International Baccalaureate:

Extended Essay

Investigation of the change in the fundamental frequency of a guitar string with a change in its temperature

Research Question: How does a change in temperature affect the fundamental frequency of a guitar string?

Word Count: 3931

Physics

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1. Introduction

Music and physics are two fields that are inseparable from each other. Musical sound is nothing but longitudinal waves produced by periodic vibrations. All musical instruments produce sound by the vibration of matter, which can be a string, an air column or a stretched membrane. This essay investigates the sound produced by the most popular stringed instrument, the guitar.

The frequency of the sound produced by a guitar string is one its most important characteristic because each note in a musical scale has a definite pitch which is determined by the frequency, for example, the note "A_4" has a fundamental frequency of 440 Hz. Therefore, the main objective of this essay is to investigate the factors that affect the fundamental frequency produced by the guitar string and attempt to answer the question, **"How does a change in temperature affect the fundamental frequency of a guitar string?"**. One of the biggest dilemmas that guitarists face is that a guitar generally goes out of tune when exposed to heat. This adverse effect caused by a change in temperature is what inspired me to write this essay.

In this essay, several established relationships will be modified to ultimately derive a relationship between a change in temperature and the fundamental frequency of a guitar string. By conducting an experiment, empirical data will be collected to first confirm a modified version of the established relationship between the fundamental frequency of a string and factors like length of the string and tension experienced by it. Then, empirical frequency for different amounts of change in temperature will be compared with the predicted values from theory, hence concluding whether the theoretical derivation is correct.

1.1 Flow of Investigation

Topic of Investigation

Temperature and Frequency of a guitar string
(Acoustics, mechanics and thermodynamics of a string)

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Acoustics

Theory	Experiment
- The structure of a guitar - Material used in the string - The fundamental frequency of the string	- D'Addario EX120 Nickel Strings Guitar -E_2, G_3 and E_4 Strings -Fender CD-60S guitar

↓

Mechanics

Theory	Experiment
- Wave equation: tension (F) and length (L) and their relationship with fundamental frequency (f) - Young's Modulus of a material	-Determining the tension in each string by varying the length and obtaining a relationship between f and 1/L

↓

Thermodynamics

Theory	Experiment
- Linear thermal expansion - Deriving a relationship between f^2 and Δ T (change in temperature) -Work function, its dependence on temperature and effect on frequency	-Varying the temperature of the string and using 'Decibel X' app, determining the frequency of sound produced

2. Guitar Fundamentals

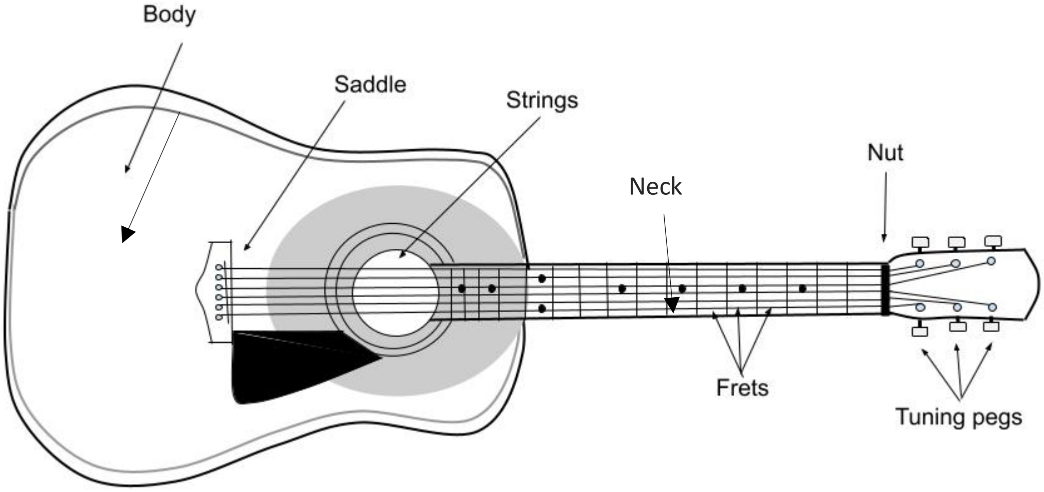


Figure 1: A diagram illustrating a guitar and its key features

As shown in the diagram, a guitar has six strings, which are named according to the note they play when plucked. From top to bottom, these names are, "E_2", "A_2", "D_3", "G_3", "B_3" and "E_4". All the strings are stretched between the **nut** and the **saddle.** A string's thickness or diameter is called **string gauge.** The tension in the strings can be increased or decreased by rotating the **tuning pegs** and the effective length of the string that is vibrating is changed by the guitarist when he/she places a finger on the "**frets**" or the divisions made on the guitar's neck.

3. Frequency of a standing wave

Since a guitar string is fixed between two points, the sound produced by it consists of standing waves. This section investigates the mechanics of a standing wave. The wave equation provides a description of all mechanical waves and the equation in one space dimension is written as[1]:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad - 3.1$$

(where 'y' is the vertical displacement of a wave at a point (m), 'x' is the horizontal distance from the origin to that point (m), t is the time at which a displacement 'y' occurs (s) and 'c' is the speed of propagation of the wave (ms^{-1}))

This equation is a second order linear partial differential equation. Partial derivatives are used when a function depends on two variables. It is simply the derivative of a function with respect to one variable while assuming the other variable is constant. In this case, 'y' is a function of both 'x' and 't'.

The equation for a standing wave in a string can be represented as[2]:

$$y\,(x,t) = A\,sin\,\omega_n t\,sin\,\frac{n\pi x}{L}$$

(where A is the amplitude of the wave (m), n is the number of the harmonic, L is the length of the string (m), and ω_n is the angular frequency of the nth harmonic (rad s^{-1}))

Using partial differentiation,

$$\left(\frac{\partial y}{\partial x}\right)_t = A\sin\omega_n t\,\frac{n\pi}{L}\cos\frac{n\pi x}{L}$$

$$\frac{\partial^2 y}{\partial x^2} = (-A\sin\omega_n t\sin\frac{n\pi x}{L})\,\frac{n^2\pi^2}{L^2} \qquad \text{- 3.2}$$

$$\left(\frac{\partial y}{\partial t}\right)_x = A\,\omega_n\cos\omega_n t\,\sin\frac{n\pi x}{L}$$

$$\frac{\partial^2 y}{\partial t^2} = (-A\sin\omega_n t\sin\frac{n\pi x}{L})\,{\omega_n}^2 \qquad \text{- 3.3}$$

The wave speed (c) is directly proportional to the square root of tension force in the string (F) and inversely proportional to the square root of linear mass density (μ). The following equation has been derived[3]:

$$c = \sqrt{\frac{F}{\mu}} \qquad \text{- 3.4}$$

Substituting equations 3.2, 3.3 and 3.4 into the wave equation (3.1) we get,

$$\omega_n{}^2 = \frac{F}{\mu}\left(\frac{n^2\pi^2}{L^2}\right)$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{F}{\mu}}$$

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}}$$

For the first harmonic (the fundamental frequency), we can write this equation for n=1:

$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{- 3.5}$$

This equation tells us that the fundamental frequency of any stretched string depends on three factors: the tension force (F), the length of the string (L) and the linear mass density (μ)

4. Young's Modulus

Young's modulus or Elastic modulus is the fundamental property of a material which can be explained as the stiffness in the material. It basically tells us how easily a material can be stretched. For a stretched string, like a guitar string, the Young's modulus is given by:

$$Y = \frac{stress}{strain}$$

(where Y is the Young's modulus, stress can be written as tension (F) per unit cross sectional area (A) and strain is the (change in length (ΔL) /original length (L)))

Therefore, we have,

$$Y = \frac{F}{A} \times \frac{L}{\Delta L} \qquad \text{- 4.1}$$

The units for Young's modulus are N m^{-2} or Pa.

5. Linear Thermal Expansion

It's a common physical fact that upon heating, metal objects can expand. Assuming the thickness remains constant, the change in length due to a temperature change can be calculated by the following equation:

$$\Delta L = \alpha L \Delta T \qquad - 5.1$$

(Where α is the coefficient of thermal expansion (K^{-1}) and has a specific value for every material, L is the original length of the string (m), ΔT is the change in temperature (K) and ΔL is the change in length of the string (m))

Thus, when a metal guitar string's temperature is increased, it tends to expand. However, the string is fixed between two points (refer to *Figure 1*), therefore this ultimately leads to a change in the tension force experienced by the string which will be discussed in detail in **Section 8.**

6. Experimental Procedure and Data Collection

An experiment is designed to study the relationship between the frequency (*f*) of three guitar strings, "E_4, G_3 and E_2", their length (L), tension (F) and the change in temperature (T).

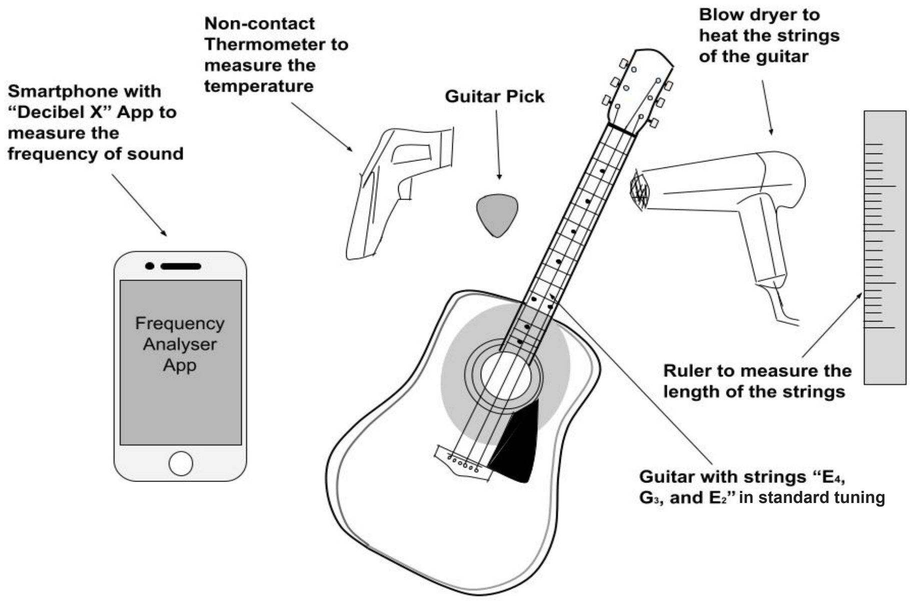


Figure 2: A diagram of materials required for the experiment

Set-up:

Figure 2 represents all the materials and equipment required for the experiment. D'Addario EXL120 Nickel Wound Electric Guitar Strings were used for the experiment. Only three strings were inserted into the Fender CD-60S guitar and their gauges or diameters are, 2.54×10^{-3} m, 4.32×10^{-4} m, and 1.06×10^{-3} m for the strings E_4, G_3, and E_2 respectively. A mobile application called "Decibel X" is used as a frequency analyser tool.[7]

Procedure:

First, using the ruler, the distances between the nut and the saddle, the first fret and the saddle, the second fret and the saddle and so on up to the 6th fret and the saddle were measured (with an uncertainty of $\pm$ 0.0005m). Then the E_4 string was plucked using the guitar pick such that it vibrates from the nut to the saddle and the frequency of the sound wave produced by it was noted down by the mobile app by manually tapping on the spectrum of the wave observed. Then a finger was placed on the first fret of the string such that the vibrating length decreases to the distance between the first fret and the saddle, and the process was repeated. Similarly, a finger was placed on the 2nd, 3rd, 4th and 5th frets of the string and the process is repeated. For each variation of length, a total of 5 frequency readings were taken by repeatedly plucking the string and tapping the frequency spectrum on the app to reduce random errors. The same process was repeated for each string.

After obtaining the value of tension in each string at room temperature (assumed as 295.5 K), the blow-dryer was used to heat each string to the temperatures of 300.5 K, 305.5 K, 310.5 K, 315.5 K, 320.5 K, 325.5 K, and 330.5 K which were measured by the non-contact thermometer. It was extremely difficult to maintain steady temperatures, and a fluctuation of about $\pm$0.5 K was observed which was taken as the uncertainty in the temperature. For each temperature, the strings were plucked without pressing any fret and the frequency of sound produced was noted. 5 repetitions were done to reduce random errors.

Sample of the raw data collected is shown below in *Table 1* and *Table 2*. Refer to **Appendix 1** and **Appendix 2** for the full tables.

Table 1: Sample data for the frequency recorded in different trials while varying the length (L) of the strings at room temperature

String	L (m) ±0.0005m	Frequency (Hz)				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
E_4	0.644	329.12	321.45	327.64	325.82	320.19
	0.611	345.34	349.07	341.53	352.24	342.41
	0.576	366.32	368.54	372.05	364.92	367.04
G_3	0.644	201.21	199.83	195.64	194.02	197.98
	0.611	213.45	204.39	210.75	208.68	201.93
	0.576	229.13	228.13	221.35	232.81	236.59
E_2	0.644	85.36	80.07	81.53	82.88	83.28
	0.611	90.22	87.69	88.25	83.52	86.83
	0.576	92.55	95.47	88.76	94.98	91.51

Table 2: Sample data for frequency recorded at different temperatures (T) while keeping the length of the strings constant

String	T (K) (+- 0.5 K)	Frequency (Hz)				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
E_4	295.5	329.12	321.45	327.64	325.82	320.19
	300.5	324.14	326.84	323.79	322.36	319.92
	305.5	321.06	325.78	320.67	323.28	321.79
G_3	295.5	201.21	199.83	195.64	194.02	197.98
	300.5	194.16	195.31	193.48	196.37	194.83
	305.5	192.27	193.82	191.34	192.99	191.76
E_2	295.5	85.36	80.07	81.53	82.88	83.28
	300.5	78.21	76.67	79.36	77.94	76.13
	305.5	70.89	75.73	73.23	71.04	71.36

7. Determining the tension in three guitar strings at room temperature

Each guitar string has the same stretched length between the nut and the saddle (refer to Figure 1). Then how do the open strings produce sounds of different frequencies? This is because each string has a different tension force acting on it and a different linear mass density. The relationship between the fundamental frequency and the tension force on the string has already been calculated (equation 3.5). This equation can be modified, and empirical data can be used to determine the tension in each of the three strings of the guitar used in the experiment.

7.1 Theoretical relationship

Using the definition of linear mass density, equation 3.5 can be written as:

$$f = \frac{1}{2L}\sqrt{\frac{FL}{m}} \qquad \because \ \mu = \frac{m}{L}$$

$$= \frac{1}{2L}\sqrt{\frac{FL}{\rho V}} \ = \frac{1}{2L}\sqrt{\frac{F}{\rho A}} \ = \frac{1}{2L}\sqrt{\frac{F}{\rho \pi r^2}}$$

$$f = \frac{1}{L}\sqrt{\frac{F}{\rho \pi d^2}} \qquad - 7.1$$

(where m, ρ, V, A, r and d represent the mass (kg), density (kg m^{-3}), volume (m^3), area of cross section (m^2), the radius of the string (m) and the diameter of the string (m) respectively)

7.2Empirical data and analysis

From *Table 1*, we can calculate the variation of frequency with length at room temperature for the three different strings. *Table 3* shows the sample data from these calculations (refer to **Appendix 3** for full table). All calculations are explained with a sample calculation.

Table 3: Sample data of length (L), reciprocal of length (1/L), average frequency (f_{avg}) and their uncertainties (Δ1/L and Δ f)

String	L (m) ±5 × 10^{-4} m	1/L (m^{-1})	Δ1/L (m^{-1})	f_{avg} (Hz)	Δ f (Hz)
E_4	**0.6440**	1.55	1.21×10^{-3}	324.84	4.46
	0.6110	1.64	1.34×10^{-3}	346.11	5.35
	0.5760	1.74	1.51×10^{-3}	367.77	3.56
G_3	**0.6440**	1.55	1.21×10^{-3}	197.74	3.59
	0.6110	1.64	1.34×10^{-3}	207.84	5.76
	0.5760	1.74	1.51×10^{-3}	229.60	7.62
E_2	**0.6440**	1.55	1.21×10^{-3}	83.42	4.64
	0.6110	1.64	1.34×10^{-3}	87.30	3.85
	0.5760	1.74	1.51×10^{-3}	92.85	3.85

Sample Calculation: **(**For string E_4, at L = 0.6440 ± 0.0005 m**)**

Reciprocal of Length

$$\frac{\mathbf{1}}{\boldsymbol{L}} = \frac{1}{0.6440} \approx 1.55 \text{ m}^{-1}$$

Calculating the uncertainty of 1/L

$$\Delta\frac{\mathbf{1}}{\boldsymbol{L}} = \left(\frac{\Delta \mathrm{L}}{L} \times 100\right) \times \frac{1}{L} \times \frac{1}{100} \approx 1.21 \times 10^{-3} \text{ m}^{-1}$$

Calculating average frequency

$$\boldsymbol{f}_{\boldsymbol{avg}} = \frac{\sum_{5}^{i=1} f_i}{5} = \frac{f_1 + f_2 + f_3 + f_4 + f_5}{5}$$

$$= \frac{329.12 + 321.45 + 327.64 + 325.82 + 320.19}{5} \approx 324.84 \text{ Hz}$$

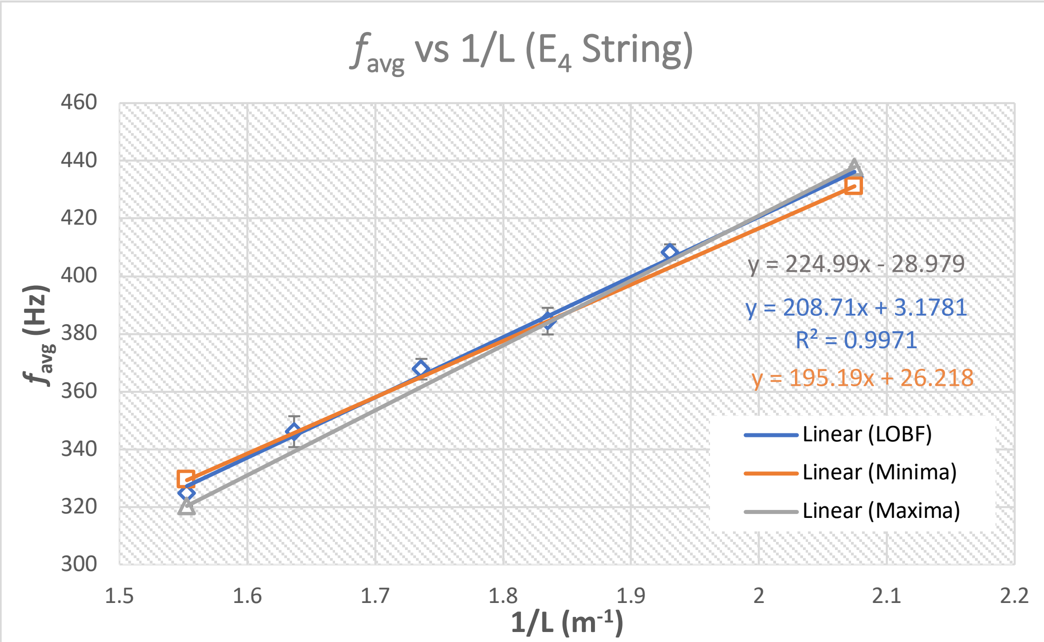
Calculating the uncertainty of frequency

The uncertainty of the device used to determine the frequency (mobile app) was negligible when compared to the human uncertainty, which was calculated as:

$$\Delta\, \boldsymbol{f} \;= \frac{range}{2} = \frac{maximum\ f - minimum\ f}{2} = \frac{329.12 - 320.19}{2} \approx 4.46 \text{ Hz}$$

A graph representing all the data points, their uncertainties, a line of best fit, maxima and minima is plotted for each string and one of these graphs is given below (refer to **Appendix** 4 and **5** for all graphs)

Graph 1: Data points of average frequency against the reciprocal of length of the string E_4



From the gradient of the graph, we can calculate the tension force being experienced by the string at room temperature.

According to equation 7.1,

$$\text{gradient} = \sqrt{\frac{F}{\rho \pi d^2}} = 208.71$$

$$\sqrt{\frac{F}{\rho \pi d^2}} = 208.712$$

For the E_4 nickel string,

$$\rho = 8908 \text{ kg m}^{-3}$$

$$d = 2.54 \times 10^{-4} \text{ m}$$

$$F = \pi(208.71)^2(8908)(0.000254)^2$$

$$\approx 78.65 \text{ N}$$

Since this is the tension at room temperature, it can be denoted as F_0.

$F_0 = 78.65 \text{ N} \pm 11.31 \text{ N}$.

Calculating the uncertainty of tension

From maxima gradient, we have,

$$224.99 = \sqrt{\frac{F_{max}}{\rho \pi d^2}}$$

$$F_{max} = \pi(224.99)^2(8908)(0.000254)^2$$

$$\approx 91.40 \text{ N}$$

From minima gradient, we have,

$$195.19 = \sqrt{\frac{F_{min}}{\rho \pi d^2}}$$

$$F_{min} = \pi(195.19)^2(8908)(0.000254)^2$$

≈ 68.79 N Uncertainty $= \frac{F_{max} - F_{min}}{2} = \frac{91.40 - 68.79}{2} \approx 11.31$ N

Similarly, the tension forces in the G_3 and E_2 strings and their uncertainty can be calculated (refer to **Appendix 4** and **Appendix 5** for the other graphs) and are given below in Table 4.

Table 4: The density (***ρ***) and diameter (d) of the three strings along with the tension (**F_0**) and uncertainty in the tension (**Δ F**) at room temperature

String	ρ (kg m^{-3})	d (m)	F_0 (N)	Δ F (N)
E4	8908	2.54×10^{-4}	78.65	11.31
G3	8908	4.32×10^{-4}	82.62	17.34
E2	8908	1.06×10^{-3}	96.17	35.56

The high uncertainty in the tension is caused by the random errors in the experiment which will be discussed further in the conclusion and evaluation section.

The y-intercept of Graph 1 was supposed to be 0 according to the theoretical relationship between 1/L and f. However, the line of best fit has the y-intercept of 3.18. The y-intercept of the maxima and minima lines can be used to calculate the uncertainty of the y-intercept, (26.22 + 28.98)/2 = 27.6. Hence, the y-intercept is 3.18 ± 27.6. While the uncertainty is pretty high due to the random error, 0 does lie in that range. Similar calculations can be made with the y-intercepts of the other graphs (refer to **Appendix 4** and **5**) for the two other strings and 0 always lies in the range of their uncertainties. Hence, the theoretical model for the relationship between length, tension and frequency has been proven by the empirical data.

8.Determining the temperature dependence of frequency

Now that the relationship between length, tension and frequency has been established, we can look into the thermodynamics of a guitar string to determine how a change in temperature will affect its fundamental frequency. First, a theoretical model has been derived which is then verified empirically using the data obtained from the experiment.

8.1 Theoretical model for the effect of temperature change on a string

The length of a string varies with the tension force as well as a temperature change. If we consider the string to be a thermodynamic system with the variables L, F and T, the variable L is a function of both the tension (F) and the temperature (T). A change in length with a change in both tension and temperature can thus be calculated using partial differentiation.

$$dL = \left(\frac{\partial L}{\partial T}\right)_F dT + \left(\frac{\partial L}{\partial F}\right)_T dF \qquad \text{- 8.1}$$

From equation 5.1 we know,

$$\Delta L = \alpha L \Delta T$$

$$\frac{\Delta L}{\Delta T} = \alpha L$$

This equation represents a change in length with a change in temperature. However, we know that length is not only dependent on temperature, but also on tension. Thus, this equation is only true when tension is considered to be constant and can be rewritten as-

$$\left(\frac{\partial L}{\partial T}\right)_F = \alpha L \qquad \text{- 8.2}$$

From equation 4.1,

$$Y = \frac{F}{A} \times \frac{L}{\Delta L}$$

The tension F is caused by the change in length and the initial tension $F_0 = 0$, when $\Delta L = 0$. Thus change in tension, $\Delta F = F - F_0 = F$. The previous equation can be rewritten as-

$$Y = \frac{\Delta F}{\Delta L} \times \frac{L}{A}$$

$$\frac{\Delta L}{\Delta F} = \frac{L}{A\,Y}$$

$$\left(\frac{\partial L}{\partial F}\right)_T = \frac{L}{A\,Y} \qquad - 8.3$$

Substituting equation 8.2 and 8.3 into equation 8.1,

$$dL = \alpha L dT + \frac{L}{AY} dF$$

$$\frac{dL}{L} = \alpha dT + \frac{1}{AY} dF \qquad - 8.4$$

We can solve equation 8.4 for *dF*, which gives,

$$dF = \frac{AY}{L} dL - AY\alpha dT$$

This equation tells us how the tension in a string varies with a change in length and temperature. However, a guitar string is fixed between two points so, the length of the string remains the same ($dL = 0$), while the tension changes because of a temperature change. After integrating the previous equation, we get,

$$\int_{F_0}^{F} dF = \int_{T_0}^{T} -AY\alpha dT$$

$$F - F_0 = -AY\alpha(T - T_0)$$

$$F = F_0 - AY\alpha\Delta T \qquad - 8.5$$

(where F_0 and F are the initial and final tension (N) and T_0 and T are the initial and final temperature(K))

The ratio of the fundamental frequency of the guitar string after changing its temperature by ΔT and the initial fundamental frequency can thus be calculated using the equation 3.5 and equation 8.5,

$$\frac{f}{f_0} = \frac{\frac{1}{2L}\left(\frac{F_0 - AY\alpha\Delta T}{\mu}\right)^{1/2}}{\frac{1}{2L}\left(\frac{F_0}{\mu}\right)^{1/2}}$$

$$= \left(1 - \frac{AY\alpha\Delta T}{F_0}\right)^{1/2}$$

Squaring both sides,

$$\left(\frac{f}{f_0}\right)^2 = 1 - \frac{AY\alpha\Delta T}{F_0}$$

$$f^2 = {f_0}^2 - \frac{AY\alpha {f_0}^2}{F_0}\Delta T \qquad \text{-8.6}$$

Equation 8.6 depicts the relationship between the square of fundamental frequency and a change in temperature of the string. We will now verify this equation using empirical data.

8.2 Empirical data and analysis

From *Table 2,* we can calculate the change in temperature, the average fundamental frequency of the string vibration, and the square of average frequency for each observation. To reduce uncertainty in the fundamental frequency values, some trials were discarded. *Table 5* shows sample data with all these quantities along with their uncertainties (refer to **Appendix 6** and **Appendix 7** for the full table and the discarded values of frequency respectively). It must be noted that in *Table 5,* ΔT represents a change in temperature while Δf and Δf^2 represent the uncertainty in the fundamental frequency and the uncertainty in the square of fundamental frequency respectively.

Table 5: Sample data of change in temperature (ΔT), average frequency (f_{avg}), square of average frequency (f^2) and the uncertainties of these quantities (Δf, Δf^2)

String	T (K) ($\pm$ 0.5 K)	ΔT (K) ($\pm$1.0 K)	f_{avg} (Hz)	Δf (Hz)	f^2 (Hz^2)	Δf^2 (Hz^2)
E_4	**295.5**	0.0	327.53	1.65	107273.72	1080.84
	300.5	5.0	323.43	0.89	104606.96	575.70
	305.5	10.0	321.70	1.30	103490.89	839.64
G_3	**295.5**	0.0	195.88	1.98	38368.97	775.68
	300.5	5.0	194.44	0.92	37808.86	355.83
	305.5	10.0	192.44	1.24	37031.61	477.24
E_2	**295.5**	0.0	81.94	1.60	6714.16	263.03
	300.5	5.0	77.24	1.04	5965.63	160.65
	305.5	10.0	71.63	1.17	5130.86	167.61

Sample Calculation

For string G_3 at 300.5 K $\pm$0.5 K,

Calculating temperature change

T= 295.5 K is considered as the room temperature (295.5 K$\approx$ 295 K)

Δ**T** = (T- 295.5) K $\pm$ (0.5+0.5) K = 5.0 K $\pm$ 1.0 K

Calculating average frequency

Refer to *Table 3* for sample calculation

Calculating uncertainty in average frequency

Refer to *Table 3* for sample calculation

Calculating the square of average frequency

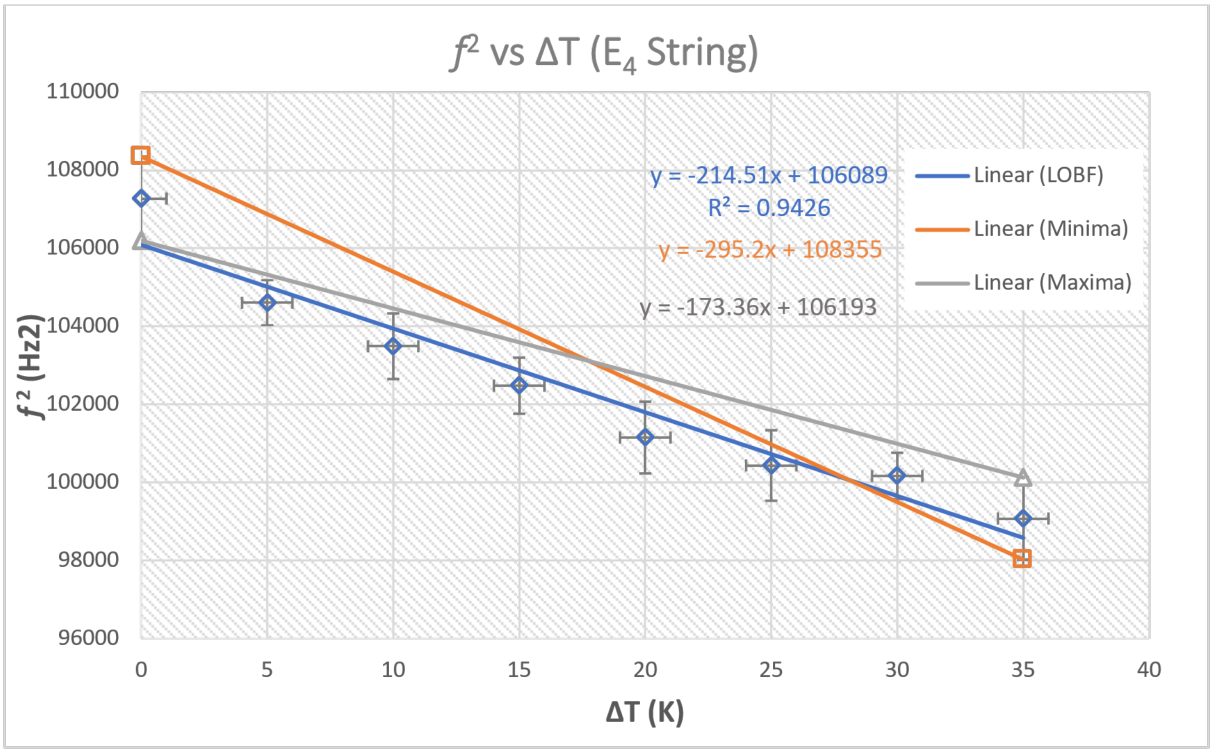
$f^2 = f_{avg} \times f_{avg} = 194.44 \times 194.44 \approx 37808.86$ Hz^2

Calculating the square of average frequency

$$\Delta f^2 = \frac{2\Delta f}{f} \times f^2 = 2 \times 0.92 \times 194.44 = 355.83 Hz^2$$

A graph representing all the data points for square of average frequency and change in temperature, their uncertainties, a line of best fit, maxima and minima is plotted for each string and all these graphs are given below.

Graph 2: Data points of squared average frequency (f^2) against the change in temperature (ΔT) for the string E_4



According to the theoretical model, $f^2 = {f_0}^2 - \frac{AY\alpha {f_0}^2}{F_0}\Delta T$. Therefore, for the graph of f^2 against ΔT, the y- intercept should give us the square of frequency at room temperature (${f_0}^2$) and the gradient should be equal to $(-\frac{AY\alpha {f_0}^2}{F_0})$.

y-intercept from *Graph 2* = 106089 (Line of best fit), and its uncertainty = $\frac{minima\ intercept - maxima\ intercept}{2} = \frac{108355-106193}{2} = 1081$. Thus, intercept = 106089.0 $\pm$ 1081.0 Hz^2.

Gradient from *Graph 2* = -214.51 (Line of best fit), and its uncertainty = $\frac{maxima\ gradient - minima\ gradient}{2} = \frac{-173.36-(-295.2)}{2}$ = 60.92. Thus, gradient = -214.51 $\pm$ 60.92 K^{-1}.

Calculation of Expected gradient from theoretical model

$$gradient = \left(-\frac{AY\alpha {f_0}^2}{F_0}\right)$$

For string E_4,

A= $\frac{\pi d^2}{4} = \frac{\pi\,(2.54 \times 10^{-4})^2}{4} = 5.07 \times 10^{-8}\ \mathrm{m}^2$

Y = 2.05×10^{11} Pa (Young's modulus of nickel)[6]

α = $1.27 \times 10^{-5}\ \mathrm{K}^{-1}$ (coefficient of linear thermal expansion for nickel)[6]

${f_0}^2$ = 107273.72 Hz^2 $\pm$ 1080.84 Hz^2 (from *Table 5*)

F_0 = 78.65 N $\pm$ 11.31 N (from *Table 4*)

theoretical gradient

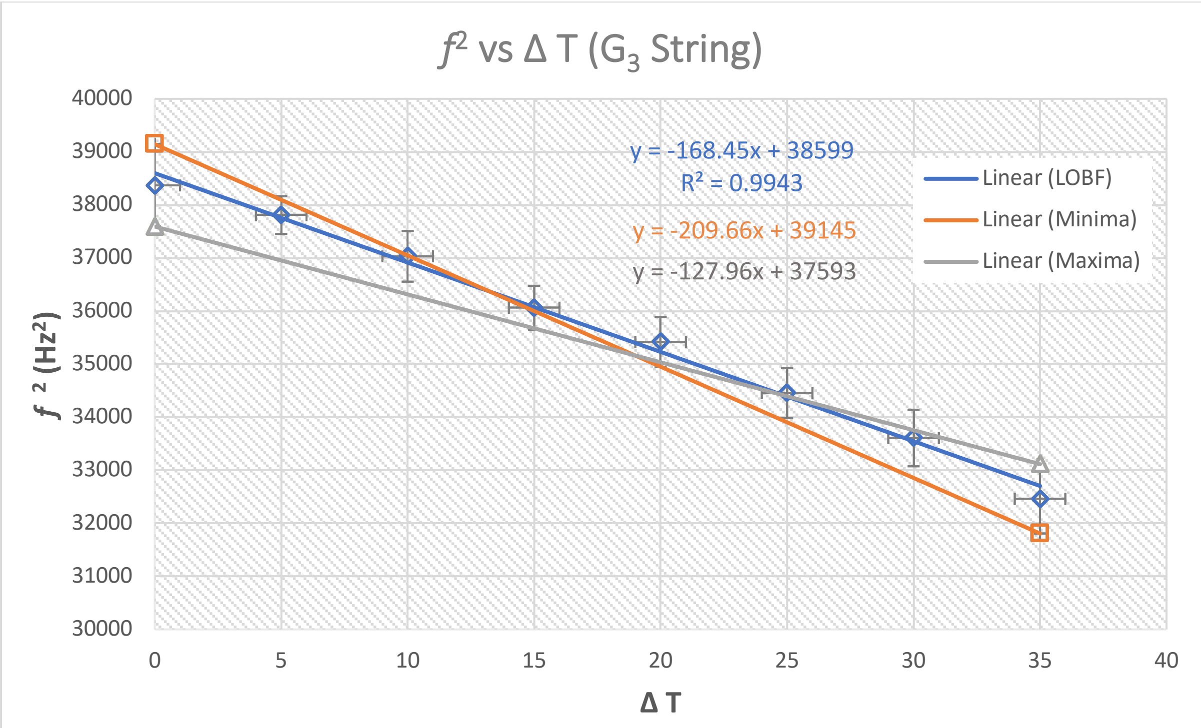
$$= -\frac{5.07 \times 10^{-8} \times 2.05 \times 10^{11} \times 1.27 \times 10^{-5} \times 107273.72}{78.65}$$

$= -$ 180.04 K^{-1}

uncertainty = $\left(\frac{\Delta\, {f_0}^2}{{f_0}^2} + \frac{\Delta F_0}{F_0}\right) \times$ gradient = $\left(\frac{1080.84}{107273.72} + \frac{11.31}{78.65}\right) \times 180.04 = 27.70\ \mathrm{K}^{-1}$

Similarly, the graphs for the string G_3 and E_2 are shown below and *Table 6* shows the comparison of theoretical and empirical values of gradient and y-intercept for each string, which have been calculated in the same way as it is done for the E_4 string.

Graph 3: Data points of squared average frequency (f^2) against the change in temperature (Δ**T**) for the string G_3



Graph 4: Data points of squared average frequency (f^2) against the change in temperature (Δ**T**) for the string G_3

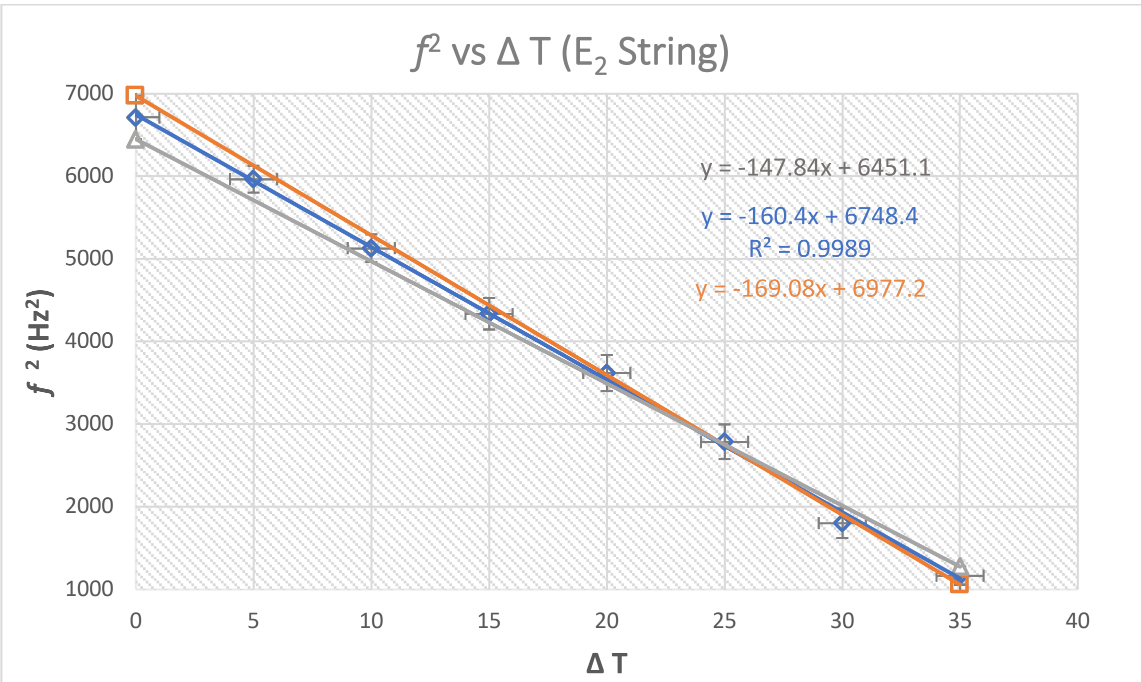


Table 6: Comparison of theoretical value of gradient (m_{theo}) and y-intercept (b_{theo}) with the empirical values (obtained from the Graph 2, 3 and 4), (m_{emp}) and (b_{emp}). The uncertainty of each value is also shown.

String	m_{theo} (K^{-1})	Δm_{theo} (K^{-1})	m_{emp} (K^{-1})	Δ m_{emp} (K^{-1})	b_{theo} (Hz^2)	Δb_{theo} (Hz^2)	b_{emp} (Hz^2)	Δ b_{emp} (Hz^2)
E_4	-180.04	27.70	-214.51	60.92	107273.72	1080.84	106089.00	1081.00
G_3	-177.22	40.78	-168.45	40.85	38368.97	775.68	38599.00	776.00
E_2	-152.25	62.26	-160.40	10.62	6714.16	263.03	6748.40	263.05

From *Table 6,* it is evident that for every string, the y-intercept of the line of best fit always lies in the range of the squared initial frequency. Moreover, even though the theoretical and empirical values of the gradient have high uncertainties (which will be addressed in the evaluation section), their values lie close to each other when their range is considered. The linear relationship predicted by the theoretical model can be clearly seen as the line of best fit is a straight line for every string with a high r^2 value that tells us that the data points lie in close proximity to the line of best fit. Thus, the data from the graphs supports the theoretical model as the square of frequency decreases constantly from the initial squared frequency when the change in temperature is increased.

9. Correction of theoretical relationship

In equation 8.6, it is assumed that the Young's modulus (Y) and the area of cross section (A) remain constant when there is a change in temperature. However, this is not true in the real world. This section will consider the effect of temperature on the Young's modulus and the area of cross section of the string and refine our model to describe the thermodynamics of a guitar string as it is.

A guitar string is essentially a stretched metal wire and all the properties of a metal are correlated to its electron behaviour which is ultimately affected by the electron work function (φ). The Young's modulus of any metal is also dependent on its electron work function according to the equation[4]:

$$Y = \beta\varphi^6 \qquad \text{- 9.1}$$

(where β is a constant of proportionality whose value depends on the crystal structure of the metal and different values for most crystal structures have been determined[5])

Electron work function can be described as the energy that is required to withdraw an electron completely from the surface of the metal. When the temperature of the metal increases, the electrons get thermally excited and thus, lesser energy would be required to withdraw them from the metal surface. This relationship between work function and temperature has been derived as[4]:

$$\varphi(T) = \varphi_0 - \gamma\frac{(k_BT)^2}{\varphi_0} \qquad \text{- 9.2}$$

(where φ_0 is the work function (J) at T=0, kB is the Boltzmann constant ($J{\cdot}K^{-1}$), γ is another material property which is dependent on the crystal structure)

These two equations (equation 9.1 and equation 9.2) allow us to determine the relationship between the Young's modulus and a change in temperature:

$$Y = \beta \left[\varphi - \gamma \frac{(k_B \Delta T)^2}{\varphi} \right]^6 \qquad - 9.3$$

(where φ is the work function of the metal at room temperature i.e., 295 K and ΔT is the change in temperature from the initial room temperature to some final temperature)

The cross-sectional area of the string (A) varies with temperature due to the thermal expansion of the string. The radius of the string expands according to the equation:

$$r = r_0(1 + \alpha\, \Delta T)$$

$$r^2 = {r_0}^2(1 + \alpha\, \Delta T)^2$$

$$A = A_0\, (1 + \alpha\, \Delta T)^2 \qquad - 9.4$$

(where r_0 is the initial radius of the string (m), A and A_0 are the initial and final areas of cross section (m^2))

Substituting equation 9.4 and equation 9.3 into equation 8.6, we get:

$$f^2 = {f_0}^2 - \frac{A_0\ \ \beta\alpha {f_0}^2}{F_0} \Delta T (1 + \alpha\, \Delta T)^2 \left[\varphi - \gamma \frac{(k_B \Delta T)^2}{\varphi} \right]^6 \qquad - 9.5$$

This is a rather complex equation and was thus not used to plot the graph of change in temperature against square of frequency. Refer to **Appendix 8** for the empirical verification of equation 9.5 and deviation in the value of frequency squared obtained by equation 8.6 and the value obtained by equation 9.5.

10. Conclusion and Evaluation

This investigation of the effect of temperature on the fundamental frequency of a guitar string has led to several interesting conclusions.

Firstly, a theoretical investigation revealed the different factors that have an impact on the frequency of a guitar string, i.e., the length, tension force, and the linear mass density. Further exploration of these factors revealed how they depend on several characteristics of the string, like its gauge and the Young's modulus and coefficient of thermal expansion for the material of the string.

Empirical evidence successfully proved this initial theoretical investigation and helped us calculate the tension in each string at room temperature. Building on this investigation, a theoretical model for the temperature dependence of frequency was developed which showed that when the string experiences an increase in temperature, the tension force experienced by it is decreased according to the equation $F=F_0 - AY\alpha\Delta T$, (if the cross-sectional area and the Young's modulus are assumed to be constant) and this decrease in tension is what leads to a decrease in the fundamental frequency of the string.

Experimental analysis of nickel strings successfully proved our theoretical model, as despite of the high uncertainties in the expected and actual values of gradients and y-intercepts of the line of best fit in *Graphs 2*, *3* and *4*, the linear relationship between the change in temperature (ΔT) and the square of frequency (f^2) was evident in the straight-lined graphs.

A more accurate version of the theoretical model was derived in **Section 9** which takes the temperature dependence of the Young's modulus and the area of cross section into account. However, the effects of the temperature dependence of these quantities on the frequency are minimal when the temperature change isn't too high (Refer to **Appendix 8**). Therefore, the initial theoretical model can be used in most cases to calculate the effect of temperature on the frequency of sound produced by not just a guitar string, but any stretched string.

Thus, it can be concluded from this investigation that the frequency of a guitar string decreases with a positive change in the temperature and this decrease in frequency is dependent on the string's diameter and the properties of its material, both of which are also affected by the change in temperature. However, there were many methodological issues and uncertainties in this investigation that must be addressed here.

The major source of uncertainties in this investigation is the random errors in the measurement of frequency. The mobile app "Decibel X" used in the experiment relied on the reaction time of the person handling the app (the person had to manually tap on the smartphone screen each time to reveal the frequency reading at an instant). This led to a high uncertainty in the calculated tension experienced by each string (*Table 4*) and a high uncertainty in the gradient of *Graphs 2*, *3* and *4*. Moreover, the app relied on the microphone of the smartphone to record the frequency which might not be as accurate as a professional microphone. To eliminate this error, a scientific frequency counter or spectrum analyser software can be used which does not rely on reaction time to obtain the reading of frequency.

Secondly, heating the guitar strings by using a hairdryer was not the most efficient method as it did not ensure that the string was uniformly heated. While the fluctuations due to the inefficiency of the equipment were recorded as uncertainties in the temperature, the readings of the temperature in *Table 2* are not an accurate depiction of the temperature of the entire string. This inefficiency in the methodology is one of the reasons that the theoretically predicted gradient for *Graphs 2, 3* and *4* varied greatly from the actual gradient (apart from the inaccuracy in the original theoretical model). A more efficient method to conduct the experiment would be creating an artificial guitar, immersing it in heated water with controlled temperature and measuring the frequency of sound produced by the vibrations of the string using an underwater frequency detecting equipment.

Moreover, there was no way to directly measure the tension force experienced by each string at room temperature which was used in theoretical calculations. We had to use indirect calculations involving length and frequency which caused a high uncertainty in the theoretical prediction of the gradients of *Graphs 2*, *3* and *4*.

The scope of data could have been increased by using strings of different materials like nylon, aluminium and steel which are all popular guitar string materials, but this investigation focussed only on nickel strings. This investigation can be extended by experimentally exploring the equation derived in **Section 9**, by comparing different string materials and the effect of temperature on their frequencies or by exploring other factors like air pressure and friction which might affect the frequency of a guitar string.

Word Count: 3931

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11. Appendix 1: Full table of *Table 1*

String	L (m) ±0.0005m	Frequency (Hz)				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
E_4	**0.644**	329.12	321.45	327.64	325.82	320.19
	0.611	345.34	349.07	341.53	352.24	342.41
	0.576	366.32	368.54	372.05	364.92	367.04
	0.545	387.16	383.21	381.74	389.62	380.38
	0.518	411.26	407.34	405.86	409.72	407.53
	0.482	434.68	432.04	436.82	437.76	431.14
G_3	**0.644**	201.21	199.83	195.64	194.02	197.98
	0.611	213.45	204.39	210.75	208.68	201.93
	0.576	229.13	228.13	221.35	232.81	236.59
	0.545	239.17	245.29	241.47	239.03	242.68
	0.518	253.09	249.73	242.39	241.56	246.14
	0.482	261.58	266.93	264.52	262.31	260.28
E_2	**0.644**	85.36	80.07	81.53	82.88	83.28
	0.611	90.22	87.69	88.25	83.52	86.83
	0.576	92.55	95.47	88.76	94.98	91.51
	0.545	97.82	99.78	100.04	97.18	93.14
	0.518	102.91	105.06	104.26	106.82	100.45
	0.482	110.60	113.87	108.66	114.09	109.95

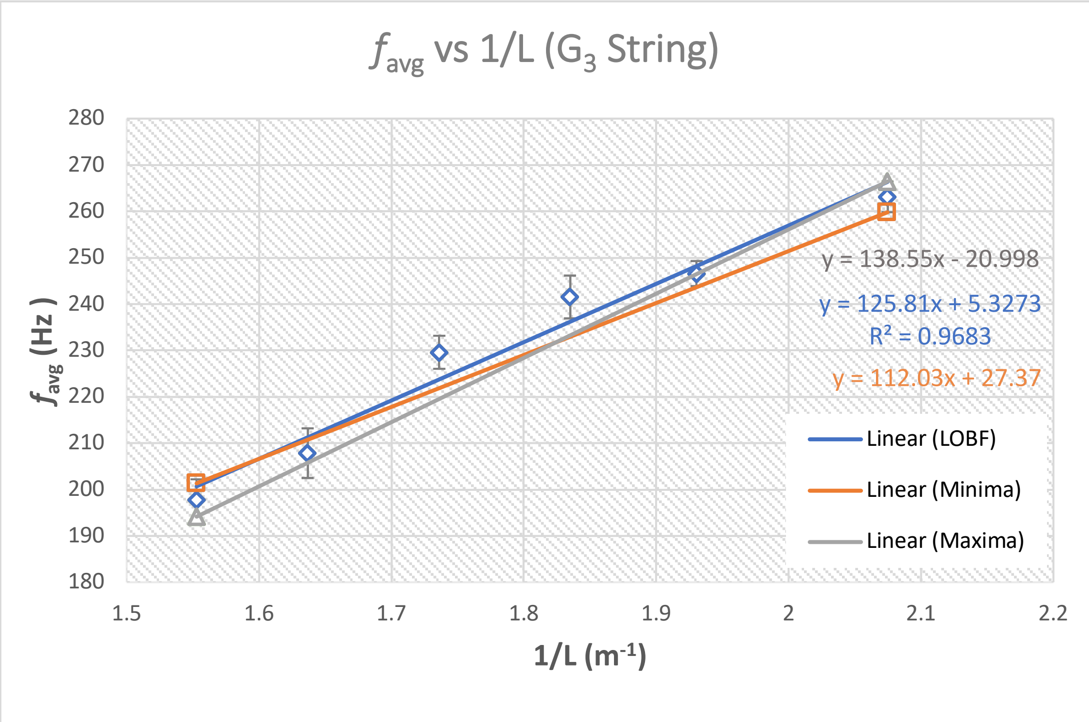
12. Appendix 2: Full table of *Table 2*

String	T (K) (+- 0.5 K)	Frequency (Hz)				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
E_4	**295.5**	329.12	321.45	327.64	325.82	320.19
	300.5	324.14	326.84	323.79	322.36	319.92
	305.5	321.06	325.78	320.67	323.28	321.79
	310.5	322.18	320.41	319.09	321.33	319.67
	315.5	319.31	317.58	321.24	318.86	316.42
	320.5	318.21	316.23	317.91	316.87	315.36
	325.5	315.83	316.12	317.71	316.89	315.91
	330.5	314.62	313.44	315.28	313.68	316.78
G_3	**295.5**	201.21	199.83	195.64	194.02	197.98
	300.5	194.16	195.31	193.48	196.37	194.83
	305.5	192.27	193.82	191.34	192.99	191.76
	310.5	188.84	191.02	190.56	191.71	189.19
	315.5	188.13	189.42	186.93	189.04	187.48
	320.5	184.57	186.24	184.82	185.32	187.11
	325.5	182.79	184.02	181.53	183.84	184.44
	330.5	181.12	178.68	179.16	179.57	182.31
E_2	**295.5**	85.36	80.07	81.53	82.88	83.28
	300.5	78.21	76.67	79.36	77.94	76.13
	305.5	70.89	75.73	73.23	71.04	71.36
	310.5	67.16	64.28	68.46	66.18	65.81
	315.5	62.48	63.06	58.83	59.21	60.17
	320.5	54.73	57.12	53.31	50.78	52.44
	325.5	45.68	42.59	45.09	40.86	41.39
	330.5	35.32	37.63	32.14	34.05	35.19

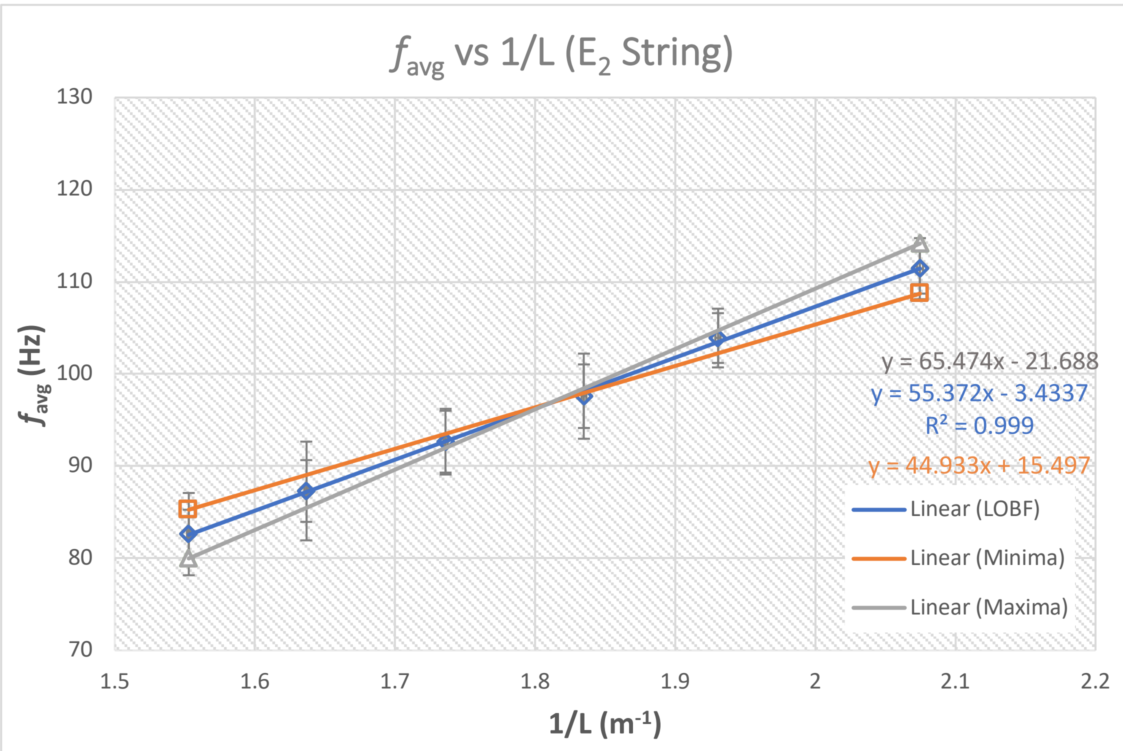
13. Appendix 3: Full Data for Table 3

String	L (m) ±5 × 10^{-4}m	1/L (m^{-1})	Δ1/L (m^{-1})	f_{avg} (Hz)	Δf (Hz)
E_4	**0.6440**	1.55	1.21×10^{-3}	324.84	4.46
	0.6110	1.64	1.34×10^{-3}	346.11	5.35
	0.5760	1.74	1.51×10^{-3}	367.77	3.56
	0.5450	1.83	1.68×10^{-3}	384.42	4.62
	0.5180	1.93	1.86×10^{-3}	408.34	2.70
	0.4820	2.07	2.15×10^{-3}	434.49	3.31
G_3	**0.6440**	1.55	1.21×10^{-3}	197.74	3.59
	0.6110	1.64	1.34×10^{-3}	207.84	5.76
	0.5760	1.74	1.51×10^{-3}	229.60	7.62
	0.5450	1.83	1.68×10^{-3}	241.53	3.13
	0.5180	1.93	1.86×10^{-3}	246.58	5.76
	0.4820	2.07	2.15×10^{-3}	263.12	3.32
E_2	**0.6440**	1.55	1.21×10^{-3}	83.424	4.64
	0.6110	1.64	1.34×10^{-3}	87.302	3.85
	0.5760	1.74	1.51×10^{-3}	92.85	3.85
	0.5450	1.83	1.68×10^{-3}	97.59	3.45
	0.5180	1.93	1.86×10^{-3}	103.90	3.18
	0.4820	2.07	2.15×10^{-3}	111.43	2.71

14. Appendix 4: f_{avg} vs 1/L Graph for G_3 String



15. Appendix 5: f_{avg} vs 1/L Graph for E_2 String



16. Appendix 6: Full Data for Table 5

String	T (K) (± 0.5 K)	ΔT (K) (±1.0 K)	f_{avg} (Hz)	Δf (Hz)	f^2 (Hz^2)	Δf^2 (Hz^2)
E_4	**295.5**	0.0	327.53	1.65	107273.72	1080.84
	300.5	5.0	323.43	0.89	104606.96	575.70
	305.5	10.0	321.70	1.30	103490.89	839.64
	310.5	15.0	320.125	1.12	102480.02	717.08
	315.5	20.0	318.0425	1.445	101151.03	919.14
	320.5	25.0	316.916	1.425	100435.76	903.21
	325.5	30.0	316.492	0.94	100167.19	595.00
	330.5	35.0	314.76	1.67	99073.86	1051.30
G_3	**295.5**	0.0	195.88	1.98	38368.97	775.68
	300.5	5.0	194.44	0.92	37808.86	355.83
	305.5	10.0	192.44	1.24	37031.61	477.24
	310.5	15.0	189.90	1.09	36062.96	413.99
	315.5	20.0	188.20	1.24	35419.24	468.62
	320.5	25.0	185.61	1.27	34451.81	471.45
	325.5	30.0	183.32	1.46	33607.69	533.47
	330.5	35.0	180.17	1.82	32460.51	654.01
E_2	**295.5**	0.0	81.94	1.60	6714.16	263.03
	300.5	5	77.24	1.04	5965.63	160.65
	305.5	10	71.63	1.17	5130.86	167.61
	310.5	15.0	65.86	1.44	4337.21	189.67
	315.5	20.0	60.17	1.82	3620.73	219.63
	320.5	25.0	52.82	1.98	2789.42	208.62
	325.5	30.0	42.48	2.12	1804.76	179.70
	330.5	35.0	34.18	1.59	1167.93	108.68

17. Appendix 7: Discarded values of frequency

String	T (K) (+- 0.5 K)	Frequency (Hz)				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
E$_4$	295.5	329.12	~~321.45~~	327.64	325.82	~~320.19~~
	300.5	324.14	~~326.84~~	323.79	322.36	~~319.92~~
	305.5	321.06	~~325.78~~	320.67	323.28	321.79
	310.5	~~322.18~~	320.41	319.09	321.33	319.67
	315.5	319.31	317.58	~~321.24~~	318.86	316.42
	320.5	318.21	316.23	317.91	316.87	315.36
	325.5	315.83	316.12	317.71	316.89	315.91
	330.5	314.62	313.44	315.28	313.68	316.78
G$_3$	295.5	~~201.21~~	~~199.83~~	195.64	194.02	197.98
	300.5	194.16	195.31	193.48	~~196.37~~	194.83
	305.5	192.27	193.82	191.34	192.99	191.76
	310.5	188.84	191.02	190.56	~~191.71~~	189.19
	315.5	188.13	189.42	186.93	189.04	187.48
	320.5	184.57	186.24	184.82	185.32	187.11
	325.5	182.79	184.02	181.53	183.84	184.44
	330.5	181.12	178.68	179.16	179.57	182.31
E$_2$	295.5	~~85.36~~	80.07	81.53	82.88	83.28
	300.5	78.21	76.67	~~79.36~~	77.94	76.13
	305.5	70.89	~~75.73~~	73.23	71.04	71.36
	310.5	67.16	64.28	~~68.46~~	66.18	65.81
	315.5	62.48	~~63.06~~	58.83	59.21	60.17
	320.5	54.73	~~57.12~~	53.31	50.78	52.44
	325.5	~~45.68~~	42.59	45.09	40.86	41.39
	330.5	35.32	~~37.63~~	32.14	34.05	35.19

The readings that have been stroked through are the ones that were discarded to reduce uncertainty.

Appendix 8: Empirical verification of Equation 9.5 and Comparison with Equation 8.6

In order to verify equation 9.5, a sample reading was chosen from Table 5 and put into the equation.

At T = 305.5 K for the E_4 string, A_0= 5.07 × 10^{-8} m^2 φ = 5.01 eV

α = 1.27 × 10^{-5} K^{-1} γ = 318

ΔT= 10.0 K F_0 = 78.65 N

$f_0{}^2$ = 107273.72 Hz β = 1.12 × 10^7 Pa eV^{-6}

According to equation 9.5,

$$f^2 = f_0{}^2 - \frac{A_0\beta\alpha f_0{}^2}{\mathrm{F}_0}\,\Delta T(1+\alpha\Delta T)^2\left[\varphi - \gamma\frac{(\mathrm{k_B}\Delta T)^2}{\varphi}\right]^6$$

Using the values of variables for the string,

$$\frac{A_0\beta\alpha f_0{}^2}{\mathrm{F}_0} = \frac{5.07\times 10^{-8}\times 1.12\times 10^{7}\times 1.27\times 10^{-5}\times 107273.72}{78.65} = 9.83\times 10^{-3}$$

$$\Delta T(1+\alpha\Delta T)^2 = 10(1+1.27\times 10^{-5}\times 10)^{\,2} = 10.002$$

$$\left[\varphi - \gamma\frac{(\mathrm{k_B}\Delta T)^2}{\varphi}\right]^6 = \left[5.01 - 318\,\frac{(1.38\times 10^{-23}\times 10)^2}{5.01}\right]^6 = 15813.44$$

Therefore, theoretically,

$f^2 = 107273.72 - (9.83\times 10^{-3})(10.002)(15813.44) = 105718.95$ Hz^2

$f = 325.14$ Hz

Empirical f = 321.70 Hz

Percent error = 1.06 %

Using equation 8.6,

$f^2 = f_0{}^2 - \frac{A_0 Y\alpha f_0{}^2}{\mathrm{F}_0}\,\Delta T = 105473.32$ Hz^2,

$f = 324.76$ Hz

Comparing with f obtained from equation 9.5,

Deviation= 0.38 Hz

This is a very minor deviation, hence the original theoretical model (equation 8.6) is applicable for most temperatures (that aren't much higher than the room temperature).