IB Subject(s): Mathematics

Extended Essay

# Volume of rings and disks:

What is the volume of a regular $n$-sided $(n \geq 3, n \in Z+)$ polygon rotated 360° around the $x$-axis to form a ring with variable radius, $h$ $(h \in R+)$?

Word Count: 3,911

# Table of Contents

## Introduction

Shapes and topology define the world we live in. From infrastructure to fashion, and everything in between, the most basic of shapes create complex, compound structures. I have always had a particular interest towards shapes in mathematics and I wished to apply them to my personal areas of interest such as fashion and engineering.

I specifically wanted to look at rings due to their intricate design. The radius would allow me to find the volume for rings of different sizes which can further help in determining the price when mass manufacturing a variety of rings (**Fig. 1**).

**Fig. 1 –** An image of a ring[1]

Other applications of this concept can be used for calculating the area of complicated designs in the creation of curved tools in mechanical engineering (**Fig. 2**). I wanted to explore the core concept in designing these rings and tools mathematically in the form of an extended essay, where I can further research about their applications.

**Fig. 2 –** An Allen key[2]

This led me to create the research question, which is –

**"What is the volume of a regular $n$-sided $(n \geq 3, n \in Z+)$ polygon rotated 360° around the $x$-axis to form a ring with variable radius, $h$ $(h \in R+)$?"**

[1] *Sonic Ring Png - Sonic The Hedgehog Rings Vector, Transparent Png, Transparent Png Image - PNGitem*. (n.d.). PNGitem.Com. https://www.pngitem.com/middle/iiwRhwR_sonic-ring-png-sonic-the-hedgehog-rings-vector/

[2] *Allen Key PNG File | PNG Mart. (n.d.). PNG Mart. http://www.pngmart.com/image/209489*

## Exploring the question

To better understand the question I will be researching, the polygons have to be graphed onto a cartesian coordinate system for visual representation. The polygons will be inscribed within a circle. As seen in **Fig. 3**, the radius of the circle which the polygons will be inscribed in will be denoted by $r$. This allows us to set the thickness of the ring which will be formed. The height of the centre of the polygon from the $x$-axis will be given by $a$. Finally, the height of the base of each polygon from the $x$-axis will differ according to the values set for $a$ and $r$, which will give us the radius of the ring.

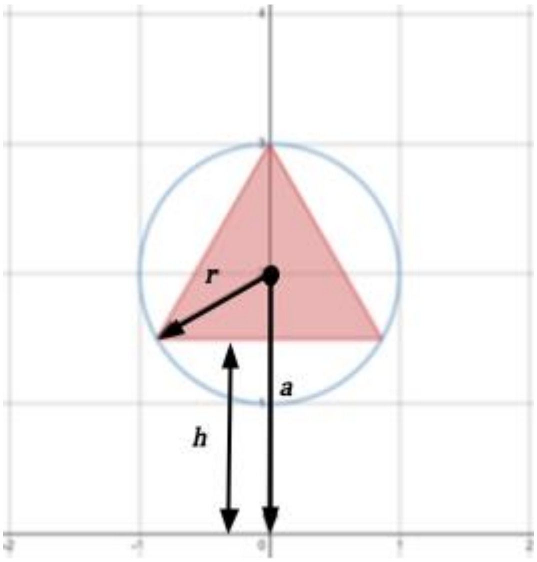



**Fig. 3 –** Triangle showing variables $a, h$, and $r$

For example, a triangle with $a = 2$ and $r = 1$ will look like **Fig. 4**.

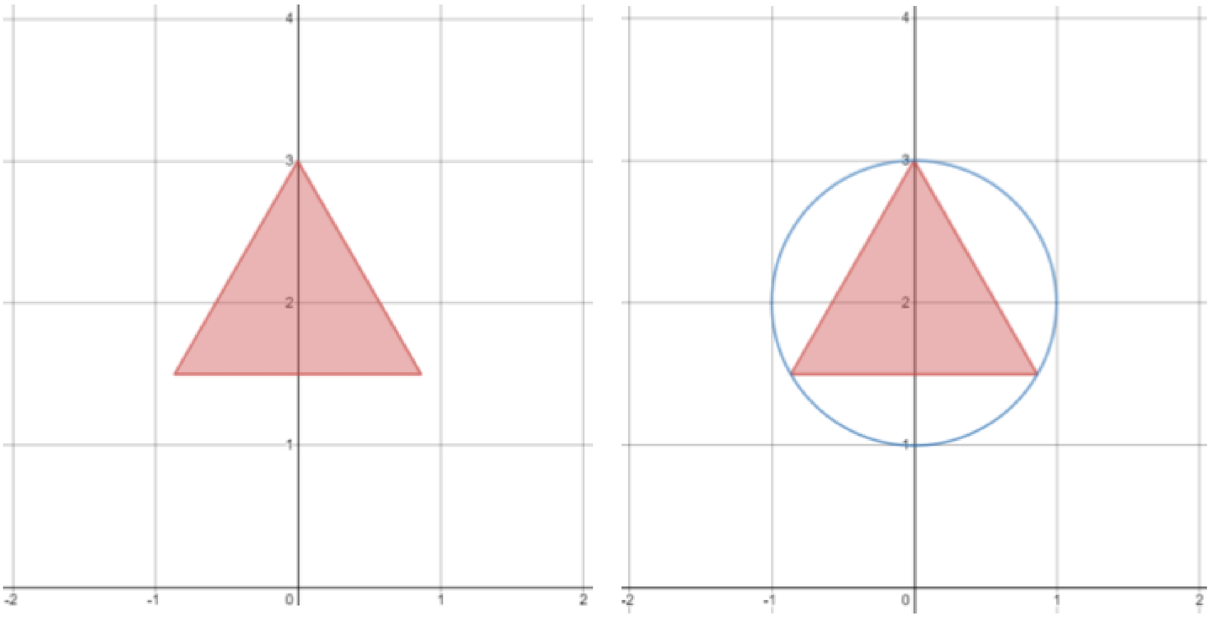


**Fig. 4 –** Regular triangle with $r = 1$ and $a = 2$

Increasing the height will make the radius of the ring larger, while increasing the radius of the circle which the polygon is inscribed in will decrease the radius of the final ring. The following figure (**Fig. 5**) illustrates a triangle with $a = 6$ and $r = 5$.

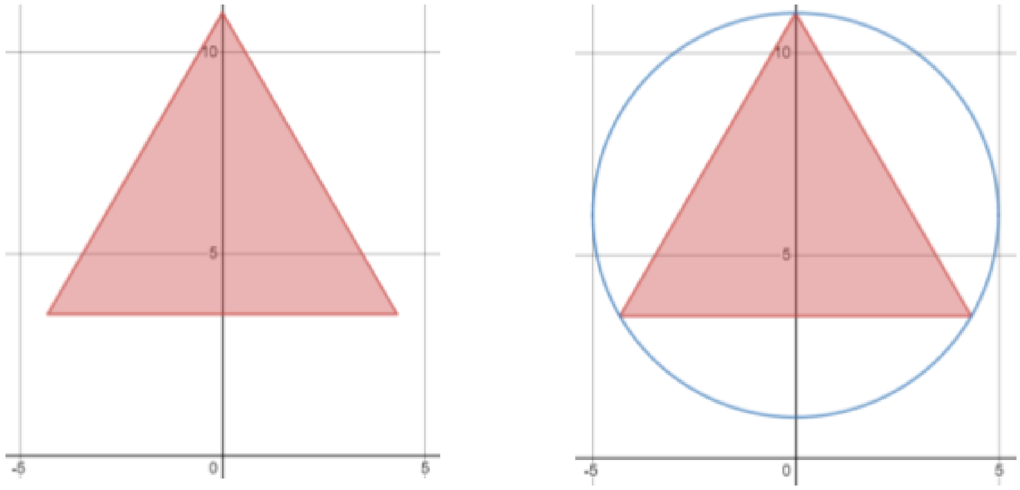


**Fig. 5 –** Regular triangle with $r = 5$ and $a = 6$

As we can see, the radius of the circle is now 5 and the height of the centre of the polygon from the $x$-axis is 6. However, adjusting the variables can pose a problem as the modified height and radius can sometimes cause the polygon to cut through the $x$-axis, which will then not result in the formation of a ring, as we can see on the right (**Fig. 6**), where $r = 5$ but $a = 1$. We must consider the circle, since as $n$ approaches infinity, the polygon turns into a circle. Thus, $a, r \in$ R and $r < a$ for the formation of a ring. If $r = a$, then when $n$ reaches infinity, the circle formed will be touching the $x$-axis and the rotation will form a solid instead of a ring.

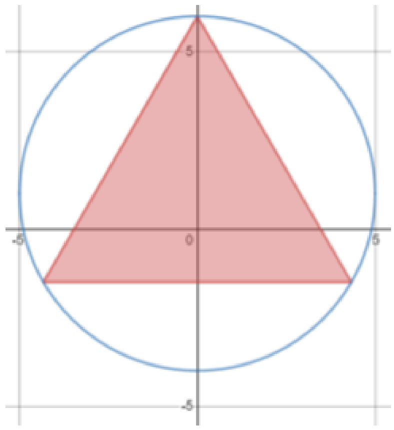


**Fig. 6 –** Regular triangle with $r = 5$ and $a = 1$

By finding a general equation of the lines of the graphed $n$-sided polygons and then using the formula for finding the volume of revolution, I will be able to create a general formula for the volume of revolution for any $n$-sided polygon.

## Finding the relationship between $\boldsymbol{a}$, the height of the centre of the polygon, and $\boldsymbol{h}$, the height of the polygon from the $\boldsymbol{x}$-axis

The variable $h$, the height of the base of the polygon from the $x$-axis, will give us the radius of the ring formed. Thus, it is important to derive it for specifying the dimensions of the ring. We can specify the height of the centre of the polygon from the $x$-axis, $a$, as seen in the previous section. In this section, I will create a relationship between $a$ and $h$. To better understand this, we can look at the following diagram (**Fig. 7**).

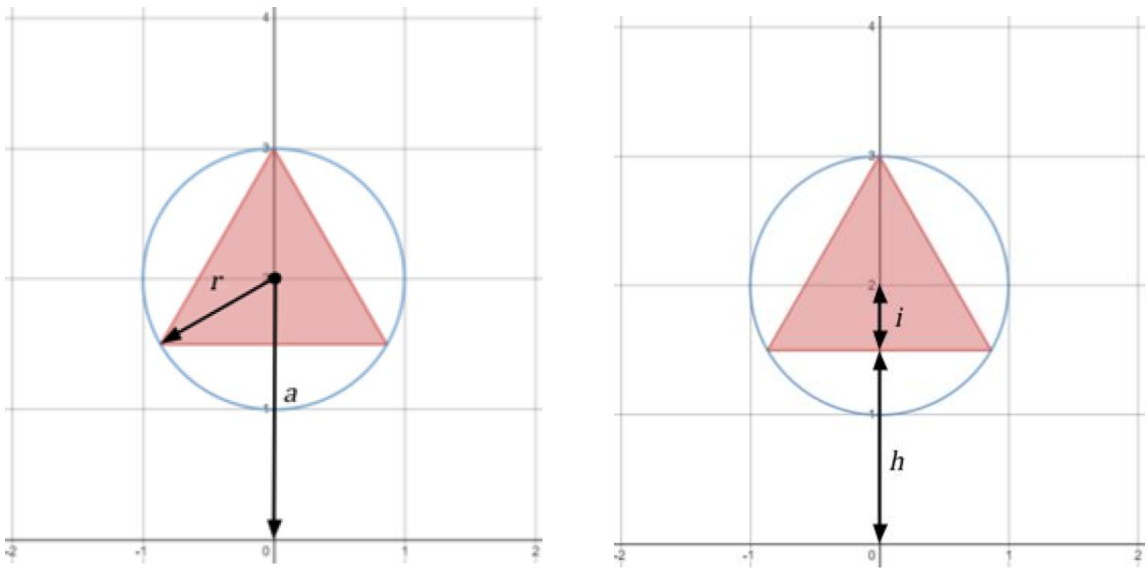



**Fig. 7 –** Regular triangle with $r = 1$ and $a = 2$. The variable $i$ is used to show the distance from the centre to the base of the polygon and the variable $h$ is used to show the distance from the base of the polygon to the $x$-axis

Here, a triangle is inscribed within a circle, where $a = 1$ and $r = 1$, but $h$ is unknown. We can find the value of $h$ by subtracting $a$ with $i$, i.e., $h = a - i$. To do this, we must consider the following isosceles triangle (**Fig. 8**).

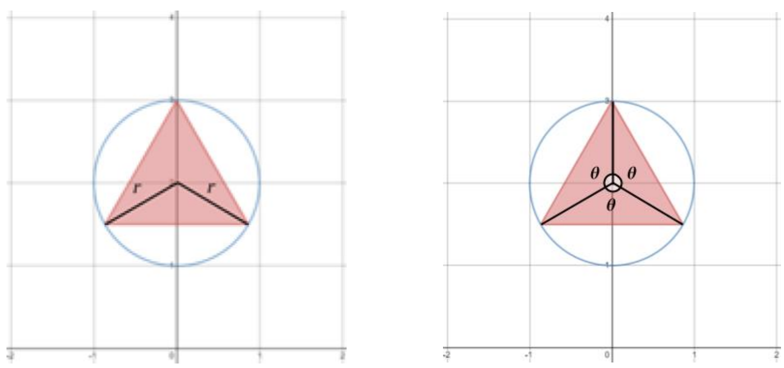
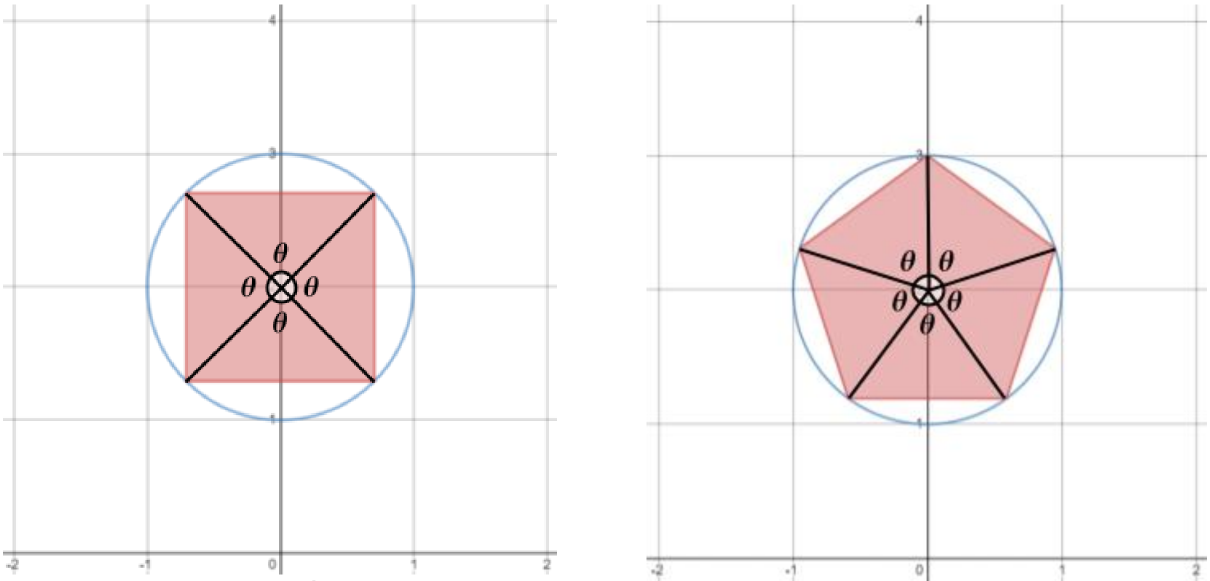



**Fig. 8 –** Isosceles triangles formed within the triangle in consideration, $n = 3$, with angle $\theta$

Since both of the lines drawn on the triangle in the first part of **Fig. 8** are radius of the circle, we can see that an isosceles triangle is formed. Moreover, three identical isosceles triangles are formed in the triangle. Hence, we can find the angle $\theta$ formed between the dotted lines. For example, for the triangle in the figure, it would be:

$$\frac{360}{3} = 120°$$

We can see this in other $n$-sided polygons as well (**Fig. 9**) –




**$n$ = 4** **$n$ = 5**

**Fig. 9 –** Isosceles triangles formed within a polygon for a square, $n = 4$, and a pentagon, $n = 5$

The number of isosceles triangles formed is equal to the number of sides of the $n$-sided polygon.

Hence the general formula for finding the angle $\theta$ is $\frac{360}{n}$.

Through the isosceles triangle theorem, we know that the bisector of angle $\theta$ will be perpendicular to the opposite side. Thus, line $i$ bisects the angle $\theta$ and creates a $90°$ angle with the opposite side.

We now have a right-angled triangle (**Fig. 10, Page 8**), and can use trigonometric functions to find the length of line $i$. We know that the hypotenuse of the isosceles triangle formed is the radius of the circle, $r$, and therefore the following can be derived:

$$\cos\frac{\theta}{2} = \frac{i}{r}$$

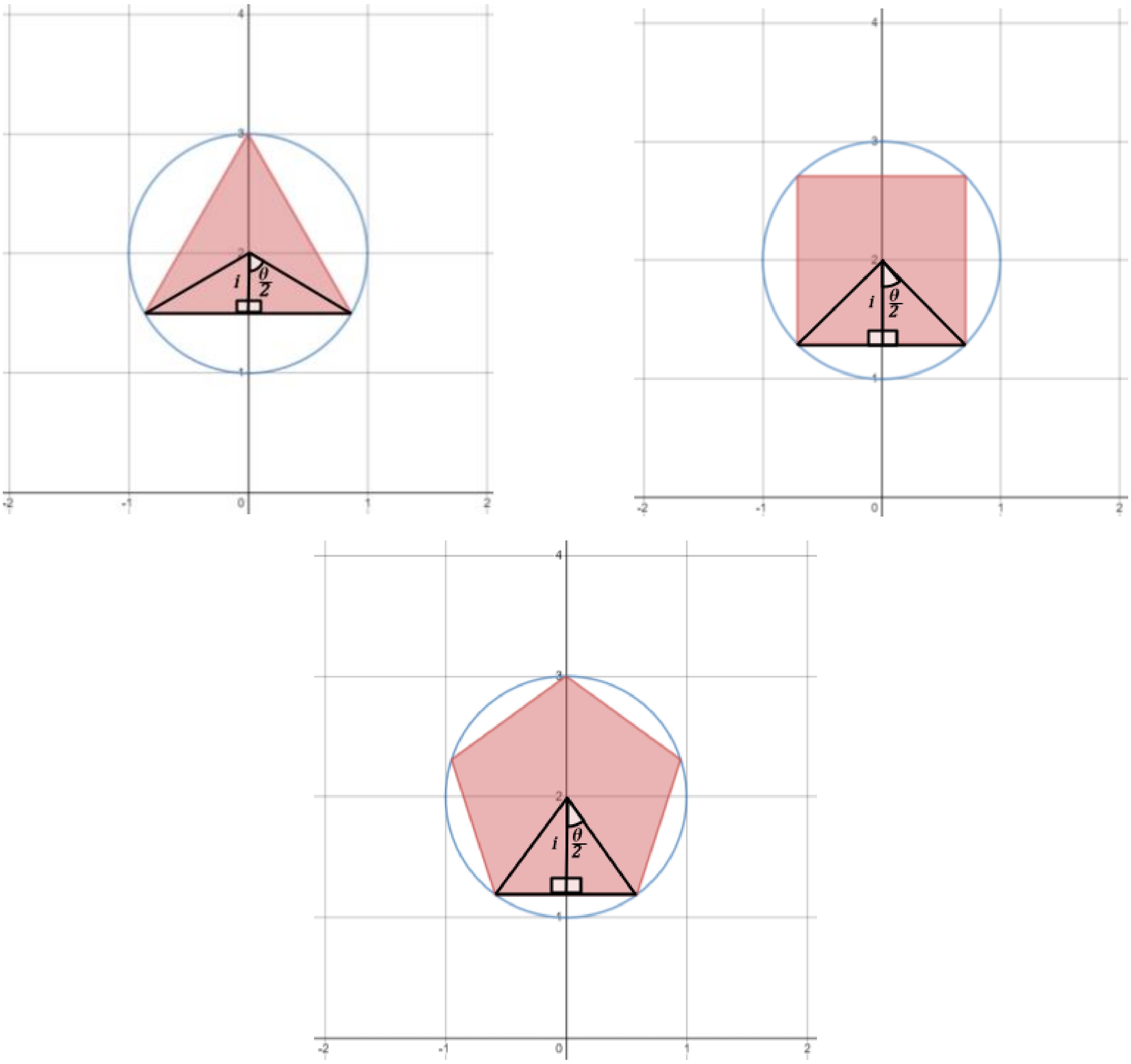


**Fig. 10 –** Line $i$ bisecting $\theta$ to give right-angled triangles for $n = 3$ to 5

Substituting $\theta$ with $\frac{360}{n}$ and rearranging to make $i$ the subject gives us:

$$r \cos \frac{360}{2n} = i$$

And finally, we can substitute this value into the earlier equation $h = a - i$, which gives us the following general equation for finding $h$, given any value of $n$, $a$, and $r$:

$$h = a - r \cos \frac{360}{2n} \qquad \textbf{(Equation 1)}$$

## Finding an equation for the lines of any $n$-sided polygon

To find a general equation for the lines, we can rotate the polygon while keeping the volume of revolution the same to simplify the general formula and make subsequent calculations easier.

To use the rotated polygon, we need to prove that the volume of revolution that is given by the sideways $n$-sided polygons is the same as the volume of revolution given by the unrotated $n$-sided polygons. The proof of this can be seen later (**Page 22**), as we first need to understand how we can derive the equations of the lines and find the volume of revolution.

Now, we will graph rotated $n$-sided polygons from $n = 3$ to $5$. The triangle in **Fig. 11** has $a = 2$ and $r = 1$. This is done so that a pattern can be created to formulate a general formula. To understand this better, we need to take a look at **Fig. 11** and **Fig. 12** below.

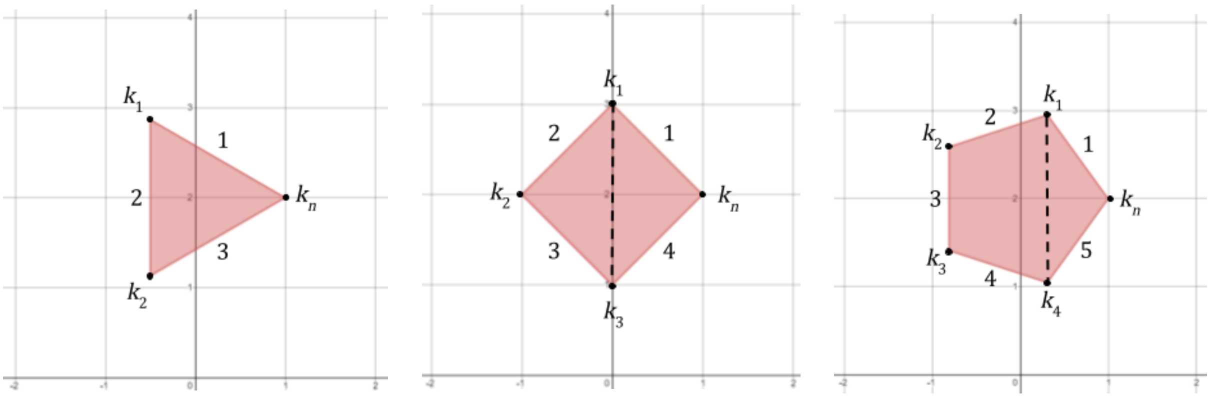



**Fig. 11 –** Rotated polygons with numbered points and lines

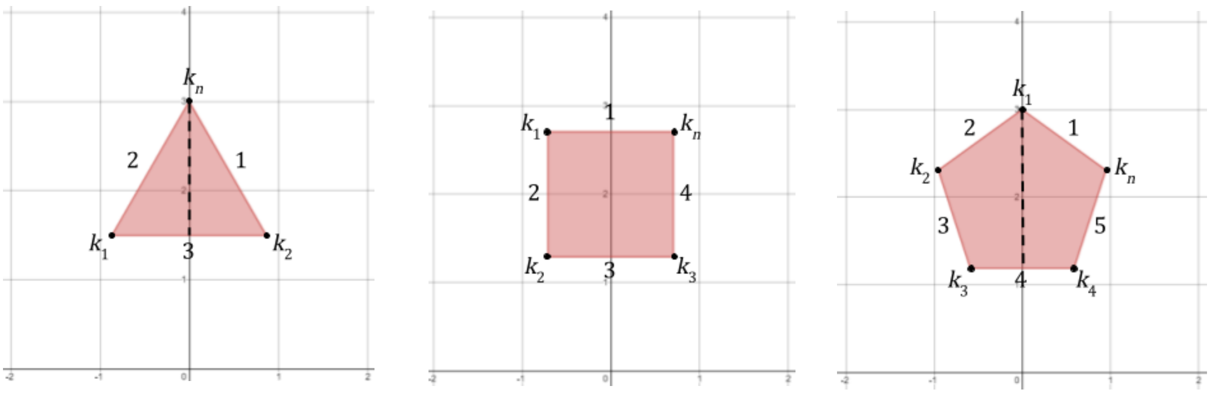



**Fig. 12 –** Unrotated polygons with numbered points and lines

In the case of the first triangle (**Fig. 11**), the first and third line fall within the same range of $x$ values, that is, they match up. However, in the second triangle (**Fig. 12**), there is no pattern that can be observed, as when integrating we must match the top lines with the bottom ones. This is done so that the area enclosed between the graphs can be found, or in my case, the volume of revolution. In this triangle, the first line and third line match up with the second line, while in the pentagon the second line matches up with the third and fourth lines and the first matches with the fourth and fifth lines. As we can see, there is no pattern. This can be seen in the other $n$-sided polygons too. In general, by keeping the polygon sideways, the first line always matches up with the $n^{th}$ line, the second line matches up with the $(n-1)^{th}$ line, and so on, until we reach the vertical line for odd polygons or the left most vertex for even polygons.

There are $n$ number of vertices in an $n$-sided polygon. For a triangle there are three vertices, $k_1, k_2, and\ k_3$ as can be seen in **Fig. 11**. We must find the $x$ and $y$ coordinates of each vertex to find the equation of the lines making up the $n$-sided polygon. We can use a unit circle and trigonometric functions to find the coordinates of each location, as the polygon is inscribed on a circle. $k_n$, or in the case of the triangle, $k_3$, will always be kept constant at the right side, as seen in **Fig. 11**.

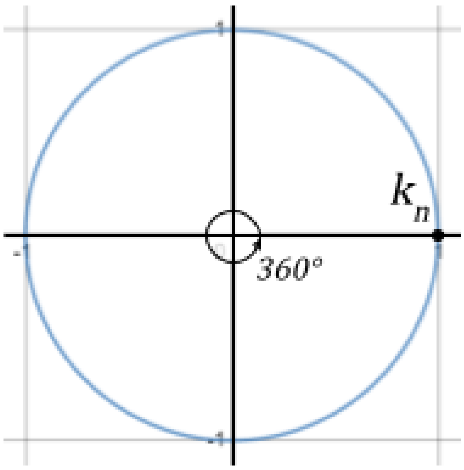



**Fig. 13 –** Unit circle showing point $k_n$ at $(cos(360), sin(360))$, i.e., (0, 1)

To find $k_n$, we must find the coordinate of the fixed point, that is, the right most vertex of the polygon. $cos(360)$ and $sin(360)$ will give us the coordinates (0, 1), as can be seen in the unit circle (**Fig. 13**). Now, this point needs to be translated vertically upwards. As it is at the same $y$-coordinate as the centre of the polygon, it will be translated upwards by $a$. Finally, it must be stretched vertically and horizontally by the scale factor $r$, as this will set the radius of the circle which the polygon is inscribed within. To make it easier to understand, we can imagine horizontally and vertically stretching the circle itself by $r$. This will set the radius of the circle. Thus, we can fix $k_n$ at this point, which will have the equation of –

$$k_n = rcos(360), rsin(360) + a$$

To make a regular polygon, the points are equally distanced on the circumference of the circle. Since we already know the formula for $k_n$, and there should be $n$ number of points for any $n$-sided polygon, we can use the properties of a unit circle to determine the rest of the points. Since there are $n$ number of points to fit in 360°, each point will be equally spaced by $\frac{360}{n}$°. As $k_n$ is at 360°, $k_1$ will be $\frac{360}{n}$° away from $k_n$ (**Fig. 14**). Hence, we can write $k_1$ as –

$$k_1 = rcos\left(360 + \frac{360}{n}\right), rsin\left(360 + \frac{360}{n}\right) + a$$

$$= rcos\left(\frac{360}{n}\right), rsin\left(\frac{360}{n}\right) + a$$

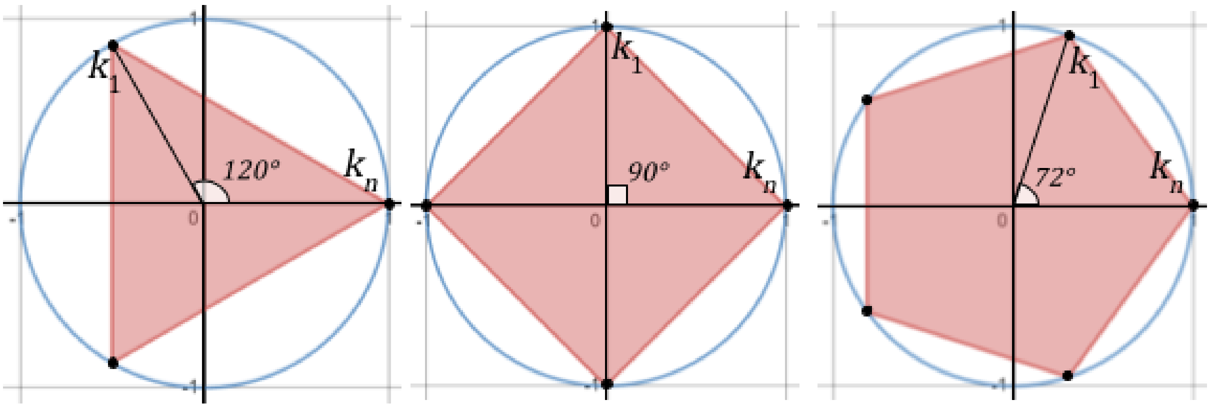



**Fig. 14 –** Polygons showing each vertex is $\frac{360}{n}$° away from each other

The vertices in an $n$-sided polygon are all $\frac{360}{n}$° apart. For example, the second point will be $\frac{360}{n}$° away from the first point, or, in other words, $\frac{720}{n}$° away from the $n^{th}$ point. The third point will be $\frac{360}{n}$° away from the second point, or, in other words, $\frac{1080}{n}$° away from the $n^{th}$ point. This pattern will continue further. Hence, we can obtain the general formula for finding point $k_t$ of the polygon, where $t$ is the vertex number of the $n$-sided polygon.

$$k_t = rcos\left(\frac{360t}{n}\right), rsin\left(\frac{360t}{n}\right) + a \qquad \textbf{(Equation 2)}$$

$$where\ t \in Z+, 0 < t \leq n, \qquad a, r \in R\ and\ r > \frac{a}{2}$$

As we can see, by plugging in $n$ for $t$, the numerator and denominator will cancel out to give us the general formula for the point, $k_n$.

To find the equation of the line we need to make a general formula for the gradient of the line connecting any two consecutive vertices. However, as we want the first line to be between $k_1$ and $k_n$, we want our first line, $L_1$, to connect vertices $k_t\ and\ k_{t-1}$. As can be seen, by plugging in $t$ as 1, we will be able to connect $k_1$ with $k_0$. Here, $k_0 = k_n$, as in a unit circle, going 360° around the circle will give the same coordinates.

The formula to find the gradient of a line is $\frac{y_2 - y_1}{x_2 - x_1}$. As we already know the $x$ and $y$ coordinates of every vertex, we can substitute our previous general formula (**Equation 2**) into this formula to get the gradient of any line, $L_t$ –

$$L_t = \frac{rsin\left(\frac{360(t)}{n}\right) + a - \left(rsin\left(\frac{360(t-1)}{\mathrm{n}}\right) + a\right)}{rcos\left(\frac{360(t)}{n}\right) - \left(rcos\left(\frac{360(t-1)}{n}\right)\right)}$$

$$= \frac{rsin\left(\frac{360(t)}{n}\right) - rsin\left(\frac{360(t-1)}{n}\right)}{rcos\left(\frac{360(t)}{n}\right) - rcos\left(\frac{360(t-1)}{n}\right)}$$

$$= \frac{r\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{r\left(cos\left(\frac{360(t)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)} \qquad \textbf{(Equation 3)}$$

$$= \frac{\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360(t)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)}$$

We can further plug this equation for the gradient (**Equation 3**) into the general equation of a line $y = mx + c$, as we have the general formula of the $x$ and $y$ coordinates of any point $k_t$ as well as the general formula of the gradient $m$. Thus, we can find $c$ as follows.

$$rsin\left(\frac{360t}{n}\right) + a = \frac{\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360(t+1)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)} \times rcos\left(\frac{360t}{n}\right) + c$$

$$c = \left(rsin\left(\frac{360t}{n}\right) + a\right) - \frac{\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360(t)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)} \times rcos\left(\frac{360t}{n}\right)$$

This will give us the final general equation of any line, $L_t$, which would be given in the formula of $y = mx + c$.

$$y = \frac{\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360(t)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)} x + \left(rsin\left(\frac{360t}{n}\right) + a\right) - \frac{\left(sin\left(\frac{360(t)}{n}\right) - sin\left(\frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360(t)}{n}\right) - cos\left(\frac{360(t-1)}{n}\right)\right)} \times rcos\left(\frac{360t}{n}\right)$$

**(Equation 4)**

## Finding the volume of revolution of an odd $n$-sided regular polygon

Now that we have the general formula to find each coordinate and each line, we can move onto finding the volume of revolution. However, we must find the volume of revolution of odd $n$-sided regular polygons and even $n$-sided polygons separately. To understand this, we need to look at **Fig. 15**.

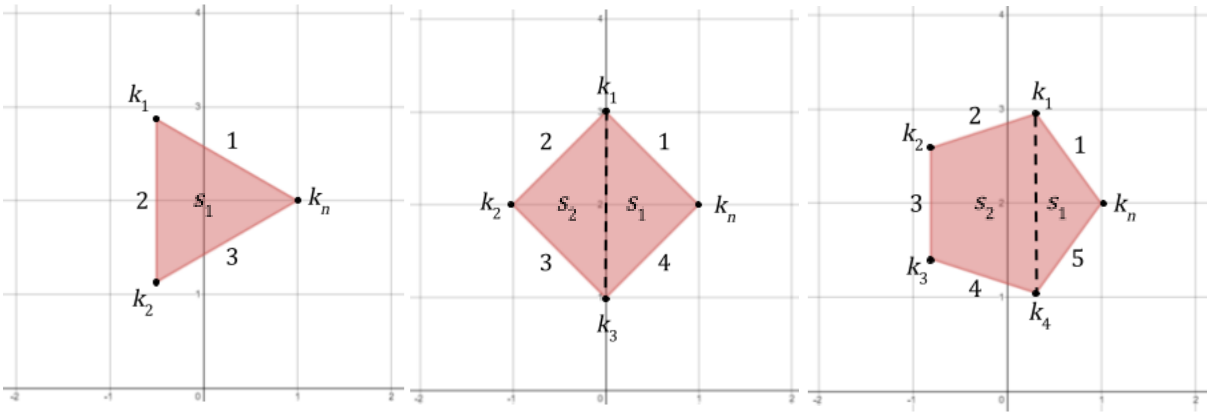



**Fig. 15 –** Segments shown for $n = 3$ to 5

As we can see, if we consider odd and even polygons together, the number of segments (seen in the figures as $S_z$) for a triangle will be 1, for a square and pentagon it will be 2, for a hexagon and heptagon it will be 3, and so on. This gives us the following pattern- 1, 2, 2, 3, 3, and so on, for an $n$-sided polygon (**Fig. 15**). We would not be able to draw out a general formula from this pattern and hence we must simplify it by making two different formulae for odd and even $n$-sided regular polygons.

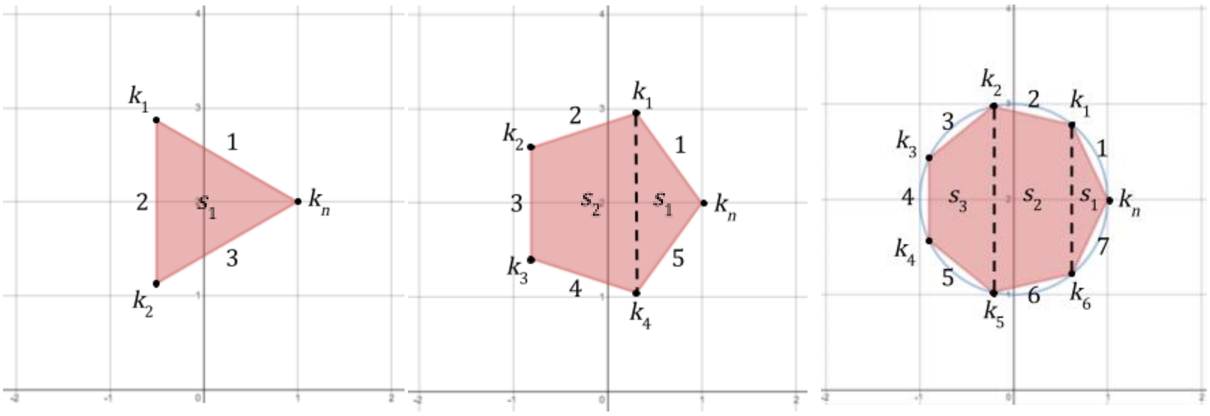



**Fig. 16 –** Segments shown for $n = 3, 5, 7$

By only looking at the segments for odd $n$-sided polygons (**Fig. 16**), a much simpler pattern appears for finding the number of times we need to find the area between the graphs – 1, 2, 3, 4, and so on.

Now we need to relate this pattern to the value of $n$, the number of sides of the polygon, to find a general formula in terms of $n$. Let's define the variable $z$ as the number of segments in the odd polygon for which we must find the volume of revolution.

When $n = 3, z = 1$

When $n = 5, z = 2$

When $n = 7, z = 3$

Using the arithmetic sequence formula, $u_n = d(n-1) + u_1$, where $d$ is the common difference between consecutive terms, $u_n$ is the $n^{th}$ term and $u_1$ is the first term. We are able to find the general formula for the pattern.

$$3, 5, 7, \ldots$$

$$n_z = 2(z-1) + 3 = 2z + 1$$

As $n_z$ is the $z^{th}$ term, it is equal to $n$ and therefore we can write this as-

$$n = 2z + 1$$

$$z = \frac{n-1}{2}, where\ z \in Z+$$

Now that we know the number of segments, we can find the general formula to find the volume of revolution of an odd $n$-sided polygon.

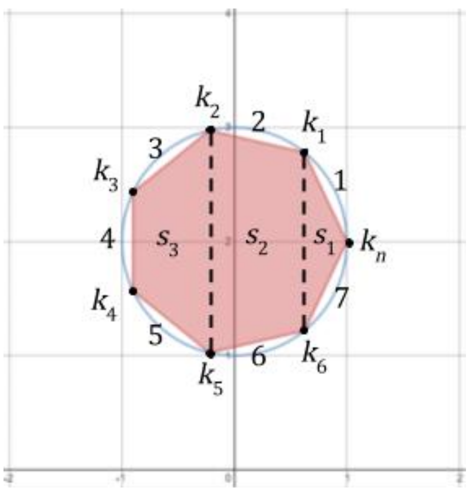



**Fig. 17 –** Heptagon with numbered segments, lines, and points

Now let's consider a regular sided heptagon (**Fig. 17**) and try to find a general pattern for the volume of revolution of an odd $n$-sided polygon. As $n = 7$,

$$z = \frac{6}{2}$$

$$z = 3$$

The volume of revolution is given by –

$$V = \pi \int_a^b f(x)^2 \, dx.$$

Looking at **Fig. 17** we can say that the volume of revolution of the first segment will be given by –

$$\pi \int_{k_1}^{k_n} (L_1)^2 - (L_n)^2 dx$$

Furthermore, the volume of revolution of the second segment will be given by –

$$\pi \int_{k_2}^{k_1} (L_2)^2 - (L_{n-1})^2 dx$$

And finally, the volume of revolution of the third segment will be given by –

$$\pi \int_{k_3}^{k_2} (L_3)^2 - (L_{n-2})^2 dx$$

**We will only use the $x$-coordinates of $k_t$ as we are setting the limits for the integration to take place on the $x$-axis.**

We must add all of them up to get our final answer, the volume of revolution of the heptagon.

$$V = \pi \int_{k_1}^{k_n} (L_1)^2 - (L_n)^2 dx + \pi \int_{k_2}^{k_1} (L_2)^2 - (L_{n-1})^2 dx + \pi \int_{k_3}^{k_2} (L_3)^2 - (L_{n-2})^2 dx$$

Taking $\pi$ common,

$$V = \pi\left(\int_{k_1}^{k_n} (L_1)^2 - (L_n)^2 dx + \int_{k_2}^{k_1} (L_2)^2 - (L_{n-1})^2 dx + \int_{k_3}^{k_2} (L_3)^2 - (L_{n-2})^2 dx\right)$$

Now, we can see a pattern emerging from looking at these three formulae. For the upper boundary, the pattern is given by –

$$k_n, k_1, k_2, \dots, k_p$$

$$where\ \ 0 < p \leq z, \qquad p \in Z+$$

But $k_n$ can be written as $k_0$ since they have the same value, as explained before. This pattern then becomes –

$$k_0, k_1, k_2, \dots, k_p$$

Just looking at the sequence formed, it is –

$$0, 1, 2, \dots, p$$

Plugging this into the arithmetic sequence formula (**Page 15**), we get –

$$u_p = 1(p-1) + 0 = p - 1$$

Plugging this back into our formula for $k$,

$$u_p = k_{p-1}$$

Doing the same for the lower boundary, we get,

$$k_1, k_2, k_3, \dots, k_p \qquad \rightarrow 1, 2, 3, \dots, p$$

$$u_p = 1(p-1) + 1 = p$$

$$u_p = k_p$$

Looking at the pattern for the top line of the area enclosed by the two lines, we get –

$$L_1, L_2, L_3 \dots, L_p \qquad \rightarrow 1, 2, 3, \dots, p$$

$$u_p = 1(p-1) + 1 = p$$

$$u_p = L_p$$

And finally, the pattern for the lower line of the area enclosed by the two lines is –

$$k_n, k_{n-1}, k_{n-2}, \dots, k_p \qquad \rightarrow n, n-1, n-\ 2, \dots, p$$

$$u_p = -1(p-1) + n = n - p + 1$$

$$u_p = L_{n-p+1}$$

Now that we have the general formula for each of the parts, we have to sum it up while incrementing $p$ from 1 to the number of sections we have, i.e., $z$. This can be done with the summation of the sequence.

$$V = \pi \sum_{p=1}^{z} \int_{k_p}^{k_{p-1}} \left(L_p\right)^2 - \left(L_{n-p+1}\right)^2 dx \quad \textbf{(Equation 5)}$$

Where the following is from **Equation 2**. We are only using the $x$ coordinates of the line, as stated earlier.

$$k_{p-1} = rcos\left(\frac{360(p-1)}{n}\right)$$

$$k_p = rcos\left(\frac{360p}{n}\right)$$

The following is from **Equation 4,**

$$L_p = \frac{\left(sin\left(\frac{360(p)}{n}\right) - sin\left(\frac{360(p-1)}{n}\right)\right)}{\left(cos\left(\frac{360(p)}{n}\right) - cos\left(\frac{360(p-1)}{n}\right)\right)}x + \left(rsin\left(\frac{360p}{n}\right) + a\right)$$
$$- \frac{\left(sin\left(\frac{360(p)}{n}\right) - sin\left(\frac{360(p-1)}{n}\right)\right)}{\left(cos\left(\frac{360(p)}{n}\right) - cos\left(\frac{360(p-1)}{n}\right)\right)} \times rcos\left(\frac{360p}{n}\right)$$

$$L_{n-p+1} = \frac{\left(sin\left(\frac{360(n-p+1)}{n}\right) - sin\left(\frac{360(n-p)}{n}\right)\right)}{\left(cos\left(\frac{360(n-p+1)}{n}\right) - cos\left(\frac{360(n-p)}{n}\right)\right)}x + \left(rsin\left(\frac{360(n-p+1)}{n}\right) + a\right)$$

$$- \frac{\left(sin\left(\frac{360(n-p+1)}{n}\right) - sin\left(\frac{360(n-p)}{n}\right)\right)}{\left(cos\left(\frac{360(n-p+1)}{n}\right) - cos\left(\frac{360(n-p)}{n}\right)\right)} \times rcos\left(\frac{360(n-p+1)}{n}\right)$$

## Finding the volume of revolution of an even $n$-sided polygon

We can now follow the same steps to find the general formula for the volume of revolution of even $n$-sided polygons. First, let's find the formula to find the number of segments in the even $n$-sided polygons, $q$, by looking at **Fig. 18**.

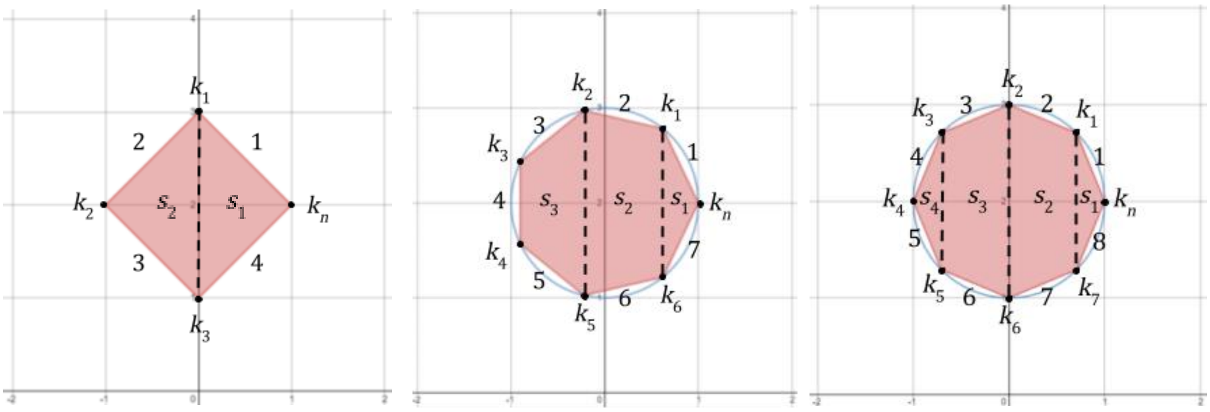



**Fig. 18 –** Segments shown for $n = 4, 6, 8$

When $n = 4, q = 2$

When $n = 6, q = 3$

When $n = 8, q = 4$

Hence the pattern formed will be –

$$4, 6, 8, \ldots$$

$$n = 2(q-1) + 2 = 2z$$

$$z = \frac{q}{2}$$

Now we will consider an octagon (**Fig. 19**) to find the pattern for the even $n$-sided polygon.

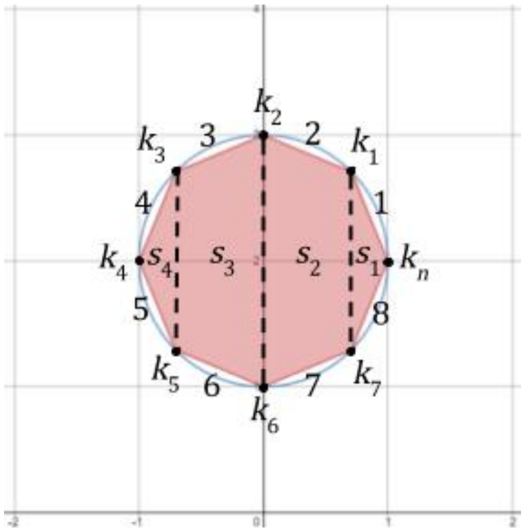



**Fig. 19 –** Octagon with segments, lines, and points numbered

As we can see in **Fig. 19**, the rest of the sequences remain the same. $L_1$corresponds with $L_n$, $L_2$ corresponds with $L_{n-1}$, and so on, just as it did for the heptagon earlier. Hence, the same general formula (**Equation 5**) to find the volume of revolution applies to the even $n$-sided polygons. The only difference is that the formula for finding the number of segments for even polygons, $q$, is different from the formula for finding the number of segments for odd polygons, $z$. Thus, we get –

$$V = \pi \sum_{p=1}^{q} \int_{k_p}^{k_{p-1}} (L_p)^2 - (L_{n-p+1})^2 dx \quad \textbf{(Equation 6)}$$

## Proving that rotating the $\boldsymbol{n}$-sided polygon sideways will have the same volume of revolution

Now that we know how to find the volume of revolution of the polygons, we can prove that the $n$-sided polygons that are rotated sideways have the same volume as the $n$-sided polygons that have not been rotated. First, we prove that the volume of revolution of the two triangles in **Fig. 11** and **12** are the same (where $a = 2$ and $r = 1$). First, finding the volume of revolution of the sideways triangle can be given by the equation we found (**Equation 5**).

$$V = \pi \sum_{p=1}^{1} \int_{rcos\left(\frac{360p}{3}\right)}^{rcos\left(\frac{360(p-1)}{3}\right)} \left( \left( \frac{\left(sin\left(\frac{360(p)}{3}\right) - sin\left(\frac{360(p-1)}{3}\right)\right)}{\left(cos\left(\frac{360(p)}{3}\right) - cos\left(\frac{360(p-1)}{3}\right)\right)} x + \left(rsin\left(\frac{360p}{3}\right) + a\right) - \frac{\left(sin\left(\frac{360(p)}{3}\right) - sin\left(\frac{360(p-1)}{3}\right)\right)}{\left(cos\left(\frac{360(p)}{3}\right) - cos\left(\frac{360(p-1)}{3}\right)\right)} \times rcos\left(\frac{360p}{3}\right) \right)^2 - \left( \frac{\left(sin\left(\frac{360(3-p+1)}{3}\right) - sin\left(\frac{360(3-p)}{3}\right)\right)}{\left(cos\left(\frac{360(3-p+1)}{3}\right) - cos\left(\frac{360(3-p)}{3}\right)\right)} x + \left(rsin\left(\frac{360(3-p+1)}{3}\right) + a\right) - \frac{\left(sin\left(\frac{360(3-p+1)}{3}\right) - sin\left(\frac{360(3-p)}{3}\right)\right)}{\left(cos\left(\frac{360(3-p+1)}{3}\right) - cos\left(\frac{360(3-p)}{3}\right)\right)} \times rcos\left(\frac{360(3-p+1)}{3}\right) \right)^2 \right) dx$$

$$V = \pi \int_{cos(120)}^{cos(0)} \left( \frac{(sin(120) - sin(0))}{(cos(120) - cos(0))} x + \ (sin(120) + 2) - \frac{(sin(120) - sin(0))}{(cos(120) - cos(0))} \times cos(120) \right)^2$$

$$- \left( \frac{(sin(360) - sin(240))}{(cos(360) - cos(240))} x + \ (sin(360) + 2) \right.$$

$$\left. - \frac{(sin(360) - sin(240))}{(cos(360) - cos(240))} \times cos(360) \right)^2 dx$$

$$V = \pi \int_{-\frac{1}{2}}^{1} \left( \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - 1\right)} x + \left(\frac{\sqrt{3}}{2} + 2\right) - \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - 1\right)} \times -\frac{1}{2} \right)^2 - \left( \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(1 + \frac{1}{2}\right)} x + \ (2) - \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(1 + \frac{1}{2}\right)} \right)^2 dx$$

$$\approx 16.32$$

Now we have to find the volume of revolution of the unrotated triangle (**Fig. 20**), but as there is no pattern, we cannot use a formula and must check it by finding the equations and then segments manually. As I have already explained how to find the equation for the coordinates and the lines in an earlier section, I will simply state them here. The $k_n$ point will be at the top at $cos(90), sin(90)$.

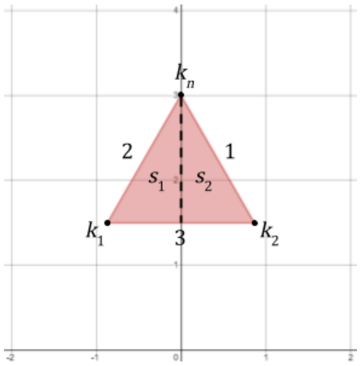



**Fig. 20 –** Segments for an unrotated regular triangle

Equation for the coordinates –

$$rcos\left(90 + \frac{360t}{n}\right), rsin\left(90 + \frac{360t}{n} + a\right)$$

Equation for the line –

$$m = \frac{\left(sin\left(90 + \frac{360t}{n}\right) - sin\left(90 + \frac{360(t-1)}{n}\right)\right)}{\left(cos\left(\frac{360t}{n}\right) - cos\left(90 + \frac{360(t-1)}{n}\right)\right)}$$

$$c = \left(rsin\left(90 + \frac{360t}{n}\right) + a\right) - \frac{\left(sin\left(90 + \frac{360(t)}{n}\right) - sin\left(90 + \frac{360(t-1)}{n}\right)\right)}{\left(cos\left(90 + \frac{360(t)}{n}\right) - cos\left(90 + \frac{360(t-1)}{n}\right)\right)} \times rcos\left(90 + \frac{360t}{n}\right)$$

$$y = \frac{\left(sin\left(90 + \frac{360t}{n}\right) - sin\left(90 + \frac{360(t-1)}{n}\right)\right)}{\left(cos\left(90 + \frac{360t}{n}\right) - cos\left(90 + \frac{360(t-1)}{n}\right)\right)} x + \left(rsin\left(90 + \frac{360t}{n}\right) + a\right)$$
$$- \frac{\left(sin\left(90 + \frac{360(t)}{n}\right) - sin\left(90 + \frac{360(t-1)}{n}\right)\right)}{\left(cos\left(90 + \frac{360(t)}{n}\right) - cos\left(90 + \frac{360(t-1)}{n}\right)\right)} \times rcos\left(90 + \frac{360t}{n}\right)$$

Now, we must find the volume of revolution of the first segment,

$$V = \pi \int_{cos\left(90+\frac{360}{3}\right)}^{cos(90+0)} \left( \frac{\left(sin\left(90 + \frac{360}{3}\right) - sin(90+0)\right)}{\left(cos\left(90 + \frac{360}{3}\right) - cos(90+0)\right)} x + \left(sin\left(90 + \frac{360}{3}\right) + 2\right) \right.$$
$$\left. - \frac{\left(sin\left(90 + \frac{360}{3}\right) - sin(90+0)\right)}{\left(cos\left(90 + \frac{360}{3}\right) - cos(90+0)\right)} \times cos\left(90 + \frac{360}{3}\right) \right)^2$$
$$- \left( \frac{\left(sin\left(90 + \frac{360 \times 2}{3}\right) - sin\left(90 + \frac{360}{3}\right)\right)}{\left(cos\left(\frac{360 \times 2}{3}\right) - cos\left(90 + \frac{360}{3}\right)\right)} x + \left(sin\left(90 + \frac{360 \times 2}{3}\right) + 2\right) \right.$$
$$\left. - \frac{\left(sin\left(90 + \frac{360 \times 2}{3}\right) - sin\left(90 + \frac{360}{3}\right)\right)}{\left(cos\left(90 + \frac{360 \times 2}{3}\right) - cos\left(90 + \frac{360}{3}\right)\right)} \times cos\left(90 + \frac{360 \times 2}{3}\right) \right)^2 dx$$

$$= \pi \int_{-\frac{\sqrt{3}}{2}}^{0} \left( \frac{\left(-\frac{1}{2} - 1\right)}{\left(-\frac{\sqrt{3}}{2}\right)} x + \left(-\frac{1}{2} + 2\right) - \frac{\left(-\frac{1}{2} - 1\right)}{\left(-\frac{\sqrt{3}}{2}\right)} \times -\frac{\sqrt{3}}{2} \right)^2$$

$$- \left( \frac{\left(-\frac{1}{2} + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)} x + \left(-\frac{1}{2} + 2\right) - \frac{\left(-\frac{1}{2} + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)} \times \frac{\sqrt{3}}{2} \right)^2 dx$$

$$= \pi \int_{-\frac{\sqrt{3}}{2}}^{0} \left( \frac{(-1.5)}{\left(-\frac{\sqrt{3}}{2}\right)} x + (1.5) - \frac{(-1.5)}{\left(-\frac{\sqrt{3}}{2}\right)} \times -\frac{\sqrt{3}}{2} \right)^2 - (1.5)^2 dx$$

$$\approx 8.162$$

Then, we must find the volume of revolution of the second segment. As the triangle is split in half to form the two segments, the volume of revolution of segment 1 equals to the volume of revolution of segment 2. Thus, it will be –

$$\approx 8.162$$

Adding up the two segments we get,

$$\approx 16.32$$

This value is equal to the value given by our formula. Hence, the volume of revolution of the rotated triangle and the regular triangle are equal. Therefore, it makes no difference if we use the rotated triangle for the general formula, as it gives the correct answer.

When $n = \infty$, a circle will be formed. Rotating the circle will make no difference, as a circle has no sides.

Since we have proved it for $n = 3$ and $n = \infty$, we can say that it should hold true for all other values in between. This can be said as the volume of revolution for increasing the value of $n$ follows a set pattern starting at 16.32 for a triangle and converging at 39.48 (**Table 1**).

## Analysis and Evaluation

As we now have our final equation, we can create a table and graph to better understand the results by considering various different values of $n$. We can further prove that the equation created is correct by setting $n$ to infinity and using the equation to find the volume of the torus. As long as we keep the dimensions constant, both equations should give us the same result as the one obtained when $n = \infty$, thus validating the equation.

We can also find $h$ as we have $a = 2$ and $r = 1$. According to **Equation 1**, it would be –

$$h = 2 - \cos \frac{360}{2n}$$

We can look at **Table 1** to see the values for $h$ and the volume of revolution of increasing values of $n$ when $a = 2$ and $r = 1$.

| $n$ | $h$ | Volume of Revolution |
|---|---|---|
| **3** | 1.5 | 16.3241942781 |
| **4** | 1.29289321881 | 25.1327412287 |
| **5** | 1.19098300563 | 29.8783216474 |
| **6** | 1.13397459622 | 32.6483885562 |
| **7** | 1.0990311321 | 34.3867445833 |
| **8** | 1.07612046749 | 35.5430635053 |
| **9** | 1.06030737921 | 36.3487829834 |
| **10** | 1.0489434837 | 36.9316366098 |
| **1,000** | 1.0000049348 | 39.4781578473 |
| **10,000** | 1.00000004935 | 39.4784150068 |

**Table 1 –** Volume of Revolution for increasing $n$ values ($a = 2$ and $r = 1$)

The equation to find the volume of a torus can be given by –

$$(2\pi R) \times (\pi r^2)$$

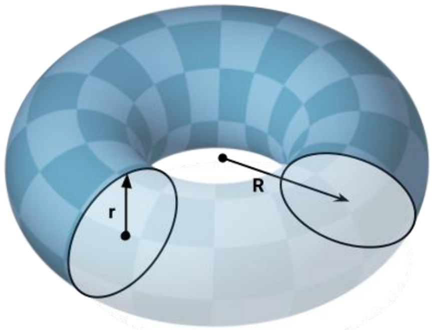



**Fig. 21 –** A diagram of a torus with its variables shown

$R$ and $r$ are seen in **Fig. 21**.

By looking at **Fig. 22**, we can see that $r$ in the formula of the torus is the same as the radius of the circle which the polygon is inscribed within, $r$. $R$ is the distance from the $x$-axis to the centre of the circle, i.e., the centre of the $n$-sided polygon, $a$.

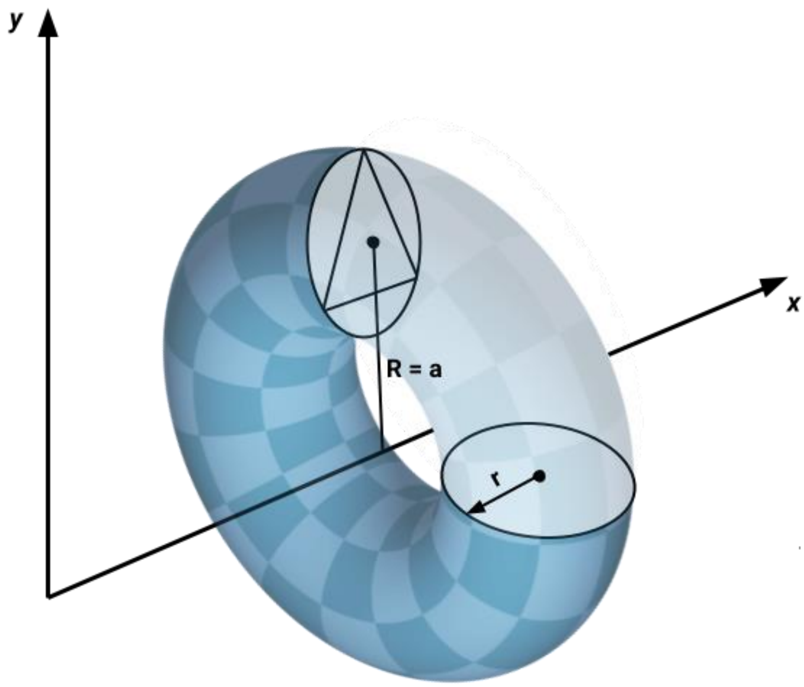



**Fig. 22 –** Comparing the variables $R$ with $a$ by looking at a torus on $x$-axis

Hence, the equation of the torus can be rewritten with my variables –

$$(2\pi a) \times (\pi r^2)$$

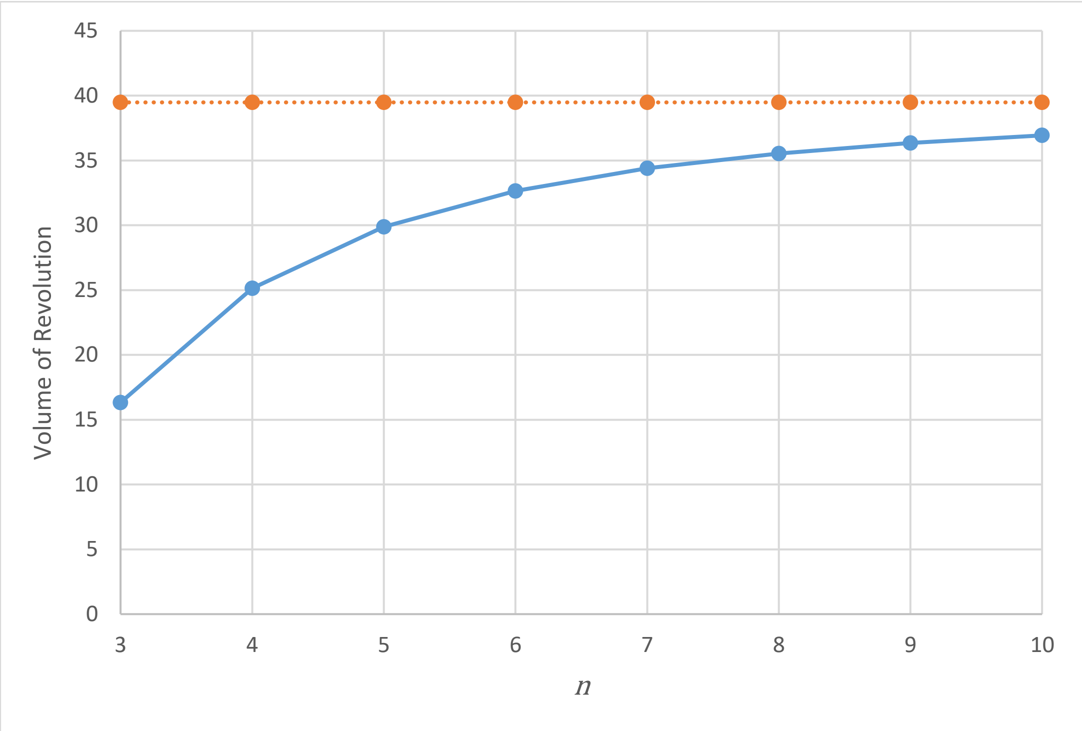
As the table was created for $a = 2$ and $r = 1$, we will use the same values here to compare the volume of revolution when $n$ is at infinity with the volume of the torus –

$$V = (4\pi) \times (\pi) = 4\pi^2$$

$$= 39.4784176 \approx 39.48$$

As we can see, when $n = 10000$, the volume of revolution is also $\approx 39.48$ (from **Table 1**). As I cannot input infinity into the calculator, I set $n$ as 10,000, because it is a large enough number to approximate infinity and makes the polygon almost indistinguishable from a circle. Thus, this can be used to estimate the volume of a torus.

The results of the equation can also be visualised in **Graph 1** –




**Graph 1 –** Volume of Revolution for increasing $n$ values ($a = 2$ and $r = 1$)

The orange line is $y = 39.4784176$, the volume of the torus. This line is an asymptote to the curve formed by the increasing $n$ values, as when $n = \infty$, it will reach this value and $h$ will be equal to 1.

This gives us a mathematical model to calculate the volume of a regular $n$-sided $(n \geq 3, n \in Z+)$ polygon rotated 360° around the $x$-axis to form a ring with variable radius, $h$ $(h \in R+)$, and therefore, answer the research question.

## Conclusion

In conclusion, I have mathematically explored and derived the formula to find the volume of revolution of a regular $n$-sided polygon rotated $360°$ around the $x$-axis. First, I set the parameters for my variables to ensure that only a ring, and not a closed figure, will be formed. I then placed the polygons in a unit circle to try and derive a general formula for the $x$ and $y$ coordinates of each and any of their vertices, $k_t$. By utilizing the coordinates and the formula for finding a gradient of a line, I then found the gradient, $m$, of any straight line formed between consecutive vertices, $L_t$. By doing so, I fit the coordinates and gradient into the general equation for a straight line to get the value of the $y$-intercept of the line, $c$. Once I had my final value, $c$, I simply input my gradient and $y$-intercept into the general formula to find the equation of each and any line, $L_t$. Next, I graphed the polygon in such a way that the volume of revolution would not change but a pattern is visible, and thus, a general formula can be formed. I achieved this by rotating the polygon 90° to the right while keeping it within the circle it was inscribed within. I then split the polygon into $z$ (for odd values of $n$) or $q$ (for even values of $n$) segments so that I could find the volume of revolution of each segment to find the total volume of revolution of the regular $n$-sided polygon. By analysing the pattern formed and plugging it into an arithmetic sequence, I then derived a general formula required for each part of the volume of revolution, i.e., the upper boundary, the lower boundary, and the lines that would be used to find the volume of revolution of the segment. To construct the final equation, I summed up the sequence, starting from $p = 1$ and ending at $z$ or $q$, and then plugged in the equations of the lines. Then I drew a table and graphed my results for clarity. Finally, I proved that the mathematical model derived is correct through using the equation of the volume of a torus and the volume of revolution for when $n = \infty$. As I cannot input infinity as a value into my sequence, I used a large value of 10,000 to approximate the circle and find the volume of revolution. Comparing this to the torus, I validated my mathematical equation that finds the volume of revolution of any regular $n$-sided polygon. Thus, I explored the subject of this extended essay and answered the research question successfully.

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