# DP mathematics: applications and interpretation

*Questionbank – 5 HL Paper 3*

*example questions and markschemes*

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# IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the organization works with schools, governments and international organizations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate and lifelong learners who understand that other people, with their differences, can also be right.


# IB learner profile

**The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.**

## As IB learners we strive to be:

### INQUIRERS
We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.

### KNOWLEDGEABLE
We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

### THINKERS
We use critical and creative thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions.

### COMMUNICATORS
We express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

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We act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

### OPEN-MINDED
We critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.

### CARING
We show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.

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We approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change.

### BALANCED
We understand the importance of balancing different aspects of our lives—intellectual, physical, and emotional—to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live.

### REFLECTIVE
We thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.

**The IB learner profile represents 10 attributes valued by IB World Schools. We believe these attributes, and others like them, can help individuals and groups become responsible members of local, national and global communities.**

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## Mathematics: applications and interpretation - HL paper 3 questions [134 marks]

**1a.** *[7 marks]*

This question will investigate the solution to a coupled system of differential equations when there is only one eigenvalue.

It is desired to solve the coupled system of differential equations

$\dot{x} = 3x + y$

$\dot{y} = -x + y$

Show that the matrix $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ has (sadly) only one eigenvalue. Find this eigenvalue and an associated eigenvector.

**1b.** *[5 marks]*

Hence, verify that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ is a solution to the above system.

**1c.** *[5 marks]*

Verify that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t+1 \end{pmatrix} e^{2t}$ is also a solution.

**1d.** *[3 marks]*

The general solution to the coupled system of differential equations is hence given by

$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + B \begin{pmatrix} t \\ -t+1 \end{pmatrix} e^{2t}$

If initially at $t = 0,\ x = 20,\ y = 10$ find the particular solution.

**1e.** *[2 marks]*

Find the values of $x$ and $y$ when $t = 2$.

**1f.** *[3 marks]*

As $t \to \infty$ the trajectory approaches an asymptote.

Find the equation of this asymptote.

**1g.** *[1 mark]*

State the direction of the trajectory, including the quadrant it is in as it approaches this asymptote.

**2a.** *[2 marks]*

This question will investigate the solution to a coupled system of differential equations and how to transform it to a system that can be solved by the eigenvector method.

It is desired to solve the coupled system of differential equations

$$\dot{x} = x + 2y - 50$$
$$\dot{y} = 2x + y - 40$$

where $x$ and $y$ represent the population of two types of symbiotic coral and $t$ is time measured in decades.

Find the equilibrium point for this system.

**2b.** *[3 marks]*

If initially $x = 100$ and $y = 50$ use Euler's method with an time increment of 0.1 to find an approximation for the values of $x$ and $y$ when $t = 1$.

**2c.** *[2 marks]*

Extend this method to conjecture the limit of the ratio $\frac{y}{x}$ as $t \to \infty$.

**2d.** *[3 marks]*

Show how using the substitution $X = x - 10,\ Y = y - 20$ transforms the system of differential equations into
$$\dot{X} = X + 2Y$$
$$\dot{Y} = 2X + Y$$

**2e.** *[8 marks]*

Solve this system of equations by the eigenvalue method and hence find the general solution for $\begin{pmatrix} x \\ y \end{pmatrix}$ of the original system.

**2f.** *[2 marks]*

Find the particular solution to the original system, given the initial conditions of part (b).

**2g.** *[2 marks]*

Hence find the exact values of $x$ and $y$ when $t = 1$, giving the answers to 4 significant figures.

**2h.** *[2 marks]*

Use part (f) to find limit of the ratio $\frac{y}{x}$ as $t \to \infty$.

**2i.** *[1 mark]*

With the initial conditions as given in part (b) state if the equilibrium point is stable or unstable.

**2j.** *[2 marks]*

If instead the initial conditions were given as $x = 20$ and $y = 10$, find the particular solution for $\begin{pmatrix} x \\ y \end{pmatrix}$ of the original system, in this case.

**2k.** *[2 marks]*

With the initial conditions as given in part (j), determine if the equilibrium point is stable or unstable.

**3a.** *[3 marks]*

This question will connect Markov chains and directed graphs.

Abi is playing a game that involves a fair coin with heads on one side and tails on the other, together with two tokens, one with a fish's head on it and one with a fish's tail on it. She starts off with no tokens and wishes to win them both. On each turn she tosses the coin, if she gets a head she can claim the fish's head token, provided that she does not have it already and if she gets a tail she can claim the fish's tail token, provided she does not have it already. There are 4 states to describe the tokens in her possession; A: no tokens, B: only a fish's head token, C: only a fish's tail token, D: both tokens. So for example if she is in state B and tosses a tail she moves to state D, whereas if she tosses a head she remains in state B.

Draw a transition state diagram for this Markov chain problem.

**3b.** *[1 mark]*

Explain why for any transition state diagram the sum of the out degrees of the directed edges from a vertex (state) must add up to +1.

**3c.** *[3 marks]*

Write down the transition matrix **M**, for this Markov chain problem.

**3d.** *[4 marks]*

Find the steady state probability vector for this Markov chain problem.

**3e.** *[1 mark]*

Explain which part of the transition state diagram confirms this.

**3f.** *[2 marks]*

Explain why having a steady state probability vector means that the matrix **M** must have an eigenvalue of $\lambda = 1$.

**3g.** *[4 marks]*

After $n$ throws the probability vector, for the 4 states, is given by $\mathbf{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}$ where the numbers represent the probability of being in that particular state, e.g. $b_n$ is the probability of being in state B after $n$ throws. Initially $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Find $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ .

**3h.** *[2 marks]*

Hence, deduce the form of $\mathbf{v}_n$.

**3i.** *[2 marks]*

Explain how your answer to part (f) fits with your answer to part (c).

**3j.** *[4 marks]*

Find the minimum number of tosses of the coin that Abi will have to make to be at least 95% certain of having finished the game by reaching state C.

**4a.** *[8 marks]*

This question will diagonalize a matrix and apply this to the transformation of a curve.

Let the matrix $M = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$.

Find the eigenvalues for $M$. For each eigenvalue find the set of associated eigenvectors.

**4b.** *[2 marks]*

Show that the matrix equation $(x \quad y)\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = (6)$ is equivalent to the Cartesian equation $\frac{5}{2}x^2 + xy + \frac{5}{2}y^2 = 6$.

**4c.** *[2 marks]*

Show that $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are unit eigenvectors and that they correspond to different eigenvalues.

**4d.** *[1 mark]*

Hence, show that $M\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

**4e.** *[2 marks]*

Let $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{R}^{-1}$.

Find matrix **R.**

**4f.** *[1 mark]*

Show that $\mathbf{M} = \mathbf{R}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{R}$.

**4g.** *[3 marks]*

Let $\mathbf{R}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$.

Verify that $(X \quad Y) = (x \quad y)\mathbf{R}^{-1}$.

**4h.** *[2 marks]*

Hence, find the Cartesian equation satisfied by $X$ and $Y$.

**4i.** *[2 marks]*

Let $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

Find the Cartesian equation satisfied by $u$ and $v$ and state the geometric shape that this curve represents.

**4j.** *[2 marks]*

State geometrically what transformation the matrix **R** represents.

**4k.** *[2 marks]*

Hence state the geometrical shape represented by

the curve in $X$ and $Y$ in part (e) (ii), giving a reason.

**4l.** *[1 mark]*

the curve in $x$ and $y$ in part (b).

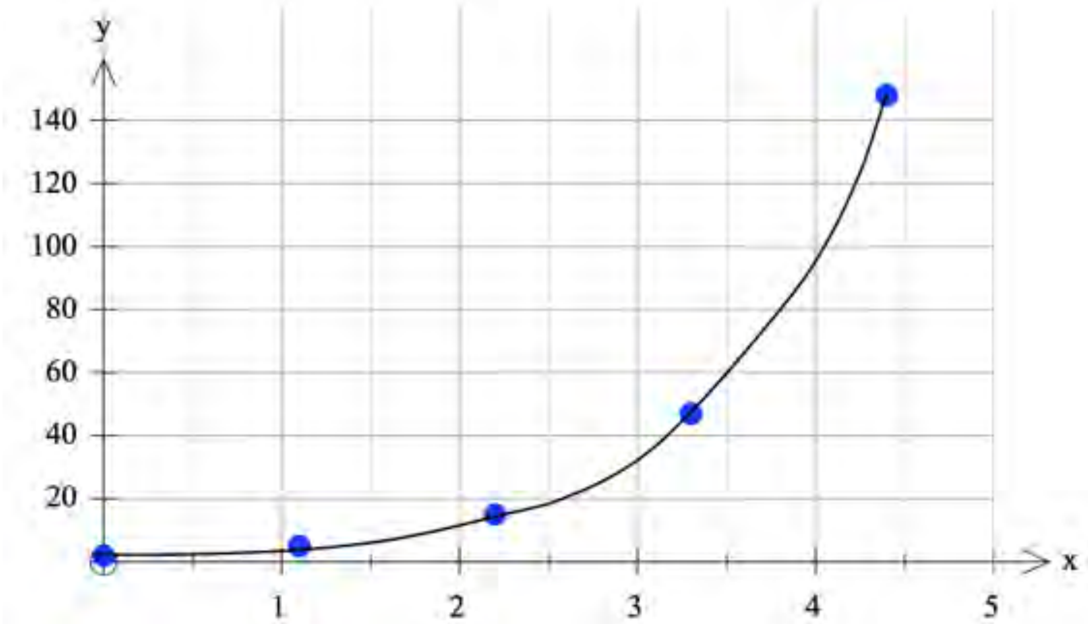
**4m.** *[2 marks]*

Write down the equations of two lines of symmetry for the curve in $x$ and $y$ in part (b).

**5a.** *[3 marks]*

*This question explores methods to determine the area bounded by an unknown curve.*

The curve $y = f(x)$ is shown in the graph, for $0 \le x \le 4.4$.




The curve $y = f(x)$ passes through the following points.

| $x$ | 0 | 1.1 | 2.2 | 3.3 | 4.4 |
|---|---|---|---|---|---|
| $y$ | 2 | 5 | 15 | 47 | 148 |

It is required to find the area bounded by the curve, the $x$-axis, the $y$-axis and the line $x =$ 4.4.

Use the trapezoidal rule to find an estimate for the area.

**5b.** *[2 marks]*

With reference to the shape of the graph, explain whether your answer to part (a)(i) will be an over-estimate or an underestimate of the area.

**5c.** *[3 marks]*

One possible model for the curve $y = f(x)$ is a cubic function.

Use all the coordinates in the table to find the equation of the least squares cubic regression curve.

**5d.** *[1 mark]*

Write down the coefficient of determination.

**5e.** *[2 marks]*

Write down an expression for the area enclosed by the cubic function, the $x$-axis, the $y$-axis and the line $x = 4.4$.

**5f.** *[2 marks]*

Find the value of this area.

**5g.** *[2 marks]*

A second possible model for the curve $y = f(x)$ is an exponential function, $y = pe^{qx}$, where $p,\ q \in \mathbb{R}$.

Show that $\ln y = qx + \ln p$.

**5h.** *[1 mark]*

Hence explain how a straight line graph could be drawn using the coordinates in the table.

**5i.** *[5 marks]*

By finding the equation of a suitable regression line, show that $p = 1.83$ and $q = 0.986$.

**5j.** *[2 marks]*

Hence find the area enclosed by the exponential function, the $x$-axis, the $y$-axis and the line $x = 4.4$.

# Mathematics: applications and interpretation - HL paper 3 questions [134 marks]

**1a.** *[7 marks]*

## Markscheme

$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(1-\lambda)+1=0$ ***M1A1***

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0$ ***A1A1***

So only one solution $\lambda = 2$ ***AGA1***

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow p+q=0$ ***M1***

So an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

***[7 marks]***

**1b.** *[5 marks]*

## Markscheme

$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So $\begin{pmatrix} 3x+y \\ -x+y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t}$ ***M1A1A1***

and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}2e^{2t}$ ***M1A1***

showing that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t}$ is a solution ***AG***

***[5 marks]***

**1c.** *[5 marks]*

## Markscheme

$\begin{pmatrix} 3x+y \\ -x+y \end{pmatrix} = \begin{pmatrix} 3t-t+1 \\ -t-t+1 \end{pmatrix}e^{2t} = \begin{pmatrix} 2t+1 \\ -2t+1 \end{pmatrix}e^{2t}$ ***M1A1***

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} e^{2t} + t2e^{2t} \\ -e^{2t} + (-t+1)2e^{2t} \end{pmatrix} = \begin{pmatrix} 2t+1 \\ -2t+1 \end{pmatrix}e^{2t}$ ***M1A1A1***

Verifying that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t+1 \end{pmatrix} e^{2t}$ is also a solution ***AG***

***[5 marks]***

**1d.** *[3 marks]*

## Markscheme

Require $\begin{pmatrix} 20 \\ 10 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow A = 20,\ B = 30$ ***M1A1***

$\begin{pmatrix} x \\ y \end{pmatrix} = 20\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + 30\begin{pmatrix} t \\ -t+1 \end{pmatrix} e^{2t}$ ***A1***

***[3 marks]***

**1e.** *[2 marks]*

## Markscheme

$t = 2 \Rightarrow x = 4370,\ y = -2730\ (3sf)$ ***A1A1***

***[2 marks]***

**1f.** *[3 marks]*

## Markscheme

As $t \to \infty,\ x \simeq 30te^{2t},\ y \simeq -30te^{2t}$ ***M1A1***

so asymptote is $y = -x$ ***A1***

***[3 marks]***

**1g.** *[1 mark]*

## Markscheme

Will approach the asymptote in the 4th quadrant, moving away from the origin. ***R1***

***[1 mark]***

**2a.** *[2 marks]*

## Markscheme

$\begin{aligned}\dot{x}=0 &\Rightarrow x+2y-50=0\\ \dot{y}=0 &\Rightarrow 2x+y-40=0\end{aligned} \quad \Rightarrow x=10,\ y=20$ ***M1A1***

***[2 marks]***

**2b.** *[3 marks]*

## Markscheme

Using $\begin{aligned}x_{n+1}&=x_n+0.1(x_n+2y_n-50)\\ y_{n+1}&=y_n+0.1(2x_n+y_n-40)\\ t_{n+1}&=t_n+0.1\end{aligned}$

Gives $x(1)\simeq 848,\ y(1)\simeq 837\ (3sf)$ ***M1A1A1***

***[3 marks]***

**2c.** *[2 marks]*

## Markscheme

By extending the table, conjecture that $\lim\limits_{t\to\infty}\frac{y}{x}=1$ ***M1A1***

***[2 marks]***

**2d.** *[3 marks]*

## Markscheme

$X=x-10,\ Y=y-20 \Rightarrow \dot{X}=\dot{x},\ \dot{Y}=\dot{y}$ ***R1***

$\begin{aligned}\dot{X}&=(X+10)+2(Y+20)-50=X+2Y\\ \dot{Y}&=2(X+10)+(Y+20)-40=2X+Y\end{aligned}$ ***M1A1AG***

***[3 marks]***

**2e.** *[8 marks]*

## Markscheme

$\begin{vmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)^2-4=0 \Rightarrow \lambda=-1 \text{ or } 3$ ***M1A1A1***

$\lambda=-1 \quad \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow q=-p$ an eigenvector is $\begin{pmatrix}1\\ -1\end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow q = p$ an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ***M1A1A1***

$\begin{pmatrix} X \\ Y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ ***A1A1***

***[8 marks]***

**2f.** *[2 marks]*

## Markscheme

$\begin{aligned} 100 &= A + B + 10 \\ 50 &= -A + B + 20 \end{aligned} \Rightarrow A = 30,\ B = 60$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix} = 30e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 60e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ ***A1***

***[2 marks]***

**2g.** *[2 marks]*

## Markscheme

$x(1) = 1226,\ y(1) = 1214\ (4sf)$ ***A1A1***

***[2 marks]***

**2h.** *[2 marks]*

## Markscheme

Dominant term is $60e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $\lim\limits_{t\to\infty} \frac{y}{x} = 1$ ***M1A1***

***[2 marks]***

**2i.** *[1 mark]*

## Markscheme

The equilibrium point is unstable. ***R1***

***[1 mark]***

**2j.** *[2 marks]*

## Markscheme

$\begin{aligned} 20 &= A + B + 10 \\ 10 &= -A + B + 20 \end{aligned} \Rightarrow A = 10,\ B = 0$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix} = 10e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ ***A1***

***[2 marks]***

**2k.** *[2 marks]*

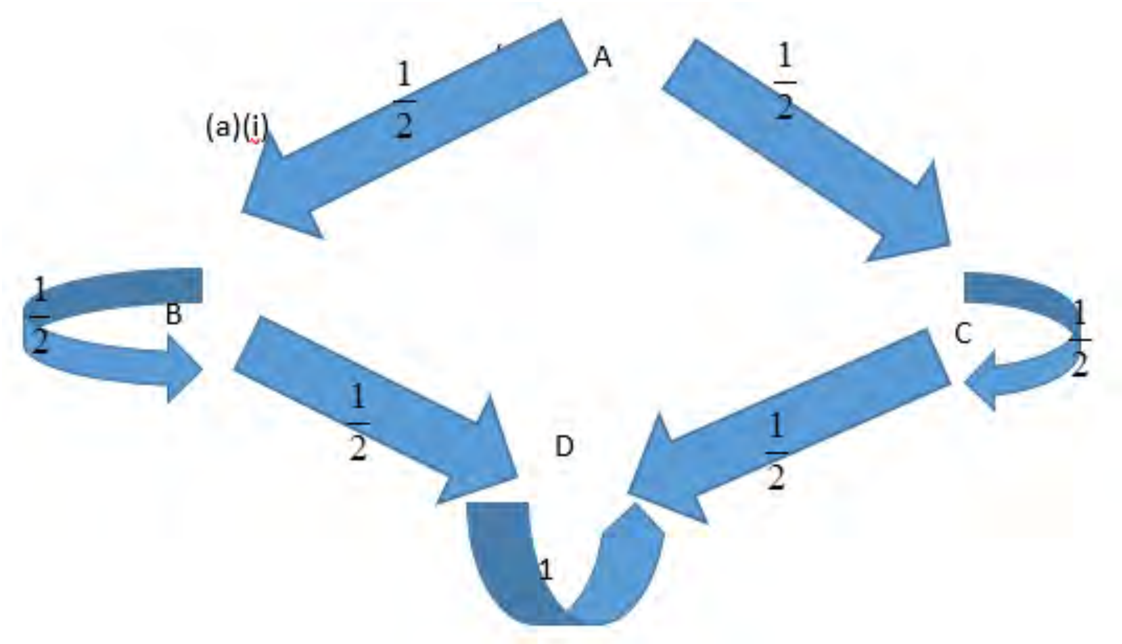
## Markscheme

As $e^{-t} \to 0$ as $t \to \infty$ the equilibrium point is stable. ***R1A1***

***[2 marks]***

**3a.** *[3 marks]*

## Markscheme




***M1A2***

***[3 marks]***

**3b.** *[1 mark]*

## Markscheme

You must leave the state along one of the edges directed out of the vertex. ***R1***

***[1 mark]***

**3c.** *[3 marks]*

## Markscheme

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$ ***M1A2***

***[3 marks]***

**3d.** *[4 marks]*

## Markscheme

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \Rightarrow 0 = w,\ \frac{w}{2}+\frac{x}{2}=x,\ \frac{w}{2}+\frac{y}{2}=y,\ \frac{x}{2}+\frac{y}{2}+z=z \quad \textbf{\textit{M1}}$$

$\Rightarrow w = 0,\ x = 0,,\ y = 0,\ z = 1$ since $w + x + y + z = 1$ so steady state vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

***A1R1A1***

***[4 marks]***

**3e.** *[1 mark]*

## Markscheme

There is a loop with probability of 1 from state D to itself. ***A1***

***[1 mark]***

**3f.** *[2 marks]*

## Markscheme

Let the steady state probability vector be **s** then **Ms** = 1**s** showing that (\lambda  = 1\) is an eigenvalue with associated eigenvector of **s**. ***A1R1***

***[2 marks]***

**3g.** *[4 marks]*

## Markscheme

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{2}{4} \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{6}{8} \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{14}{16} \end{pmatrix} \qquad \textbf{\textit{A1A1A1A1}}$$

***[4 marks]***

**3h.** *[2 marks]*

## Markscheme

$$\mathbf{v}_n = \begin{pmatrix} 0 \\ \frac{1}{2^n} \\ \frac{1}{2^n} \\ \frac{2^n-2}{2^n} \end{pmatrix}$$ ***A2***

***[2 marks]***

**3i.** *[2 marks]*

## Markscheme

$\lim_{n\to\infty} \mathbf{v}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ the steady state probability vector ***M1R1***

***[2 marks]***

**3j.** *[4 marks]*

## Markscheme

Require $\frac{2^n-2}{2^n} \geq 0.95 \Rightarrow \frac{2}{2^n} \leq 0.05 \Rightarrow n = 6$ (e.g. by use of table) ***R1M1A2***

***[4 marks]***

**4a.** *[8 marks]*

## Markscheme

$$\begin{vmatrix} \frac{5}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2}-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{5}{2}-\lambda\right)^2 - \left(\frac{1}{2}\right)^2 = 0 \Rightarrow \frac{5}{2}-\lambda = \pm\frac{1}{2} \Rightarrow \lambda = 2 \text{ or } 3 \qquad \textbf{\textit{M1M1A1A1}}$$

$$\lambda = 2 \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow q = -p \quad \text{eigenvalues are of the form } t\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \textbf{\textit{M1A1}}$$

$$\lambda = 3 \quad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow q = p \quad \text{eigenvalues are of the form } t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \textbf{\textit{M1A1}}$$

***[8 marks]***

**4b.** *[2 marks]*

## Markscheme

$$(x \ \ y)\begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow \left(\frac{5}{2}x + \frac{1}{2}y \quad \frac{1}{2}x + \frac{5}{2}y\right)\begin{pmatrix} x \\ y \end{pmatrix} = (6) \qquad \textbf{\textit{M1A1}}$$

$$\Rightarrow \left(\frac{5}{2}x^2 + \frac{1}{2}xy + \frac{1}{2}xy + \frac{5}{2}y^2\right) = (6) \Rightarrow \frac{5}{2}x^2 + xy + \frac{5}{2}y^2 = 6. \qquad \textbf{\textit{AG}}$$

***[2 marks]***

**4c.** *[2 marks]*

## Markscheme

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ corresponding to } \lambda = 2, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponding to } \lambda = 3 \qquad \textbf{\textit{R1R1}}$$

***[2 marks]***

**4d.** *[1 mark]*

## Markscheme

$$M\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ and } M\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 3\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow M\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad \textbf{\textit{A1AG}}$$

***[1 mark]***

**4e.** *[2 marks]*

## Markscheme

Determinant is 1. $\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ ***M1A1***

***[2 marks]***

**4f.** *[1 mark]*

## Markscheme

$\mathbf{MR}^{-1} = \mathbf{R}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ so post multiplying by $\mathbf{R}$ gives $\mathbf{M} = \mathbf{R}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{R}$ ***M1AG***

***[1 mark]***

**4g.** *[3 marks]*

## Markscheme

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Rightarrow (X \quad Y) = \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \quad \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)$$

***M1A1***

and $(x \quad y)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \quad \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)$ completing the proof ***A1AG***

***[3 marks]***

**4h.** *[2 marks]*

## Markscheme

$$(x \quad y)\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow (x \quad y)\mathbf{R}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{R}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow (X \quad Y)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = (6)$$

$\Rightarrow (2X^2 + 3Y^2) = (6) \Rightarrow \frac{X^2}{3} + \frac{Y^2}{2} = 1$ ***M1A1***

***[2 marks]***

**4i.** *[2 marks]*

## Markscheme

$\frac{X}{\sqrt{3}} = u,\ \frac{Y}{\sqrt{2}} = v \Rightarrow u^2 + v^2 = 1$, a circle (centre at the origin radius of 1) ***A1A1***

***[2 marks]***

**4j.** *[2 marks]*

## Markscheme

A rotation about the origin through an angle of 45° anticlockwise. ***A1A1***

***[2 marks]***

**4k.** *[2 marks]*

## Markscheme

an ellipse, since the matrix represents a vertical and a horizontal stretch ***R1A1***

***[2 marks]***

**4l.** *[1 mark]*

## Markscheme

an ellipse ***A1***

***[1 mark]***

**4m.** *[2 marks]*

## Markscheme

$y = x, y = -x$ ***A1A1***

***[2 marks]***

**5a.** *[3 marks]*

## Markscheme

Area $= \frac{1.1}{2}(2 + 2(5 + 15 + 47) + 148)$ ***M1A1***

Area = 156 units$^2$ ***A1***

***[3 marks]***

**5b.** *[2 marks]*

## Markscheme

The graph is concave up, ***R1***

so the trapezoidal rule will give an overestimate. ***A1***

***[2 marks]***

**5c.** *[3 marks]*

## Markscheme

$f(x) = 3.88x^3 - 12.8x^2 + 14.1x + 1.54$ ***M1A2***

***[3 marks]***

**5d.** *[1 mark]*

## Markscheme

$R^2 = 0.999$ ***A1***

***[1 mark]***

**5e.** *[2 marks]*

## Markscheme

Area $= \int_0^{4.4}(3.88x^3 - 12.8x^2 + 14.1x + 1.54)\,dx$ ***A1A1***

***[2 marks]***

**5f.** *[2 marks]*

## Markscheme

Area = 145 units$^2$ (Condone 143–145 units$^2$, using rounded values.) ***A2***

***[2 marks]***

**5g.** *[2 marks]*

## Markscheme

$\ln y = \ln(pe^{qx})$ ***M1***

$\ln y = \ln p + \ln(e^{qx})$ ***A1***

$\ln y = qx + \ln p$ ***AG***

***[2 marks]***

**5h.** *[1 mark]*

## Markscheme

Plot $\ln y$ against $p$. ***R1***

***[1 mark]***

**5i.** *[5 marks]*

## Markscheme

Regression line is $\ln y = 0.986x + 0.602$ ***M1A1***

So $q$ = gradient = 0.986 ***R1***

$p = e^{0.602} = 1.83$ ***M1A1***

***[5 marks]***

**5j.** *[2 marks]*

## Markscheme

Area = $\int_0^{4.4} 1.83e^{0.986x}dx = 140$ units$^2$ ***M1A1***

***[2 marks]***