**Optimising The Surface Area Of A Package To Reduce It's Cost of Packaging**

Fish has been part of my family's staple diet and over the years I have grown to love tuna fish in particular. In my city there is a scarcity of fresh and good quality tuna, so residents often have to rely on it's supply through imported canned tuna. My family consumes about 1.5 kilograms of tuna a month which is equivalent to 13 cans (approximately) according to our favourite brand, John West[1].


Figure 1: Picture of John West canned Tuna

One day while my mother was preparing a tuna dish for my family, I noticed that she opened the can, took the fish out and simply threw the can into the dustbin. This made me realise, that tin packaging is mainly used to protect a product which has no use once opened.

Tin food has been popular in western countries for many years but the food cans market in my country has seen a significant growth in the last 5 to 10 years. Being a business management student, I realised in order for a business to survive in todays highly

[1] "Homepage – John West UK." https://www.john-west.co.uk/. Accessed 11 Apr. 2021.

competitive market, it is important for them to minimise the cost of raw material of packaging. This is done to improve profitability and attain a competitive advantage.

I soon started to research about the different shapes of cans and noticed myself getting interested and involved into this topic. Detailed research lead to the formation of the research question, “How can businesses reduce the cost of packaging by optimising the surface area of a package?”. I will answer this question by considering two different shapes of a cans to find out their surface area and volume. I will then use the volume of the can in figure 1 to consider the surface area and other parameters as variables. After that, I will construct an equation that will indicate the relationship between the parameter variables and the surface area. Lastly, I will apply the optimisation process to find out the optimum/least surface area to determine which shape will have the least cost of packaging.

**Process of calculation:**

The two shapes of tin cans that I will be considering are, cylindrical shaped and elliptical cylindrical shaped as they are the most popular can shapes seen in supermarkets. The volume is considered to be 200 $cm^3$ (volume of can in Figure 1) which will be fixed during the whole investigation.

**Case 1: Cylindrical Shaped can**

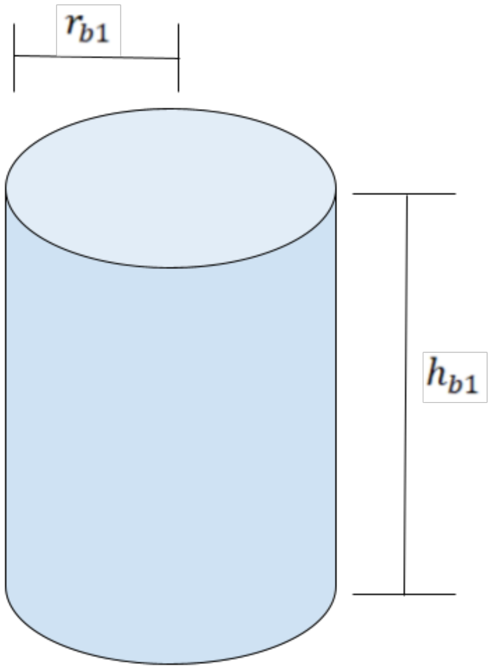


Figure 2: Cylindrical Shaped Can

Since I am considering the can to be cylindrical,

The can's radius = $r_{b1}$

The can's height = $h_{b1}$

Hence, the volume[2] = $V_{b1} = \pi r_{b1}^2 h_{b1}$

Or, $h_{b1} = \dfrac{V_{b1}}{\pi r_{b1}^2}$

As stated previously, the fixed volume of the can = 200 $cm^3$

So, $h_{b1} = \dfrac{200}{\pi r_{b1}^2}$

[2] "Volume and Surface Area of Combined Solids - Solved ... - BrainKart." 6 Jun. 2019, https://www.brainkart.com/article/Volume-and-Surface-Area-of-Combined-Solids_39428/. Accessed 11 Apr. 2021.

And surface area [3] = $S_{b1} = 2\pi r_{b1}(r_{b1} + h_{b1})$

I will now insert the value the of height into the equation of the surface area. After replacing the value, I get:

$$S_{b1} = 2\pi r_{b1}(r_{b1} + \frac{V_{b1}}{\pi r_{b1}^2})$$

$$S_{b1} = 2\pi r_{b1}{}^2 + \frac{400}{r_{b1}}$$

Now I will try to obtain the dimension where the surface area will be minimum for the fixed amount of volume. For that, I have to perform optimisation to obtain the least surface area possible.

By performing the 1st order-derivative, I get:

$$\frac{dS_{b1}}{dr_{b1}} = 4\pi r_{b1} - \frac{400}{r_{b1}{}^2}$$

Now to get the values of the radius, I have to compare the equation with zero.

$$\frac{dS_{b1}}{dr_{b1}} = 4\pi r_{b1} - \frac{400}{r_{b1}{}^2} = 0$$

$$4\pi r_{b1} - \frac{400}{r_{b1}{}^2} = 0$$

$$4\pi r_{b1} = \frac{400}{r_{b1}{}^2}$$

[3] "Volume & Surface Area of Cylinder Calculator - Ncalculators." https://ncalculators.com/geometry/cylinder-calculator.htm. Accessed 11 Apr. 2021.

$r_{b1}{}^3 = 31.831$

$r_{b1} = 3.169\ cm$

Now I need to find out if the surface of radius 3.169 cm will be maximum or minimum. Therefore, the result after performing the 2nd order derivative is:

$$\frac{d^2S_{b1}}{dr_{b1}{}^2} = 4\pi + \frac{800}{r_{b1}{}^3}$$

Now I will apply the value of $r_{b1}$ in the result and observe if the value will be greater than or less than 0. So,

$$\frac{d^2S_{b1}}{dr_{b1}{}^2} = 4\pi + \frac{800}{(3.169)^3}$$

$$\frac{d^2S_{b1}}{dr_{b1}{}^2} = 4\pi + \frac{800}{31.831}$$

Here I can see that the value of, $\frac{d^2S_{b1}}{dr_{b1}{}^2}$ is greater than zero for $r_{b1}$.

$$\frac{d^2S_{b1}}{dr_{b1}{}^2} = 37.699 > 0$$

So, the surface area will be minimum for the value of $r_{b1}$. Therefore, to get the optimum, I have to replace the value of $r_{b1}$ into the equation of surface area.

The surface area = $S_{b1} = 2\pi(3.169)^2 + \frac{400}{3.169}$

$S_{b1} = 189.322\ square\ cm$

And the height will be = $h_{b1} = \dfrac{200}{\pi(3.169)^2}$

$$h_{b1} = 6.339\ cm$$

Now I will insert some values into the equation of the surface area to check if my result is correct.

Table 1:

| Radius (cm) | Surface area ($cm^2$) |
|---|---|
| 1 | 406.283 |
| 2 | 225.133 |
| 3 | 189.882 |
| 3.169 | 189.322 |
| 3.5 | 191.255 |
| 4 | 200.531 |
| 5 | 237.079 |
| 6 | 292.861 |
| 7 | 365.019 |
| 8 | 452.124 |

By observing the values I can say that for radius = 3.169 cm, the surface area will be optimum/the least.

This paper helped me become a better reflector. I started this paper by simply calculating the optimised value of the can. I realised that this has limitations as my calculations might turn out to be incorrect. This resulted in verifying my result so my overall conclusion is as accurate and objective as possible.

Cans are mainly made of tin. Considering that the price of a tin sheet is = x rupee per square cm, the cost of a the can is:

= $189.322 \times x = 189.322x$ rupee per square cm

**Case 2: Elliptical Cylindrical Shaped Can**

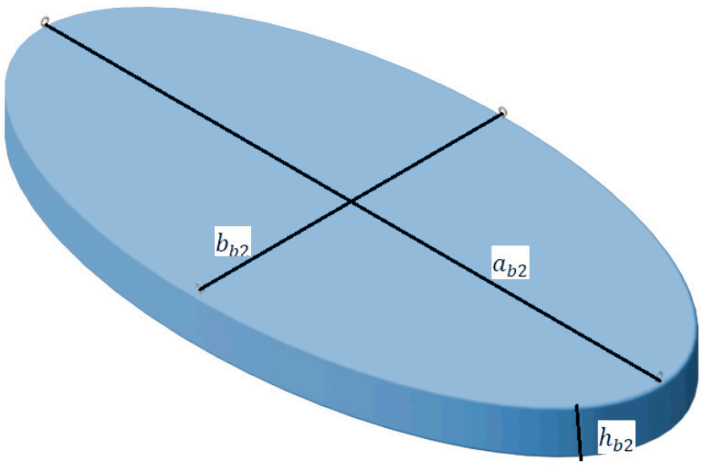


Figure 3: Elliptical Cylindrical Shaped Can

Since I am considering the can to be elliptical cylindrical, therefore,

The can has a semi-major axis = $2a_{b2}$

The can has a semi-minor axis = $2b_{b2}$

The can's has a height = $h_{b2}$

I assume that, $a_{b2} = 2b_{b2}$ (This assumption limits the type of elliptical cylindrical shaped cans explored as there are cans that are $a_{b2} = 3b_{b2}$ etcetera. But since $a_{b2} = 2b_{b2}$ is the most commonly seen elliptical cylindrical shaped in the market, it has been selected)

Hence, the can's volume [4] = $V_{b2} = \pi a_{b2} b_{b2} h_{b2}$

By replacing the value of $a_{b2}$, I get,

$$V_{b2} = 2\pi b_{b2}^2 h_{b2}$$

$$h_{b2} = \frac{V_{b2}}{2\pi b_{b2}^2}$$

As stated previously, the fixed volume of the can = 200 $cm^3$

So, $h_{b2} = \dfrac{200}{2\pi b_{b2}^2}$

$$h_{b2} = \frac{100}{\pi b_{b2}^2}$$

And surface area[5] = $S_{b2} = 2\pi a_{b2} b_{b2} + 2\pi h_{b2}\sqrt{\dfrac{{a_{b2}}^2 + {b_{b2}}^2}{2}}$

I assumed that, $a_{b2} = 2b_{b2}$

Hence, the surface area = $S_{b2} = 2\pi 2b_{b2} b_{b2} + 2\pi h_{b2}\sqrt{\dfrac{(2b_{b2})^2 + {b_{b2}}^2}{2}}$

$$S_{b2} = 4\pi {b_{b2}}^2 + 3.162\pi b_{b2} h_{b2}$$

I will now place the value of the height into the equation of the surface area. After replacing the value, I get:

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[4] "Ellipse Formula | Area, Perimeter & Volume of an Ellipse - Byjus." https://byjus.com/ellipse-formula/. Accessed 11 Apr. 2021.

[5] Hase, Michael. *Mathematics Analysis and approaches HL*

$$S_{b2} = 4\pi {b_{b2}}^2 + 3.162\pi b_{b2}(\frac{100}{\pi b_{b2}^2})$$

By opening the brackets and expanding we get:

$$S_{b2} = 4\pi {b_{b2}}^2 + \frac{316.2}{b_{b2}}$$

Now I will try to obtain the dimension for which the surface area will be minimum for the fixed volume. For that, I have to perform optimisation to obtain the least surface area possible.

Performing the 1st order-derivative, I get,

$$\frac{dS_{b2}}{db_{b2}} = 8\pi b_{b2} - \frac{316.2}{{b_{b2}}^2}$$

Now to calculate values of the radius, I have to compare the above equation with zero.

$$\frac{dS_{b2}}{db_{b2}} = 8\pi b_{b2} - \frac{316.2}{{b_{b2}}^2} = 0$$

$$\frac{dS_{b2}}{db_{b2}} = 8\pi b_{b2} - \frac{316.2}{{b_{b2}}^2} = 0$$

$$8\pi b_{b2} = \frac{316.2}{{b_{b2}}^2}$$

$${b_{b2}}^3 = 12.581$$

$$b_{b2} = 2.326\ cm$$

Now I need to find out if the surface of radius 2.326 cm will be maximum or minimum. Therefore, the result after performing the 2nd order derivative is:

$$\frac{d^2S_{b1}}{dr_{b1}^2} = 8\pi + \frac{632.4}{b_{b2}^{\ 3}}$$

Now I will apply the value of $b_{b2}$ in the above equation and observe if the value will be greater than or less than 0. So,

$$\frac{d^2S_{b2}}{db_{b2}^{\ 2}} = 8\pi + \frac{632.4}{(2.326)^3}$$

Or, $\frac{d^2S_{b2}}{db_{b2}^{\ 2}} = 8\pi + \frac{632.4}{12.581}$

Or, $\frac{d^2S_{b2}}{db_{b2}^{\ 2}} = 75.399 > 0$

Here I can see that the value of, $\frac{d^2S_{b2}}{db_{b2}^{\ 2}}$ is greater than zero for $b_{b2}$. So, I can say, for the value of $b_{b2}$, the surface area will be minimum.

Length of the sub minor axis = $2b_{b2} = 4.652\ cm$

Length of the sub-major axis = $4b_{b2} = 9.304\ cm$

To get the optimum I have to replace the value of $b_{b2}$ in the equation of surface area.

Therefore, the surface area = $S_{b2} = S_{b1} = 4\pi(2.326)^2 + \frac{316.2}{2.326}$

$S_{b2} = 203.929\ square\ cm$

And the height will be = $h_{b2} = \dfrac{100}{\pi(2.326)^2}$

$h_{b2} = 5.883\ cm$

Now I will put some values into the equation of the surface area to check if my result is correct.

Table 2:

| Radius (cm) | Surface area ($cm^2$) |
|---|---|
| 1 | 328.966 |
| 2 | 208.465 |
| 2.326 | 204.015 |
| 2.5 | 205.099 |
| 3 | 218.564 |
| 4 | 280.161 |
| 5 | 377.439 |
| 6 | 505.123 |
| 7 | 660.952 |
| 8 | 843.797 |

By observing the values I can say that for radius 2.324 cm, the surface area will be optimum/the least.

The cans are mainly made of tin. So considering that the price of tin sheet is = x rupee per centimetre square. Therefore, the cost of the packaging is

$=204.015 \times x = 204.015x$ rupee per square cm

**Conclusion:**

The raw material cost for John West would be $14.693x$ rupee square cm more if they were to produce an elliptical cylindrical shaped can. Therefore by comparing the two results I can say that John West is using the the best possible method of packaging which is a cylindrical shaped can. Therefore, it is recommended that every business in the food cans market should use a cylindrical shaped can compared to an elliptical cylindrical shaped can to reduce the cost of packaging.

The limitation of this investigation is that not every shape of can was considered. There are various other shapes of cans like figure 3 and 4 which may have a lower raw material cost. This is subject to further research.

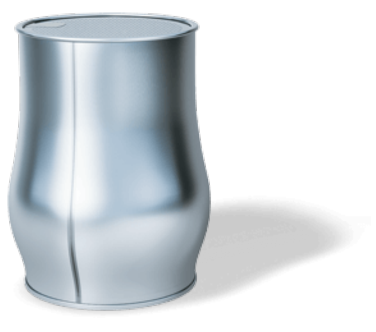

Figure 4[6]

Figure 5[7]

6 "1.5.5 - Assembly Lines for Shaped Cans | Soudronic." https://www.soudronic.com/products/industrial-solutions/assembly-lines-shaped-cans. Accessed 18 Apr. 2021.

7 "2-litre 10-litre tin can, square tin can, engine oil can ... - Global Sources." https://www.globalsources.com/si/AS/Suzhou-Innovation/6008848578772/pdtl/Tin-can/1163956088.htm. Accessed 11 Apr. 2021.

Another limitation of this investigation is that reducing the cost of packaging material is not always the key, some businesses deliberately use more packaging to create an image to the consumers that it contains more of their product. Though unethical, tricking consumers into this may lead to higher levels of profit.

When I was first introduced to optimisation, I did not know its practical application. Being able to apply concepts learnt theoretically was extremely interesting to me and opened new doors of exploration. This investigation turned me into a more knowledgeable student. Not only did I learn about the math behind packaging for a business, but I was also intrigued to learn more about its effects in other areas of knowledge like the environment. I live in one of the world's most polluted countries where reducing waste is not only important but critical. If one minimalises the amount of packaging used, it will lead to less use of natural resources which is better for the sustainability for our future generation. Overall, using different areas of mathematics together made me realise how it can be used to solve real life problems, broadening my perspective towards mathematics.

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- Hase, Michael. *Mathematics Analysis and approaches HL*

- "Homepage – John West UK." https://www.john-west.co.uk/. Accessed 11 Apr. 2021.

## Appendix

**Attempting to calculate the actual prices of packaging.**

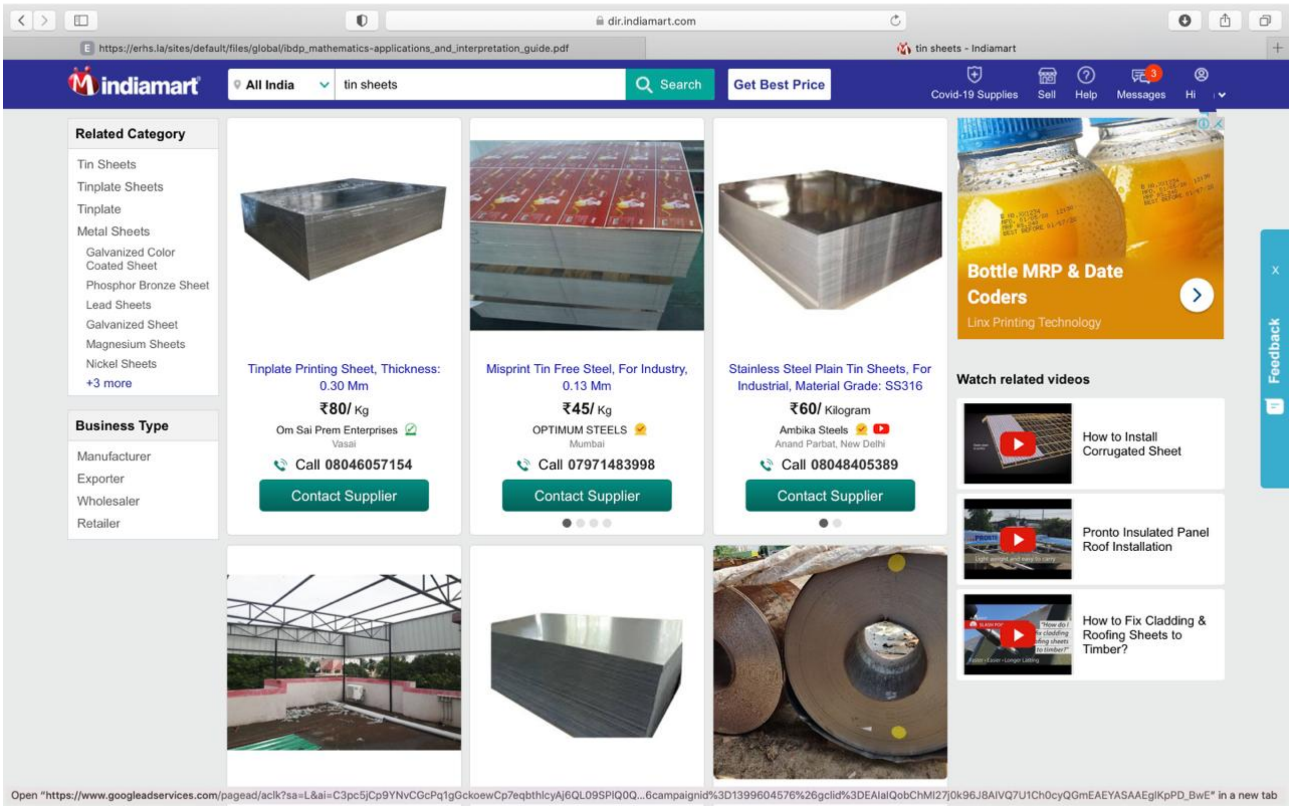
Since I got the following two equations:

$189.322 \times x = 189.322x$ rupee per square cm

=$204.015 \times x = 204.015x$ rupee per square cm

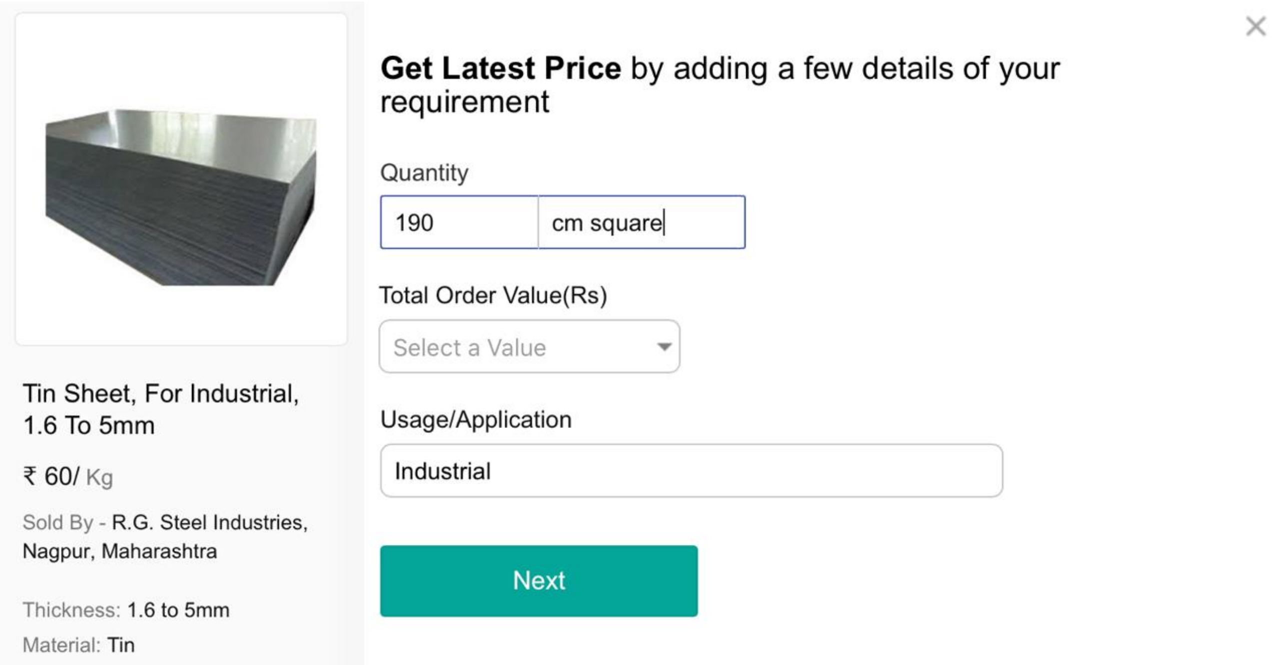
For my knowledge, I thought it would be interesting to find an approximate of the actual price of packaging. So I decided to do some research on IndiaMART[8] which is one of India's most popular e-commerce company that provides B2C, B2B and C2C sales services via its web portal.

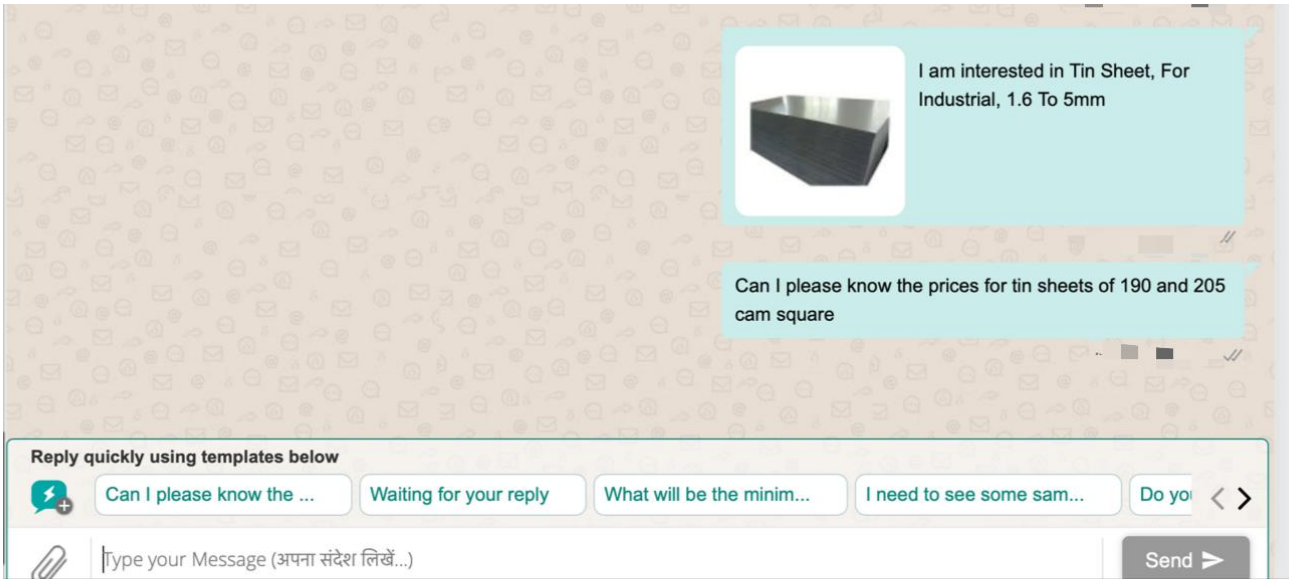
After some research I realised that sellers sold tin sheets by weight (kg):




[8] My IndiaMART." https://my.indiamart.com/. Accessed 11 Apr. 2021.

So I decided to raise a query on what their prices are is in terms of cm square:




But for several days I did not get a response back.

Even if I did, this method would have a lot of limitation as this would be a vague approximation. Firstly, big companies like Johan West would buy tin in a lot of bulk therefore the price per cm square would be a lot cheaper. Secondly, I am looking at prices for tin in India, John West in based in the UK therefore tin prices would be different there. Lastly, knowing the price of just one of their raw materials (tin) is not enough. Packaging has more components such as labelling etcetera that add to the cost of packaging.