Exploring Cricket Ball Trajectories

1. Introduction

I picked up a cricket bat when I was 8 years old, and ever since then, cricket has been a passion of mine. The maximum scoring shot in cricket is called a 'six', which involves hitting the ball with enough force so that it falls outside the boundary line without bouncing on the ground at any instant before it. Whenever I watch a cricket match, I am fascinated by how professional batsmen know exactly how to hit the ball in order to land it past the boundary. However, not all attempts to hit a six turn out to be successful, because the ball often falls behind the boundary line, or ends up in the hands of a fielder (player from the opposite team on the field). This made me wonder, "What would be the perfect way to hit a six?". Thus, I was inspired to conduct this mathematical exploration centred around the mathematical modelling of the trajectory of a cricket ball by considering the factors that affect the aerodynamics of the ball when it is hit by the batsman with the objective of scoring a six.

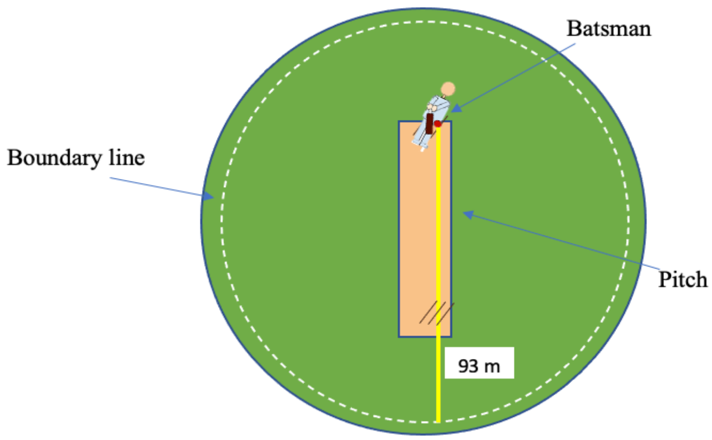

Figure 1: A diagram representing a cricket field

2. Factors affecting the motion of a cricket ball

After a batsman hits the ball, the ball comes to a rest, then reverses its direction, and is set into a parabolic motion. Apart from the velocity at which the batsman swings his bat and the angle at which it hits the ball, several other factors impact the trajectory of the ball. The gravitational force acts downward on the ball during the course of its motion, and air resistance also plays an important role in slowing down the ball as it exerts a drag force on the ball[1]. Apart from these factors, some other forces like lift force and side force also act on the ball due to something termed as the Magnus effect. However, their effect on the trajectory is much more complex in nature. These forces are more prominent when the ball is being pitched to the batsman compared to when the ball is hit by the batsman. Therefore, they have been neglected in this investigation.

3. Objective of the Investigation

In this investigation, I will firstly model the trajectory of a cricket ball in a projectile moving towards the boundary, assuming that the ball's motion occurs in vacuum and thus this model will only account for the force with which the ball is hit and the gravitational force that acts on it during its motion. This will be used to explore several features of the trajectory, like its maximum height. Then, I will model the effect of drag force (air resistance) on the trajectory of the ball to obtain a more realistic theoretical trajectory of the cricket ball. Using the final model of the trajectory, I will determine the optimum conditions required for the batsman to hit a six. Since the cricket boundary line is not exactly circular in shape, I will take the average distance from the batsman's position to the boundary which, according to the International Cricket Council[3], should be 81.50 m. The aim of this investigation is to understand the motion of a cricket ball as it is hit by the batsman with the intention of scoring a six and determine the conditions that will lead to a successful shot.

4. Vacuum Model

For the initial model, let's assume that after the ball leaves the bat, it travels in vacuum. The drag force (air resistance) is assumed to be negligible in order to obtain a simplistic model. Therefore, the only forces acting on the ball are:

- The initial force provided by the bat
- The gravitation force (F_g)

Although the acceleration due to the gravitational force[4] is not constant at all instants during the motion of the ball (varies between 9.764 ms^{-2} and 9.834 ms^{-2}), here, I have assumed it to be constant at 9.8 ms^{-2}. According to the official ICC rules[5], the bowler has to ensure that the ball reaches the batsman below his waist (0.9- 1.0 m), thus, the central height between the waist and the ground (0.5 m) was taken as the initial height of the ball. In addition to that, I conducted an experiment where a professional bowler was asked to bowl the ball 50 times, and it was observed that the average height at which the ball landed near the batsman was 0.47 m $\approx$ 0.5 m.

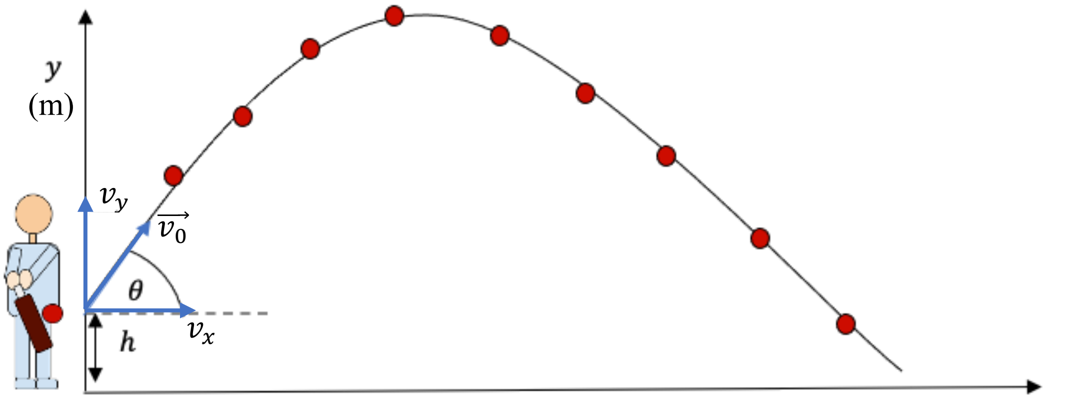

Figure 2: Illustration of the trajectory of the cricket ball in vacuum

The predicted theoretical trajectory was plotted on a cartesian plane, using which I will derive the equations for horizontal and vertical motion of the ball during its flight in vacuum.

The following symbols need to be defined before I derive the equations of motion:

h = distance between ground and the ball when it hits the bat (m)

g = gravitational acceleration (9.8 ms^{-2})

$\boldsymbol{a_x}$ = horizontal acceleration (ms^{-2})

$\boldsymbol{a_y}$ = vertical acceleration (ms^{-2})

$\overrightarrow{\boldsymbol{v_0}}$ = vector of the initial velocity (ms^{-1})

$|\overrightarrow{\boldsymbol{v_0}}| = \boldsymbol{v_0}$ = initial velocity or the magnitude of the vector $\overrightarrow{v_0}$ (ms^{-1})

$\boldsymbol{v_x}$ = horizontal component of $\overrightarrow{v_0}$ (ms^{-1})

$\boldsymbol{v_y}$ = vertical component of $\overrightarrow{v_0}$ (ms^{-1})

t = time (s)

θ = angle between horizontal and direction of initial velocity (rad) $(0 < \theta < \frac{\pi}{2})$

4.1 Horizontal Motion

- Since it is assumed that no external forces are acting on the ball in the horizontal direction, the horizontal acceleration, $a_x = 0$ (uniform motion).
- Therefore, horizontal velocity, v_x is constant throughout the motion, $v_x = v_0 \cos\theta$
- Horizontal displacement, x, at time t can be calculated by integrating the equation for horizontal velocity as the integral of velocity gives us acceleration. Thus,

$$x(t) = \int v_0 \cos\theta \; dt = (v_0 \cos\theta)t \qquad [1]$$

4.2 Vertical Motion

- In the vertical direction, the force of gravity acts downward on the ball. Therefore, vertical acceleration, $a_y = -g$.
- According to the equations for free falling bodies[6], initial vertical velocity = $v_0 \sin\theta$ and vertical velocity at time t: $v_y = v_0 \sin\theta - gt$
- Vertical displacement can be calculated just like horizontal displacement i.e., by integrating the vertical velocity equation.

$$y(t) = \int v_0 \sin\theta - gt \; dt = (v_0 \sin\theta)\, t - \frac{1}{2}gt^2 + c$$

If I substitute $t = 0$, we know that $y(t) = h = c$;

$$\Rightarrow \; y(t) = (v_0 \sin\theta)\, t - \frac{1}{2}gt^2 + h \qquad [2]$$

4.3 Parabolic Equation

The equations [1] and [2] can be manipulated to obtain a parabolic equation in terms of x and y in the form of $y = ax^2 + bx + c$

From equation [1], a formula for t can be derived;

$$t = \frac{x}{v_0 \cos\theta}$$

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[6] The University of Tennessee. (2013). *Freely Falling Objects*. Introduction to Physics I

Substituting this into equation [2],

$$y = \frac{x\, v_0 \sin\theta}{v_0 \cos\theta} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta}\right)^2 + h$$

$$\therefore y = -\frac{g \sec^2\theta}{2{v_0}^2} \cdot x^2 + \tan\theta \cdot x + h \qquad [3]$$

($y = ax^2 + bx + c$; where $a = \frac{-g \sec^2\theta}{2{v_0}^2}, b = \tan\theta, c =$ h; a < 0 which indicates that the parabola faces downwards as we can see in the Figure 2).

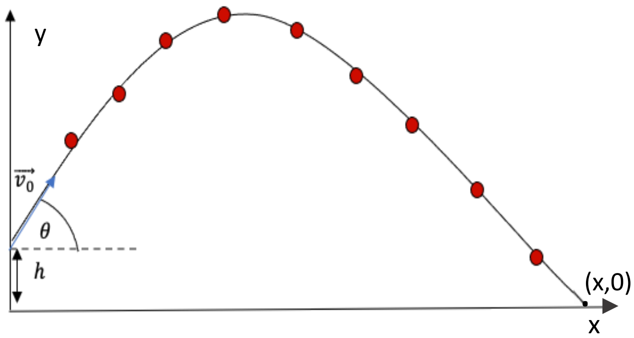
4.4 Equation for angle θ required to reach maximum horizontal distance

When the ball finally lands on the ground, covering its maximum horizontal distance, the coordinates on the Cartesian Plane would be (x,0), since the height from ground would be 0. Substituting y with 0 in equation [3];

$$0 = x^2 \cdot \frac{-g \sec^2\theta}{2{v_0}^2} + x \cdot \tan\theta + h$$

Using the trigonometric identity $\sec^2\theta = \tan^2\theta + 1$;

$$0 = \frac{-gx^2}{2{v_0}^2} \tan^2\theta + x \cdot \tan\theta + h - \frac{gx^2}{2{v_0}^2}$$



Let $\tan\theta$ = b;

$$0 = \frac{-gx^2}{2{v_0}^2} \mathrm{b}^2 + bx + \left(h - \frac{gx^2}{2{v_0}^2}\right)$$

Using the quadratic formula;

$$b = \frac{-x \pm \sqrt{x^2 - 4\left(-\frac{gx^2}{2{v_0}^2}\right)\left(h - \frac{gx^2}{2{v_0}^2}\right)}}{2\left(-\frac{gx^2}{2{v_0}^2}\right)}$$

$$\Rightarrow b = \frac{{v_0}^2 \pm {v_0}^2 \sqrt{\left(1 - 2\left(\frac{g}{{v_0}^2}\right)\left(-h + \frac{gx^2}{2{v_0}^2}\right)\right)}}{gx}$$

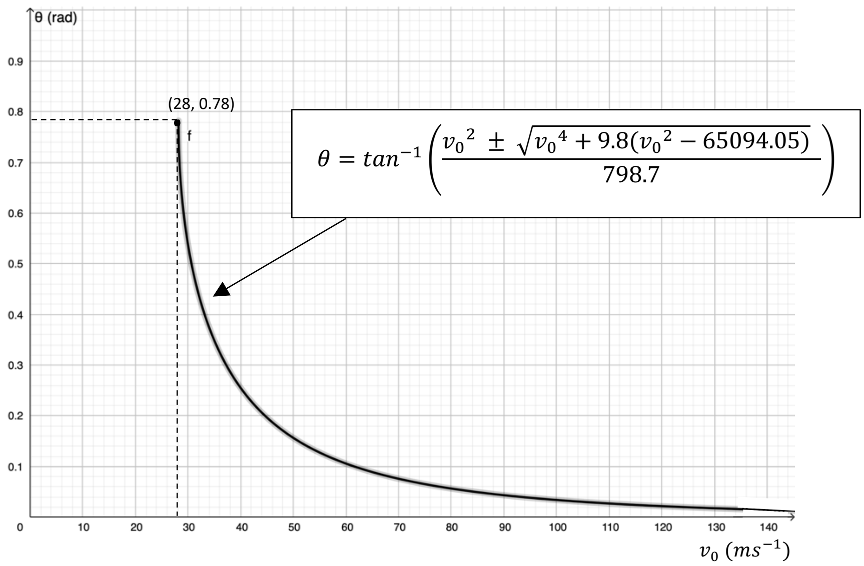
$$\Rightarrow b = \frac{{v_0}^2 \pm \sqrt{{v_0}^4 - g(-2{v_0}^2 h + gx^2)}}{gx}$$

Substituting b = tan θ :

$$\tan\theta = \frac{{v_0}^2 \pm \sqrt{{v_0}^4 - g(-2{v_0}^2 h + gx^2)}}{gx}$$

$$\therefore \theta = tan^{-1}\left(\frac{{v_0}^2 \pm \sqrt{{v_0}^4 + g(2{v_0}^2 h - gx^2)}}{gx}\right) \qquad [4]$$

In order to make sure that the ball reaches the boundary line of the ground, the maximum horizontal displacement (x) required is 81.5 m (length of the boundary). By substituting the values of 'x' as 81.5 m , 'h' as 0.5 m and 'g' as 9.8 ms^{-2} in the equation [4], we can plot the graph of θ vs v_0 to gain a better understanding of the angle between the ball and the horizontal and the corresponding velocity that must be provided to it in order to hit a six.



Graph 1: Relationship between the angle to the horizontal and the initial velocity of the ball

From graph 1, we can see that the minimum initial velocity needed to hit a six is approximately 28 ms^{-1} and the corresponding angle required is 0.78 radians or approximately 45°. It must be noted however that these conditions are only applicable when the ball is travelling in vacuum which is not the case in the real world. The angle required between the horizontal and the ball reduces exponentially as the initial velocity provided to the ball increases. This makes sense to me as a batsman as intuitively I try to hit the ball at a lower angle but with greater force in order to send it beyond the boundary. If the angle is too high, the ball's vertical motion will be greater than its horizontal motion and therefore, it will never reach the horizontal distance of the boundary.

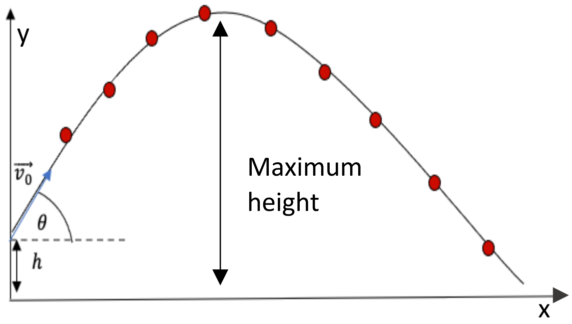
4.5 Maximum height of the cricket ball

In order to find the maximum height reached by the cricket ball, I will determine the maxima point of the graph of the trajectory of the ball in the cartesian plane. From equation [3],

$$y = x^2 \cdot \frac{-g\sec^2\theta}{2{v_0}^2} + x\tan\theta + h$$

At the maxima, the gradient is 0, thus the first derivative is 0

$$\frac{dy}{dx} = 0 = \frac{-xg\sec^2\theta}{{v_0}^2} + \tan\theta$$



$$\Rightarrow x = \frac{{v_0}^2\ \tan\theta}{g\sec^2\theta}$$

Using basic trigonometric identities;

$$\Rightarrow x = \frac{{v_0}^2\ \sin\theta\ \cos^2\theta}{g\cos\theta}$$

$$\Rightarrow x = \frac{{v_0}^2\ \sin\theta\ \cos\theta}{g}$$

From trigonometric identities, we know; $\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow x = \frac{{v_0}^2\ \sin 2\theta}{2g}$$

Substituting x in equation [3];

$$\Rightarrow y = \frac{{v_0}^2\ \sin\theta\cos\theta\tan\theta}{g} - \frac{g{v_0}^4\ \sin^2\theta\cos^2\theta\sec^2\theta}{2g^2{v_0}^2} + h$$

$$\Rightarrow y = \frac{{v_0}^2\ \sin^2\theta}{g} - \frac{{v_0}^2\ \sin^2\theta}{2g} + h = \frac{{v_0}^2\ \sin^2\theta}{2g} + h$$

Thus, the maximum height that the ball reaches is $y = \left(\frac{{v_0}^2 \sin^2\theta}{2g} + h\right)$. This is at a horizontal distance of $\mathrm{x} = \left(\frac{{v_0}^2 \sin 2\theta}{2g}\right)$ from the point where it starts its motion from.

I have successfully calculated equations for several features of the trajectory of the cricket ball when it is hit with the objective of scoring a six. Using similar equations, the height of a 'six' shot and the distance covered by it are calculated by the professional analysts who analyse the trajectory of the ball in a real cricket match using technology.

The equations derived for the features of the ball are highly complex and the mathematics behind the shot is not considered while hitting the ball but understanding the mathematics does provide great insight into the concept of trajectories. However, these trajectories aren't an accurate representation of the cricket ball as there are some other factors at play that affect the motion of the ball.

5. Air Resistance

We often underestimate the impact of air resistance on the motion of bodies travelling through air, while in fact air is heavier than you may expect. One cubic meter of air is approximately 8 times heavier than a cricket ball[1]. When the cricket ball is travelling through air, air resistance opposes its motion, causing the ball to move slower than expected, resulting in a lower maximum height and overall displacement. This opposing force is termed as the drag force and is often roughly equal to the weight of the ball or sometimes about 1.7 times the weight[1]! This force depends on several factors, including the density of air, the speed of the ball, and the cross-sectional area of the ball. Drag coefficient (C_d) is a dimensionless quantity that is used in the drag equation. A high drag coefficient indicates a higher drag force and vice versa.

While playing cricket, I never considered the impact of air resistance or drag force on the flight of the ball or its trajectory. Since cricket balls are pretty heavy, I always thought that air plays no role in the motion of the ball. However, according to experimental data, air resistance can reduce ball speed significantly.

The drag force is given by the following equation[8]:

$$\boldsymbol{F_d = \frac{1}{2} C_d \rho v^2 A}$$

In this equation, the following symbols must be defined:

$\boldsymbol{F_d}$ = drag force (N)

$\boldsymbol{C_d}$ = drag coefficient.

The value of this coefficient is not constant as it depends on the type of air flow (turbulent/laminar), direction of flow, the ball's speed and even the roughness of the cricket ball. This property of the ball is used by bowlers to provide swing to the ball because the ball that is rough on one side, but smooth on the other has a different drag coefficient and experiences different air flow. While this concept is too complex to grasp at the level of this investigation, the drag coefficient for standard cricket ball has been experimentally calculated[7] to be between 0.5 and 0.45, depending on the condition of the ball (assuming that the ball speed is greater than 28 ms^{-1}).

$\boldsymbol{v}$ = velocity of the ball (ms^{-1})

The velocity in the vertical direction (v_y) and in the horizontal direction (v_x) will be considered separately since the force acts in all directions.

$\boldsymbol{A}$ = area of cross section of the cricket ball (m^2)

The diameter of a standard cricket ball[9] is 0.072 m. Therefore, radius (r) = 0.036 m. Cross sectional area can be calculated by πr^2 . ∴ A = 0.00407 m^2.

$\boldsymbol{\rho}$ = density of air (kgm^{-3})

According to the International Standard Atmosphere[10], the density of air is approximately 1.275 kg/m^3 .

[7]Alam, F.; Brooy, R.; Watkins, S. & Subic A. (2007) *An experimental study of cricket ball aerodynamics*
[8]Werner, A. (2007). *Flight Model of a Golf Ball*

5.1 Horizontal Motion

Previously, the acceleration in the horizontal direction was assumed to be 0, however, now we know that drag force acts backwards on the ball and provides negative acceleration.

$$F_d = -\frac{1}{2} C_d \rho {v_x}^2 A$$

$$a_x = -\frac{\rho A}{2m} C_d {v_x}^2$$

where 'm' is the mass of the ball which is 0.156 kg for a standard cricket ball[7].

$$a_x = -(k\, C_d {v_x}^2) \qquad [5]$$

$k = \dfrac{\rho A}{2m}$ is taken as a constant

We know that the initial horizontal velocity at time t= 0, is $v_0 \cos \theta$. However, the horizontal acceleration is a function of the horizontal velocity and I can't simply use Newton's equations of motion to calculate the horizontal velocity at time 't'. Since acceleration is the derivative of velocity with respect to time, equation [5] can be rewritten as;

$$\frac{dv_x}{dt} = -k\, C_d {v_x}^2$$

Bringing both v_x variables to one side;

$$\frac{dv_x}{{v_x}^2} = -k\, C_d dt$$

We know that at t=0, $v_x = v_0 \cos \theta$. Thus, we can find the definite integral on both sides of the equation;

$$\int_{v_0 \cos \theta}^{v_x} \frac{1}{{v_x}^2} dv_x = \int_0^t -k\, C_d dt$$

$$\Rightarrow \int_{v_0 \cos \theta}^{v_x} \frac{1}{{v_x}^2} dv_x = -k\, C_d \int_0^t 1\, dt$$

$$\Rightarrow \left[-\frac{1}{v_x}\right]_{v_0 \cos \theta}^{v_x} = [-k\, C_d t]_0^t$$

$$\Rightarrow -\frac{1}{v_x} + \frac{1}{v_0 \cos \theta} = -k\, C_d t$$

$$\therefore v_x = \frac{1}{k\, C_d t + \frac{1}{v_0 \cos\theta}}$$

I can find the horizontal displacement by integrating the equation for horizontal velocity derived above with respect to time;

$$x = \int \frac{1}{k\, C_d t + \frac{1}{v_0 \cos\theta}}\, dt \qquad [6]$$

I can integrate this equation using the u-substitution method,

$$u = k\, C_d t + \frac{1}{v_0 \cos\theta} \text{ , } du = kC_d\, dt \text{ , } dt = \frac{du}{kC_d}$$

Substituting the values of *dt* and $\left(k\, C_d t + \frac{1}{v_0 \cos\theta}\right)$ into the equation [6],

$$x = \int \frac{1}{kC_d} \cdot \frac{1}{u} du = \frac{1}{kC_d} \ln u + c$$

Undoing substitution $u = k\, C_d t + \frac{1}{v_0 \cos\theta}$ into the above equation;

$$x = \frac{1}{kC_d} \ln\left(k\, C_d t + \frac{1}{v_0 \cos\theta}\right) + c$$

We know that at time t=0, the horizontal displacement, x= 0. Thus,

$$0 = \frac{1}{kC_d} \ln\left(\frac{1}{v_0 \cos\theta}\right) + c$$

$$c = -\frac{1}{kC_d} \ln\left(\frac{1}{v_0 \cos\theta}\right)$$

Substituting c into the equation for the horizontal motion;

$$x = \frac{1}{kC_d}\left(\ln\left(k\, C_d t + \frac{1}{v_0 \cos\theta}\right) - \ln\left(\frac{1}{v_0 \cos\theta}\right)\right)$$

Using the law of logs;

$$x = \frac{1}{K}\left(\ln(Kt\, v_0 \cos\theta + 1)\right) \qquad [7]$$

Here $K = kC_d$ is taken as a constant for simplicity. This equation for the horizontal displacement provides an approximate depiction of the effect of air resistance on the horizontal motion of the ball. In reality, the drag force in the horizontal direction is directly proportional to the instantaneous velocity, not the horizontal velocity which was assumed in this derivation. However, using the instantaneous velocity in the equation makes it impossible to algebraically solve the equation and thus, this alternative approach was used.

5.2 Vertical Motion

Acceleration in the vertical direction, a_y was initially assumed to be -g, but now we know that drag force also acts on the ball in the vertical direction and provides negative acceleration;

$$a_y = -k\,(C_d {v_y}^2) - g$$

Acceleration can be written as the derivative of velocity with respect to time;

$$\frac{dv_y}{dt} = -K{v_y}^2 - g$$

$K = k\,C_d$ is taken as a constant for simplicity. Rewriting the equation;

$$\frac{dv_y}{K{v_y}^2 + g} = -1\,dt$$

We know that, at t= 0 ; $v_y = v_0 \sin\theta$. Taking the definite integral on both sides of the equation;

$$\int_{v_0 \sin\theta}^{v_y} \frac{1}{K{v_y}^2 + g} dv_y = \int_0^t -1\,dt$$

$$\Rightarrow \frac{1}{K}\int_{v_0 \sin\theta}^{v_y} \frac{1}{{v_y}^2 + \left(\sqrt{\frac{g}{K}}\right)^2}\,dv_y = \int_0^t -1\,dt$$

Using the standard integral $\int \frac{1}{\left(\sqrt{\frac{g}{K}}\right)^2 + {v_y}^2}\,dv_y = \frac{1}{\sqrt{\frac{g}{K}}}\arctan\left[\frac{v_y}{\sqrt{\frac{g}{K}}}\right] + C$;

$$\Rightarrow \left[\frac{\tan^{-1}\left(\frac{v_y\sqrt{K}}{\sqrt{g}}\right)}{K\left(\sqrt{\frac{g}{K}}\right)}\right]_{v_0 \sin\theta}^{v_y} = [-t]_0^t$$

$$\Rightarrow \frac{\tan^{-1}\left(v_y\sqrt{\frac{K}{g}}\right)}{\sqrt{Kg}} - \frac{\tan^{-1}\left(v_0 \sin\theta\,\sqrt{\frac{K}{g}}\right)}{\sqrt{Kg}} = -t$$

$$\Rightarrow \tan^{-1}\left(v_y\sqrt{\frac{K}{g}}\right) = \tan^{-1}\left(v_0 \sin\theta \sqrt{\frac{K}{g}}\right) - t\sqrt{Kg}$$

Applying tan to the both sides of the equation;

$$\Rightarrow v_y\sqrt{\frac{K}{g}} = \tan\left(\tan^{-1}\left(v_0\sin\theta\sqrt{\frac{K}{g}}\right) - t\sqrt{Kg}\right)$$

Using the compound angle trigonometric identity for tan;

$$v_y = \sqrt{\frac{g}{K}}\left(\frac{v_0\sin\theta\sqrt{\frac{K}{g}} - \tan(t\sqrt{Kg})}{1 + v_0\sin\theta\sqrt{\frac{K}{g}}\tan(t\sqrt{Kg})}\right)$$

Thus, I have successfully determined the equation for vertical velocity. In order to find the equation for vertical displacement, y, I will integrate this equation with respect to time.

$$y = \sqrt{\frac{g}{K}}\int \frac{v_0\sin\theta\sqrt{\frac{K}{g}} - \tan(t\sqrt{Kg})}{1 + v_0\sin\theta\sqrt{\frac{K}{g}}\tan(t\sqrt{Kg})}\,dt$$

In this equation, let $\left(v_0\sin\theta\sqrt{\frac{K}{g}}\right) = a$ and $\sqrt{Kg} = b$, where a and b are constants.

Rewriting the equation;

$$y = -\frac{\sqrt{g}}{\sqrt{K}}\int \frac{\tan(bt) - a}{a\tan(bt) + 1}\,dt$$

Solving the integral using u-substitution;

$$u = bt, du = b\,dt, dt = \frac{du}{b}$$

$$y = -\frac{\sqrt{g}}{b\sqrt{K}}\int \frac{\tan(u) - a}{a\tan(u) + 1}\,du \qquad [8]$$

Solving the integral $\int \frac{\tan(u) - a}{a\tan(u) + 1}\,du$;

$$= \int\left(\frac{\frac{1}{a}(a\tan(u) + 1)}{a\tan(u) + 1} + \frac{-a - \frac{1}{a}}{a\tan(u) + 1}\right)du$$

$$= \left(-a - \frac{1}{a}\right) \int \frac{1}{a\tan(u) + 1} du \; + \frac{1}{a} \int 1 \, du$$

Using the identity $\tan^2\theta + 1 = \sec^2\theta$;

$$= \left(-a - \frac{1}{a}\right) \int \frac{\sec^2(u)}{\tan^2(u) + 1} \left(\frac{1}{a\tan(\,u) + 1}\right) du \; + \frac{1}{a} \int 1 \, du$$

Substituting $v = \tan(u)$, $dv = \sec^2(u)\, du$, $du = \frac{1}{\sec^2(u)} dv$;

$$= \left(-a - \frac{1}{a}\right) \int \frac{1}{(\mathrm{v}^2 + 1)(av + 1)} \, dv \; + \frac{1}{a} \int 1 \, du$$

Performing partial fraction decomposition;

$$= \left(-a - \frac{1}{a}\right) \int \left(\frac{a^2}{(a^2 + 1)(av + 1)} - \frac{av - 1}{(a^2 + 1)(v^2 + 1)}\right) dv \; + \frac{1}{a} \int 1 \, du$$

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a^2}{a^2 + 1} \int \left(\frac{1}{av + 1}\right) dv - \frac{1}{a^2 + 1} \int \left(\frac{av - 1}{v^2 + 1}\right) dv \right) + \frac{1}{a} \int 1 \, du$$

Substituting $z = av + 1$, $dz = a\, dv$, $dv = \frac{dz}{a}$;

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a\ \ln(z)}{a^2 + 1} - \frac{1}{a^2 + 1} \int \left(\frac{av - 1}{v^2 + 1}\right) dv\right) + \frac{1}{a} \int 1 \, du$$

Undoing the substitution $z = av + 1$, and expanding $\left(\frac{av-1}{v^2+1}\right)$;

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a\ln(av + 1)}{a^2 + 1} - \frac{1}{a^2 + 1} \int \left(\frac{av}{v^2 + 1} - \frac{1}{v^2 + 1}\right) dv\right) + \frac{1}{a} \int 1 \, du$$

Substituting $p = v^2 + 1$, $dp = 2v\, dv$, $dv = \frac{dp}{2v}$, and using the standard integral $\int \left(\frac{1}{v^2+1}\right) dv =$

arctan v;

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a\ln(av + 1)}{a^2 + 1} - \frac{a}{a^2 + 1} \int \frac{1}{2p} dp + \frac{\arctan(v)}{a^2 + 1}\right) + \frac{1}{a} \int 1 \, du$$

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a\ln(av + 1)}{a^2 + 1} - \frac{a\ln p}{2(a^2 + 1)} + \frac{\arctan(v)}{a^2 + 1}\right) + \frac{1}{a} \int 1 \, du$$

Undoing the substitution $p = v^2 + 1$ and integrating the constant 1;

$$= \left(-a - \frac{1}{a}\right) \left(\frac{a\ln(av + 1)}{a^2 + 1} - \frac{a\ln\,(v^2 + 1)}{2(a^2 + 1)} + \frac{\arctan(v)}{a^2 + 1}\right) + \frac{u}{a}$$

Undoing the substitution $v = \tan(u)$ and $u = \mathrm{bt}$;

$$= \left(-\frac{a^2+1}{a}\right)\left(\frac{a\ln(a\tan(bt)+1)}{a^2+1} - \frac{a\ln\left(\tan^2(bt)+1\right)}{2(a^2+1)} + \frac{\arctan(\tan(bt))}{a^2+1}\right) + \frac{bt}{a}$$

Simplifying the equation;

$$= \frac{\ln\left(\tan^2(bt)+1\right)}{2} - \ln(a\tan(bt)+1)$$

Substituting this into equation [8];

$$y = \frac{\sqrt{g}\left(\ln(a\tan(bt)+1) - \frac{\ln\left(\tan^2(bt)+1\right)}{2}\right)}{b\sqrt{K}} + C$$

Using trigonometric identities and law of logs;

$$\frac{\ln\left(\tan^2 bt+1\right)}{2} = \frac{\ln(\sec^2 bt)}{2} = \frac{\ln(\sec(bt)) + \ln(\sec(bt))}{2} = \ln\left(\frac{1}{\cos(bt)}\right) = \ln 1 - \ln(\cos(bt))$$

We can expand the domain by taking the absolute value of logarithms.

$$y = \frac{\sqrt{g}\left(\ln(|a\tan(bt)+1|) + \ln(|\cos(bt)|)\right)}{b\sqrt{K}} + C$$

Substituting the values for a and b into the equation $\left(v_0 \sin\theta\sqrt{\frac{K}{g}} = a \text{ and } \sqrt{Kg} = b\right)$;

$$y = \frac{\ln\left|\frac{v_0\sin\theta\sqrt{K}\tan\left(\sqrt{Kg}t\right)}{\sqrt{g}} + 1\right| + \ln\left(\left|\cos\left(\sqrt{Kg}t\right)\right|\right)}{K} + C$$

We know that at t = 0, y = 0.5 (initial height of the ball is assumed to be 0.5 m) $\therefore C = 0.5$

$$y = \frac{\ln\left|\frac{v_0\sin\theta\sqrt{K}\tan\left(\sqrt{Kg}t\right)}{\sqrt{g}} + 1\right| + \ln\left(\left|\cos\left(\sqrt{Kg}t\right)\right|\right)}{K} + 0.5 \qquad [9]$$

5.3 Corrected Parabolic Equation

Now I will correct the parabolic equation by manipulating the new derived formulas for ‘x’ and ‘y’ to come up with a new equation for the trajectory of the ball that takes air resistance into account.

From equation [7], an equation for time ‘t’ with respect to ‘x’ can be determined.

$$x = \frac{1}{K}\left(\ln(Kt\, v_0 \cos\theta + 1\,)\right)$$

$$Kx = \ln(Kt\, v_0 \cos\theta + 1)$$

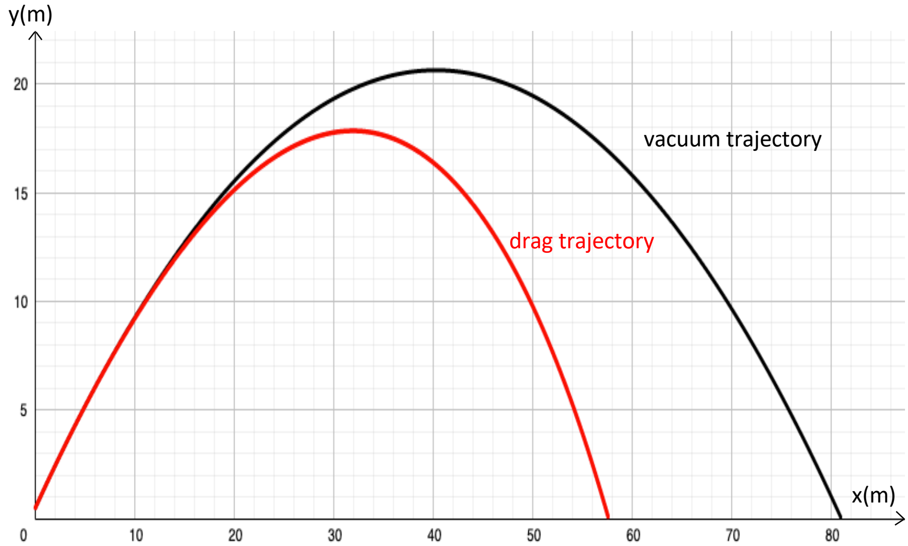
$$e^{Kx} = e^{\ln(Kt\, v_0 \cos\theta + 1)} = Kt\, v_0 \cos\theta + 1$$

$$t = \frac{e^{Kx} - 1}{Kv_0 \cos\theta}$$

In equation [9], this value of t can be substituted. The absolute values are removed since all constants and variables are positive.

$$y = \frac{\ln\left(v_0 \sin\theta \sqrt{\frac{K}{g}} \tan\left(\frac{\sqrt{Kg}(e^{Kx} - 1)}{Kv_0 \cos\theta}\right) + 1\right) + \ln\left(\cos\left(\frac{\sqrt{Kg}(e^{Kx} - 1)}{Kv_0 \cos\theta}\right)\right)}{K} + 0.5 \qquad [10]$$

Equation [10] is not in the form of a standard parabola equation. Therefore, a cricket ball's trajectory in reality is not parabolic. This is easily observed in the cricket field as the ball never follows an actual parabolic path. This corrected equation can be used to plot the trajectory of the ball on a cartesian plane. The value of constant K is taken as 0.008 after assuming the value of drag coefficient (C_d) to be 0.48 (as it varies[6] between 0.45 and 0.5). The following graph shows the trajectory of ball hit with initial velocity 28 ms^{-1} and at an angle of 45° from the horizontal (according to graph 1, 28 ms^{-1} is the minimum initial velocity required to hit a six in vacuum).

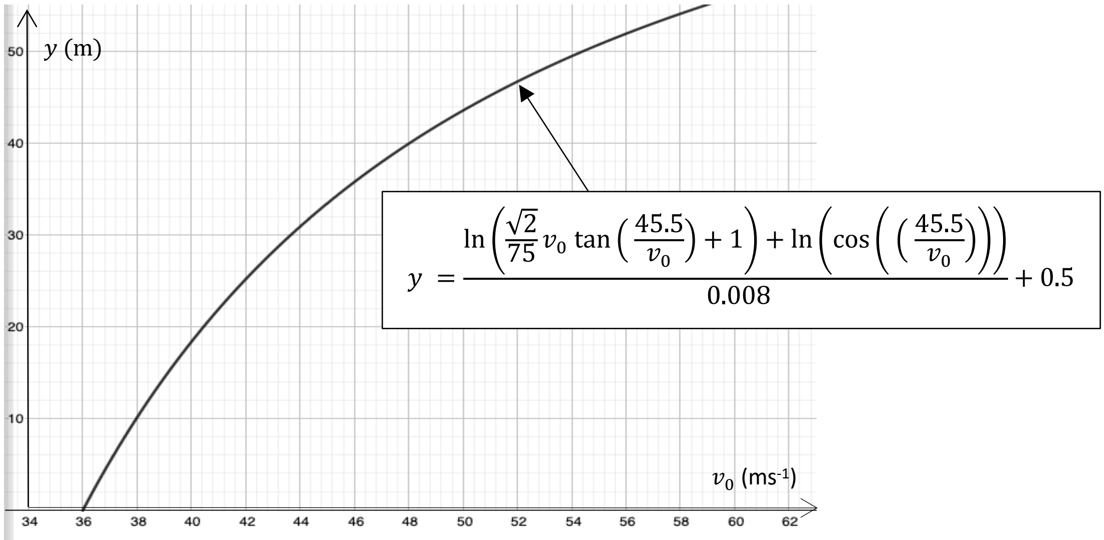


Graph 2: The graph of the theoretical predictions of the trajectory of ball when it is hit at angle of 45° and with an initial speed of 28 ms^{-1}. The trajectory for the ball in vacuum and the ball experiencing air resistance/drag force are plotted.

Graph 2 clearly shows the impact of air resistance on the theoretical trajectory of the ball. The maximum height achieved by the ball is lower compared to the vacuum trajectory, and much to my surprise, the maximum horizontal distance is significantly lesser (by about 25 m). It is evident from the graph that while the initial speed of 28 m/s and an angle of 45° would not be enough to hit a six in a real life scenario.

5.4 Velocity required to reach point (81.5,y) on the cartesian plane

In order to find the minimum initial velocity required to hit a six, we need to substitute x = 81.5 in equation [10] and let us take the same initial angle, i.e. 45°. After substituting these values into equation [10], we can plot the graph of y vs v_0.



Graph 3: The graph of initial velocity vs vertical distance at the point (81.5,y) on the cartesian plane.

It must be noted the graph is only plotted for the values of v_0 greater than 28 ms^{-1} because for initial velocities less than that, the assumed value for the drag coefficient would be invalid.

Graph 3 shows that the minimum initial velocity required to reach (81.5, 0) is 36 ms^{-1}. Any velocity more than that will ensure that the ball crosses the boundary line. This value is much greater than 28 ms^{-1} which was the initial velocity required to hit a six in vacuum. Therefore, air resistance makes it much harder for the batsman to hit a six. However, these values assume that the initial angle at which the ball is hit is 45°. For any angle lower than that, the initial velocity required would be greater, thus these values were considered to be appropriate.

6. Swinging the bat to hit a six

I decided to research about the perfect way of hitting a ball so that I could apply my findings of initial vertical velocity to find out the perfect way to hit a six. Much to my surprise, hitting a ball with a bat has very complex mathematics and physics behind it. The average bowling speed[12] for a fast bowler is about 38.89 m/s while the horizontal distance between the ball and the batsman, at the moment of release, is approximately 18. Thus, the batsman only has about 0.46 seconds to react to the ball and try to hit a six. If the ball perpendicularly strikes the centre of percussion of the bat while the batsman plays his shot, the least amount of power is required and calculations can easily be performed. Batsmen tend to figure out where this point approximately lies on their bat, so, for the purposes of this investigation, let us assume that the ball hits the bat, making a right angle with the centre of percussion.

The following equation has been derived for the velocity at which the ball is released from the bat after being hit[14]:

$$\boldsymbol{v = [-muh^2 + Mk^2\omega(1+e)h + Muk^2e][mh^2 + Mk^2]^{-1}}$$

Here, *v* represents the initial speed given to the ball, which, according to my trajectory model, should be 36 ms^{-1} in order to hit a six. *M* and *m* are the masses of the bat and the ball respectively[14] which can be taken as 1.17 kg and 0.156 kg. The incident ball speed, *u*, can be taken as 38.9 m/s as previously discussed. *e* is the coefficient of restitution, *k* is the radius of gyration about the axis of rotation and *h* is the distance of the centre of percussion from the axis of rotation. The experimentally calculated values for these can be used, which are[14]; $e = 0.53$, $k = 0.42$ m , $h = 0.49$ m. We need to calculate ω , which is the angular velocity that the bat must be swung with. Substituting these values into the equation;

$$\omega = \frac{v(mh^2 + Mk^2) + muh^2 - Muk^2e}{Mk^2(1+e)h} = 38.65 \text{ rad/s}$$

The linear velocity can be calculated as $k\omega = 16.23$ m/s

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[14]Brearley, M. N.; Burns, J. C & Demestre, N. J., (1990), *Int. J. Math. Educ. Sci. Technol*

7. Conclusion

After having investigated the trajectory of the cricket ball during its flight from the batsman's bat to the cricket ground, and then using the model to find out the velocity at which the bat must be swung in order to accomplish the task of hitting a six, I have come to the conclusion that hitting a six is no easy task. The batsman has 0.46 seconds to observe the kind of bowl being bowled, choose his type of shot, then accelerate the bat to up to 16 m/s (acceleration of 3.5 times the gravitational acceleration!) and then also make sure that the cricket ball strikes the 'sweet spot' of the bat. However, there were many complex characteristics of the trajectory of the ball that were simplified for this investigation.

While the model employing the quadratic drag force obtained is much more realistic than a zero-drag theory that is generally studied in high school (even in my IB Physics class), other forces like side force and spin force were completely neglected in the final theoretical model which can lead to inaccuracies if the technique is applied to a practical situation. Moreover, the drag coefficient was assumed to be constant throughout the motion. This is not true in reality as the coefficient varies with the speed of airflow, but this variation is negligible enough to be neglected for our purposes. Moreover, as pointed out earlier, the gravitational acceleration is not constant and may change with the direction of force acting over a long distance and while the cricket ground isn't too big (81.5 m radius was assumed), that distance is still significant enough to cause minor variations in the gravitational acceleration.

Further applications of this work can be to firstly, model the effect of forces like side force and spin force on the trajectory of the cricket ball and then, also experimentally analyse the trajectory using video analysis software in order to compare the theoretical model with experimental data. This wasn't possible for me to do due to lack of appropriate resources, but my final theoretical predictions were based on experimental data from (Brearley, Burns & Demestre)[14] which shows that the theoretical data can be applied to real situations.

8. Critical Reflection

Despite all its limitations and weaknesses, this investigation helped me a lot as a batsman to understand the process of hitting a shot and trying to score a six. I could finally understand what factors I could control in order to make sure I hit the perfect shot and what mistakes I could avoid making. Overall, it made me more aware of the underlying technical aspects of a ball's flight. Therefore, other cricket players and enthusiasts can gain a lot of crucial information from this investigation which would help them improve their style of playing. It would especially help beginner players as they don't intuitively know what speed and angle should be provided to the ball in order to hit that perfect six and not end up getting caught. Apart from cricket, the theoretical model for the trajectory can also be applied to other ball sports like tennis, baseball and golf too, but they will all have different values for coefficients like 'K' and 'C_d' (drag coefficient) due to every ball being off different dimensions. A golf ball's trajectory is particularly intriguing since the ball has holes all over its body which increases the effect of drag and spin force which can be further explored by expanding the work done in this investigation. Thus, not only is this investigation extremely useful to cricket players and enthusiasts, but also to players of other ball sports. Moreover, other projectiles can be investigated in a similar way, like trajectories of a bullet, a rocket or unguided bombs and the theoretical model derived here can be applied to those conditions which could help researchers gain more insight into those projectiles.

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