector that is inscribed in another circle.



$r =$ radius of small circle $\quad L =$ radius of sector $\quad R =$ radius of big circle $\quad \theta = \dfrac{\text{sector angle}}{2}$

$A_1 =$ area of small circle $\quad A_2 =$ area of sector $\quad A_3 =$ area of big circle

(a) (i) Show that $\dfrac{A_2}{A_3} = \dfrac{4\theta}{\pi}\cos^2\theta$.

(ii) Find the values of $\dfrac{A_2}{A_3}$ when $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{\pi}{4}$. [6]

Naim guesses that $\dfrac{A_2}{A_3}$ is maximized when $\theta = \dfrac{1}{2}\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) = \dfrac{5\pi}{24}$.

(b) (i) Find, to five decimal places, the value of $\dfrac{A_2}{A_3}$ when $\theta = \dfrac{5\pi}{24}$.

(ii) Determine if Naim's guess is correct or not. [4]

(c) (i) Show that $\dfrac{A_1}{A_3} = \left(\dfrac{2\cos\theta\sin\theta}{1+\sin\theta}\right)^2$.

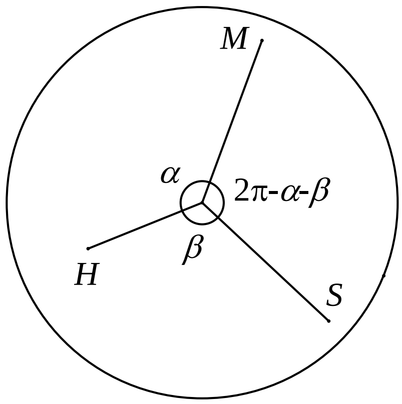
(ii) Find the maximum value of $\dfrac{A_1}{A_3}$ to four decimal places. [6]

Renee guesses that $\dfrac{A_1}{A_3}$ is maximized when the circles are concentric.

(d) Determine if Renee's guess is correct or not. [8]

4. [Maximum mark: 29]

This question asks you to investigate the perimeter of a triangle on a clock.



The lengths of the hour, minute and second hands are 3, 4, 4, respectively.
H, M, S are the tips of the hour, minute and second hands, respectively.
$P =$ perimeter of triangle HMS
$t =$ time in seconds after midnight

(a) Express P in terms of α and β. [4]

(b) Find the angle (in radians) swept between midnight and t by the:

(i) second hand

(ii) minute hand

(iii) hour hand [6]

(c) Express P in terms of t. [5]

(d) (i) Show that the three hands coincide only at midnight and noon.

(ii) Find the period of P as a function of t. [8]

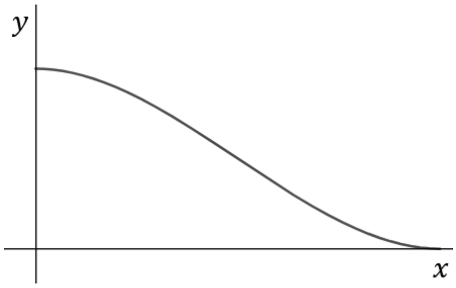
(e) (i) Let $I =$ perimeter of triangle HMS if the hands can be positioned independently.
Find the maximum value of I to four decimal places.

(ii) Approximate the maximum value of P to two decimal places. [6]

5. [Maximum mark: 29]

This question asks you to investigate the horizontal distance from the airport, l m, at which an airplane should start its descent, given the following conditions:

I. As the airplane descends, it follows a cubic curve, $y = ax^3 + bx^2 + cx + d$.



y is the airplane's vertical distance in m from the horizontal ground.
x is the airplane's horizontal distance in m from the point where it begins its descent.

II. The airplane's cruising altitude is 6 km.

III. The airplane's horizontal velocity is a constant 100 m s^{-1}.

IV. For passenger comfort, vertical acceleration ranges from -0.1 m s^{-2} to 0.1 m s^{-2}.

(a) Find d and c. [3]

(b) By considering the point at which $x = l$, or otherwise, show that $al^3 + bl^2 + 6000 = 0$ and $3al + 2b = 0$. [2]

(c) (i) Sketch a graph of the airplane's vertical velocity against time.

(ii) Sketch a graph of the airplane's vertical acceleration against time.

(iii) Determine the airplane's vertical acceleration at the beginning of its descent.

(iv) Express the airplane's vertical acceleration in terms of a, b and x.

(v) Show that $b = -5 \times 10^{-6}$. [11]

(d) Find l. [4]

(e) If the airplane can follow any curve, find the minimum value of l. [9]

6. [Maximum mark: 23]

This question asks you to investigate the roots of a polynomial.

$f(x) = ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ.

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) Factorise $ax^3 + bx^2 + cx + d$, then expand your result with α, β, γ. [3]

(b) (i) Show that $S_1 a + b = 0$.

(ii) By expanding $(\alpha + \beta + \gamma)^2$, or otherwise, show that $S_2 a + S_1 b + 2c = 0$.

(iii) By considering $\alpha^{n-3} f(\alpha)$, $\beta^{n-3} f(\beta)$, $\gamma^{n-3} f(\gamma)$, or otherwise, show that $S_n a + S_{n-1} b + S_{n-2} c + S_{n-3} d = 0$. [7]

Now you are given that $x^3 - x + d = 0$, $d \neq 0$, has roots α, β, γ.

(c) Find, in terms of d, the value of:

(i) $\alpha + \beta + \gamma$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

(iii) $\alpha^3 + \beta^3 + \gamma^3$

(iv) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(v) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma}$ [9]

(d) Find, in terms of d, an equation with roots $\alpha + \beta$, $\beta + \gamma$, $\alpha + \gamma$. [4]

7a. *[6 marks]*

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \ n \in \mathbb{N}$.

Find the exact values of I_0, I_1 and I_2.

7**b.** *[5 marks]*

Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}, \ n \geq 2$.

7c. *[1 mark]*

Explain where the condition $n \geq 2$ was used in your proof.

7**d.** *[2 marks]*

Hence, find the exact values of I_3 and I_4.

7**e.** *[4 marks]*

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx, \ n \in \mathbb{N}$.

Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$.

7f. *[2 marks]*

Hence, find the exact values of J_5 and J_6

7**g.** *[3 marks]*

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \ n \in \mathbb{N}$.

Find the exact values of T_0 and T_1.

7**h.** *[3 marks]*

Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}, \ n \geq 2$.

7i. *[1 mark]*

Explain where the condition $n \geq 2$ was used in your proof.

7**j.** *[2 marks]*

Hence, find the exact values of T_2 and T_3.

8. [Maximum mark: 28]

This question asks you to investigate definite integrals of powers of $(1+x^2)^{-1}$.

(a) On the same set of axes, sketch and label $y=(1+x^2)^{-1}$, $y=(1+x^2)^{-2}$ and $y=(1+x^2)^{-3}$ for $0<x<1$. [3]

(b) Find the exact value of $\int_0^1 (1+x^2)^{-1}\,\mathrm{d}x$. [3]

(c) By substituting $x=\tan\theta$, or otherwise, find the exact value of $\int_0^1 (1+x^2)^{-2}\,\mathrm{d}x$. [8]

Let $I_n=\int_0^1 (1+x^2)^{-n}\,\mathrm{d}x$.

(d) (i) By expressing $(1+x^2)^{-n}$ as $(1+x^2)^{-n}(1)$, or otherwise, show that

$$I_{n+1}=\left(1-\frac{1}{2n}\right)I_n+\frac{2^{-n-1}}{n} \quad \text{for } n\geq 1.$$

(ii) Find the exact value of $\int_0^1 (1+x^2)^{-3}\,\mathrm{d}x$. [10]

(e) Find the exact value of $\int_0^1 (x^2-2x+2)^{-3}\,\mathrm{d}x$. [4]

9. [Maximum mark: 26]

This question asks you to investigate possible applications of the curve $y = \dfrac{\ln x}{x}$.

(a) For the curve $y = \dfrac{\ln x}{x}$, find the

(i) x-intercept(s)

(ii) asymptote(s)

(iii) coordinates of the stationary point(s) and their nature [9]

(b) Sketch the curve $y = \dfrac{\ln x}{x}$. [3]

(c) By considering two points on the curve $y = \dfrac{\ln x}{x}$, show that $\pi^{\mathrm{e}} < \mathrm{e}^{\pi}$. [3]

(d) Given that $a^b = b^a$ where $a, b \in \mathbb{R}$ and $a < b$, find the ranges of a and b. [5]

(e) Given that $\log_a b = \log_b a$ where $a, b \in \mathbb{R}$ and $a < b$, find the ranges of a and b. [6]

10. [Maximum mark: 32]

This question asks you to investigate functions of the form $y = \frac{f(x)}{x}$, where $y = f(x)$ goes through the origin and is concave up.

(a) In each part below, show that $y = f(x)$ goes through the origin and is concave up, then sketch $y = \frac{f(x)}{x}$.

(i) $f(x) = \mathrm{e}^x - 1$

(ii) $f(x) = -\ln(x+1)$

(iii) $f(x) = \sec x - 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ [9]

(b) Make a conjecture about $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x}\right), \ x \neq 0$. [1]

Let the tangent to $y = f(x)$ at $x \neq 0$ be $y = mx + c$.

(c) (i) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x}\right) = -\frac{c}{x^2}$ for $x \neq 0$.

(ii) Show that $y = \frac{f(x)}{x}$ is increasing for $x \neq 0$. [5]

(d) Show that $y = \frac{\arcsin x}{x}$ is increasing for $0 < x < 1$. [3]

(e) (i) By expressing $\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$ as $\mathrm{e}^{g(x)}$, or otherwise, show that $y = \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}, \ 0 < a < b$, is increasing for $x \neq 0$.

(ii) Find the range of $y = \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}, \ 0 < a < b$. [12]

(f) Determine, with justification, whether $y = xf(x)$ is increasing for all x in its domain. [2]

11. [Maximum mark: 24]

This question asks you to investigate sums and products of cosines and sines.

For this question, answers obtained only by technology will not receive marks.

(a) (i) Simplify $\sin(A+B)+\sin(A-B)$.

(ii) By multiplying and dividing by $2\sin\frac{\pi}{7}$, or otherwise, simplify

$$\cos\frac{0\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}.$$

(iii) Simplify $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{7\pi}{7}$. [8]

(b) (i) Solve $\sin 7\theta = 0$.

(ii) Show that $\dfrac{\sin 7\theta}{\sin\theta} = 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1$ for $\sin\theta \neq 0$.

(iii) Solve $64x^6 - 80x^4 + 24x^2 - 1 = 0$.

(iv) Deduce the value of $\left(\cos\frac{\pi}{7}\right)\left(\cos\frac{2\pi}{7}\right)\left(\cos\frac{3\pi}{7}\right)\dots\left(\cos\frac{7\pi}{7}\right)$. [12]

(c) (i) Find the value of $\left(\sin\frac{\pi}{100}\right)\left(\sin\frac{2\pi}{100}\right)\left(\sin\frac{3\pi}{100}\right)\dots\left(\sin\frac{100\pi}{100}\right)$.

(ii) Approximate $\sin\frac{\pi}{100}+\sin\frac{2\pi}{100}+\sin\frac{3\pi}{100}+\cdots+\sin\frac{100\pi}{100}$. [4]

12. [Maximum mark: 27]

This question asks you to investigate arithmetic and geometric sequences that have three matching terms. For example, the arithmetic sequence $1, -2, -5, -8$ and the geometric sequence $1, -2, 4, -8$ have the same u_1, u_2 and u_4.

(a) Find a geometric sequence that has three matching terms with the arithmetic sequence $1, \frac{5}{8}, \frac{1}{4}, -\frac{1}{8}$. [1]

Suppose an arithmetic sequence with common difference d, and a geometric sequence with common ratio r, have the same u_1, u_{p+1} and u_{q+1} where $0 < p < q$.

(b) (i) Sketch graphs of $y = u_1 r^{x-1}$ for $x > 0$, for the cases $r > 1$, $r = 1$ and $0 < 1 < r$.

(ii) Find the possible positive values of r, justifying your answer. [5]

(c) Show that $p(r^q - 1) - q(r^p - 1) = 0$. [4]

Let $f(r) = p(r^q - 1) - q(r^p - 1)$, where p is even and q is odd.

(d) (i) Show that $f(r)$ has a root between -1 and 0.

(ii) Show that $f(r)$ cannot have more than one negative root. [10]

(e) (i) Evaluate $\lim\limits_{q \to \infty} \dfrac{r^q - 1}{r^2 - 1}$ for the negative root r.

(ii) Show that the negative root r can be arbitrarily close to -1. [7]

1.

(a) Draw another ladder that makes an "X" with the first ladder. *M*1

As the ladders slide, their midpoint moves along the circle $x^2 + y^2 = \left(\frac{l}{2}\right)^2$. *A*1

(b) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\sqrt{l^2 - y^2}$ *A*1

$\int \frac{1}{\sqrt{l^2 - y^2}} \mathrm{d}y = \int -\mathrm{d}t$ *A*1

$\arcsin \frac{y}{l} = -t + c$ *A*1

$t = 0, \ \ y = l \qquad \Rightarrow \qquad c = \frac{\pi}{2}$ *A*1

$y = l \sin\left(\frac{\pi}{2} - t\right)$ *A*1

$y = 0 \quad \Rightarrow \quad t = \frac{\pi}{2}$ s *A*1

(c) (i) $x^2 + y^2 = l^2$

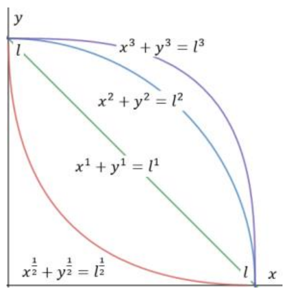
$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ *M*1*A*1

$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{x}{y}$ *A*1

As $y \to 0$, $\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right| \to \infty$ m s^{-1} *M*1*A*1

(ii) Not realistic. In real life, the ladder will separate from the wall before it hits the floor. *A*2

(d) (i)



*A*4

(ii) $nx^{n-1} + ny^{n-1}y' = 0 \qquad \Rightarrow \qquad y' = -a^{n-1}b^{1-n}$ *M*1*A*1

$b = -a^{n-1}b^{1-n}a + k$

$k = b^{1-n}(a^n + b^n) = b^{1-n}l^n$

$y = -a^{n-1}b^{1-n}x + b^{1-n}l^n$ *M*1*A*1*AG*

(iii) Assume the equation of the envelope curve is of the form $x^n + y^n = l^n$. *M*1

Make the distance between the tangent's axes intercepts equal to l. (*M*1)

$(a^{1-n}l^n)^2 + (b^{1-n}l^n)^2 = l^2$ *A*1

$a^{2-2n} + b^{2-2n} = l^{2-2n}$ *A*1

$a^n + b^n = l^n$ *A*1

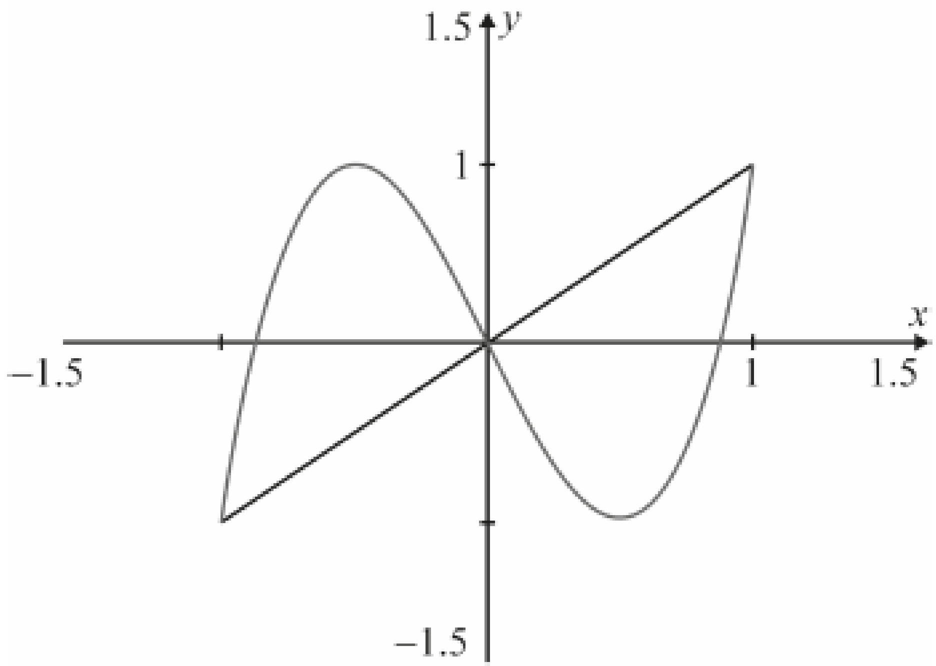
$n = 2 - 2n \quad \Rightarrow \quad n = \frac{2}{3} \quad \Rightarrow$ A solution for n exists, so the assumption is correct. *M*1

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$ *A*1

2a. *[2 marks]*

correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

2b. *[3 marks]*

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 3, 1 local maximum point and 1 local minimum point

n = 5, 2 local maximum points and 2 local minimum points

n = 7, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

[3 marks]

2**c.** *[1 mark]*

$\frac{n-1}{2}$ local minimum points ***A1***

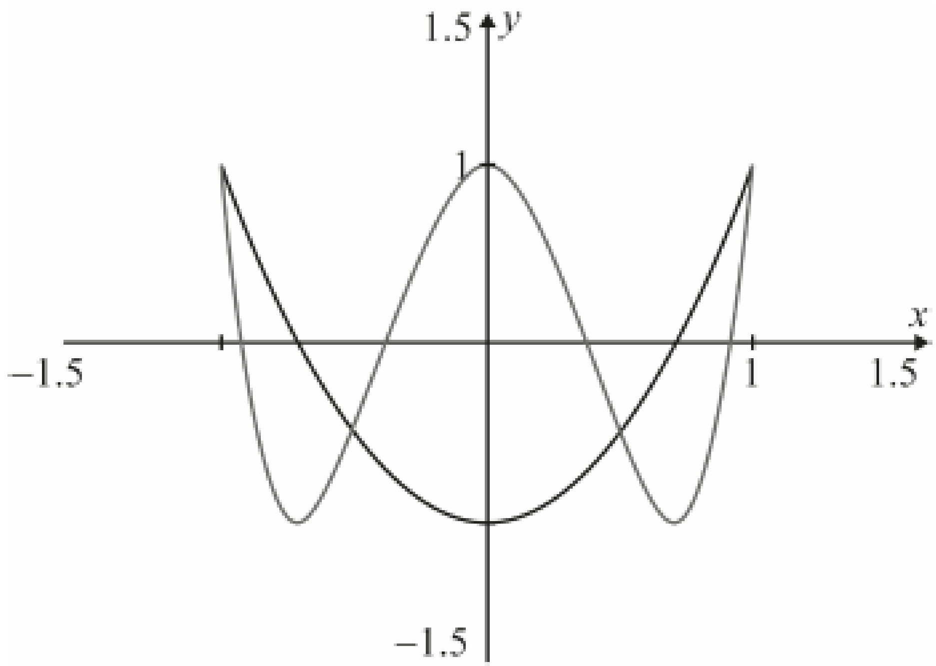
Note: Allow follow through from an incorrect local maximum formula expression.

[1 mark]

2d. *[2 marks]*

correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

2e. *[3 marks]*

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 2, 0 local maximum point and 1 local minimum point

n = 4, 1 local maximum points and 2 local minimum points

n = 6, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

[3 marks]

2**f.** *[1 mark]*

$\frac{n}{2}$ local minimum points ***A1***

[1 mark]

2**g.** *[4 marks]*

$f_n(x) = \cos\left(n \arccos(x)\right)$

$f_n{}'(x) = \frac{n\sin\left(n\arccos(x)\right)}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n{}'(x) = 0 \Rightarrow n\sin\left(n\arccos(x)\right) = 0$ ***M1***

$n\arccos(x) = k\pi \ \ (k \in \mathbb{Z}^+)$ ***A1***

leading to

$x = \cos\frac{k\pi}{n} \ \ (k \in \mathbb{Z}^+ \text{ and } 0 < k < n)$ ***AG***

[4 marks]

2h. *[2 marks]*

$f_2(x) = \cos(2\arccos x)$

$= 2\left(\cos(\arccos x)\right)^2 - 1$ ***M1***

stating that $\left(\cos(\arccos x)\right) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

2**i.** *[2 marks]*

$f_{n+1}(x) = \cos\left((n+1)\arccos x\right)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A + B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

2**j.** *[3 marks]*

$f_{n-1}(x) = \cos\left((n-1)\arccos x\right)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

[3 marks]

2**k.** *[2 marks]*

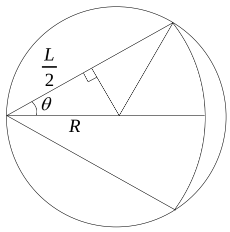
$f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

3.

(a) (i)



$$\cos\theta = \frac{\frac{L}{2}}{R} \quad \Rightarrow \quad L = 2R\cos\theta \quad \Rightarrow \quad \frac{A_2}{A_3} = \frac{\frac{1}{2}L^2(2\theta)}{\pi R^2} = \frac{4\theta}{\pi}\cos^2\theta$$ *M1A3AG*

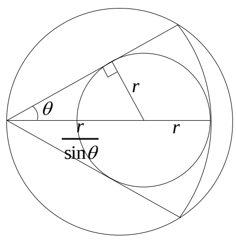
(ii) $\theta = \frac{\pi}{6} \Rightarrow \frac{A_2}{A_3} = \frac{1}{2}$ $\qquad$ $\theta = \frac{\pi}{4} \Rightarrow \frac{A_2}{A_3} = \frac{1}{2}$ *A2*

(b) (i) 0.52451 *A1*

(ii) GDC: At $\left(\frac{A_2}{A_3}\right)_{\max}$, $\theta = 0.65327\ldots \neq \frac{5\pi}{24}$. *M1A1*

Naim’s guess is incorrect. *A1*

(c) (i)

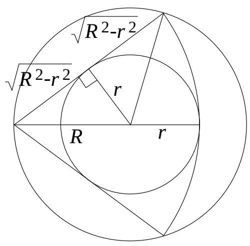


$$L = \frac{r}{\sin\theta} + r$$ *M1A1*

$$2R\cos\theta = \frac{r}{\sin\theta} + r \quad \Rightarrow \quad \frac{r}{R} = \frac{2\cos\theta}{\frac{1}{\sin\theta} + 1} = \frac{2\cos\theta\sin\theta}{1+\sin\theta} \quad \Rightarrow \quad \frac{A_1}{A_3} = \left(\frac{2\cos\theta\sin\theta}{1+\sin\theta}\right)^2$$ *M1A2AG*

(ii) GDC: $\left(\frac{A_1}{A_3}\right)_{\max} = 0.3607$ *A1*

(d)



If circles are concentric, $L = 2\sqrt{R^2 - r^2} = R + r$ *M1A1*

$4R^2 - 4r^2 = R^2 + 2rR + r^2 \quad \Rightarrow \quad 3R^2 - 2rR - 5r^2 = 0 \quad \Rightarrow \quad R = \frac{2r + \sqrt{4r^2 + 60r^2}}{6} = \frac{5}{3}r$ *M1A2*

$\frac{A_1}{A_3} = \left(\frac{r}{R}\right)^2 = \frac{9}{25} \neq 0.3607$ *M1A1*

Renee’s guess is incorrect. *A1*

4.

(a) $P=\sqrt{3^2+4^2-2(3)(4)\cos\alpha}+\sqrt{3^2+4^2-2(3)(4)\cos\beta}+\sqrt{4^2+4^2-2(4)(4)\cos(2\pi-\alpha-\beta)}$ $(M1)$

$=\sqrt{25-24\cos\alpha}+\sqrt{25-24\cos\beta}+\sqrt{32-32\cos(\alpha+\beta)}$ $A3$

(b) (i) $\dfrac{t\pi}{30}$ $M1A1$

(ii) $\dfrac{t\pi}{1800}$ $M1A1$

(iii) $\dfrac{t\pi}{21600}$ $M1A1$

(c) $\alpha=\dfrac{t\pi}{1800}-\dfrac{t\pi}{21600}=\dfrac{11\pi}{21600}t$ $M1A1$

$\beta=\dfrac{t\pi}{21600}-\dfrac{t\pi}{30}=-\dfrac{719\pi}{21600}t$ $M1A1$

$P=\sqrt{25-24\cos\left(\dfrac{11\pi}{21600}t\right)}+\sqrt{25-24\cos\left(\dfrac{719\pi}{21600}t\right)}+\sqrt{32-32\cos\left(\dfrac{59\pi}{1800}t\right)}$ $A1$

(d) (i) When the three hands coincide after midnight, $\dfrac{t\pi}{30}=\dfrac{t\pi}{1800}+2m\pi=\dfrac{t\pi}{21600}+2n\pi,\ \ m,n\in\mathbb{Z}^+$. $M1$

$720t=12t+43200m=t+43200n$

$t=\dfrac{43200}{708}m=\dfrac{43200}{719}n=\dfrac{43200}{11}(n-m)$ $A1$

$\dfrac{m}{n-m}=\dfrac{708}{11}$ $A1$

11 is prime

$(n-m)_{\min}=11$ $A1$

$t_{\min}=43200=12$ hours $A1$

The three hands coincide only at midnight and noon. AG

(ii) P reaches its minimum value only at midnight and noon. $A1$

Period is not more than 12 hours. $A1$

Period $=12$ hours $A1$

(e) (i) At $I_{\max}$, $\alpha=\beta$. $A1$

$I=2\sqrt{25-24\cos\alpha}+\sqrt{32-32\cos 2\alpha}$ $M1$

GDC: $I_{\max}=19.1109$ $A1$

(ii) GDC: P exceeds 19.105, e.g. $P(1302.620)\approx 19.108$ $M1$

$19.105<P_{\max}<19.111$ $A1$

$P_{\max}=19.11$ to two decimal places $A1$

5.

(a) $d = 6000$ $A1$

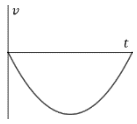
$y' = 3ax^2 + 2bx + c$ $(M1)$

$c = 0$ $A1$

(b) At $x = l,\ y = 0 \quad \Rightarrow \quad 0 = al^3 + bl^2 + 6000$ $M1AG$

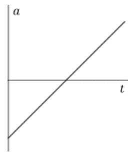
At $x = l,\ y' = 0 \quad \Rightarrow \quad 0 = 3al^2 + 2bl \quad \Rightarrow \quad 0 = 3al + 2b$ $M1AG$

(c) (i)



$A2$

(ii)



$A2$

(iii) $-0.1\ \mathrm{m\,s^{-2}}$ $A1$

(iv) vertical velocity $= \dfrac{\mathrm{d}y}{\mathrm{d}t} = \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right) = (3ax^2 + 2bx)100$ $M1A1$

vertical acceleration $= \dfrac{\mathrm{d}v}{\mathrm{d}t} = \left(\dfrac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right) = (6ax + 2b)100^2$ $M1A1$

(v) $x = 0$, vertical acceleration $= -0.1$: $\quad -0.1 = (2b)100^2$ $M1A1$

$b = -5 \times 10^{-6}$ AG

(d) $0 = 3al + 2(-5 \times 10^{-6})$ $M1$

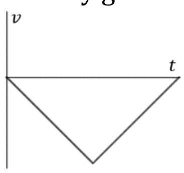
$a = \dfrac{1}{(3 \times 10^5)l}$ $A1$

$0 = \dfrac{1}{(3 \times 10^5)l}(l^3) + (-5 \times 10^{-6})l^2 + 6000$ $M1$

$l = 60000$ $A1$

(e) On the graph of vertical velocity against time, the area enclosed by the curve and the t-axis equals vertical distance travelled (6 km) and is therefore constant.

For any given l, the maximum magnitude of acceleration is minimized when the graph has this shape:



$R1A1$

So for $0 \le x \le \dfrac{l}{2}$ the plane's path is $y = 6000 - kx^2$. $A1$

$\left(\dfrac{l}{2}, 3000\right) \quad \Rightarrow \quad k = \dfrac{12000}{l^2} \quad \Rightarrow \quad y = 6000 - \dfrac{12000}{l^2}x^2$ $M1A1$

vertical velocity $= -\dfrac{24000}{l^2}x(100)$ $A1$

vertical acceleration $= -\dfrac{24000}{l^2}(100)^2 \ge -0.1$ $M1A1$

$l_{\min} = 20000\sqrt{6}$ $A1$

6.

(a) $\dfrac{\sqrt{2}}{2} < r < \dfrac{\sqrt{10}}{2}$ $A2$

(b) Assume $\sqrt{2}$ is rational, i.e. $\sqrt{2} = \dfrac{p}{q}$ where p and q are coprime integers. $A1$

$p^2 = 2q^2$ $A1$

p is even, so $p = 2k$, $k \in \mathbb{Z}$. $A1$

$(2k)^2 = 2q^2$ $A1$

$q^2 = 2k^2$ $A1$

q is even. $A1$

p and q are not coprime, contradiction. $A1$

$\sqrt{2}$ is irrational. AG

(c) Assume x, y, z are rational and not all 0. $A1$

Square both sides: $2x^2 + 2\sqrt{2}xy + y^2 = 3z^2$ $A1$

If $x \neq 0$ and $y \neq 0$, then $\sqrt{2} = \dfrac{3z^2 - 2x^2 - y^2}{2xy} \in \mathbb{Q}$, contradiction $M1A1$

If $x = 0$ and $y \neq 0$, then $y = \sqrt{3}z \Rightarrow \sqrt{3} = \dfrac{y}{z} \in \mathbb{Q}$, contradiction $M1A1$

If $y = 0$ and $x \neq 0$, then $\sqrt{2}x = \sqrt{3}z \Rightarrow \sqrt{1.5} = \dfrac{x}{z} \in \mathbb{Q}$, contradiction $M1A1$

$\therefore$ $x = y = z = 0$ is the only rational solution. AG

(d) Assume $(\sqrt{2}, \sqrt{3})$ is equidistant to two distinct lattice points (a, b) and (c, d). $A1$

$(\sqrt{2} - a)^2 + (\sqrt{3} - b)^2 = (\sqrt{2} - c)^2 + (\sqrt{3} - d)^2$ $M1$

$2 - 2\sqrt{2}a + a^2 + 3 - 2\sqrt{3}b + b^2 = 2 - 2\sqrt{2}c + c^2 + 3 - 2\sqrt{3}d + d^2$ $M1$

$\sqrt{2}(c - a) + \dfrac{a^2 + b^2 - c^2 - d^2}{2} = \sqrt{3}(b - d)$ $A1$

From (c), $c - a = b - d = 0 \Rightarrow a = c$, $b = d$ $R1$

(a, b) and (c, d) are not distinct, contradiction $A1$

$(\sqrt{2}, \sqrt{3})$ cannot be equidistant to two distinct lattice points (a, b) and (c, d). AG

(e) Let $(\sqrt{2}, \sqrt{3})$ be the centre of a circle. $A1$

Increase the radius. The circle will never touch two lattice points at the same time. $A2$

So the radius can be increased until the circle encloses exactly n lattice points. AG

7a. *[6 marks]*

$I_0 = \int_0^{\frac{\pi}{2}} 1\ dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ ***M1A1***

$I_1 = \int_0^{\frac{\pi}{2}} \sin x\ dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$ ***M1A1***

$I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x\ dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\ dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ ***M1A1***

[6 marks]

7**b.** *[5 marks]*

$u = \sin^{n-1}x$ $v = -\cos x$

$\frac{du}{dx} = (n-1)\sin^{n-2}x \cos x$ $\frac{dv}{dx} = \sin x$

$I_n = [-\sin^{n-1}x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}x \cos^2 x\ dx$ ***M1A1A1***

$= 0 + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}x(1-\sin^2 x)\ dx = (n-1)(I_{n-2} - I_n)$ ***M1A1***

$\Rightarrow nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{(n-1)}{n} I_{n-2}$ ***AG***

[6 marks]

7**c.** *[1 mark]*

need $n \geq 2$ so that $\sin^{n-1}\frac{\pi}{2} = 0$ in $[-\sin^{n-1} x \cos x]_0^{\frac{\pi}{2}}$ ***R1***

[1 mark]

7**d.** *[2 marks]*

$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$ $I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}$ ***A1A1***

[2 marks]

7**e.** *[4 marks]*

$x = \frac{\pi}{2} - u \Rightarrow \frac{dx}{du} = -1$ ***A1***

$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_{\frac{\pi}{2}}^0 -\cos^n\left(\frac{\pi}{2} - u\right) du = -\int_{\frac{\pi}{2}}^0 \sin^n u\, du = \int_0^{\frac{\pi}{2}} \sin^n u\, du =$
I_n ***M1A1A1AG***

[4 marks]

7f. *[2 marks]*

$J_5 = I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ $\qquad J_6 = I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}$ ***A1A1***

[2 marks]

7**g.** *[3 marks]*

$T_0 = \int_0^{\frac{\pi}{4}} 1\, dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$ ***A1***

$T_1 = \int_0^{\frac{\pi}{4}} \tan dx = [-\ln|\cos x|]_0^{\frac{\pi}{4}} = -\ln \frac{1}{\sqrt{2}} = \ln\sqrt{2}$ ***M1A1***

[3 marks]

7**h.** *[3 marks]*

$T_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\sec^2 x - 1)\, dx$ ***M1***

$\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - T_{n-2} = \frac{1}{n-1} - T_{n-2}$ ***A1A1AG***

[3 marks]

7**i.** *[1 mark]*

need $n \geq 2$ so that the powers of tan in $\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx$ are not negative ***R1***

[1 mark]

7**j.** *[2 marks]*

$T_2 = 1 - T_0 = 1 - \frac{\pi}{4}$ ***A1***

$T_3 = \frac{1}{2} - T_1 = \frac{1}{2} - \ln\sqrt{2}$ ***A1***

[2 marks]

8.

(a) $ax^3 + bx^2 + cx + d$

$= a(x-\alpha)(x-\beta)(x-\gamma)$ *A1*

$= a(x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma)$ *A2*

(b) (i) $b = -a(\alpha+\beta+\gamma)$ *A1*

$b = -aS_1$ *A1*

$S_1 a + b = 0$ *AG*

(ii) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\beta\gamma+\alpha\gamma)$ *A1*

$S_1{}^2 = S_2 + 2\left(\frac{c}{a}\right)$ *A1*

$S_1\left(-\frac{b}{a}\right)a = S_2 a + 2c$ *M1*

$S_2 a + S_1 b + 2c = 0$ *AG*

(iii) Add the equations $\alpha^{n-3}f(\alpha) = 0,\ \ \beta^{n-3}f(\beta) = 0,\ \ \gamma^{n-3}f(\gamma) = 0$ *M2*

$S_n a + S_{n-1}b + S_{n-2}c + S_{n-3}d = 0$ *AG*

(c) (i) $S_1 = 0$ *A1*

(ii) $S_2 + 0S_1 - 2 = 0 \quad \Rightarrow \quad S_2 = 2$ *M1A1*

(iii) $S_3 + 0S_2 - S_1 + S_0 d = 0 \quad \Rightarrow \quad S_3 = -3d$ *M1A1*

(iv) $S_2 + 0S_1 - S_0 + S_{-1}d = 0 \quad \Rightarrow \quad S_{-1} = \frac{1}{d}$ *M1A1*

(v) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = 0$ *M1A1*

(d) $\alpha+\beta+\gamma = 0$, so roots are $-\alpha,\ -\beta,\ -\gamma.$ *M1A1*

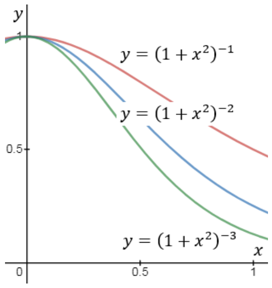
Reflect in y-axis. *(M1)*

$(-x)^3 - (-x) + d = 0$

$x^3 - x - d = 0$ *A1*

9.

(a)



*A*3

(b) $\int_0^1 (1+x^2)^{-1}\,\mathrm{d}x = [\arctan x]_0^1 = \frac{\pi}{4}$ *M*1*A*2

(c) $x = \tan\theta \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ *A*1

$\int (1+x^2)^{-2}\,\mathrm{d}x = \int (1+\tan^2\theta)^{-2}\sec^2\theta\,\mathrm{d}\theta$ *A*1

$= \int \cos^2\theta\,\mathrm{d}\theta$ *A*1

$= \int \frac{\cos 2\theta + 1}{2}\mathrm{d}\theta$ *A*1

$= \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + c$ *A*1

$\int_0^1 (1+x^2)^{-2}\,\mathrm{d}x = \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{4}}$ *A*1

$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$ *M*1

$= \frac{\pi}{8} + \frac{1}{4}$ *A*1

(d) (i) $I_n = \int_0^1 (1+x^2)^{-n}\,(1)\mathrm{d}x$

$= [x(1+x^2)^{-n}]_0^1 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,\mathrm{d}x$ *M*1*A*2

$= 2^{-n} + 2n\int_0^1 (1+x^2-1)(1+x^2)^{-n-1}\,\mathrm{d}x$ *M*2

$= 2^{-n} + 2n(I_n - I_{n+1})$ *A*1

$2nI_{n+1} = (2n-1)I_n + 2^{-n}$ *A*1

$I_{n+1} = \left(1 - \frac{1}{2n}\right)I_n + \frac{2^{-n-1}}{n}$ *AG*

(ii) $I_3 = \left(1 - \frac{1}{4}\right)I_2 + \frac{2^{-3}}{2} = \frac{3}{4}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{16} = \frac{3\pi}{32} + \frac{1}{4}$ *M*1*A*2

(e) $\int_0^1 (x^2 - 2x + 2)^{-3}\,\mathrm{d}x = \int_0^1 (1 + (x-1)^2)^{-3}\,\mathrm{d}x = \int_{-1}^0 (1+x^2)^{-3}\,\mathrm{d}x$ *A*2

$= \int_0^1 (1+x^2)^{-3}\,\mathrm{d}x = \frac{3\pi}{32} + \frac{1}{4}$ *A*2

10.

(a) (i) $x = 1$ *A*1

(ii) $x = 0$ *A*1

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

$y = 0$ *A*1

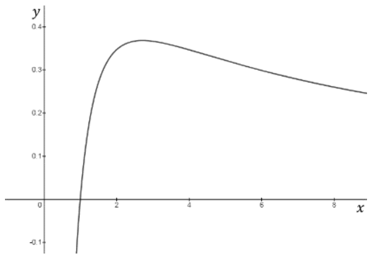
(iii) $y' = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = 0$ *A*1

$x = \mathrm{e}$ *A*1

$$y'' = \frac{x^2\left(-\frac{1}{x}\right) - 2x(1 - \ln x)}{x^4} = \frac{-\mathrm{e}}{\mathrm{e}^4} < 0$$ *M*1*A*1

$\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$ maximum *A*2

(b)



*A*3

(c) $\dfrac{\ln\pi}{\pi} < \dfrac{\ln \mathrm{e}}{\mathrm{e}}$ *A*1

$\mathrm{e}\ln\pi < \pi\ln\mathrm{e}$ *A*1

$\ln\pi^{\mathrm{e}} < \ln\mathrm{e}^{\pi}$ *A*1

$\pi^{\mathrm{e}} < \mathrm{e}^{\pi}$ *AG*

(d) $\ln a^b = \ln b^a$ *A*1

$b\ln a = a\ln b$ (*A*1)

$\dfrac{\ln a}{a} = \dfrac{\ln b}{b}$ *A*1

$1 < a < \mathrm{e}, \quad b > \mathrm{e}$ *A*2

(e) $\dfrac{\ln b}{\ln a} = \dfrac{\ln a}{\ln b}$ *A*1

$(\ln a)^2 = (\ln b)^2$

$\ln a = -\ln b$ *A*1

$a = \dfrac{1}{b}$ *A*1

$a < b \quad \Rightarrow \quad a < \dfrac{1}{a}, \quad b > \dfrac{1}{b}$ *A*1

$a^2 < 1, \quad b^2 > 1$

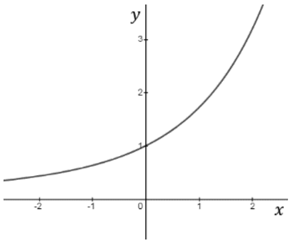
$0 < a < 1, \quad b > 1$ *A*2

11.

(a) (i) $f(0) = e^0 - 1 = 0$ *A*1

$f'(x) = e^x$

$f''(x) = e^x > 0$ *A*1

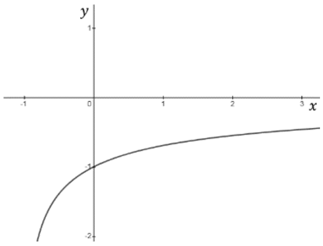


*A*1

(ii) $f(0) = -\ln(0+1) = 0$ *A*1

$f'(x) = -\dfrac{1}{x+1}$

$f''(x) = \dfrac{1}{(x+1)^2} > 0$ *A*1

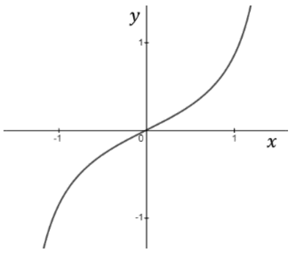


*A*1

(iii) $f(0) = \sec 0 - 1 = 0$ *A*1

$f'(x) = (\sec x)(\tan x)$

$f''(x) = \sec^3 x + (\sec x)(\tan^2 x) > 0$ *A*1



*A*1

(b) $\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right) > 0$ *A*1

(c) (i) $\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right) = \dfrac{xf'(x) - f(x)}{x^2}$ *M*1

$= \dfrac{xm - (mx + c)}{x^2}$ *A*1

$= -\dfrac{c}{x^2}$ *AG*

(ii) $y = mx + c$ is tangent to $y = f(x)$ at $x \neq 0$ and $y = f(x)$ is concave up. *R*1

$y = mx + c$ is below $y = f(x)$ *A*1

$c < 0$ *A*1

$\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right) = -\dfrac{c}{x^2} > 0$ *AG*

(d) $\arcsin 0 = 0$ *A*1

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\arcsin x = \frac{x}{(1-x^2)^{\frac{3}{2}}} > 0$$ *A*2

$y = \dfrac{\arcsin x}{x}$ is increasing for $0 < x < 1$. *AG*

(e) (i) $$\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \mathrm{e}^{\ln\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}} = \mathrm{e}^{\frac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}}$$ *A*1

$$\ln\left(\frac{a^0+b^0}{2}\right) = 0$$ *A*1

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{a^x+b^x}{2}\right) = \frac{a^x\ln a + b^x\ln b}{a^x+b^x}$$ *A*1

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left(\frac{a^x+b^x}{2}\right) = \frac{(a^x+b^x)(a^x(\ln a)^2 + b^x(\ln b)^2) - (a^x\ln a + b^x\ln b)^2}{(a^x+b^x)^2}$$ *M*1

$$= \frac{a^x b^x(\ln a - \ln b)^2}{(a^x+b^x)^2} > 0$$ *A*1

$\dfrac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}$ is increasing for $x \neq 0$. *A*1

$\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ is increasing for $x \neq 0$. *AG*

(ii) $$\lim_{x\to\infty}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to\infty}\mathrm{e}^{\frac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}} = \lim_{x\to\infty}\mathrm{e}^{\left(\frac{a^x\ln a + b^x\ln b}{a^x+b^x}\right)} = \lim_{x\to\infty}\mathrm{e}^{\left(\frac{\left(\frac{a}{b}\right)^x\ln a + \ln b}{\left(\frac{a}{b}\right)^x+1}\right)} = b$$ *M*1*A*2

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\mathrm{e}^{\left(\frac{\left(\frac{a}{b}\right)^x\ln a + \ln b}{\left(\frac{a}{b}\right)^x+1}\right)} = \sqrt{ab}$$ *A*1

$$\lim_{x\to -\infty}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to -\infty}\mathrm{e}^{\left(\frac{\ln a + \left(\frac{b}{a}\right)^x\ln b}{1+\left(\frac{b}{a}\right)^x}\right)} = a$$ *A*1

$$a < \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} < b$$ *A*1

(f) No. E.g. $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(x(\mathrm{e}^x - 1)\big) = x\mathrm{e}^x + \mathrm{e}^x - 1 = -1 < 0$ when $x = -1$. *A*1*R*1

12.

(a) (i) $\sin(A+B)+\sin(A-B)=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$ $(A1)$

$= 2\sin A\cos B$ $A1$

(ii) $\cos\frac{0\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$=\frac{1}{2\sin\frac{\pi}{7}}\left(2\sin\frac{\pi}{7}\cos\frac{0\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{2\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{4\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}\right)$$

$$=\frac{1}{2\sin\frac{\pi}{7}}\left(2\sin\frac{\pi}{7}+\left(\sin\left(\frac{\pi}{7}+\frac{2\pi}{7}\right)+\sin\left(\frac{\pi}{7}-\frac{2\pi}{7}\right)\right)+\left(\sin\left(\frac{\pi}{7}+\frac{4\pi}{7}\right)+\sin\left(\frac{\pi}{7}-\frac{4\pi}{7}\right)\right)\right.$$

$$\left.+\left(\sin\left(\frac{\pi}{7}+\frac{6\pi}{7}\right)+\sin\left(\frac{\pi}{7}-\frac{6\pi}{7}\right)\right)\right)$$ $M1A1$

$$=\frac{1}{2\sin\frac{\pi}{7}}\left(2\sin\frac{\pi}{7}+\sin\left(\frac{3\pi}{7}\right)-\sin\left(\frac{\pi}{7}\right)+\sin\left(\frac{5\pi}{7}\right)-\sin\left(\frac{3\pi}{7}\right)+\sin\left(\frac{7\pi}{7}\right)-\sin\left(\frac{5\pi}{7}\right)\right)$$ $A1$

$$=\frac{1}{2}$$ $A1$

(iii) $\sum_{k=0}^{7}\cos\frac{k\pi}{7}=0 \quad\Rightarrow\quad \cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{7\pi}{7}=-\frac{1}{2}$ $R1A1$

(b) (i) $\theta=\frac{k\pi}{7},\ k\in\mathbb{Z}$ $A2$

(ii) $\frac{\sin 7\theta}{\sin\theta}=\frac{\operatorname{Im}(\cos\theta+\mathrm{i}\sin\theta)^7}{\sin\theta}$ $A1$

$=\frac{1}{s}(7c^6s-35c^4s^3+21c^2s^5-s^7)$ $M1A1$

$=7c^6-35c^4(1-c^2)+21c^2(1-c^2)^2-(1-c^2)^3$ $M1$

$=7c^6-35c^4(1-c^2)+21c^2(1-2c^2+c^4)-(1-3c^2+3c^4-c^6)$ $A1$

$=64\cos^6\theta-80\cos^4\theta+24\cos^2\theta-1$ AG

(iii) $x=\cos\frac{k\pi}{7},\ k=1,2,3,4,5,6$ $A2$

(iv) $\left(\cos\frac{\pi}{7}\right)\left(\cos\frac{2\pi}{7}\right)\left(\cos\frac{3\pi}{7}\right)\ldots\left(\cos\frac{7\pi}{7}\right)=(\text{product of roots})\left(\cos\frac{7\pi}{7}\right)=\left(-\frac{1}{64}\right)(-1)=\frac{1}{64}$ $A3$

(c) (i) 0 $A1$

(ii) $\sum_{k=1}^{100}\sin\frac{k\pi}{100}\approx 100\times(\text{average value of }\sin x\text{ from }x=0\text{ to }x=\pi)$ $A1$

$=100\frac{\int_0^{\pi}\sin x\,\mathrm{d}x}{\pi}$ $A1$

$=\frac{200}{\pi}$ $A1$

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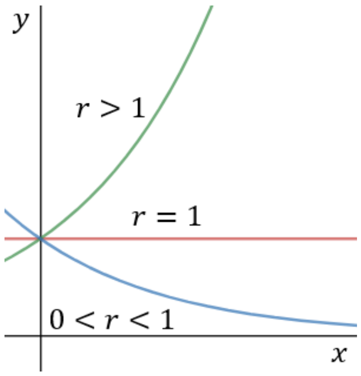
(a) $1, \quad -\frac{1}{2}, \frac{1}{4}, \quad -\frac{1}{8}$ *A*1

(b) (i)



*A*3

(ii) If $0 < r < 1$ or $r > 1$, a straight line cannot intersect $y = u_1 r^{x-1}$ at three points. *R*1

$\therefore$ The only possible positive value of r is 1. *A*1

(c) $u_{p+1} = u_1 + pd = u_1 r^p$ *A*1

$u_{q+1} = u_1 + qd = u_1 r^q$ *A*1

$d = \frac{u_1 r^p - u_1}{p} = \frac{u_1 r^q - u_1}{q}$ *M*1*A*1

$p(r^q - 1) - q(r^p - 1) = 0$ *AG*

(d) (i) $f(-1) = -2p < 0$ *A*2

$f(0) = q - p > 0$ *A*2

$f(r)$ is continuous. *A*1

$\therefore$ $f(r)$ has a root between -1 and 0. *AG*

(ii) Assume $f(r)$ has more than one negative root. *A*1

$f'(r) = 0$ for some $r < 0$. *A*1

$f'(r) = pqr^{q-1} - pqr^{p-1} = pqr^{p-1}(r^{q-p} - 1) = 0$ *M*1*A*1

$r = 0, 1$, contradiction *A*1

$\therefore$ $f(r)$ cannot have more than one negative root. *AG*

(e) (i) $\frac{r^q - 1}{r^2 - 1} = \frac{q}{2}$ *A*1

$\lim_{q \to \infty} \frac{r^q - 1}{r^2 - 1} = \lim_{q \to \infty} \frac{q}{2} = \infty$ *M*1*A*1

(ii) $-2 \leq \lim_{q \to \infty} (r^q - 1) \leq -1$ *M*1*A*1

$\lim_{q \to \infty} (r^2 - 1) = 0$ *A*1

$\lim_{q \to \infty} r = -1$ *A*1

r can be arbitrarily close to -1. *AG*