HAESE MATHEMATICS

Mathematics

analysis and approaches HL 2



Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with

IB Diploma Programme

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Specialists in mathematics education

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MATHEMATICS: ANALYSIS AND APPROACHES HL

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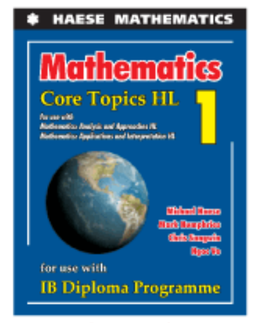
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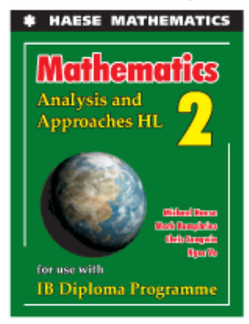
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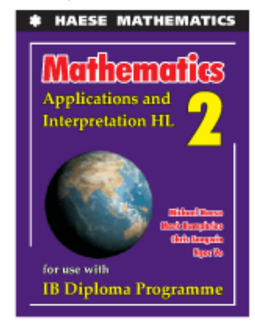
FOREWORD

This book has been written for the International Baccalaureate Diploma Programme course *Mathematics: Analysis and Approaches HL*, for first teaching in August 2019, and first assessment in May 2021.

HL Mathematics







This book is designed to complete the course in conjunction with the **Mathematics: Core Topics HL** textbook. It is expected that students will start using this book approximately 6-7 months into the two-year course, upon the completion of the Mathematics: Core topics HL textbook.

The *Mathematics: Analysis and Approaches* courses have a focus on algebraic rigour, and the book has been written with this focus in mind. The material is presented in a clear, easy-to-follow style, free from unnecessary distractions, while effort has been made to contextualise questions so that students can relate concepts to everyday use.

Each chapter begins with an Opening Problem, offering an insight into the application of the mathematics that will be studied in the chapter. Important information and key notes are highlighted, while worked examples provide step-by-step instructions with concise and relevant explanations. Discussions, Activities, and Investigations are used throughout the chapters to develop understanding, problem solving, and reasoning. A large number of more difficult questions have been included to challenge the HL student.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

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PMH, MAH, CS, NV

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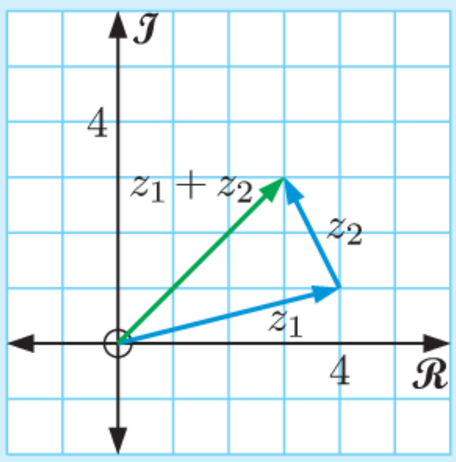
Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Example 2 Self Tutor

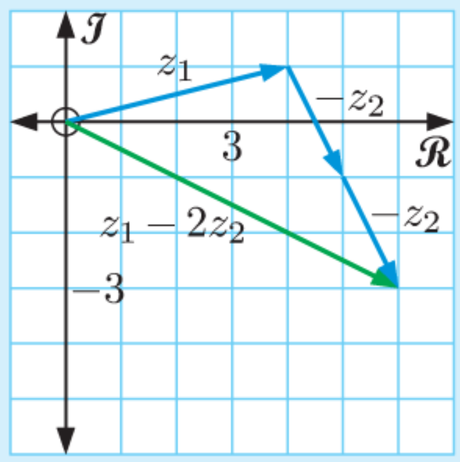
Suppose $z_1 = 4 + i$ and $z_2 = -1 + 2i$. Find using both algebra and vectors:

a $z_1 + z_2$ **b** $z_1 - 2z_2$ **c** $2z_1 + z_2^*$

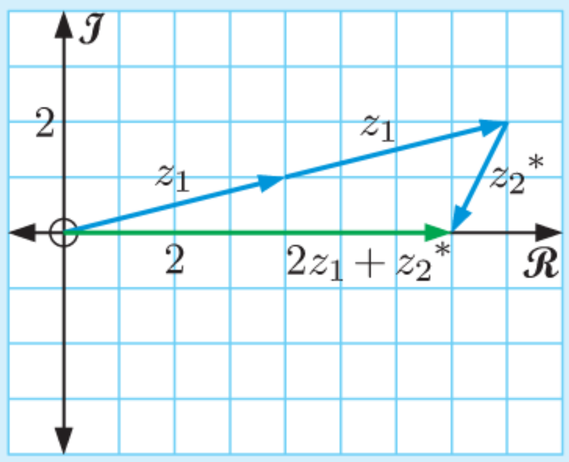
a
$$\begin{aligned} & z_1 + z_2 \\ &= (4 + i) + (-1 + 2i) \\ &= 4 + i - 1 + 2i \\ &= 3 + 3i \end{aligned}$$



b
$$\begin{aligned} & z_1 - 2z_2 \\ &= (4 + i) - 2(-1 + 2i) \\ &= 4 + i + 2 - 4i \\ &= 6 - 3i \end{aligned}$$



c
$$\begin{aligned} & 2z_1 + z_2^* \\ &= 2(4 + i) + (-1 - 2i) \\ &= 8 + 2i - 1 - 2i \\ &= 7 \end{aligned}$$



See **Chapter 14, Complex numbers**, p. 370

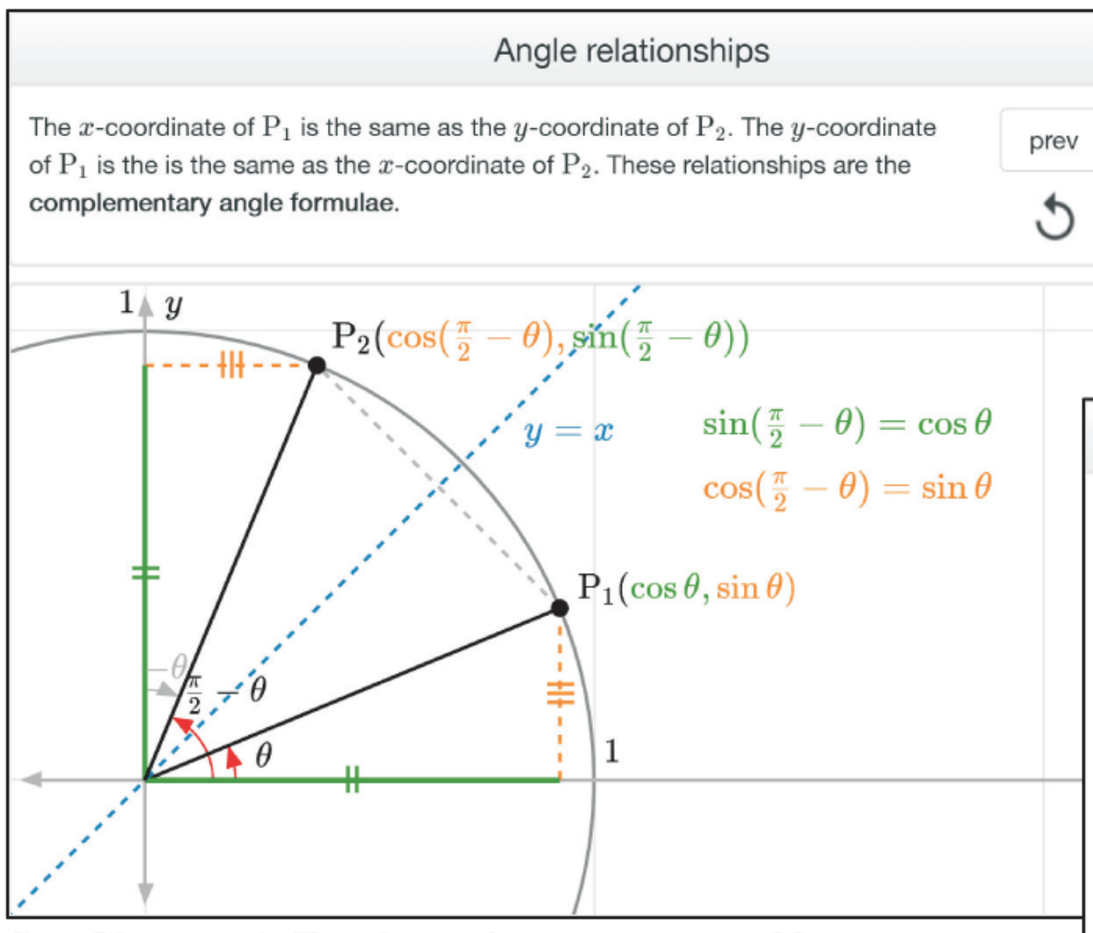
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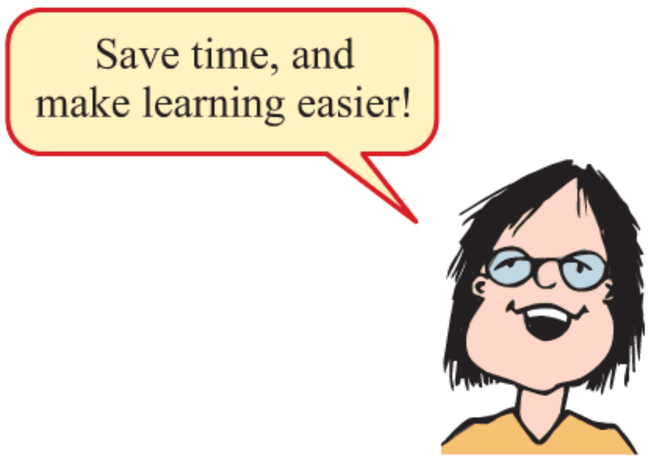
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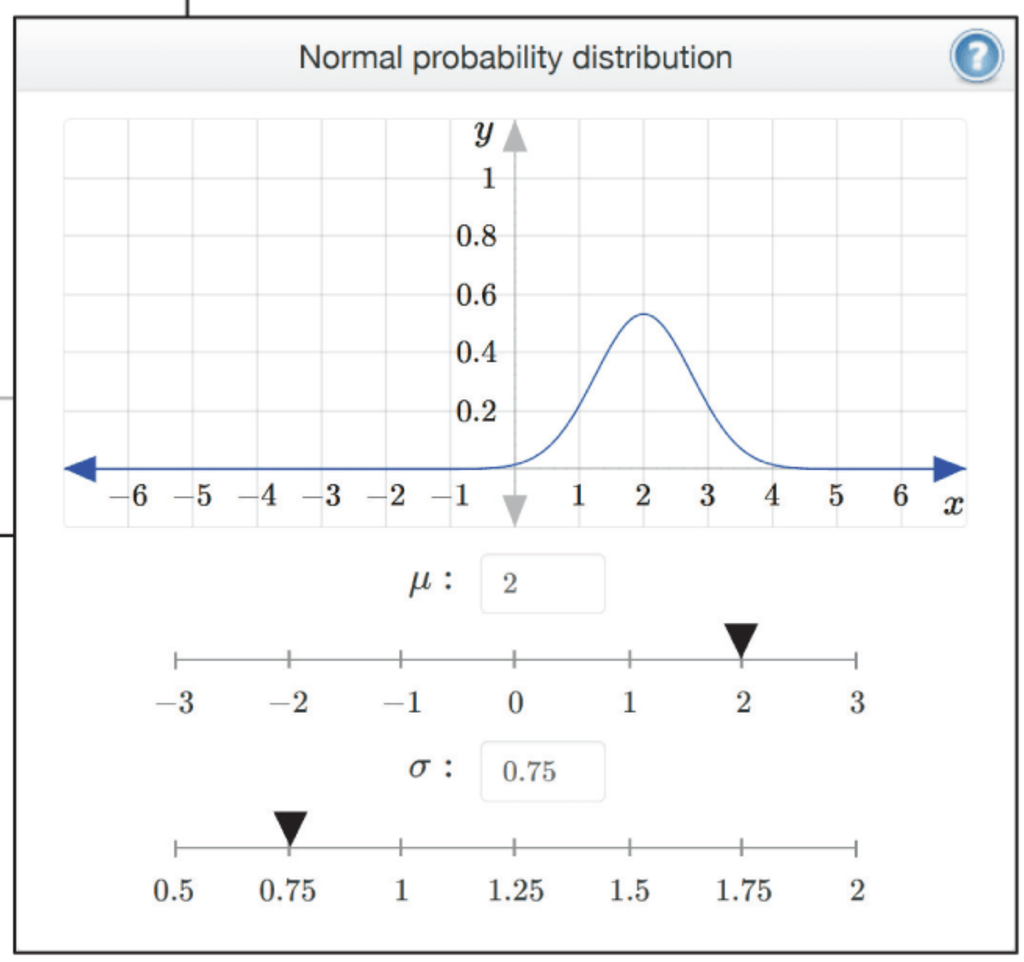


- interactive demonstrations to illustrate and animate concepts
- games and other tools for practising your skills
- graphing and statistics packages which are fast, powerful alternatives to using a graphics calculator
- printable pages to save class time.



See **Chapter 1, Further trigonometry**, p. 23





See **Chapter 28, Continuous random variables**, p. 780

GRAPHICS CALCULATOR INSTRUCTIONS

Graphics calculator instruction booklets are available for the **Casio fx-CG50**, **TI-84 Plus CE**, **TI-*n*spire**, and the **HP Prime**. Click on the relevant icon below.









When additional calculator help may be needed, specific instructions are available from icons within the text.

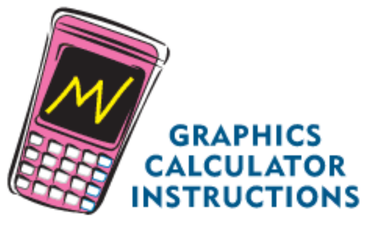


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SYMBOLS AND NOTATION USED IN THIS COURSE

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}'$	the set of irrational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers $\{a + bi \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathcal{Re}(z)$	the real part of z
$\mathcal{Im}(z)$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}}$, $\sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a
$a^{\frac{1}{2}}$, $\sqrt{a}$	a to the power $\frac{1}{2}$, square root of a
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x & \text{for } x \geqslant 0, \quad x \in \mathbb{R} \\ -x & \text{for } x < 0, \quad x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to
$\not> $	is not greater than
$\not< $	is not less than

$[a, b]$	the closed interval $a \leqslant x \leqslant b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\prod_{i=1}^{n} u_i$	$u_1 \times u_2 \times .... \times u_n$
$n!$	$n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1$
$\binom{n}{r}$ or ${}^nC_r$	the rth binomial coefficient, $r = 0, 1, 2, ...$ in the expansion of $(a+b)^n$
$f : A \to B$	f is a function under which each element of set A has an image in set B
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\frac{dy}{dx}$	the derivative of y with respect to x
$\frac{d}{dx}(....)$	the derivative of .... with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\frac{d^ny}{dx^n}$	the nth derivative of y with respect to x
$f^{(n)}(x)$	the nth derivative of $f(x)$ with respect to x
$M_n(x)$	the nth Maclaurin polynomial
$\int y\, dx$	the indefinite integral of y with respect to x

$\int_a^b y\,dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	the logarithm in base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
cosec, sec, cot	the reciprocal circular functions
$\text{cis}\,\theta$	$\cos\theta + i\sin\theta$
A(x, y)	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
PB(A, B)	the perpendicular bisector of [AB]
$\widehat{A}$	the angle at A
C$\widehat{\text{A}}$B	the angle between [CA] and [AB]
$\triangle$ABC	the triangle whose vertices are A, B, and C
$\parallel$	is parallel to
$\perp$	is perpendicular to
v	the vector **v**
$\overrightarrow{\text{AB}}$	the displacement vector represented in magnitude and direction by the directed line segment from A to B
a	the position vector $\overrightarrow{\text{OA}}$
i, **j**, **k**	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\,\mathbf{a}\,\rvert$	the magnitude of vector **a**
$\lvert\,\overrightarrow{\text{AB}}\,\rvert$	the magnitude of $\overrightarrow{\text{AB}}$

$\mathbf{a} \bullet \mathbf{b}$	the scalar product of **a** and **b**
$\mathbf{a} \times \mathbf{b}$	the vector product of **a** and **b**
P(A)	probability of event A
P(A')	probability of the event "not A"
P$(A \mid B)$	probability of the event A given B
$x_1, x_2, \ldots$	observations of a variable
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, x_3, \ldots$ occur
$p_1, p_2, \ldots$	probabilities with which the observations $x_1, x_2, x_3, \ldots$ occur
P$(X = x)$, $P(x)$	probability mass function of the discrete random variable X
$f(x)$	probability density function of the continuous random variable X
E(X)	the expected value of the random variable X
Var (X)	the variance of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
B(n, p)	binomial distribution with n trials and probability of success p
N(μ, σ^2)	normal distribution with mean μ and variance σ^2
$\sim$	is distributed as
z	standardised normal z-score, $z = \dfrac{x - \mu}{\sigma}$
r	Pearson's product-moment correlation coefficient

THEORY OF KNOWLEDGE

Theory of Knowledge is a Core requirement in the International Baccalaureate Diploma Programme.

Students are encouraged to think critically and challenge the assumptions of knowledge. Students should be able to analyse different ways of knowing and areas of knowledge, while considering different cultural and emotional perceptions, fostering an international understanding.

The activities and discussion topics in the below table aim to help students discover and express their views on knowledge issues.

Chapter 3: Logarithms p. 78	**IS MATHEMATICS AN INVENTION OR A DISCOVERY?**
Chapter 9: Reasoning and proof p. 219	**DEFINITIONS**
Chapter 9: Reasoning and proof p. 221	**AXIOMS**
Chapter 13: Vector applications p. 337	**INDEPENDENT DEVELOPMENT OF MATHEMATICS**
Chapter 15: Limits p. 403	**ZENO'S PARADOXES**
Chapter 19: Applications of differentiation p. 537	**SNELL'S LAW**
Chapter 26: Bivariate statistics p. 727	**MATHEMATICAL EXTRAPOLATION**
Chapter 26: Bivariate statistics p. 732	**LINEAR REGRESSION**

THEORY OF KNOWLEDGE

Since the 1970s and 1980s, wage discrimination between men and women has been a topic of debate. During that time, Conway and Roberts[1] published a study which used linear regression to show that on average, women with the same qualifications as men were paid less. This would seem to imply that given the same salary, women would be more qualified. However, when the regression was applied the other way, the opposite conclusion was observed.

1. Can you explain why this occurred?
2. Are these two questions necessarily equivalent?
   - "Given the same qualifications, do men and women earn the same wage?"
   - "Given the same wage, do men and women have the same qualifications?"
3. Is it necessary to consider both regression lines in order to conclude whether discrimination has occurred?
4. Should people be paid according to their qualifications, the job they do, or their capability in doing that job?

[1] Conway, Delores A. and Harry V. Roberts (1983). "Reverse Regression, Fairness, and Employment Discrimination". In: *Journal of Business & Economic Statistics* 1.1, pp. 75-85.

See **Chapter 26**, **Bivariate statistics**, p. 732

USEFUL FORMULAE

TRIGONOMETRIC IDENTITIES

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

DOUBLE ANGLE IDENTITIES

- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
- $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

COMPOUND ANGLE IDENTITIES

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

LAWS OF LOGARITHMS

For $a, b, c, m, n > 0,\ a, c \neq 1$:

- $a^x = b \iff x = \log_a b$
- $\log_a m + \log_a n = \log_a(mn)$
- $\log_a m - \log_a n = \log_a\left(\dfrac{m}{n}\right)$
- $m\log_a b = \log_a(b^m)$
- $\log_b a = \dfrac{\log_c a}{\log_c b}$

SUM AND PRODUCT OF ROOTS OF A POLYNOMIAL

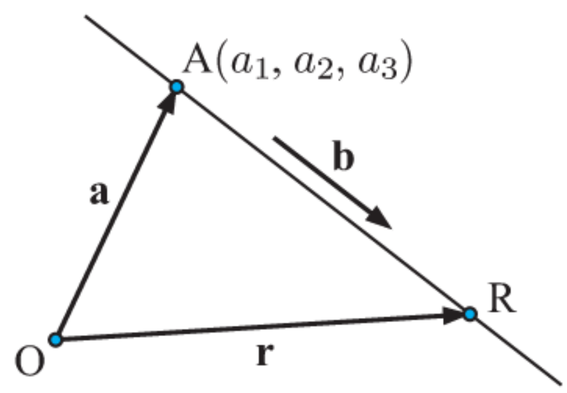
The roots of the polynomial equation $\sum_{r=0}^{n} a_r x^r = 0,\ a_n \neq 0$ have sum $= -\dfrac{a_{n-1}}{a_n}$ and product $= \dfrac{(-1)^n a_0}{a_n}$.

COUNTING AND THE BINOMIAL THEOREM

- $\binom{n}{r} = \begin{cases} \dfrac{n!}{r!(n-r)!}, & n, r \in \mathbb{Z}^+ \\ \dfrac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}, & n \in \mathbb{Q},\ r \in \mathbb{Z}^+ \end{cases}$
- $(a+b)^n = \begin{cases} \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r, & n \in \mathbb{Z}^+ \\ \sum_{r=0}^{\infty} \binom{n}{r} a^{n-r}b^r, & n \in \mathbb{Q} \end{cases}$

VECTOR FORMULAE

- For vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:
  - $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
  - $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
  - $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
  - $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$
  - $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- The line passing through point A with position vector $\mathbf{a}$ in the direction $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ has equation:



  - $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \quad \lambda \in \mathbb{R}$ (vector equation)
  - $x = a_1 + \lambda b_1$
    $y = a_2 + \lambda b_2, \quad \lambda \in \mathbb{R}$ (parametric equations)
    $z = a_3 + \lambda b_3$
  - $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$ (Cartesian equation)

COMPLEX NUMBERS

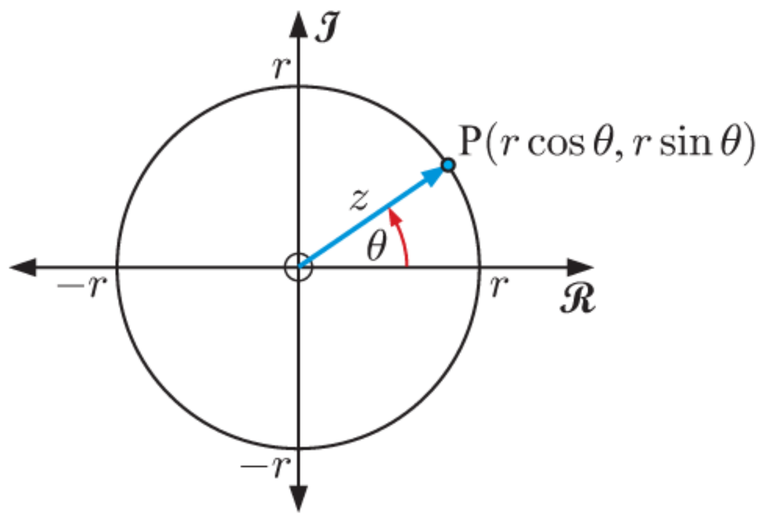
- The complex number $z = a + bi$ has modulus $|z| = \sqrt{a^2 + b^2}$.
- The complex conjugate of z is $z^* = a - bi$.

PROPERTIES OF MODULUS

- $|z^*| = |z|$
- $|z|^2 = zz^*$
- $|z_1z_2| = |z_1||z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$
- $|z_1z_2z_3....z_n| = |z_1||z_2||z_3|....|z_n|$ and $|z^n| = |z|^n$ for $n \in \mathbb{Z}^+$.

POLAR AND EULER FORM

$z = r\,\text{cis}\,\theta$ (polar form)
$= re^{i\theta}$ (Euler form)



PROPERTIES OF $\text{cis}\,\theta$

- $\text{cis}\,\theta \times \text{cis}\,\phi = \text{cis}(\theta + \phi)$
- $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} = \text{cis}(\theta - \phi)$
- $\text{cis}(\theta + k2\pi) = \text{cis}\,\theta, \quad k \in \mathbb{Z}$
- $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,n\theta$ {De Moivre's theorem}

STANDARD DERIVATIVES

Function	*Derivative*
x^n	nx^{n-1}
e^x	e^x
$a^x, \quad a > 0$	$a^x \ln a$
$\ln x$	$\frac{1}{x}, \quad x > 0$
$\log_a x, \quad a > 0$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Function	*Derivative*
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$
$\arctan x$	$\frac{1}{1+x^2}$

RULES OF DIFFERENTIATION

- If $y = uv$ then $\frac{dy}{dx} = u'v + uv'$ {product rule}
- If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}
- If $y = g(u)$ where $u = f(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}
- If $y = f(x)$ then $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$ {first principles}

STANDARD INTEGRALS

Function	*Integral*
k, a constant	$kx + c$
$x^n, \quad n \neq -1$	$\frac{x^{n+1}}{n+1} + c$
e^x	$e^x + c$
$a^x \ln a, \quad a > 0, \quad a \neq 1$	$a^x + c$
$\frac{1}{x}$	$\ln \lvert x \rvert + c, \quad x \neq 0$
$\frac{1}{x \ln a}, \quad a > 0, \quad a \neq 1$	$\log_a \lvert x \rvert + c$

Function	*Integral*
$\cos x$	$\sin x + c$
$-\sin x$	$\cos x + c$
$\sec^2 x$	$\tan x + c$
$-\operatorname{cosec} x \cot x$	$\operatorname{cosec} x + c$
$\sec x \tan x$	$\sec x + c$
$-\operatorname{cosec}^2 x$	$\cot x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\frac{-1}{\sqrt{1-x^2}}$	$\arccos x + c$
$\frac{1}{1+x^2}$	$\arctan x + c$

RULES OF INTEGRATION

- $\int uv' \, dx = uv - \int u'v \, dx$ {integration by parts}
- $\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du$ {integration by substitution}

DEFINITE INTEGRALS

- If $f(x) > 0$ on $a \leqslant x \leqslant b$, the area between $y = f(x)$ and the x-axis on $a \leqslant x \leqslant b$ is $\displaystyle\int_a^b f(x)\,dx$.
- If $f(x)$ is invertible, the area between $y = f(x)$ and the y-axis on $c \leqslant y \leqslant d$ is $\displaystyle\int_c^d \left|\, f^{-1}(y) \,\right| dy = \int_c^d |\, x \,|\, dy$.
- The volume of revolution when $y = f(x)$ is revolved around:
  - the x-axis on $a \leqslant x \leqslant b$ is $\pi \displaystyle\int_a^b y^2\,dx$
  - the y-axis on $c \leqslant y \leqslant d$ is $\pi \displaystyle\int_c^d x^2\,dy$.

KINEMATICS

For an object with displacement s, velocity v, and acceleration a:

- $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$
- Distance travelled from $t = t_1$ to $t = t_2 = \displaystyle\int_{t_1}^{t_2} |\, v(t) \,|\, dt$
- Displacement from $t = t_1$ to $t = t_2 = \displaystyle\int_{t_1}^{t_2} v(t)\,dt$

IMPORTANT MACLAURIN SERIES

- $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots.$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots.$
- $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots., \quad -1 < x \leqslant 1$
- $(1 + x)^p = \displaystyle\sum_{k=0}^{\infty} \frac{p(p-1)\ldots.(p-k+1)}{k!}\,x^k, \quad |\, x \,| < 1, \;\; p \in \mathbb{R}$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots.$
- $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots., \quad |\, x \,| \leqslant 1$

MEASURES OF CENTRE AND SPREAD

Statistic	*Discrete random variable*	*Continuous random variable*
$\mu = \mathrm{E}(X)$	$\sum x_i p_i$	$\displaystyle\int_a^b x\,f(x)\,dx$
$\sigma^2 = \mathrm{Var}(X)$ $= \mathrm{E}[(X-\mu)^2]$ $= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$	$\sum (x_i - \mu)^2 p_i$ $= \sum {x_i}^2 p_i - \mu^2$	$\displaystyle\int_a^b (x-\mu)^2\,f(x)\,dx$ $= \displaystyle\int_a^b x^2\,f(x)\,dx - \mu^2$
σ	$\sqrt{\mathrm{Var}(X)}$	

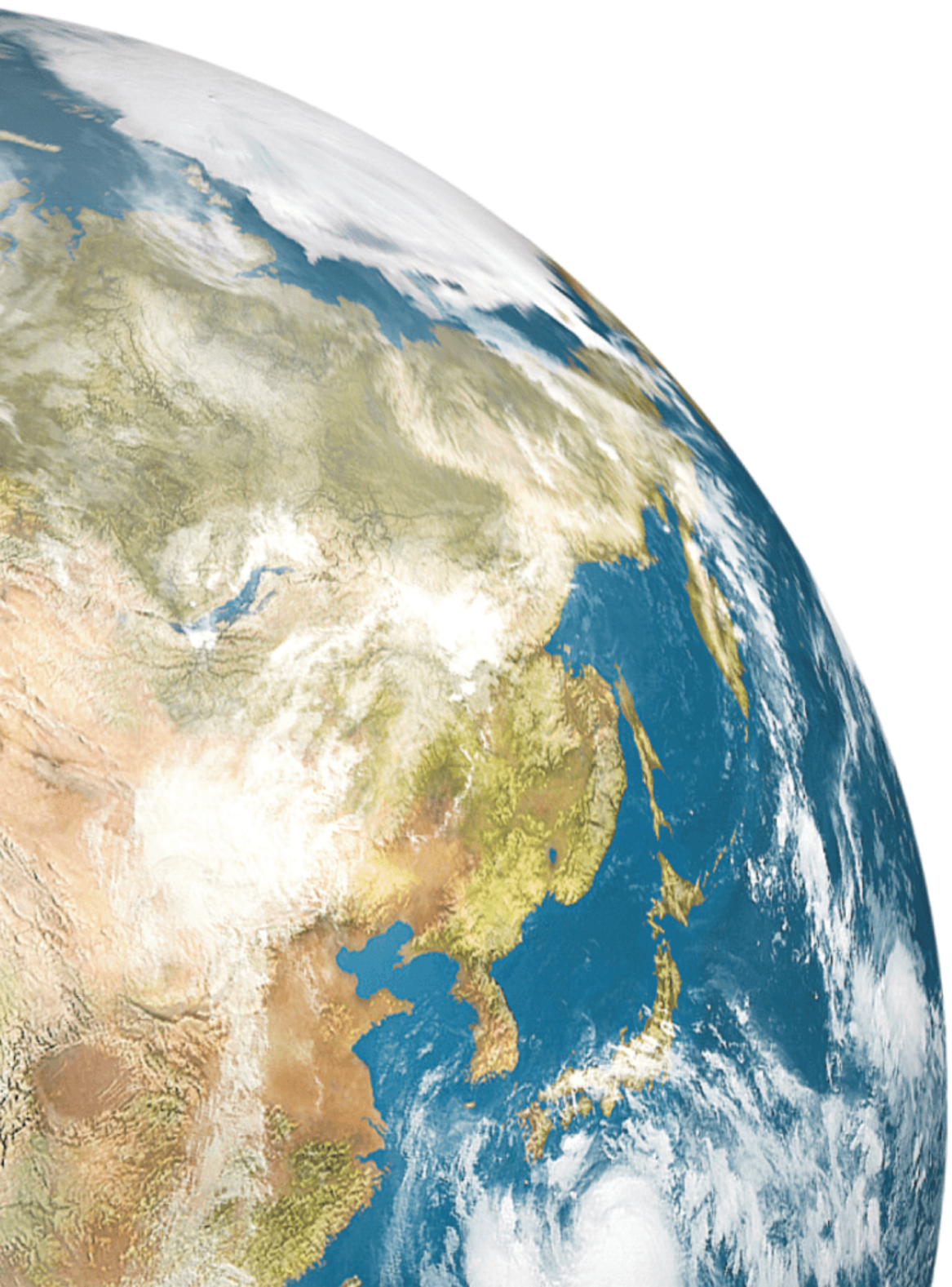
- If $X \sim \mathrm{B}(n, p)$, then $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$.

Chapter 1

Further trigonometry

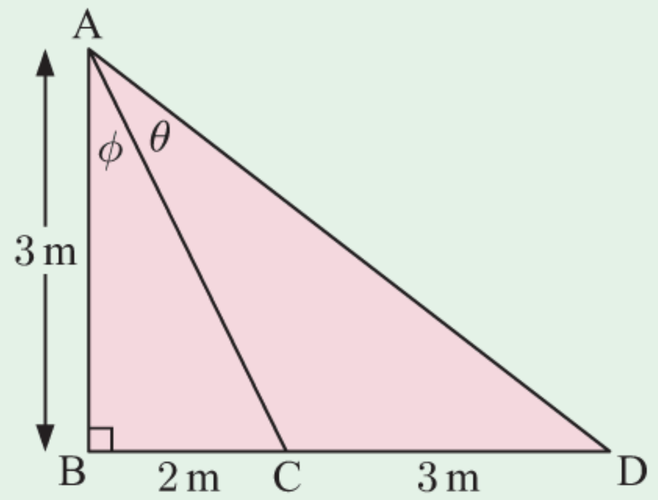
Contents:

A Reciprocal trigonometric functions
B Inverse trigonometric functions
C Algebra with trigonometric functions
D Double angle identities
E Compound angle identities



OPENING PROBLEM

Consider the triangle alongside.



Things to think about:

a Can you find $\tan\theta$ using a combination of Pythagoras' theorem, the cosine rule, and your knowledge of the unit circle?

b Write down the ratios $\tan\phi$ and $\tan(\theta+\phi)$. Is there an easier way to calculate $\tan\theta$ exactly, which involves these ratios?

A RECIPROCAL TRIGONOMETRIC FUNCTIONS

We define the functions cosec x, secant x, and cotangent x as the **reciprocal** functions of $\sin x$, $\cos x$, and $\tan x$ respectively:

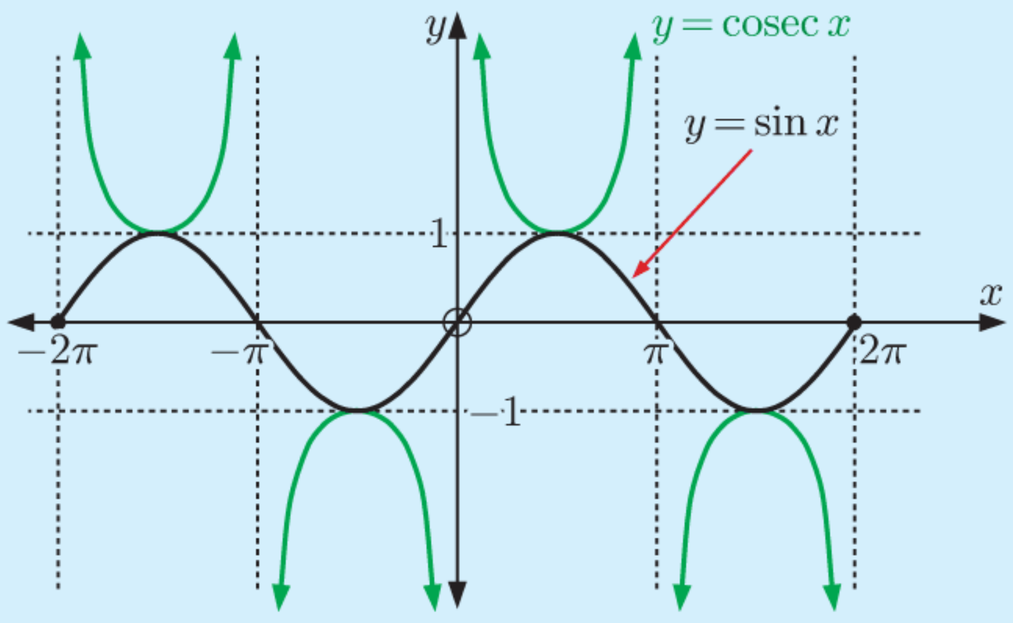
$$\operatorname{cosec} x = \frac{1}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

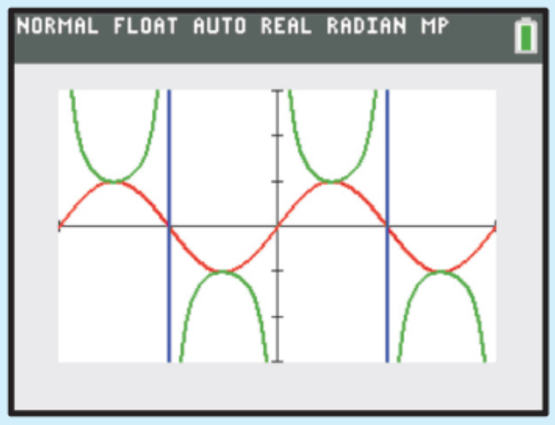
For example, $\operatorname{cosec}\frac{\pi}{6} = \dfrac{1}{\sin\frac{\pi}{6}} = \dfrac{1}{\frac{1}{2}} = 2$.

Example 1 — Self Tutor

Using the graph of $y = \sin x$, sketch the graph of $y = \dfrac{1}{\sin x} = \operatorname{cosec} x$ for $-2\pi \leqslant x \leqslant 2\pi$.
Check your answer using technology.

- The zeros of $y = \sin x$ become vertical asymptotes of $y = \dfrac{1}{\sin x}$.
- The local maxima of $y = \sin x$ become local minima of $y = \dfrac{1}{\sin x}$.
- The local minima of $y = \sin x$ become local maxima of $y = \dfrac{1}{\sin x}$.
- When $\sin x = 1$, $\dfrac{1}{\sin x} = 1$ and when $\sin x = -1$, $\dfrac{1}{\sin x} = -1$.





EXERCISE 1A

1 Without using a calculator, find:

a $\operatorname{cosec}\frac{\pi}{3}$ **b** $\cot\frac{2\pi}{3}$ **c** $\sec\frac{5\pi}{6}$ **d** $\cot\pi$

2 Without using a calculator, find $\operatorname{cosec}x$, $\sec x$, and $\cot x$ given:

a $\sin x = \frac{3}{5}$, $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$

3 Find the other *five* trigonometric ratios if:

a $\cos x = \frac{3}{4}$ and $\frac{3\pi}{2} < x < 2\pi$ **b** $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$

c $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$ **d** $\operatorname{cosec}x = 2$ and $\frac{\pi}{2} < x < \pi$

e $\tan\beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$ **f** $\cot\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Using the graph of $y = \cos x$, sketch the graph of $y = \dfrac{1}{\cos x} = \sec x$ for $-2\pi \leqslant x \leqslant 2\pi$. Check your answer using technology.

5 Using the graph of $y = \tan x$, sketch the graph of $y = \dfrac{1}{\tan x} = \cot x$ for $-2\pi \leqslant x \leqslant 2\pi$.

6 Use technology to sketch $y = \sec x$ and $y = \operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ on the same set of axes for $-2\pi \leqslant x \leqslant 2\pi$. Explain your answer.

7 Solve exactly for $0 \leqslant x \leqslant 2\pi$:

a $\sec x = 2$ **b** $\operatorname{cosec}x = -\sqrt{2}$

c $\sqrt{3}\sec 2x = -2$ **d** $\operatorname{cosec}\left(x + \frac{\pi}{6}\right) + \sqrt{2} = 0$

e $\cot x + 1 = 0$ **f** $\cot\left(2x - \frac{\pi}{4}\right) - \sqrt{3} = 0$

8 Solve exactly for $0 \leqslant x \leqslant 2\pi$:

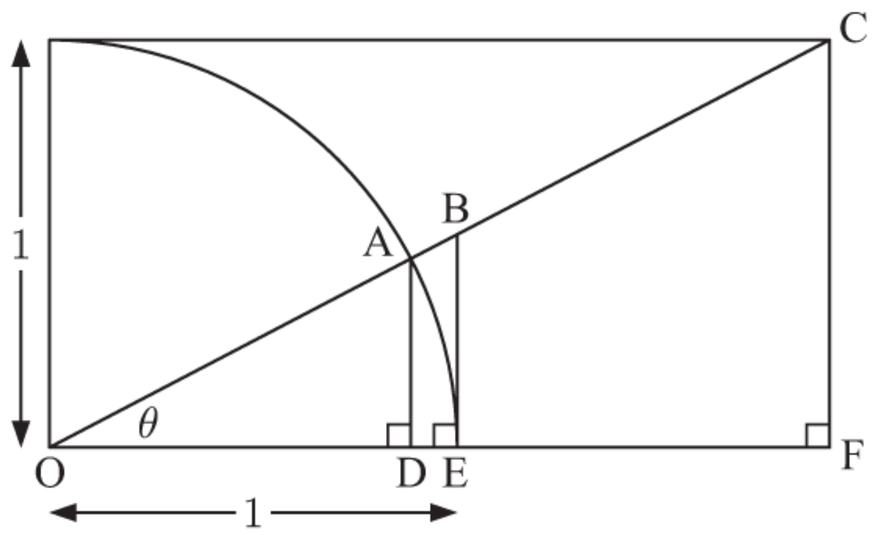
a $2\sin x + \operatorname{cosec}x = 3$ **b** $2\cos x = \sec x$

c $\tan^2 x - 2 - 3\cot^2 x = 0$ **d** $4\sin x = \sqrt{3}\operatorname{cosec}x + 2 - 2\sqrt{3}$

9 Consider the diagram alongside.
Identify a line segment which has length:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

d $\sec\theta$ **e** $\operatorname{cosec}\theta$ **f** $\cot\theta$



HISTORICAL NOTE — ASTRONOMY AND TRIGONOMETRY

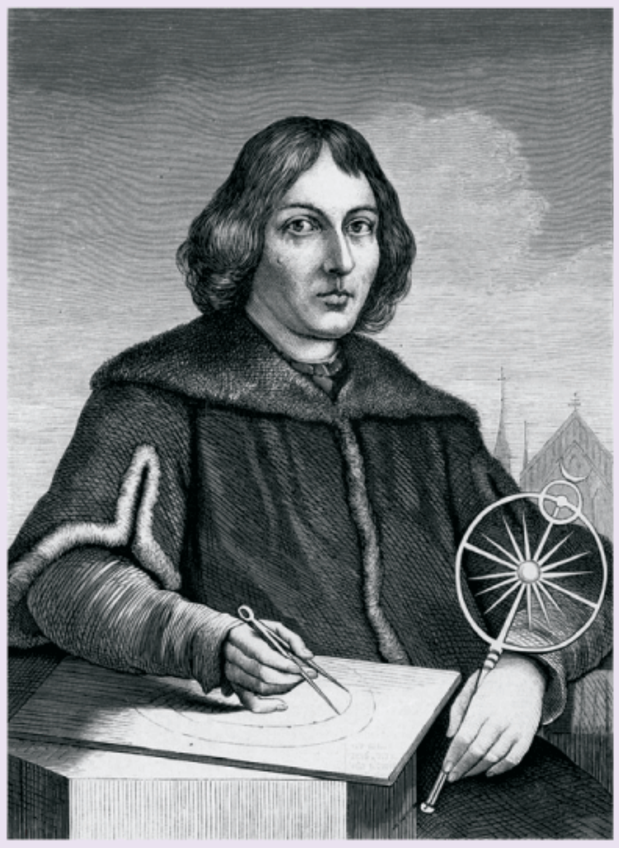
The Greek astronomer **Hipparchus** (140 BC) is credited with being the founder of trigonometry. To aid his astronomical calculations, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest*, which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 5th and 6th Century AD, constructed a table of the lengths of half-chords of a circle with radius one unit. This was the first table of **sine** values.

In the late 16th century, **Georg Joachim de Porris**, also known as **Rheticus**, produced comprehensive and remarkably accurate tables of all six trigonometric ratios. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

Rheticus was the only student of **Nicolaus Copernicus**, and helped his tutor publish his work *De revolutionibus orbium coelestium* (On the Revolutions of the Heavenly Spheres).

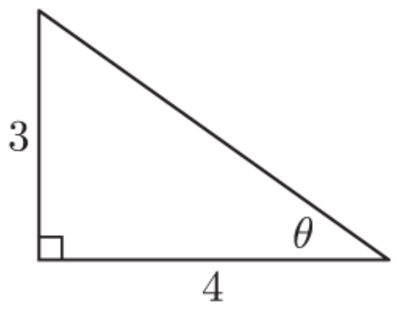


Nicolaus Copernicus

B INVERSE TRIGONOMETRIC FUNCTIONS

In many problems we need to know what angle results in a particular trigonometric ratio. We have previously seen this for right angled triangle problems and when using the cosine and sine rules.

For example:



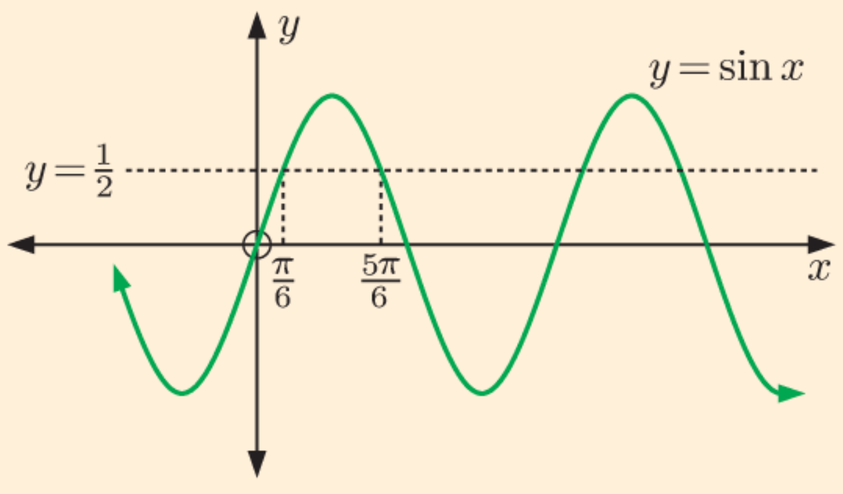
$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{4}$$
$$\therefore\ \theta = \tan^{-1}\left(\tfrac{3}{4}\right)$$
$$\therefore\ \theta \approx 36.9^\circ$$

Rather than writing the inverse trigonometric functions as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, which can be confused with reciprocals, mathematicians more formally refer to these functions as the **inverse trigonometric functions** **arcsine**, **arccosine**, and **arctangent**.

INVESTIGATION 1 INVERSE TRIGONOMETRIC FUNCTIONS

The function $y = \sin x$ fails the horizontal line test on its natural domain. For example, when $y = \frac{1}{2}$, there is more than one corresponding value of x, since $\sin x = \frac{1}{2}$ has infinitely many solutions, including $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Therefore, if we want to find inverse functions for $\sin x$, $\cos x$, and $\tan x$, we will need to apply suitable domain restrictions.



What to do:

1 Draw the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ on the domain $-3\pi \leqslant x \leqslant 3\pi$.

2 Copy and complete the following table, indicating whether each function is one-to-one on the given restricted domain.

Restricted domain	$\sin x$	$\cos x$	$\tan x$
$0 \leqslant x \leqslant 2\pi$			
$-\pi \leqslant x \leqslant \pi$			
$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$			
$0 \leqslant x \leqslant \pi$			
$-\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$			
$\pi \leqslant x \leqslant 2\pi$			

3 Discuss which domain restriction you think would be most suitable for the inverse function of:

a $\sin x$ **b** $\cos x$ **c** $\tan x$.

4 The function $f(x) = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ has inverse $f^{-1}(x) = \arcsin x$.

Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arcsin x$ on the same set of axes.

5 The function $f(x) = \cos x,\ 0 \leqslant x \leqslant \pi$ has inverse $f^{-1}(x) = \arccos x$.

Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arccos x$ on the same set of axes.

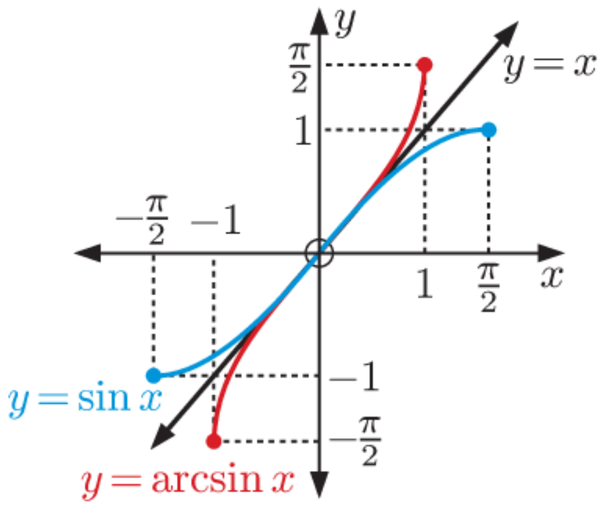
6 The function $f(x) = \tan x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ has inverse $f^{-1}(x) = \arctan x$.

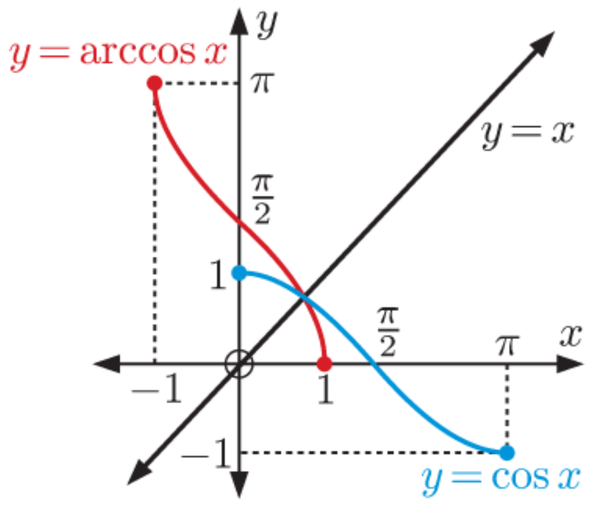
Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arctan x$ on the same set of axes.

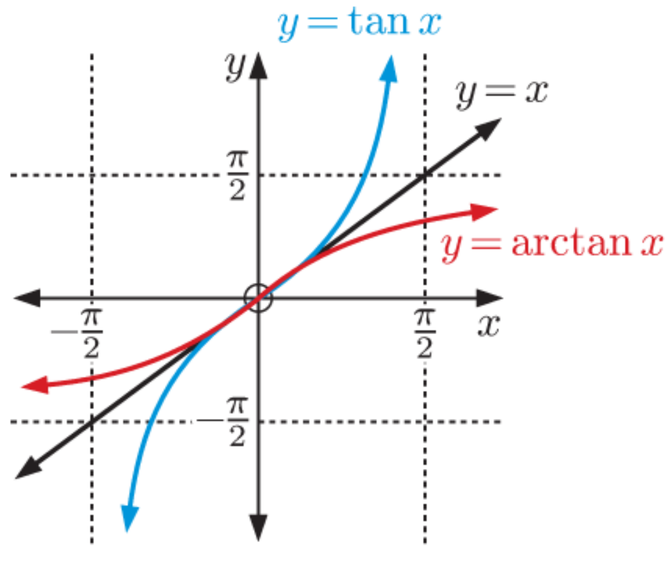
The **inverse trigonometric functions** are defined as:

Function	*Definition*	*Range*
$y = \arcsin x$	$x = \sin y, \quad -1 \leqslant x \leqslant 1$	$-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$
$y = \arccos x$	$x = \cos y, \quad -1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$y = \arctan x$	$x = \tan y, \quad x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these functions are illustrated below, along with the corresponding graphs of $\sin x$, $\cos x$, and $\tan x$ on their restricted domains.







EXERCISE 1B

1 Find, giving your answer in radians:

a $\arccos 1$ **b** $\arcsin(-1)$ **c** $\arctan 1$ **d** $\arctan(-1)$

e $\arcsin \frac{1}{2}$ **f** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **g** $\arctan \sqrt{3}$ **h** $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

i $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **j** $\sin^{-1}(-0.767)$ **k** $\cos^{-1} 0.327$ **l** $\tan^{-1}(-50)$

2 Find the invariant point for the inverse transformation from:

a $y = \sin x$ to $y = \arcsin x$

b $y = \tan x$ to $y = \arctan x$

c $y = \cos x$ to $y = \arccos x$.

An **invariant point** is a point on the function that does not move when a transformation is applied.



3 **a** State the equations of the asymptotes of $y = \arctan x$.

b Do the functions $y = \arcsin x$ and $y = \arccos x$ have vertical asymptotes? Explain your answer.

4 Simplify:

a $\arcsin\left(\sin \frac{\pi}{3}\right)$ **b** $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

c $\tan(\arctan(0.3))$ **d** $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

e $\arctan(\tan \pi)$ **f** $\arcsin\left(\sin \frac{4\pi}{3}\right)$

Remember to think about domain and range.



Example 2 **Self Tutor**

Find, where possible, the exact solutions of:

a $\arctan x = \frac{\pi}{3}$ **b** $\arccos(x-1) = \frac{2\pi}{3}$ **c** $\arcsin x = \frac{2\pi}{3}$

a The range of $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. $\frac{\pi}{3}$ is within the range.

$\therefore \quad x = \tan \frac{\pi}{3} = \sqrt{3}$

b The range of $y = \arccos(x-1)$ is $0 \leqslant y \leqslant \pi$. $\frac{2\pi}{3}$ is within the range.

$\therefore \quad x - 1 = \cos \frac{2\pi}{3}$

$\therefore \quad x - 1 = -\frac{1}{2}$

$\therefore \quad x = \frac{1}{2}$

c The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and $\frac{2\pi}{3}$ is outside this range.

$\therefore$ no solution exists, even though we can find $\sin \frac{2\pi}{3}$.

5 Find, where possible, the exact solutions of:

a $\arctan x = \frac{\pi}{4}$ **b** $\arcsin x = -\frac{\pi}{3}$ **c** $\arccos x = \frac{3\pi}{4}$

d $\arcsin(x+1) = \frac{\pi}{6}$ **e** $\arccos x = -\frac{\pi}{4}$ **f** $\arctan(x - \sqrt{3}) = -\frac{\pi}{3}$

ACTIVITY 1 — $\arctan x$

Carl Friedrich Gauss used his **Gaussian hypergeometric series** to analyse the continued fraction:

$$\cfrac{x}{1+\cfrac{(1x)^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\cfrac{(4x)^2}{9+\ldots.}}}}}$$

What to do:

1. Evaluate the fraction with $x = 1$ for as many levels as necessary for the answer to be accurate to 5 decimal places. You may wish to use a spreadsheet.
2. Compare your result with $\arctan 1$.
3. Compare the continued fraction and $\arctan x$ for another value of x of your choosing.

C ALGEBRA WITH TRIGONOMETRIC FUNCTIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers. $\tan\theta$ is also real whenever it is defined. The algebra of trigonometry is therefore identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$ when we wish to do simplification, and so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

Example 3 — Self Tutor

Simplify:

a $3\cos\theta + 4\cos\theta$ b $\tan\alpha - 3\tan\alpha$

a $3\cos\theta + 4\cos\theta = 7\cos\theta$ {compare with $3x + 4x = 7x$}

b $\tan\alpha - 3\tan\alpha = -2\tan\alpha$ {compare with $x - 3x = -2x$}

ANGLE RELATIONSHIPS

The **negative angle formulae** are established by reflection in the x-axis:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

The **supplementary angle formulae** are established by reflection in the y-axis:

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

The **complementary angle formulae** are established by reflection in the line $y = x$:

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$



The tangent definition $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ enables us to calculate the tangent ratio in each case.

Example 4 **Self Tutor**

Simplify:

a $\sin(-\theta) + 2\sin\theta$ **b** $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos(\pi - \theta)}$

a
$$\begin{aligned}&\sin(-\theta) + 2\sin\theta\\ &= -\sin\theta + 2\sin\theta\\ &= \sin\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\cos(\frac{\pi}{2} - \theta)}{\cos(\pi - \theta)}\\ &= \frac{\sin\theta}{-\cos\theta}\\ &= -\tan\theta\end{aligned}$$

EXERCISE 1C.1

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$ **e** $\tan\theta - 3\tan\theta$ **f** $2\cos^2\theta - 5\cos^2\theta$

2 Simplify:

a $3\tan x - \dfrac{\sin x}{\cos x}$ **b** $\dfrac{\sin^2 x}{\cos^2 x}$ **c** $\tan x \cos x$

d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

3 Simplify:

a $\tan x \cot x$ **b** $\sin x \operatorname{cosec} x$ **c** $\operatorname{cosec} x \cot x$

d $\sin x \cot x$ **e** $\dfrac{\cot x}{\operatorname{cosec} x}$

4 Simplify:

a $3\cos\theta - \cos(-\theta)$ **b** $\tan(-\theta)$ **c** $\sin(-\theta) + \cos\left(\frac{\pi}{2} - \theta\right)$

d $\tan(\pi - \theta)$ **e** $\tan\left(\frac{\pi}{2} - \theta\right)$ **f** $\sin\left(\frac{\pi}{2} - \theta\right) - \cos(\pi - \theta)$

g $\dfrac{\sin(-\theta)}{\cos(\pi - \theta)}$ **h** $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos(-\theta)}$ **i** $\dfrac{\sin(\pi - \theta) - \sin(-\theta)}{\cos(-\theta)}$

THE PYTHAGOREAN IDENTITY

The **Pythagorean identity** is established by applying Pythagoras' theorem on the unit circle:

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

We commonly use rearrangements of these formulae such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

Using these definitions we can also derive the identities:

$$\boldsymbol{\tan^2\theta + 1 = \sec^2\theta} \quad \text{and} \quad \boldsymbol{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

Proof (for the first case):

Using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \{\text{dividing each term by } \cos^2\theta\}$$

$$\therefore \quad \tan^2\theta + 1 = \sec^2\theta$$

Example 5 — Self Tutor

Simplify:

a $2 - 2\sin^2\theta$ **b** $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned} &2 - 2\sin^2\theta \\ &= 2(1 - \sin^2\theta) \\ &= 2\cos^2\theta \end{aligned}$$

b
$$\begin{aligned} &\cos^2\theta\sin\theta + \sin^3\theta \\ &= \sin\theta(\cos^2\theta + \sin^2\theta) \\ &= \sin\theta \end{aligned}$$

EXERCISE 1C.2

1 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$ **b** $-2\sin^2\theta - 2\cos^2\theta$ **c** $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$ **e** $4 - 4\cos^2\theta$ **f** $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$ **h** $\sin^2\theta - 1$ **i** $2\cos^2\theta - 2$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$ **k** $\dfrac{1 - \cos^2\theta}{\sin\theta}$ **l** $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

2 Prove that $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

Example 6 — Self Tutor

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned} &(\cos\theta - \sin\theta)^2 \\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \quad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\} \\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta \\ &= 1 - 2\cos\theta\sin\theta \end{aligned}$$

3 Expand and simplify, if possible:

a $(1 + \sin\theta)^2$ **b** $(\sin\alpha - 2)^2$ **c** $(\tan\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$ **e** $(\sin\beta - \cos\beta)^2$ **f** $-(2 - \cos\alpha)^2$

4 Simplify:

a $1 - \sec^2\beta$ **b** $\dfrac{\tan^2\theta(\cot^2\theta + 1)}{\tan^2\theta + 1}$

c $\cos^2\alpha(\sec^2\alpha - 1)$ **d** $(\sin x + \tan x)(\sin x - \tan x)$

e $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$ **f** $(1 + \operatorname{cosec}\theta)(\sin\theta - \sin^2\theta)$

g $\sec A - \sin A\tan A - \cos A$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 7 Self Tutor

Factorise:

a $\cos^2\alpha - \sin^2\alpha$ **b** $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ $\{a^2 - b^2 = (a+b)(a-b)\}$

b $\tan^2\theta - 3\tan\theta + 2$

$= (\tan\theta - 2)(\tan\theta - 1)$ $\{x^2 - 3x + 2 = (x-2)(x-1)\}$

EXERCISE 1C.3

1 Factorise:

a $1 - \sin^2\theta$ **b** $\sin^2\alpha - \cos^2\alpha$ **c** $\tan^2\alpha - 1$

d $2\sin^2\beta - \sin\beta$ **e** $2\cos\phi + 3\cos^2\phi$ **f** $3\sin^2\theta - 6\sin\theta$

g $\tan^2\theta + 5\tan\theta + 6$ **h** $2\cos^2\theta + 7\cos\theta + 3$ **i** $6\cos^2\alpha - \cos\alpha - 1$

j $3\tan^2\alpha - 2\tan\alpha$ **k** $\sec^2\beta - \operatorname{cosec}^2\beta$ **l** $2\cot^2 x - 3\cot x + 1$

m $2\sin^2 x + 7\sin x\cos x + 3\cos^2 x$

Example 8 Self Tutor

Simplify:

a $\dfrac{2 - 2\cos^2\theta}{1 + \cos\theta}$ **b** $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

a $\dfrac{2 - 2\cos^2\theta}{1 + \cos\theta}$

$= \dfrac{2(1 - \cos^2\theta)}{1 + \cos\theta}$

$= \dfrac{2\cancel{(1 + \cos\theta)}(1 - \cos\theta)}{\cancel{(1 + \cos\theta)}}$

$= 2(1 - \cos\theta)$

b $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{\cancel{(\cos\theta - \sin\theta)}}{(\cos\theta + \sin\theta)\cancel{(\cos\theta - \sin\theta)}}$

$= \dfrac{1}{\cos\theta + \sin\theta}$

2 Simplify:

a $\dfrac{1 - \sin^2\alpha}{1 - \sin\alpha}$ **b** $\dfrac{\tan^2\beta - 1}{\tan\beta + 1}$ **c** $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi + \sin\phi}$

d $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$ **e** $\dfrac{\sin\alpha + \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$ **f** $\dfrac{3 - 3\sin^2\theta}{6\cos\theta}$

g $1 - \dfrac{\cos^2\theta}{1 + \sin\theta}$ **h** $\dfrac{1 + \cot\theta}{\operatorname{cosec}\theta} - \dfrac{\sec\theta}{\tan\theta + \cot\theta}$ **i** $\dfrac{\tan^2\theta}{\sec\theta - 1}$

3 Show that:

a $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$ **b** $(\sin\theta + 4\cos\theta)^2 + (4\sin\theta - \cos\theta)^2 = 17$

c $(1 - \cos\theta)\left(1 + \dfrac{1}{\cos\theta}\right) = \tan\theta\sin\theta$ **d** $\left(1 + \dfrac{1}{\sin\theta}\right)(\sin\theta - \sin^2\theta) = \cos^2\theta$

e $\sec A - \cos A = \tan A \sin A$

f $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha} + \dfrac{\sin\alpha}{1-\cot\alpha} = \sin\alpha + \cos\alpha$

h $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

i $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\sin\theta}{1+\cos\theta} = 2\cot\theta$

j $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

GRAPHING PACKAGE

Check these simplifications by graphing both sides of the equations on the same set of axes.

4 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\cos^2 x = \sin x + 1$

b $\sin^2 x = 2 - \cos x$

c $2\cos^2 x = 3\sin x$

d $2\tan^2 x + 3\sec^2 x = 7$

5 For $-\pi \leqslant x \leqslant \pi$, find the exact solutions of $3\sec 2x = \cot 2x + 3\tan 2x$.

D DOUBLE ANGLE IDENTITIES

INVESTIGATION 2 DOUBLE ANGLE IDENTITIES

What to do:

1 Copy and complete this table using your calculator. Include extra lines for angles of your choice.

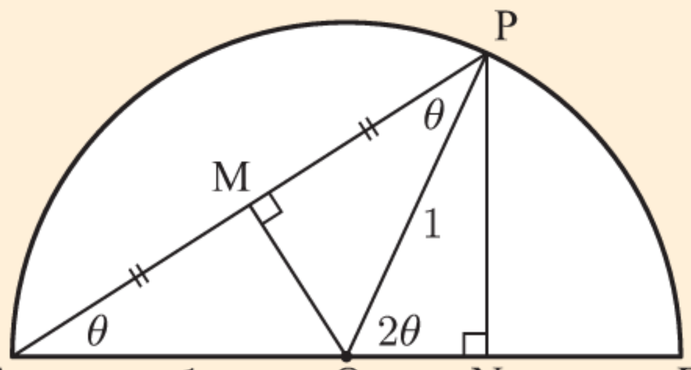
θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta - \sin^2\theta$
0.631						
57.81°						
−3.697						
⋮						

2 Write down any discoveries from your table of values.

3 In the diagram alongside, the semi-circle has radius 1 unit, and $\widehat{PAB} = \theta$.

$\widehat{APO} = \theta$ {△AOP is isosceles}

$\widehat{PON} = 2\theta$ {exterior angle of a triangle}



a Find in terms of θ, the lengths of:

i [OM] **ii** [AM]

iii [ON] **iv** [PN]

b Use △ANP and the lengths in **a** to show that:

i $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$

ii $\cos\theta = \dfrac{1+\cos 2\theta}{2\cos\theta}$

c Hence deduce that:

i $\sin 2\theta = 2\sin\theta\cos\theta$

ii $\cos 2\theta = 2\cos^2\theta - 1$

d For what values of θ have we proven the identities in **c**?

The formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$ found in the **Investigation** are in fact true for all angles θ.

Using $\cos^2\theta = 1 - \sin^2\theta$, we can also show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

and $\cos 2\theta = 1 - 2\sin^2\theta$.

Using the definition of tangent, we find

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

So, the **double angle identities** are:

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

Example 9 — **Self Tutor**

Given that $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$ find:

a $\sin 2\alpha$ **b** $\cos 2\alpha$ **c** $\tan 2\alpha$

a
$$\begin{aligned}
&\sin 2\alpha \\
&= 2\sin\alpha\cos\alpha \\
&= 2\left(\tfrac{3}{5}\right)\left(-\tfrac{4}{5}\right) \\
&= -\tfrac{24}{25}
\end{aligned}$$

b
$$\begin{aligned}
&\cos 2\alpha \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \left(-\tfrac{4}{5}\right)^2 - \left(\tfrac{3}{5}\right)^2 \\
&= \tfrac{7}{25}
\end{aligned}$$

c
$$\begin{aligned}
&\tan 2\alpha \\
&= \frac{\sin 2\alpha}{\cos 2\alpha} \\
&= \frac{-\frac{24}{25}}{\frac{7}{25}} \quad \{\text{using } \mathbf{a}, \mathbf{b}\} \\
&= -\tfrac{24}{7}
\end{aligned}$$

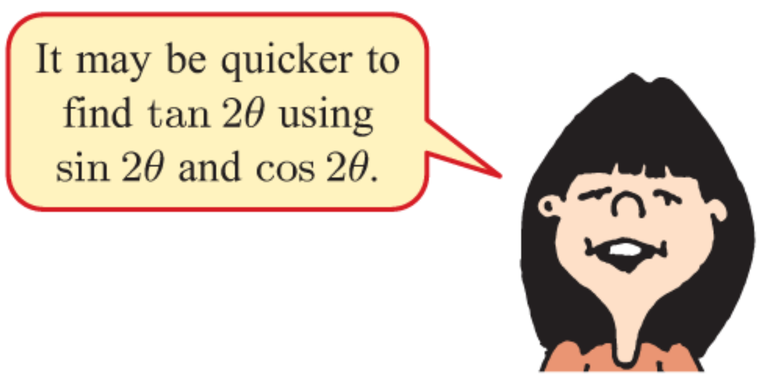
EXERCISE 1D

1 For $\theta = 30°$, verify that:

a $\sin 2\theta = 2\sin\theta\cos\theta$ **b** $\cos 2\theta = \cos^2\theta - \sin^2\theta$ **c** $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

2 If $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$ find the exact value of:

a $\sin 2\theta$ **b** $\cos 2\theta$ **c** $\tan 2\theta$



3 **a** If $\cos A = \frac{1}{3}$, find $\cos 2A$.

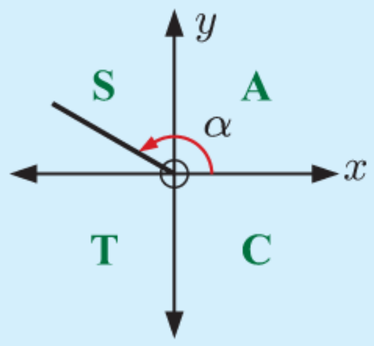
b If $\sin\phi = -\frac{2}{3}$, find $\cos 2\phi$.

Example 10

Self Tutor

If $\sin\alpha = \frac{5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$, find the exact value of: **a** $\sin 2\alpha$ **b** $\tan 2\alpha$.

α is in quadrant 2, so $\cos\alpha$ is negative.



Now $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \cos^2\alpha + \frac{25}{169} = 1$

$\therefore \cos^2\alpha = \frac{144}{169}$

$\therefore \cos\alpha = \pm\frac{12}{13}$

$\therefore \cos\alpha = -\frac{12}{13}$ $\{\text{as } \frac{\pi}{2} < \alpha < \pi\}$

a $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\therefore \sin 2\alpha = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right)$

$= -\frac{120}{169}$

b $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = -\frac{5}{12}$

$\therefore \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$= \frac{2\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2}$

$= \frac{-\frac{5}{6}}{1-\frac{25}{144}}$

$= -\frac{120}{119}$

4 If $\sin\alpha = -\frac{2}{3}$ where $\pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

a $\cos\alpha$ **b** $\sin 2\alpha$ **c** $\tan 2\alpha$

5 If $\cos\beta = \frac{2}{5}$ where $270° < \beta < 360°$, find the exact value of:

a $\sin\beta$ **b** $\sin 2\beta$ **c** $\tan 2\beta$

Example 11

Self Tutor

If α is acute and $\cos 2\alpha = \frac{3}{4}$ find the exact value of: **a** $\cos\alpha$ **b** $\sin\alpha$.

a $\cos 2\alpha = 2\cos^2\alpha - 1$

$\therefore \frac{3}{4} = 2\cos^2\alpha - 1$

$\therefore \cos^2\alpha = \frac{7}{8}$

$\therefore \cos\alpha = \pm\frac{\sqrt{7}}{2\sqrt{2}}$

$\therefore \cos\alpha = \frac{\sqrt{7}}{2\sqrt{2}}$

$\{\text{as } \alpha \text{ is acute, } \cos\alpha > 0\}$

b $\sin\alpha = \sqrt{1-\cos^2\alpha}$

$\{\text{as } \alpha \text{ is acute, } \sin\alpha > 0\}$

$\therefore \sin\alpha = \sqrt{1-\frac{7}{8}}$

$\therefore \sin\alpha = \sqrt{\frac{1}{8}}$

$\therefore \sin\alpha = \frac{1}{2\sqrt{2}}$

6 If α is acute and $\cos 2\alpha = -\frac{7}{9}$, find without a calculator:

a $\cos\alpha$ **b** $\sin\alpha$

7 If θ is obtuse and $\cos 2\theta = -\frac{1}{3}$, find the exact value of:

a $\cos\theta$ **b** $\sin\theta$

8 Find the exact value of $\tan A$ if:

a $\tan 2A = \frac{21}{20}$ and A is obtuse **b** $\tan 2A = -\frac{12}{5}$ and A is acute.

9 Find the exact value of $\tan\frac{\pi}{8}$.

Example 12 **Self Tutor**

Use an appropriate double angle identity to simplify:

a $3\sin\theta\cos\theta$ **b** $4\cos^2 2B - 2$

a
$$\begin{aligned} & 3\sin\theta\cos\theta \\ &= \tfrac{3}{2}(2\sin\theta\cos\theta) \\ &= \tfrac{3}{2}\sin 2\theta \end{aligned}$$

b
$$\begin{aligned} & 4\cos^2 2B - 2 \\ &= 2(2\cos^2 2B - 1) \\ &= 2\cos 2(2B) \\ &= 2\cos 4B \end{aligned}$$

10 Use an appropriate double angle identity to simplify:

a $2\sin\alpha\cos\alpha$ **b** $4\cos\alpha\sin\alpha$ **c** $\sin\alpha\cos\alpha$

d $2\cos^2\beta - 1$ **e** $1 - 2\cos^2\phi$ **f** $1 - 2\sin^2 N$

g $2\sin^2 M - 1$ **h** $\cos^2\alpha - \sin^2\alpha$ **i** $\sin^2\alpha - \cos^2\alpha$

j $2\sin 2A\cos 2A$ **k** $2\cos 3\alpha\sin 3\alpha$ **l** $2\cos^2 4\theta - 1$

m $1 - 2\cos^2 3\beta$ **n** $1 - 2\sin^2 5\alpha$ **o** $2\sin^2 3D - 1$

p $\cos^2 2A - \sin^2 2A$ **q** $\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)$ **r** $2\sin^2 3P - 2\cos^2 3P$

11 Find the exact value of $\left[\cos\frac{\pi}{12} + \sin\frac{\pi}{12}\right]^2$.

12 Show that:

a $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ **b** $\cos^4\theta - \sin^4\theta = \cos 2\theta$

Check these identities by graphing both sides of the equations on the same set of axes.

GRAPHING PACKAGE

13 Solve exactly for x where $0 \leqslant x \leqslant 2\pi$:

a $\sin 2x + \sin x = 0$ **b** $\sin 2x - 2\cos x = 0$ **c** $\sin 2x + 3\sin x = 0$

14 **a** Use a double angle identity to show that:

i $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ **ii** $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

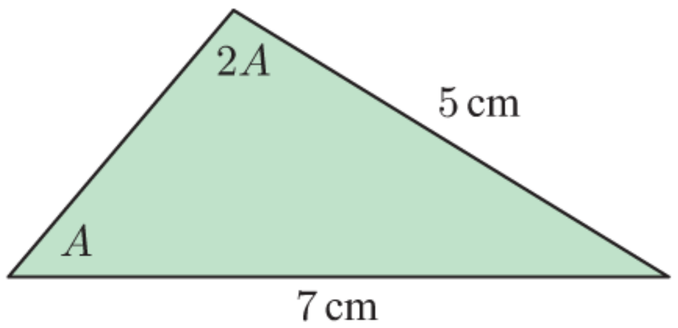
b Hence show that:

i $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos\theta$ **ii** $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\theta$

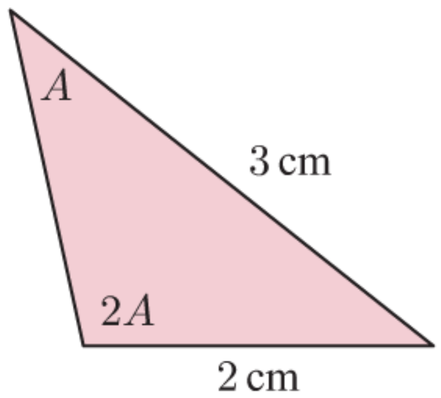
15 Solve $\sin\theta\cos\theta = \frac{1}{4}$ for $-\pi \leqslant \theta \leqslant \pi$.

16 Find the exact value of $\cos A$ in the diagram:

a



b



17 Prove that:

a $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot\theta$ **b** $\dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$ **c** $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$

d $\operatorname{cosec} 2\theta = \tan\theta + \cot 2\theta$ **e** $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$

18 Solve for $0 \leqslant x \leqslant 2\pi$, giving exact answers:

a $\cos 2x - \cos x = 0$ **b** $\cos 2x + 3\cos x = 1$ **c** $\cos 2x + \sin x = 0$

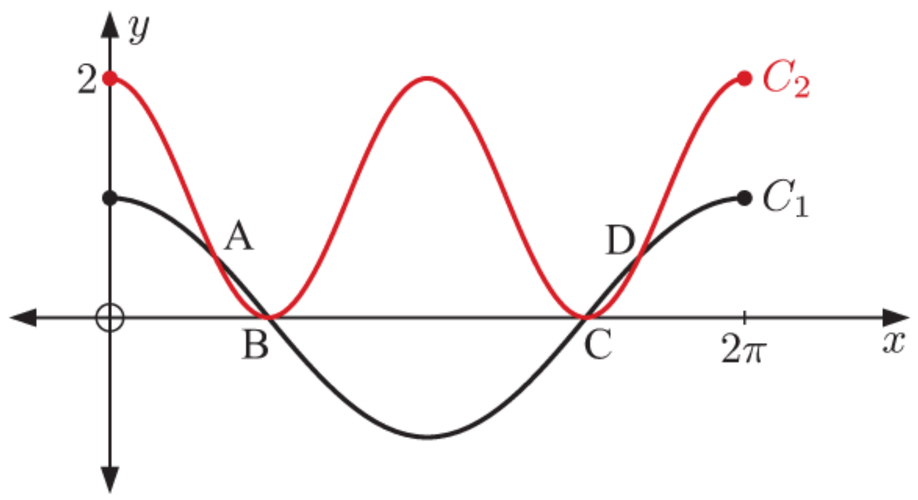
d $\sin 4x = \sin 2x$ **e** $2\cos 2x + 9\sin x = 7$ **f** $\sin x + \cos x = \sqrt{2}$

g $2\cos^2 x = 3\sin x$ **h** $\sin 2x + \cos x - 2\sin x - 1 = 0$

19 The curves $y = \cos x$ and $y = \cos 2x + 1$ are graphed alongside for $0 \leqslant x \leqslant 2\pi$.

a Identify each curve.

b Find the exact coordinates of A, B, C, and D.



ACTIVITY 2 PARAMETRIC EQUATIONS

In a **parametric equation**, the variables x and y are each expressed in terms of a third variable, called the **parameter**.

Click on the icon to obtain an Activity about parametric equations.



E COMPOUND ANGLE IDENTITIES

INVESTIGATION 3 ANGLE SUM AND DIFFERENCE IDENTITIES

What to do:

1 Copy and complete for angles A and B in radians or degrees. Include some angles of your own choosing.

A	B	$\cos A$	$\cos B$	$\cos(A - B)$	$\cos A - \cos B$	$\cos A \cos B + \sin A \sin B$
$47°$	$24°$					
$138°$	$49°$					
3^c	2^c					
$\vdots$	$\vdots$					

2 Write down any discoveries from your table of values.

3 Copy and complete for four pairs of angles A and B of your own choosing:

A	B	$\sin A$	$\sin B$	$\sin(A + B)$	$\sin A + \sin B$	$\sin A \cos B + \cos A \sin B$
$\vdots$	$\vdots$					

4 Write down any discoveries from your table of values.

If A and B are **any** two angles then:

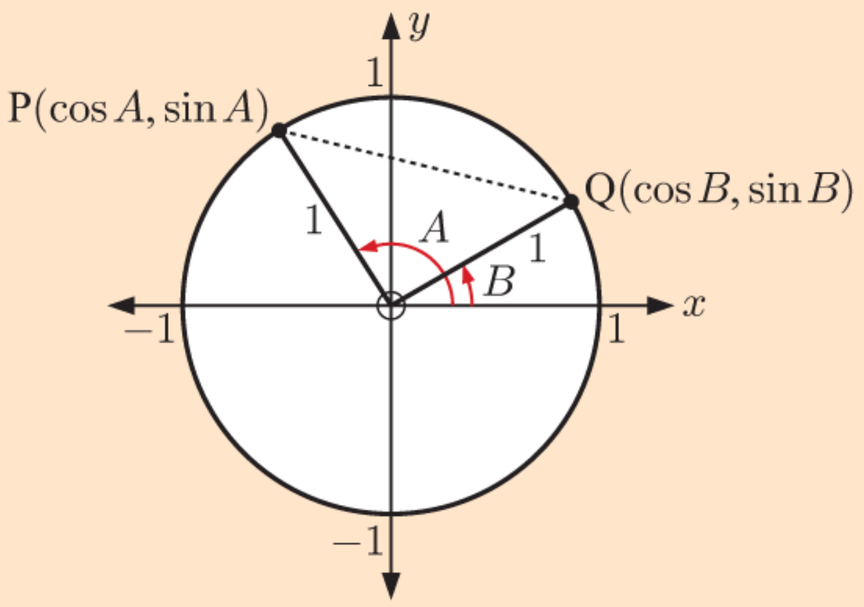
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

These are known as the **compound angle identities**. There are many ways of establishing them, but many are unsatisfactory as the arguments limit the angles A and B to being acute.

Proof:

Consider $\mathrm{P}(\cos A, \sin A)$ and $\mathrm{Q}(\cos B, \sin B)$ as any two points on the unit circle, as shown.

Angle POQ is $A - B$.



Using the distance formula:

$$\mathrm{PQ} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$\therefore \quad (\mathrm{PQ})^2 = \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$
$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B) \qquad \ldots\ (1)$$

But, by the Cosine Rule in $\triangle$POQ,

$$(\mathrm{PQ})^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$$
$$= 2 - 2\cos(A - B) \qquad \ldots\ (2)$$

$$\therefore \quad \cos(A - B) = \cos A \cos B + \sin A \sin B \qquad \{\text{comparing (1) and (2)}\}$$

From this formula the other formulae can be established:

- $\cos(A + B) = \cos(A - (-B))$
$= \cos A \cos(-B) + \sin A \sin(-B)$
$= \cos A \cos B + \sin A(-\sin B)$
$\{\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta\}$
$= \cos A \cos B - \sin A \sin B$

- $\sin(A - B)$
$= \cos\left(\frac{\pi}{2} - (A - B)\right)$
$= \cos\left(\left(\frac{\pi}{2} - A\right) + B\right)$
$= \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$
$= \sin A \cos B - \cos A \sin B$

- $\sin(A + B)$
$= \sin(A - (-B))$
$= \sin A \cos(-B) - \cos A \sin(-B)$
$= \sin A \cos B - \cos A(-\sin B)$
$= \sin A \cos B + \cos A \sin B$

- $\tan(A+B)$
$= \dfrac{\sin(A+B)}{\cos(A+B)}$
$= \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$
$= \dfrac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$
$= \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

- $\tan(A-B)$
$= \tan(A+(-B))$
$= \dfrac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$
$= \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
$\{\tan(-B) = -\tan B\}$

Example 13 — Self Tutor

Expand and simplify $\sin(270^\circ + \alpha)$.

$\sin(270^\circ + \alpha)$
$= \sin 270^\circ\cos\alpha + \cos 270^\circ\sin\alpha$ {compound angle identity}
$= -1 \times \cos\alpha + 0 \times \sin\alpha$
$= -\cos\alpha$

EXERCISE 1E

1 Use your calculator to verify all six of the compound angle identities for $A = 57^\circ$ and $B = 21^\circ$.

2 Expand and simplify:

a $\sin(90^\circ + \theta)$ **b** $\cos(90^\circ + \theta)$ **c** $\sin(180^\circ - \theta)$

d $\cos(\pi + \alpha)$ **e** $\sin(2\pi - A)$ **f** $\cos\left(\frac{3\pi}{2} - \theta\right)$

g $\tan\left(\frac{\pi}{4} + \theta\right)$ **h** $\tan\left(\theta - \frac{3\pi}{4}\right)$ **i** $\tan(\pi + \theta)$

3 Expand, then simplify and write your answer in the form $A\sin\theta + B\cos\theta$:

a $\sin\left(\theta + \frac{\pi}{3}\right)$ **b** $\cos\left(\frac{2\pi}{3} - \theta\right)$ **c** $\cos\left(\theta + \frac{\pi}{4}\right)$ **d** $\sin\left(\frac{\pi}{6} - \theta\right)$

Example 14 — Self Tutor

Simplify $\cos 3\theta\cos\theta - \sin 3\theta\sin\theta$.

$\cos 3\theta\cos\theta - \sin 3\theta\sin\theta$
$= \cos(3\theta + \theta)$ {compound angle identity in reverse}
$= \cos 4\theta$

4 Simplify using the compound angle identities in reverse:

a $\cos 2\theta\cos\theta + \sin 2\theta\sin\theta$ **b** $\sin 2A\cos A + \cos 2A\sin A$

c $\cos A\sin B - \sin A\cos B$ **d** $\sin\alpha\sin\beta + \cos\alpha\cos\beta$

e $\sin\phi\sin\theta - \cos\phi\cos\theta$ **f** $2\sin\alpha\cos\beta - 2\cos\alpha\sin\beta$

g $\dfrac{\tan 2\theta - \tan\theta}{1 + \tan 2\theta\tan\theta}$ **h** $\dfrac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

5 Use the compound angle identities to prove the double angle identities:

a $\sin 2\theta = 2\sin\theta\cos\theta$ **b** $\cos 2\theta = \cos^2\theta - \sin^2\theta$ **c** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Example 15 **Self Tutor**

Without using your calculator, show that $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ\cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)\\ &= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

6 Without using your calculator, show that:

a $\cos 75^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\sin 105^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ **c** $\cos\frac{13\pi}{12} = \dfrac{-\sqrt{6}-\sqrt{2}}{4}$

7 Find the exact value of:

a $\tan\frac{5\pi}{12}$ **b** $\tan 105^\circ$

8 If $\tan A = \frac{2}{3}$ and $\tan B = -\frac{1}{5}$, find the exact value of $\tan(A+B)$.

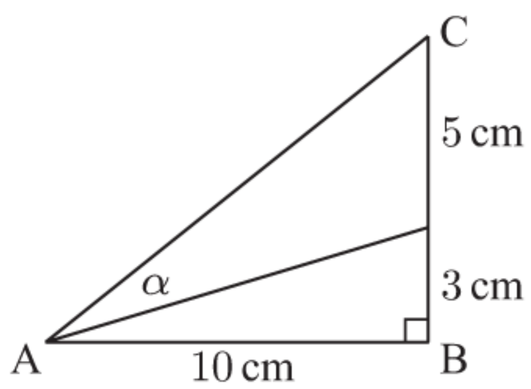
9 If $\tan A = \frac{3}{4}$, find $\tan\left(A + \frac{\pi}{4}\right)$.

10 If $\sin A = -\frac{1}{3}$, $\pi \leqslant A \leqslant \frac{3\pi}{2}$ and $\cos B = \frac{1}{\sqrt{5}}$, $0 \leqslant B \leqslant \frac{\pi}{2}$, find $\tan(A+B)$.

11 Simplify, giving your answer exactly: $\dfrac{\tan 80^\circ - \tan 20^\circ}{1 + \tan 80^\circ \tan 20^\circ}$

12 If $\tan(A+B) = \frac{3}{5}$ and $\tan B = \frac{2}{3}$, find the exact value of $\tan A$.

13 Find the exact value of $\tan\alpha$:



14 Find exactly the tangent of the acute angle between two lines with gradients $\frac{1}{2}$ and $\frac{2}{3}$.

15 Simplify:

a $\cos(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$

b $\sin(\theta-2\phi)\cos(\theta+\phi) - \cos(\theta-2\phi)\sin(\theta+\phi)$

c $\cos\alpha\cos(\beta-\alpha) - \sin\alpha\sin(\beta-\alpha)$

d $\tan\left(A+\frac{\pi}{4}\right)\tan\left(A-\frac{\pi}{4}\right)$ **e** $\dfrac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$

16 If $\sin A = \frac{2}{3}$, $\frac{\pi}{2} \leqslant A \leqslant \pi$ and $\cos B = -\frac{4}{5}$, $\pi \leqslant B \leqslant \frac{3\pi}{2}$, find:

a $\tan(A+B)$ **b** $\tan 2A$.

17 Find $\tan A$ if $\tan(A-B)\tan(A+B) = 1$.

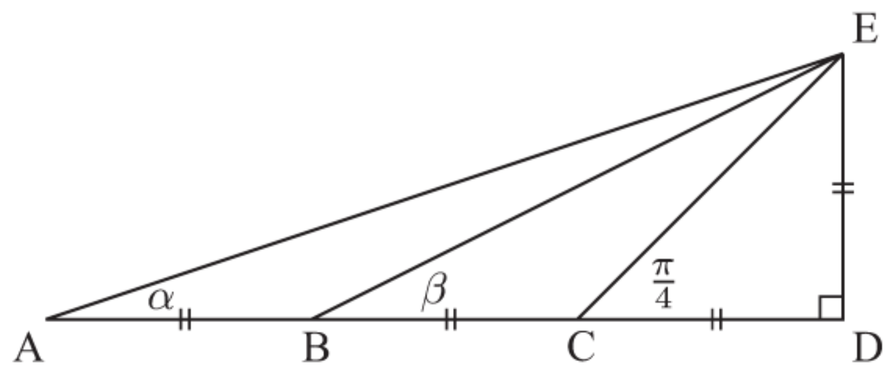
18 Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$, and $\tan C$.
Hence show that if A, B, and C are the angles of a triangle, then
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

19 Show that:

a $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) = \cos\theta - \sin\theta$ **b** $2\cos\left(\theta - \frac{\pi}{3}\right) = \cos\theta + \sqrt{3}\sin\theta$

c $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$ **d** $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

20 Prove that, in the given figure, $\alpha + \beta = \frac{\pi}{4}$.



21 **a** Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ by replacing 3θ by $(2\theta + \theta)$.

b Hence solve the equation $8\cos^3\theta - 6\cos\theta + 1 = 0$ for $-\pi \leqslant \theta \leqslant \pi$.

22 **a** Write $\sin 3\theta$ in the form $a\sin^3\theta + b\sin\theta$ where $a, b \in \mathbb{Z}$.

b Hence solve the equation $\sin 3\theta = \sin\theta$ for $0 \leqslant \theta \leqslant 3\pi$.

Example 16 **Self Tutor**

Suppose $\sin x - \sqrt{3}\cos x = k\cos(x+b)$ for $k > 0$ and $0 < b < 2\pi$. Find k and b.

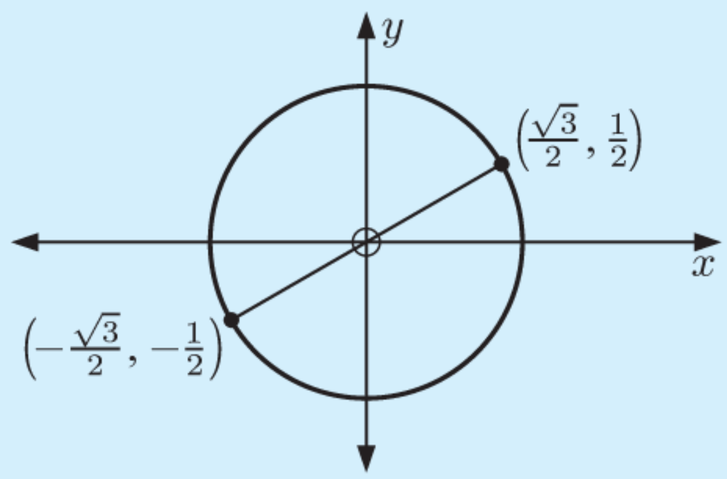
$$\begin{aligned}\sin x - \sqrt{3}\cos x &= k\cos(x+b)\\ &= k[\cos x\cos b - \sin x\sin b]\\ &= k\cos x\cos b - k\sin x\sin b\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$, $k\cos b = -\sqrt{3}$ .... (1) and $-k\sin b = 1$ .... (2)

$\frac{-k\sin b}{k\cos b} = -\frac{1}{\sqrt{3}}$ {dividing (2) by (1)}

$\therefore \tan b = \frac{1}{\sqrt{3}}$

$\therefore b = \frac{\pi}{6}$ or $\frac{7\pi}{6}$



Substituting $b = \frac{\pi}{6}$ into (1) gives $k \times \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore k = -2$

We reject this solution as $k > 0$.

Substituting $b = \frac{7\pi}{6}$ into (1) gives $k \times -\frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore k = 2$

So, $k = 2$ and $b = \frac{7\pi}{6}$.

23 Suppose $\sqrt{3}\sin x + \cos x = k\sin(x+a)$ for $k > 0$ and $0 < a < 2\pi$. Find k and a.

24 **a** Write $2\cos x + 2\sin x$ in the form $k\cos(x+a)$ for $k > 0$, $0 < a < 2\pi$.

b Hence solve the equation $2\cos x + 2\sin x = \sqrt{2}$ for $0 \leqslant x \leqslant 2\pi$.

25 **a** Find a sequence of transformations which map the graph of $y = \sin x$ onto the graph of $y = \cos x + 3\sin x$.

b Find the greatest and least values of $(\cos x + 3\sin x)^2 + 2$.

26 Use the basic definition of periodicity to show algebraically that the period of $f(x) = \sin(nx)$ is $\frac{2\pi}{n}$, for all $n > 0$.

27 **a** Write $2\cos x - 5\sin x$ in the form $k\cos(x+b)$ for $k > 0$, $0 < b < 2\pi$.

b Hence solve $2\cos x - 5\sin x = -2$ for $0 \leqslant x \leqslant \pi$.

c Given that $t = \tan\frac{x}{2}$, prove that $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$.

d Hence solve $2\cos x - 5\sin x = -2$ for $0 \leqslant x \leqslant \pi$.

28 Use $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$ to show that:

a $\arctan 5 - \arctan\frac{2}{3} = \frac{\pi}{4}$ **b** $\arctan\frac{1}{5} + \arctan\frac{2}{3} = \frac{\pi}{4}$

29 Find the exact value of:

a $\arctan\frac{4}{3} - 2\arctan\frac{1}{2}$ **b** $4\arctan\frac{1}{5} - \arctan\frac{1}{239}$

30 **a** Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

b Hence show that $\sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

c Hence write the following as sums:

i $\sin 3\theta\cos\theta$ **ii** $\sin 6\alpha\cos\alpha$ **iii** $2\sin 5\beta\cos\beta$

iv $4\cos\theta\sin 4\theta$ **v** $6\cos 4\alpha\sin 3\alpha$ **vi** $\frac{1}{3}\cos 5A\sin 3A$

31 **a** Show that $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$.

b Hence show that $\cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$.

c Hence write the following as a *sum* of cosines:

i $\cos 4\theta\cos\theta$ **ii** $\cos 7\alpha\cos\alpha$ **iii** $2\cos 3\beta\cos\beta$

iv $6\cos x\cos 7x$ **v** $3\cos P\cos 4P$ **vi** $\frac{1}{4}\cos 4x\cos 2x$

32 **a** Show that $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$.

b Hence show that $\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

c Hence write the following as a *difference* of cosines:

i $\sin 3\theta\sin\theta$ **ii** $\sin 6\alpha\sin\alpha$ **iii** $2\sin 5\beta\sin\beta$

iv $4\sin\theta\sin 4\theta$ **v** $10\sin 2A\sin 8A$ **vi** $\frac{1}{5}\sin 3M\sin 7M$

33 The **products to sums formulae** are:

$$\sin A\cos B = \tfrac{1}{2}\sin(A+B) + \tfrac{1}{2}\sin(A-B) \quad \ldots. (1)$$

$$\cos A\cos B = \tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B) \quad \ldots. (2)$$

$$\sin A\sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B) \quad \ldots. (3)$$

a What formulae result if we replace B by A in each of these formulae?

b Suppose $A+B=S$ and $A-B=D$.

i Show that $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$.

ii Using the substitutions $A+B=S$ and $A-B=D$, show that equation (1) becomes $\sin S+\sin D=2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$ .... (4)

iii By replacing D by $(-D)$ in (4), show that $\sin S-\sin D=2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$.

c What formula results when the substitution $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$ is made into (2)?

d What formula results when the substitution $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$ is made into (3)?

34 The **factor formulae** are:

$$\sin S+\sin D=2\sin\left(\tfrac{S+D}{2}\right)\cos\left(\tfrac{S-D}{2}\right) \qquad \cos S+\cos D=2\cos\left(\tfrac{S+D}{2}\right)\cos\left(\tfrac{S-D}{2}\right)$$

$$\sin S-\sin D=2\cos\left(\tfrac{S+D}{2}\right)\sin\left(\tfrac{S-D}{2}\right) \qquad \cos S-\cos D=-2\sin\left(\tfrac{S+D}{2}\right)\sin\left(\tfrac{S-D}{2}\right)$$

Use these formulae to convert the following to products:

a $\sin 5x+\sin x$ **b** $\cos 8A+\cos 2A$ **c** $\cos 3\alpha-\cos\alpha$

d $\sin 5\theta-\sin 3\theta$ **e** $\cos 7\alpha-\cos\alpha$ **f** $\sin 3\alpha+\sin 7\alpha$

g $\cos 2B-\cos 4B$ **h** $\sin(x+h)-\sin x$ **i** $\cos(x+h)-\cos x$

35 In triangle ABC it is known that $\sin A=\cos B+\cos C$. Show that the triangle is right angled.

HISTORICAL NOTE

In the late sixteenth century, there were no calculators. Instead, values of trigonometric functions for different angles were calculated by hand and recorded in tables. The values could then readily be used for trigonometric applications, and also in other surprising ways. For example, the compound angle identities could be used to quickly multiply two numbers together.

Using the compound angle identities, we can show that

$$\cos A\times\cos B=\frac{\cos(A+B)+\cos(A-B)}{2} \qquad \text{.... } (*)$$

Now, suppose you need to find 0.1362×0.4573. By consulting trigonometric tables, it could be found that $\cos 82.172^\circ\approx 0.1362$ and $\cos 62.787^\circ\approx 0.4573$.

So, 0.1362×0.4573

$\approx \cos 82.172^\circ\times\cos 62.787^\circ$

$\approx \dfrac{\cos(82.172^\circ+62.787^\circ)+\cos(82.172^\circ-62.787^\circ)}{2}$ {using $(*)$}

$\approx \dfrac{\cos(144.959^\circ)+\cos(19.385^\circ)}{2}$

$\approx \dfrac{-0.8187+0.9433}{2}$ {using trigonometric tables}

≈ 0.0623

This method may seem complicated, but in the late sixteenth century it was much faster than performing long multiplication!

ACTIVITY 3

A **trigonometric series** is the sum of a sequence of trigonometric expressions which follow a rule.

In this Investigation we will explore patterns formed by a trigonometric series.

What to do:

1 Consider the function $f(x) = \sin x + \dfrac{\sin 3x}{3}$.

a Show that $f(x) = \frac{2}{3}\sin x(2\cos^2 x + 1)$.

Hint: Write $\sin 3x$ as $\sin(2x + x)$.

b Hence find the x-intercepts of $y = f(x)$ on $-4\pi \leqslant x \leqslant 4\pi$.

c Use the graphing package to sketch $y = f(x)$ on $-4\pi \leqslant x \leqslant 4\pi$. Discuss the similarities and differences between this graph and the graph of $y = \sin x$.

GRAPHING PACKAGE



2 **a** Write the function $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$ in terms of $\sin x$ and $\cos x$.

b Hence find the x-intercepts of this function on $-4\pi \leqslant x \leqslant 4\pi$.

c Use the graphing package to sketch the graph of the function on $-4\pi \leqslant x \leqslant 4\pi$. Compare your graph with the graphs in **1**.

3 Use the graphing package to graph on $-4\pi \leqslant x \leqslant 4\pi$:

a $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7}$

b $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9} + \dfrac{\sin 11x}{11}$

4 Predict the graph of $f(x) = \sum_{k=1}^{\infty} \dfrac{\sin(2k-1)}{2k-1}$.

REVIEW SET 1A

1 Without using a calculator, find:

a $\operatorname{cosec} \frac{\pi}{4}$ **b** $\cot \frac{5\pi}{6}$ **c** $\sec \frac{5\pi}{3}$

2 Find, giving your answer in radians:

a $\arccos \frac{1}{\sqrt{2}}$ **b** $\arctan \frac{1}{\sqrt{3}}$ **c** $\arcsin\left(-\frac{1}{2}\right)$

3 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sec x = \sqrt{2}$ **b** $\sqrt{3}\cos x \operatorname{cosec} x + 1 = 0$

4 Simplify:

a $3\cos(-\theta) - 2\cos\theta$ **b** $\cos\left(\frac{3\pi}{2} - \theta\right)$ **c** $\sin\left(\theta + \frac{\pi}{2}\right)$

d $\dfrac{1 - \cos^2\theta}{1 + \cos\theta}$ **e** $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$ **f** $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

5 If $\sin\alpha = -\frac{3}{4}$, $\pi \leqslant \alpha \leqslant \frac{3\pi}{2}$, find the exact value of:

a $\cos\alpha$ **b** $\sin 2\alpha$ **c** $\cos 2\alpha$ **d** $\tan 2\alpha$ **e** $\cos\frac{\alpha}{2}$ **f** $\sin\frac{\alpha}{2}$

6 Show that:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$

b $\sin\theta\cos(\phi - \theta) + \cos\theta\sin(\phi - \theta) = \sin\phi$

c $\dfrac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1} = \tan\alpha$

d $\operatorname{cosec} 2x + \cot 2x = \cot x$

7 Find the exact value of:

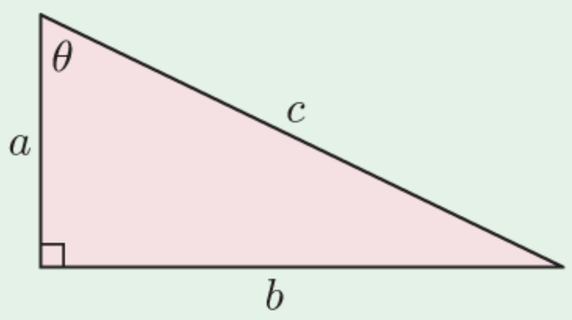
a $\cos 165^\circ$

b $\tan\frac{\pi}{12}$

8 From ground level, a shooter is aiming at targets on a vertical brick wall. At the current angle of elevation of his rifle, he will hit a target 20 m above ground level. If he doubles the angle of elevation of the rifle, he will hit a target 45 m above ground level. How far is the shooter from the wall?

9 Solve for $0 \leqslant x \leqslant 2\pi$: $\sec^2 x = \tan x + 1$

10



For the diagram alongside, prove that:

a $\sin 2\theta = \dfrac{2ab}{c^2}$

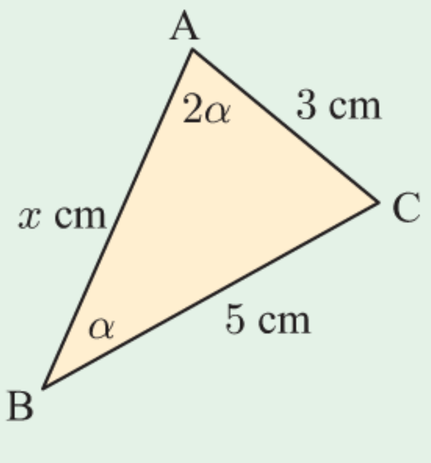
b $\cos 2\theta = \dfrac{a^2 - b^2}{c^2}$

11 Consider triangle ABC shown.

a Show that $\cos\alpha = \frac{5}{6}$.

b Show that x is a solution of $3x^2 - 25x + 48 = 0$.

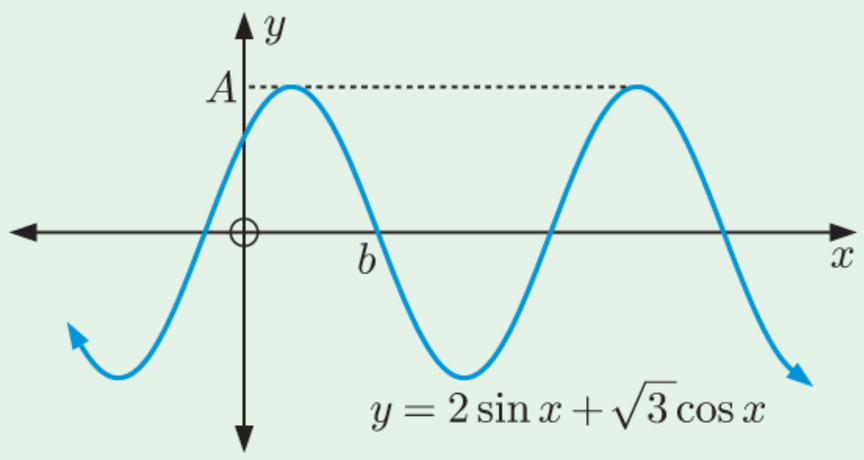
c Find x by solving the equation in **b**.



12 If α and β are the other angles of a right angled triangle, show that $\sin 2\alpha = \sin 2\beta$.

13 Write $3\sin x - 5\cos x$ in the form $k\cos(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

14 The graph of $y = 2\sin x + \sqrt{3}\cos x$ is shown alongside.



a Write $2\sin x + \sqrt{3}\cos x$ in the form $k\sin(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

b Hence find:

i the exact value of A

ii b correct to 2 decimal places.

15 Find $\arctan\frac{1}{7} + 2\arctan\frac{1}{3}$.

REVIEW SET 1B

1 Find the other *five* trigonometric ratios if $\cos x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$.

2 **a** Sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ on the same set of axes for $-2\pi \leqslant x \leqslant 2\pi$.

b State a transformation which maps $y = \sec x$ onto $y = \operatorname{cosec} x$ for all $x \in \mathbb{R}$.

3 Solve $\cot x = \sqrt{3}$ for $-\pi \leqslant x \leqslant \pi$.

4 Solve exactly:

a $\arcsin x = \frac{\pi}{3}$ **b** $\arctan(x-2) = \frac{\pi}{6}$

5 Simplify:

a $\operatorname{cosec} x \tan x$ **b** $\dfrac{\tan x}{\sec x}$ **c** $\sec x - \tan x \sin x$

6 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$ **c** $5 - 5\sin^2\theta$

d $\dfrac{\sin^2\theta - 1}{\cos\theta}$ **e** $\dfrac{\tan\theta + \cot\theta}{\sec\theta}$ **f** $\cos^2\theta(\tan\theta + 1)^2 - 1$

7 If $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$, find:

a $\sin 2A$ **b** $\cos 2A$ **c** $\tan 2A$

8 Show that:

a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$ **b** $\left(1 + \dfrac{1}{\cos\theta}\right)\left(\cos\theta - \cos^2\theta\right) = \sin^2\theta$

9 Show that $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ using a suitable double angle identity.

10 Solve:

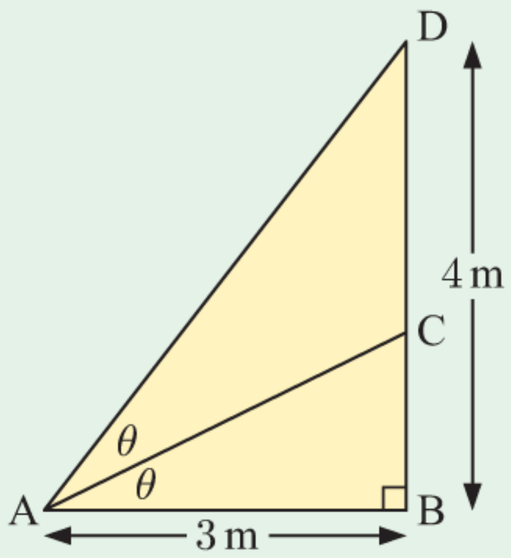
a $\sqrt{3}\cos x + \sin 2x = 0$ for $-\pi \leqslant x \leqslant \pi$ **b** $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \sqrt{3}$ for $0 < \theta < \frac{\pi}{2}$.

11 Given $\sin\theta = \frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin\left(\theta + \frac{\pi}{6}\right)$.

12 Consider the figure in the **Opening Problem** on page **18**. Find $\tan\theta$ using the ratios $\tan\phi$ and $\tan(\theta+\phi)$.

13 Write $3\sin x + 4\cos x$ in the form $k\sin(x+a)$, where $k > 0$ and $0 < a < 2\pi$.

14 Find exactly the length of [BC].



15 **a** Show that $\dfrac{1}{1+\sqrt{2}\sin x} + \dfrac{1}{1-\sqrt{2}\sin x} = 2\sec 2x$.

b Hence explain why $\dfrac{1}{1+\sqrt{2}\sin x} + \dfrac{1}{1-\sqrt{2}\sin x} = 1$ has no solutions.

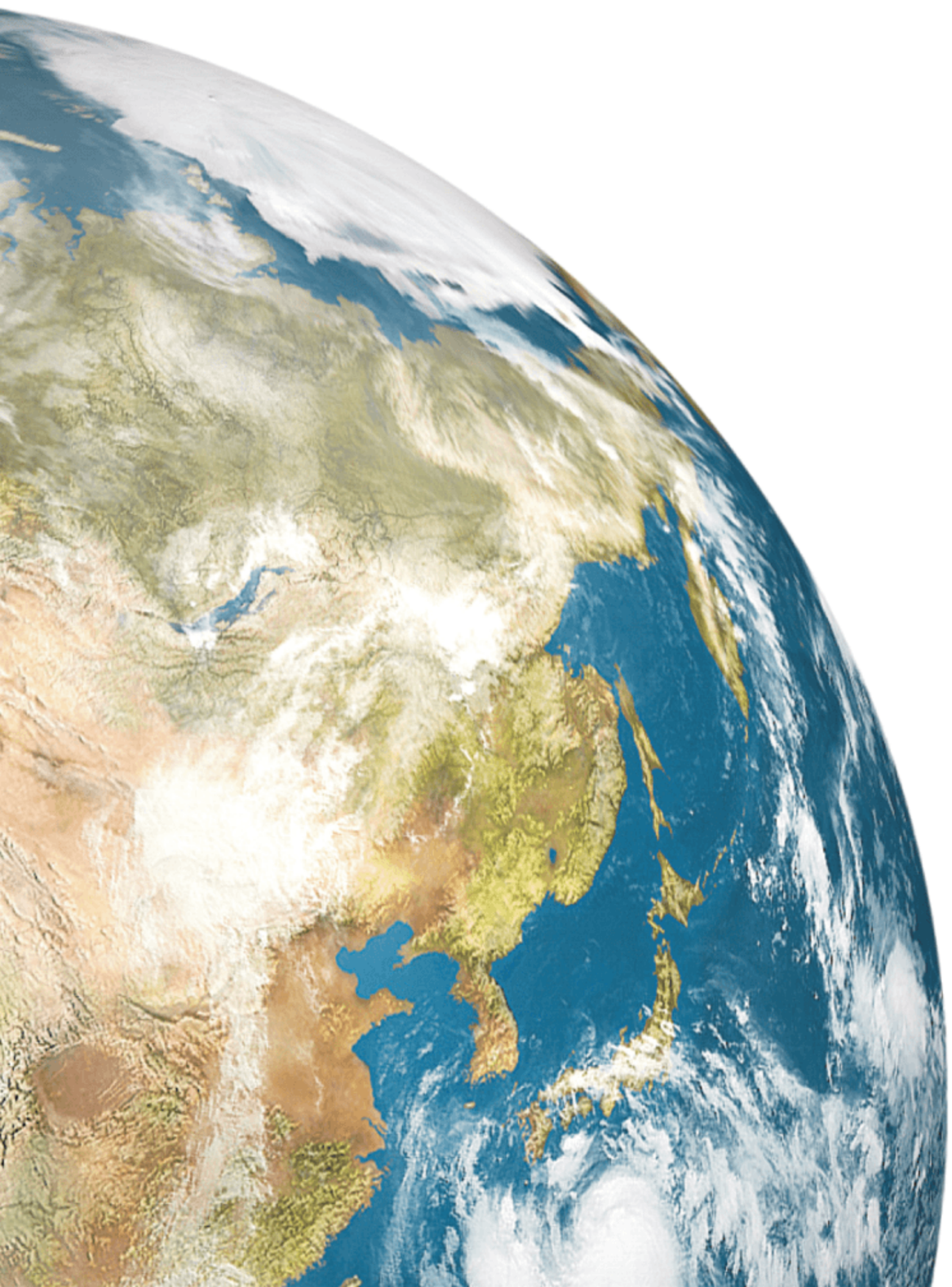
16 Prove that if A, B, and C are the angles of a triangle, then $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

Chapter 2

Exponential functions

Contents:

A Rational exponents
B Algebraic expansion and factorisation
C Exponential equations
D Exponential functions
E Growth and decay
F The natural exponential



OPENING PROBLEM

At an antiques fair, Bernard purchases a clock for £500 and a vase for £400. The clock increases in value by 5% each year, and the vase increases in value by 7% each year.



Things to think about:

a What is the value of each item 1 year after purchase?

b Can you write a formula for the value of each item t years after purchase?

c Which item is more valuable 15 years after purchase?

d How can we determine when the items are equal in value?

We have seen previously how exponents are used to indicate when a number is raised to a power.

For a positive integer exponent, the exponent tells us how many of the base are multiplied together.

Any non-zero base to the power 0 is defined as 1, to give consistency to the exponent laws.

For a negative integer exponent, we take the reciprocal of the corresponding positive integer power.

$$\begin{cases} 2^3 = 2 \times 2 \times 2 & = 8 \\ 2^2 = 2 \times 2 & = 4 \\ 2^1 = 2 & = 2 \end{cases}$$

$$2^0 = 1 \qquad = 1$$

$$\begin{cases} 2^{-1} = \frac{1}{2} & = \frac{1}{2} \\ 2^{-2} = \frac{1}{2 \times 2} & = \frac{1}{4} \\ 2^{-3} = \frac{1}{2 \times 2 \times 2} & = \frac{1}{8} \end{cases}$$

In this Chapter we give meaning to exponents which are **rational**, allowing us to start filling in the gaps between the integer exponents. This will allow us to consider **exponential functions** for which the variable appears in an exponent.

A RATIONAL EXPONENTS

Using the definition $a^n = a \times a \times .... \times a$, the **laws of exponents** such as $a^n \times a^m = a^{n+m}$ can be proven for any integers n and m.

For a positive base a, we choose to define a raised to a rational exponent so that the laws of exponents still hold.

So, for any $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}

and $\sqrt{a} \times \sqrt{a} = a$ also.

Likewise, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$

and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$ also.

By direct comparison, we conclude that $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ and $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$.

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads "the nth root of a" for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$

$= a^{\frac{m}{n}}$

$\therefore$ $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{for} \quad a > 0, \ n \in \mathbb{Z}^+, \ m \in \mathbb{Z}}$

Example 1 **Self Tutor**

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2} = 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}} = \dfrac{1}{2^{\frac{1}{2}}} = 2^{-\frac{1}{2}}$

c $\sqrt[5]{4} = (2^2)^{\frac{1}{5}} = 2^{2\times\frac{1}{5}} = 2^{\frac{2}{5}}$

EXERCISE 2A

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write in the form a^k, where a is a prime number and k is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Write in the form x^k, where k is rational:

a $\sqrt{x}$ **b** $x\sqrt{x}$ **c** $\dfrac{1}{\sqrt{x}}$ **d** $x^2\sqrt{x}$ **e** $\dfrac{1}{x\sqrt{x}}$

5 Use your calculator to find, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $4^{-\frac{3}{5}}$ **c** $\sqrt[4]{8}$ **d** $\sqrt[5]{27}$ **e** $\dfrac{1}{\sqrt[3]{7}}$



GRAPHICS CALCULATOR INSTRUCTIONS

6 Write *without* rational exponents:

a $5^{\frac{1}{3}}$ **b** $3^{-\frac{1}{2}}$ **c** $3^{\frac{5}{2}}$ **d** $m^{\frac{3}{2}}$ **e** $x^{\frac{7}{2}}$

Example 2 — Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a
$$\begin{aligned} & 8^{\frac{4}{3}} \\ &= (2^3)^{\frac{4}{3}} \\ &= 2^{3\times\frac{4}{3}} && \{(a^m)^n = a^{mn}\} \\ &= 2^4 \\ &= 16 \end{aligned}$$

b
$$\begin{aligned} & 27^{-\frac{2}{3}} \\ &= (3^3)^{-\frac{2}{3}} \\ &= 3^{3\times-\frac{2}{3}} \\ &= 3^{-2} \\ &= \tfrac{1}{9} \end{aligned}$$

7 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

B ALGEBRAIC EXPANSION AND FACTORISATION

We can use the standard rules of algebra, together with the laws of exponents, to simplify expressions containing rational or variable exponents:

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)(a-b) &= a^2-b^2 \\ (a+b)^2 &= a^2+2ab+b^2 \end{aligned}$$

Example 3 — Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$$\begin{aligned} & x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}}) \\ &= x^{-\frac{1}{2}}\times x^{\frac{3}{2}} + x^{-\frac{1}{2}}\times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\times 3x^{-\frac{1}{2}} && \{\text{each term is multiplied by } x^{-\frac{1}{2}}\} \\ &= x^1 + 2x^0 - 3x^{-1} && \{\text{adding indices}\} \\ &= x+2-\frac{3}{x} \end{aligned}$$

EXERCISE 2B

1 Simplify:

a $x^{\frac{1}{2}}\times x^{-\frac{1}{2}}$ **b** $x^{\frac{3}{2}}\times x^{-\frac{1}{2}}$ **c** $x^2\times x^{-\frac{3}{2}}$

2 Expand and simplify:

a $x^2(x^3 + 2x^2 + 1)$ **b** $2^x(2^x + 1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d $7^x(7^x + 2)$ **e** $3^x(2 - 3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$

g $2^{-x}(2^x + 5)$ **h** $5^{-x}(5^{2x} + 5^x)$ **i** $x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}})$

j $3^x(3^x + 5 + 3^{-x})$ **k** $x^{-\frac{1}{2}}(2x^2 - x + 5x^{\frac{1}{2}})$ **l** $2^{2x}(2^x - 3 - 2^{-2x})$

Example 4 Self Tutor

Expand and simplify:

a $(2^x + 3)(2^x + 1)$ **b** $(7^x + 7^{-x})^2$

a
$$\begin{aligned}&(2^x + 3)(2^x + 1)\\ &= 2^x \times 2^x + 2^x + 3 \times 2^x + 3\\ &= 2^{2x} + 4 \times 2^x + 3\end{aligned}$$

b
$$\begin{aligned}&(7^x + 7^{-x})^2\\ &= (7^x)^2 + 2 \times 7^x \times 7^{-x} + (7^{-x})^2\\ &= 7^{2x} + 2 \times 7^0 + 7^{-2x}\\ &= 7^{2x} + 2 + 7^{-2x}\end{aligned}$$

3 Expand and simplify:

a $(2^x - 1)(2^x + 3)$ **b** $(3^x + 2)(3^x + 5)$ **c** $(5^x - 2)(5^x - 4)$

d $(2^x + 3)^2$ **e** $(3^x - 1)^2$ **f** $(4^x + 7)^2$

g $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$ **h** $(2^x + 3)(2^x - 3)$ **i** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$

j $\left(x + \frac{2}{x}\right)^2$ **k** $(7^x - 7^{-x})^2$ **l** $(5 - 2^{-x})^2$

Example 5 Self Tutor

Factorise:

a $2^{n+3} + 2^n$ **b** $2^{n+3} + 8$ **c** $2^{3n} + 2^{2n}$

a
$$\begin{aligned}&2^{n+3} + 2^n\\ &= 2^n 2^3 + 2^n\\ &= 2^n(2^3 + 1)\\ &= 2^n \times 9\end{aligned}$$

b
$$\begin{aligned}&2^{n+3} + 8\\ &= 2^n 2^3 + 8\\ &= 8(2^n) + 8\\ &= 8(2^n + 1)\end{aligned}$$

c
$$\begin{aligned}&2^{3n} + 2^{2n}\\ &= 2^{2n} 2^n + 2^{2n}\\ &= 2^{2n}(2^n + 1)\end{aligned}$$

4 Factorise:

a $5^{2x} + 5^x$ **b** $3^{n+2} + 3^n$ **c** $7^n + 7^{3n}$

d $5^{n+1} - 5$ **e** $6^{n+2} - 6$ **f** $4^{n+2} - 16$

g $2^{2n} - 2^{n+3}$ **h** $2^{n+1} + 2^{n-1}$ **i** $4^{n+1} + 2^{2n-1}$

Example 6 — Self Tutor

Factorise:

a $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a
$$\begin{aligned} & 4^x - 9 \\ &= (2^x)^2 - 3^2 && \{\text{compare } a^2 - b^2 = (a+b)(a-b)\} \\ &= (2^x + 3)(2^x - 3) \end{aligned}$$

b
$$\begin{aligned} & 9^x + 4(3^x) + 4 \\ &= (3^x)^2 + 4(3^x) + 4 && \{\text{compare } a^2 + 4a + 4\} \\ &= (3^x + 2)^2 && \{\text{as } a^2 + 4a + 4 = (a+2)^2\} \end{aligned}$$

5 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$

d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$

g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

6 Factorise:

a $(2^x)^2 - 2^x - 2$ **b** $(3^x)^2 + 3^x - 6$ **c** $4^x - 7(2^x) + 12$

d $4^x + 9(2^x) + 18$ **e** $4^x - 2^x - 20$ **f** $9^x + 9(3^x) + 14$

g $9^x + 4(3^x) - 5$ **h** $25^x + 5^x - 2$ **i** $49^x - 7^{x+1} + 12$

Example 7 — Self Tutor

Simplify:

a $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a
$$\begin{aligned} & \frac{6^n}{3^n} \\ &= \frac{2^n \cancel{3^n}}{\cancel{3^n}_{1}} \\ &= 2^n \end{aligned} \quad \textbf{or} \quad \begin{aligned} & \frac{6^n}{3^n} \\ &= \left(\tfrac{6}{3}\right)^n \\ &= 2^n \end{aligned}$$

b
$$\begin{aligned} & \frac{4^n}{6^n} \\ &= \frac{\cancel{2^n} 2^n}{\cancel{2^n} 3^n} \\ &= \frac{2^n}{3^n} \end{aligned} \quad \textbf{or} \quad \begin{aligned} & \frac{4^n}{6^n} \\ &= \left(\tfrac{4}{6}\right)^n \\ &= \left(\tfrac{2}{3}\right)^n \end{aligned}$$

7 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$ **e** $\dfrac{35^x}{7^x}$

f $\dfrac{6^a}{8^a}$ **g** $\dfrac{24^k}{9^k}$ **h** $\dfrac{5^{n+1}}{5^n}$ **i** $\dfrac{5^{n+1}}{5}$

Example 8 — Self Tutor

Simplify:

a $\dfrac{3^n + 6^n}{3^n}$ **b** $\dfrac{2^{m+2} - 2^m}{2^m}$ **c** $\dfrac{2^{m+3} + 2^m}{9}$

a
$$\frac{3^n + 6^n}{3^n}$$
$$= \frac{3^n + 2^n 3^n}{3^n}$$
$$= \frac{\cancel{3^n}(1 + 2^n)}{\cancel{3^n}_1}$$
$$= 1 + 2^n$$

b
$$\frac{2^{m+2} - 2^m}{2^m}$$
$$= \frac{2^m 2^2 - 2^m}{2^m}$$
$$= \frac{\cancel{2^m}(4 - 1)}{\cancel{2^m}_1}$$
$$= 3$$

c
$$\frac{2^{m+3} + 2^m}{9}$$
$$= \frac{2^m 2^3 + 2^m}{9}$$
$$= \frac{2^m(8 + 1)}{\cancel{9}_1}$$
$$= 2^m$$

8 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

9 Simplify:

a $2^n(n + 1) + 2^n(n - 1)$ **b** $3^n\left(\dfrac{n-1}{6}\right) - 3^n\left(\dfrac{n+1}{6}\right)$

C EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 3**. However, in some cases we can solve the equation algebraically.

If both sides of an exponential equation are written as powers with the same base numbers, we can **equate indices**.

So, if $a^x = a^k$ then $x = k$.

For example, if $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

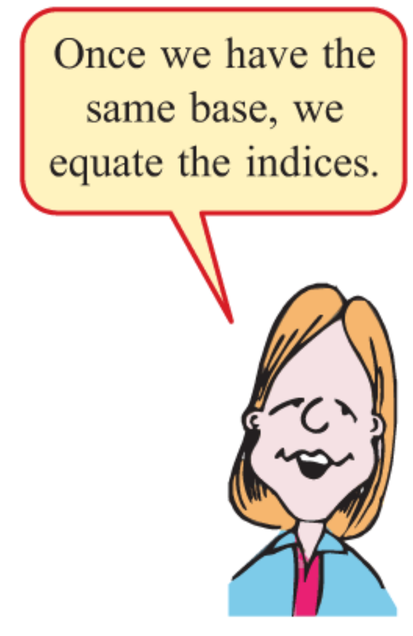
Example 9 **Self Tutor**

Solve for x:

a $2^x = 16$ **b** $3^{x+2} = \frac{1}{27}$

a
$$2^x = 16$$
$$\therefore\ 2^x = 2^4$$
$$\therefore\ x = 4$$

b
$$3^{x+2} = \frac{1}{27}$$
$$\therefore\ 3^{x+2} = 3^{-3}$$
$$\therefore\ x + 2 = -3$$
$$\therefore\ x = -5$$



EXERCISE 2C

1 Solve for x:

a $2^x = 32$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $2^x = \sqrt{2}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{\sqrt{5}}$

Example 10 — Self Tutor

Solve for x:

a $4^x = 8$ **b** $9^{x-2} = \frac{1}{3}$

a
$$4^x = 8$$
$$\therefore\ (2^2)^x = 2^3$$
$$\therefore\ 2^{2x} = 2^3$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

b
$$9^{x-2} = \tfrac{1}{3}$$
$$\therefore\ (3^2)^{x-2} = 3^{-1}$$
$$\therefore\ 3^{2(x-2)} = 3^{-1}$$
$$\therefore\ 2(x-2) = -1$$
$$\therefore\ 2x - 4 = -1$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $\left(\frac{1}{2}\right)^{x+1} = 8$ **l** $\left(\frac{1}{3}\right)^{x+2} = \sqrt{27}$

m $81^x = 27^{-x}$ **n** $\left(\frac{1}{4}\right)^{1-x} = 32$ **o** $\left(\frac{1}{7}\right)^x = \sqrt[3]{49}$ **p** $\left(\frac{1}{3}\right)^{x+1} = 243$

3 Solve for x, if possible:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = \left(\frac{1}{3}\right)^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

d $3^{x+2} \times 9^x = 27$ **e** $\left(\frac{1}{2}\right)^{x-1} \times 8^x = 4^{-x}$ **f** $\left(\frac{1}{5}\right)^{x^2} \times 25^x = \frac{1}{125}$

4 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 28$ **c** $4 \times 3^{x+2} = 12$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times \left(\frac{1}{3}\right)^x = 36$ **f** $5 \times \left(\frac{1}{2}\right)^x = 20$

Example 11 — Self Tutor

Solve for x: $4^x + 2^x - 20 = 0$

$$4^x + 2^x - 20 = 0$$
$$\therefore\ (2^x)^2 + 2^x - 20 = 0 \quad \{\text{compare } a^2 + a - 20 = 0\}$$
$$\therefore\ (2^x - 4)(2^x + 5) = 0 \quad \{a^2 + a - 20 = (a-4)(a+5)\}$$
$$\therefore\ 2^x = 4 \text{ or } 2^x = -5$$
$$\therefore\ 2^x = 2^2 \quad \{2^x \text{ cannot be negative}\}$$
$$\therefore\ x = 2$$

5 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

g $3^x - 1 = 6(3^{-x})$ **h** $2(4^x) - 5(2^x) + 2 = 0$

i $4(9^x) - 35(3^x) = 9$ **j** $4^{x+1} + 2 = 9(2^x)$

k $3^{2x-1} = 3^x + 18$ **l** $4^x + 2^{x+\frac{1}{2}} = 4$


GRAPHICS CALCULATOR INSTRUCTIONS

Check your answers using technology.

6 Solve simultaneously: $4^x = 8^y$ and $9^y = \dfrac{243}{3^x}$.

D EXPONENTIAL FUNCTIONS

We have already seen how to evaluate a^n for any $n \in \mathbb{Q}$.

But how do we evaluate a^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

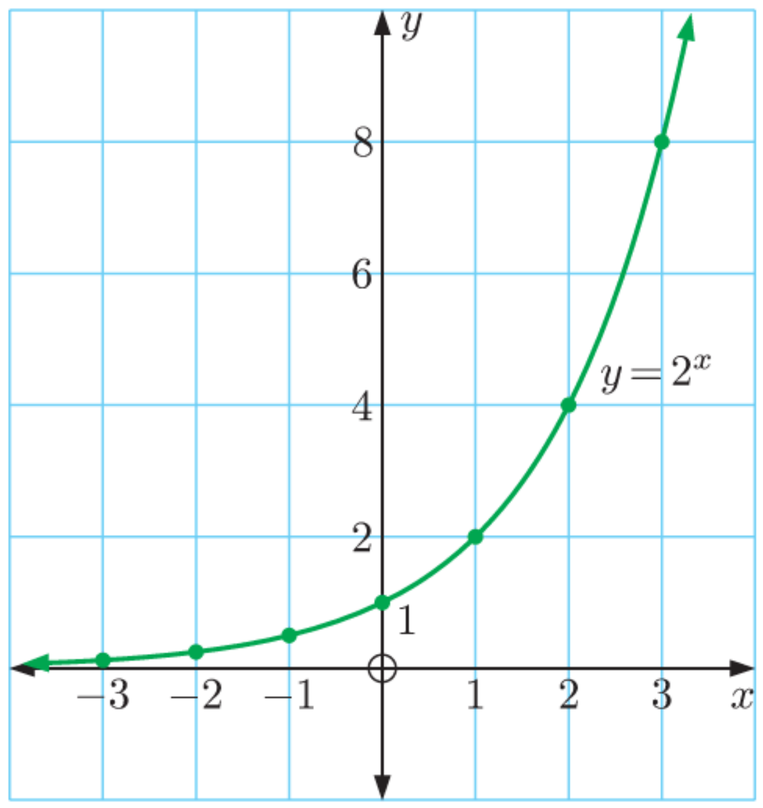
To answer this question, we can study the graphs of exponential functions.

The most simple **exponential function** has the form $y = a^x$ where $a > 0$, $a \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above. However, it never touches the x-axis, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0^+$.

$y = 0$ is therefore a **horizontal asymptote**.

Plotting $y = a^x$ for $x \in \mathbb{Q}$ suggests a smooth, continuous curve. This allows us to complete the curve for all $x \in \mathbb{R}$, giving meaning to a^x for irrational values of x.

INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

In this Investigation we examine the graphs of various families of exponential functions. You can use the **graphing package** or your calculator.


GRAPHING PACKAGE

What to do:

1 **a** State the transformation which maps $y = a^x$ to $y = a^x + k$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$ on the same set of axes.

2 **a** State the transformation which maps $y = a^x$ to $y = a^{x-h}$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$ on the same set of axes.

3 **a** State the transformation which maps $y = a^x$ to $y = p \times a^x$, $p > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$ on the same set of axes.

4 **a** State the transformation which maps $y = a^x$ to $y = -a^x$.

b Predict what the graph of $y = -2^x$ will look like, and check your answer using technology.

5 **a** State the transformation which maps $y = a^x$ to $y = a^{qx}$, $q > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Notice that $2^{2x} = (2^2)^x = 4^x$ and $2^{3x} = (2^3)^x = 8^x$.

Check your predictions by graphing $y = 2^x$, $y = 4^x$, and $y = 8^x$ on the same set of axes.

6 **a** State the transformation which maps $y = a^x$ to $y = a^{-x}$.

b Notice that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$.

Predict what the graph of $y = \left(\frac{1}{2}\right)^x$ will look like, and check your answer using technology.

From your **Investigation** you should have discovered that:

For the general exponential function $\boldsymbol{y = p \times a^{x-h} + k}$ where $a > 0$, $a \neq 1$, $p \neq 0$:

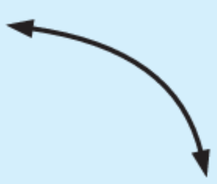
- a controls how steeply the graph increases or decreases.
- h controls horizontal translation.
- k controls vertical translation.
- The equation of the horizontal asymptote is $y = k$.
- If $p > 0$, $a > 1$ the function is increasing.



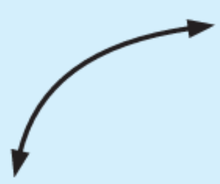
- If $p > 0$, $0 < a < 1$ the function is decreasing.



- If $p < 0$, $a > 1$ the function is decreasing.

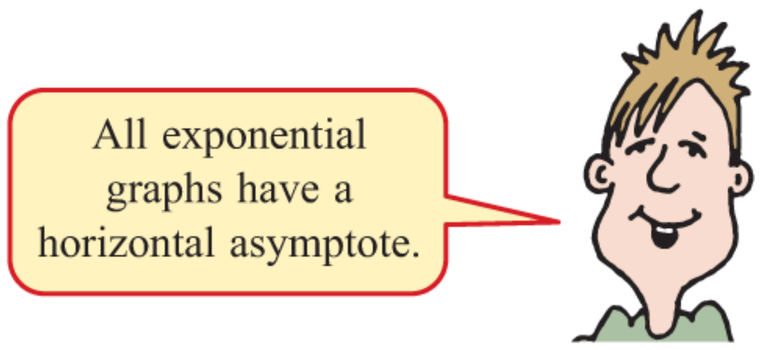


- If $p < 0$, $0 < a < 1$ the function is increasing.



We can sketch the graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points.



Example 12 **Self Tutor**

Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

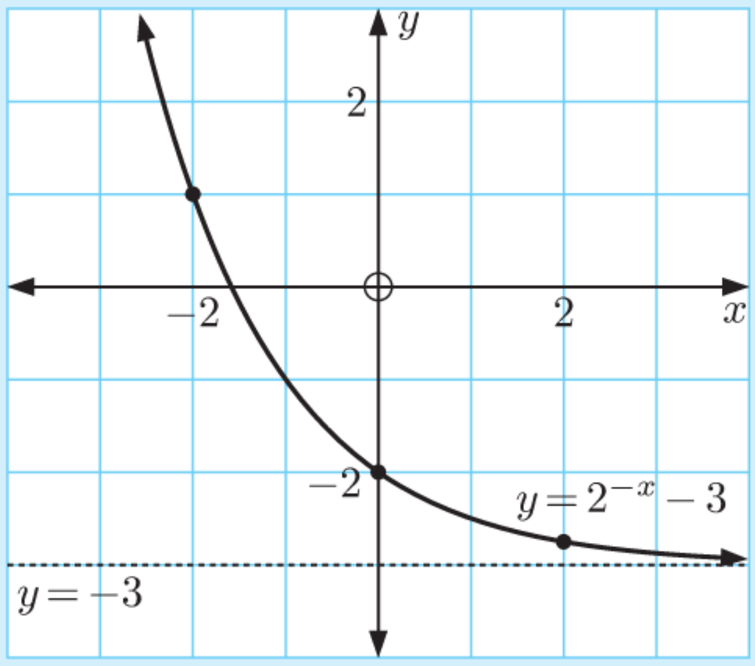
For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$



The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

EXERCISE 2D

1 Consider the graph of $y = 2^x$ alongside.

a Use the graph to estimate the value of:

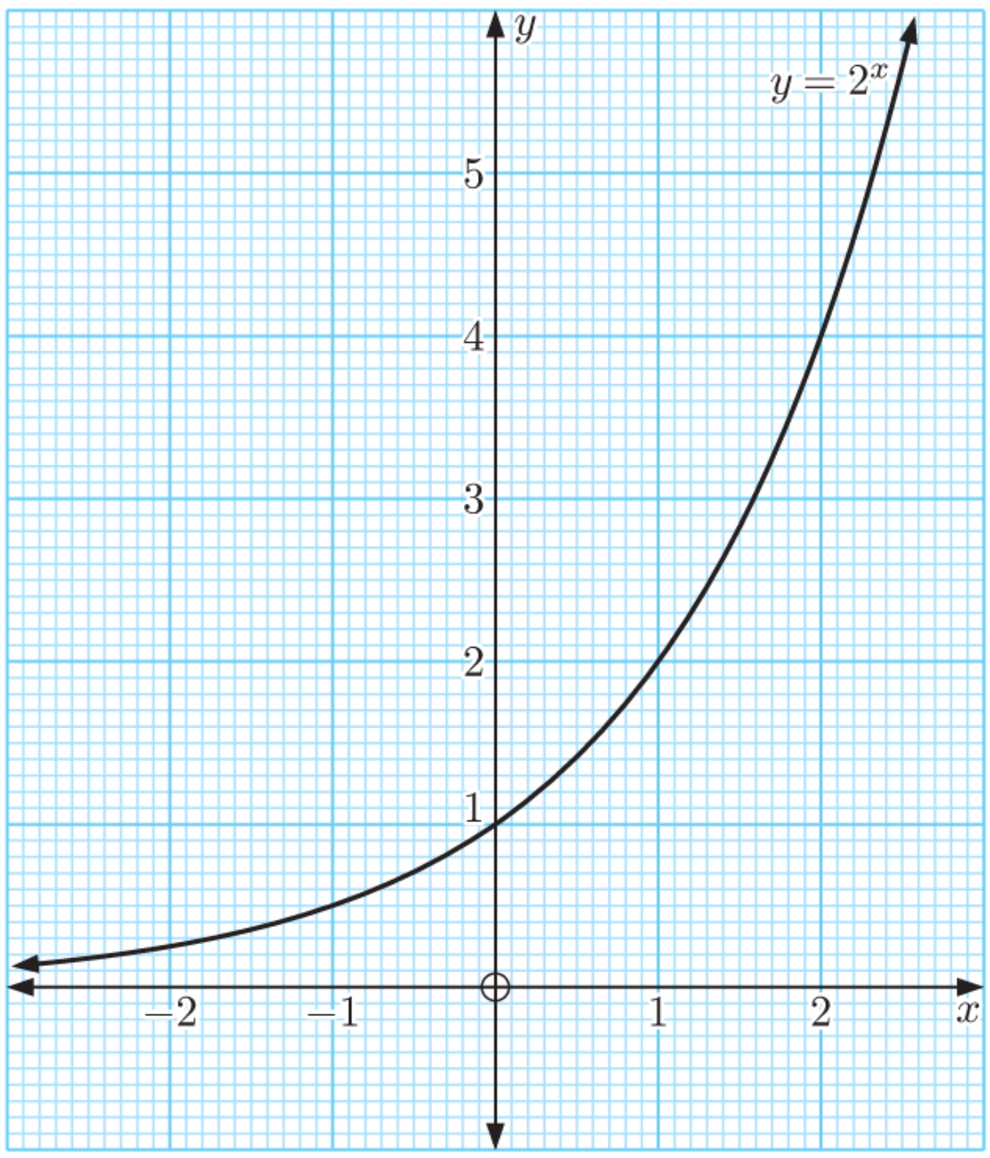
i $2^{\frac{1}{2}}$ or $\sqrt{2}$ **ii** $2^{0.8}$

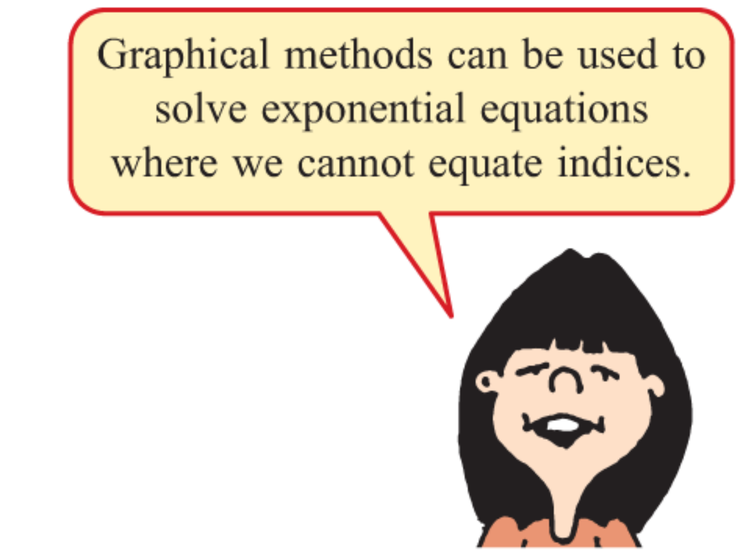
iii $2^{1.5}$ **iv** $2^{-\sqrt{2}}$

b Use the graph to estimate the solution to:

i $2^x = 3$ **ii** $2^x = 0.6$

c Use the graph to explain why $2^x = 0$ has no solutions.



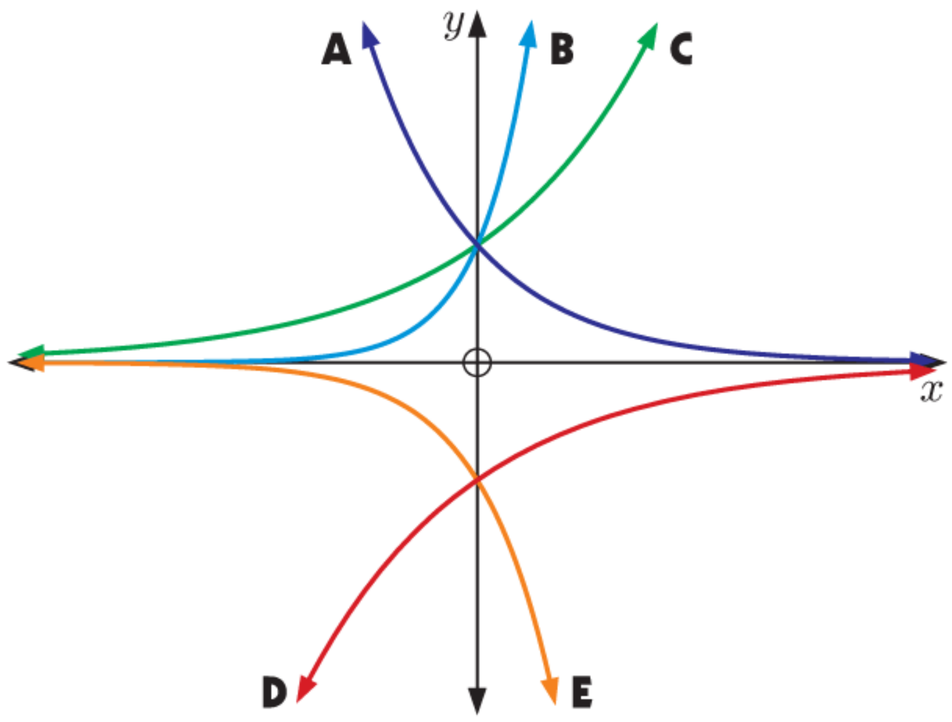


2 Match each function with its graph:

a $y = 2^x$ **b** $y = 10^x$

c $y = -5^x$ **d** $y = \left(\frac{1}{3}\right)^x$

e $y = -\left(\frac{1}{2}\right)^x$



3 Use a transformation to help sketch each pair of functions on the same set of axes:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

4 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

5 State the equation of the horizontal asymptote of:

a $y = 3^{-x}$ **b** $y = 2^x - 1$ **c** $y = 3 - 2^{-x}$

d $y = 4 \times 2^x + 2$ **e** $y = 5 \times 3^{x+2}$ **f** $y = -2 \times 3^{1-x} - 4$

6 Consider the exponential function $f(x) = 3^x - 2$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

7 Consider the function $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$.

a Find: **i** $g(0)$ **ii** $g(2)$ **iii** $g(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

8 Consider the function $h(x) = -2^{x-3} + 1$.

a Find: **i** $h(0)$ **ii** $h(3)$ **iii** $h(6)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

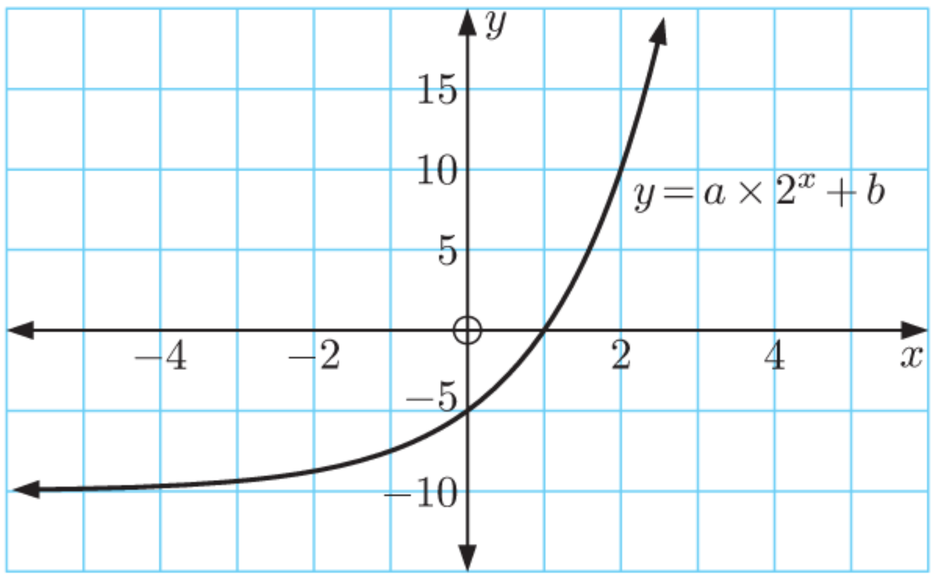
9 For each of the functions below:

i Sketch the graph of the function.
ii State the domain and range.
iii Use your calculator to find the value of y when $x = \sqrt{2}$.
iv Discuss the behaviour of y as $x \to \pm\infty$.
v Determine the horizontal asymptote.

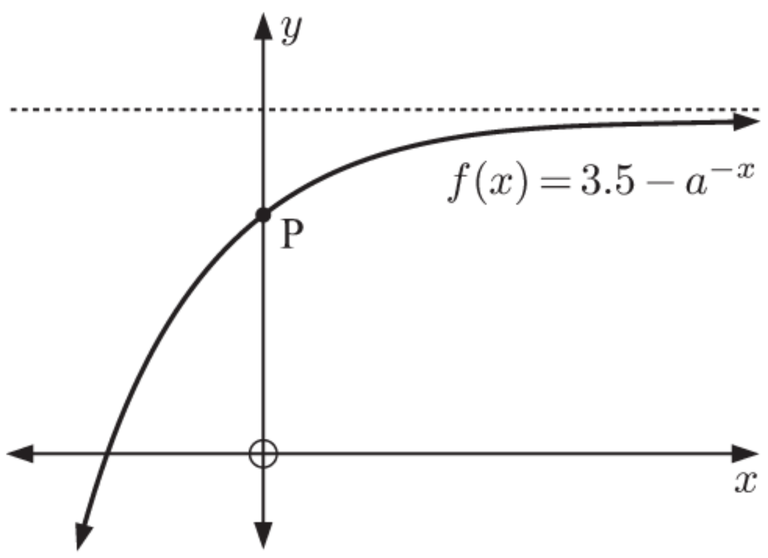
a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

10 The graph alongside shows the curve $y = a \times 2^x + b$, where a and b are constants.

a Find the values of a and b.
b Find y when $x = 6$.



11



This graph shows the function $f(x) = 3.5 - a^{-x}$, where a is a positive constant.
The point $(-1, 2)$ lies on the graph.

a Write down the coordinates of P.
b Find the value of a.
c Find the equation of the horizontal asymptote.

12 Find the domain and range of:

a $y = 2^{x^2+1}$ **b** $y = \dfrac{1}{3^x - 1}$ **c** $y = \sqrt{5^x - 5}$

13 Let $f(x) = 3^x - 9$ and $g(x) = \sqrt{x}$.

a Find $(f \circ g)(x)$, and state its domain and range.
b Find $(g \circ f)(x)$, and state its domain and range.
c Solve:
i $(f \circ g)(x) = 0$ **ii** $(g \circ f)(x) = 3\sqrt{2}$

14 Suppose $f(x) = 2^x - 3$ and $g(x) = 1 + 2^{-x}$.

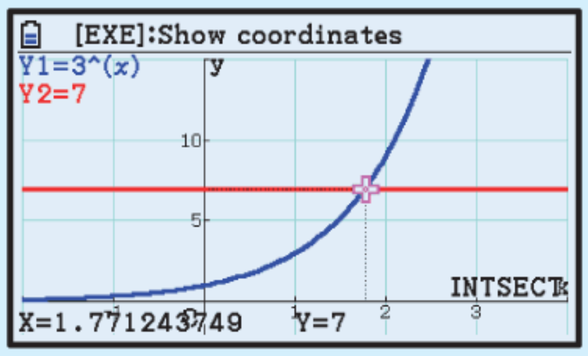
a For each function, find the:
i horizontal asymptote **ii** range **iii** y-intercept.
b Graph the functions on the same set of axes.
c Find the exact y-coordinate of the point where the graphs intersect.

Example 13 **Self Tutor**

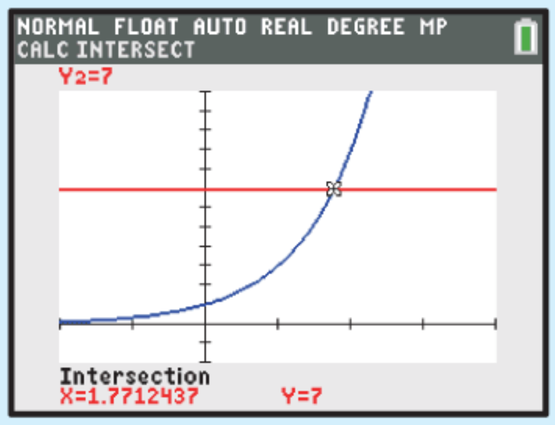
Use technology to solve the equation $3^x = 7$.

We graph $Y_1 = 3^X$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.

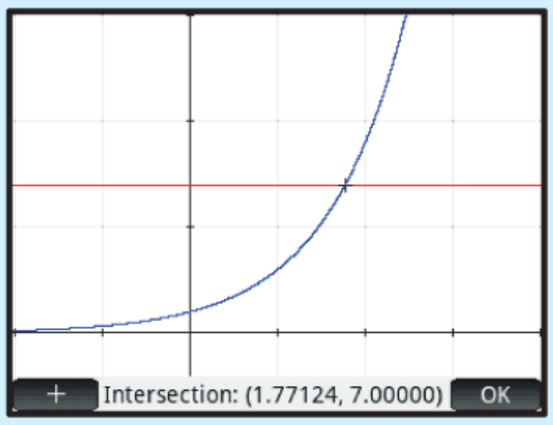
Casio fx-CG50



TI-84 Plus CE



HP Prime



The solution is $x \approx 1.77$.

15 Use technology to solve:

a $2^x = 11$ **b** $3^x = 15$ **c** $4^x + 5 = 10$

d $3^{x+2} = 4$ **e** $5 \times 2^x = 18$ **f** $3^{-x} = 0.9$

g $2 \times 3^{x-2} = 168$ **h** $26 \times (0.95)^x = 2$ **i** $2000 \times (1.03)^x = 5000$

DISCUSSION

For the exponential function $y = a^x$, why do we choose to specify $a > 0$?

What would the graph of $y = (-2)^x$ look like? What is its domain and range?

E GROWTH AND DECAY

In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.



Populations of animals, people, and bacteria usually *grow* in an exponential way.

Radioactive substances, cooling, and items that depreciate in value, usually *decay* exponentially.

For the exponential function $y = p \times a^{x-h} + k$ where $a, p > 0$, $a \neq 1$, we see:

- growth if $a > 1$
- decay if $a < 1$.

GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}

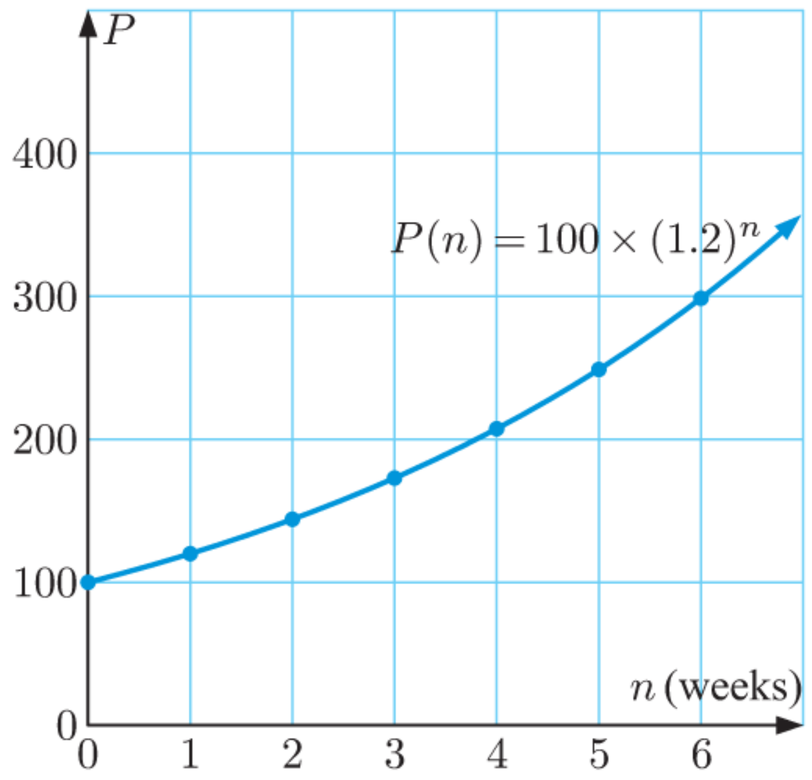
$P_1 = P_0 \times 1.2 = 100 \times 1.2$

$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$

$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$, $n \in \mathbb{Z}$, which is a geometric sequence.

However, while the population of mice must always be an integer, we expect that the population will grow continuously throughout the year, rather than in big, discrete jumps. We therefore expect it will be well approximated by the corresponding exponential function $P(n) = 100 \times (1.2)^n$, $n \in \mathbb{R}$.



Example 14 **Self Tutor**

A scientist is modelling a grasshopper plague. The area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after: **i** 5 weeks **ii** 10 weeks.

c Draw the graph of the affected area over time.

d Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

a $A(0) = 1000 \times 1.15^0 = 1000$

$\therefore$ the original affected area was 1000 hectares.

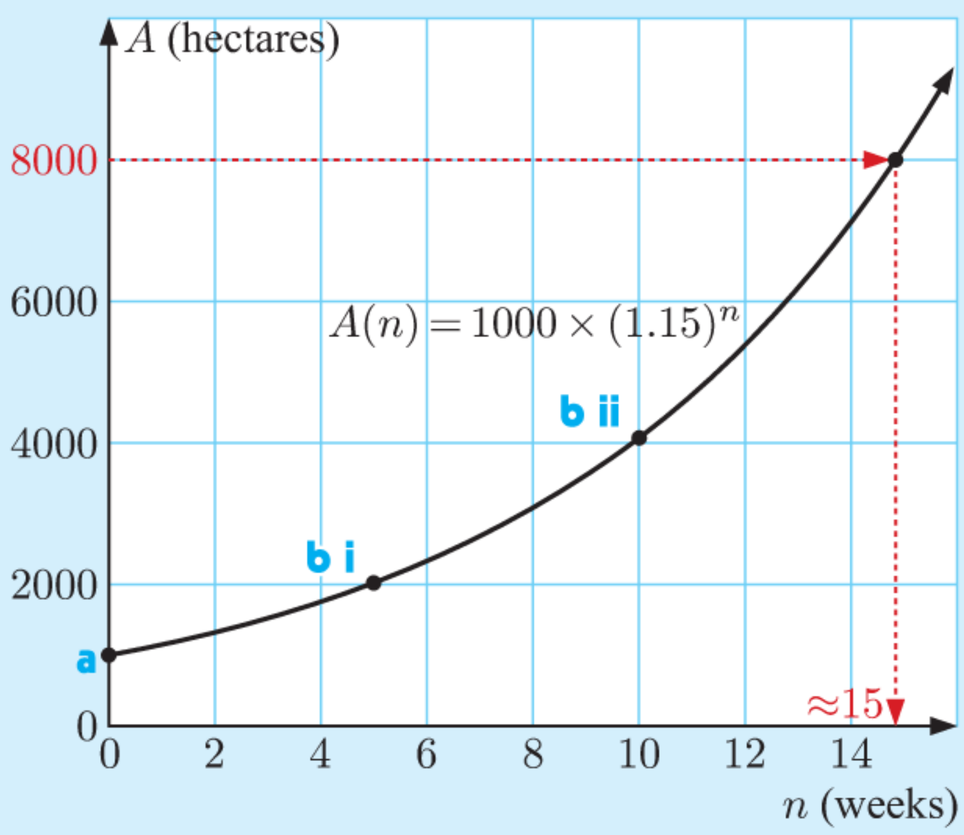
b **i** $A(5) = 1000 \times 1.15^5 \approx 2010$

The affected area is about 2010 hectares.

ii $A(10) = 1000 \times 1.15^{10} \approx 4050$

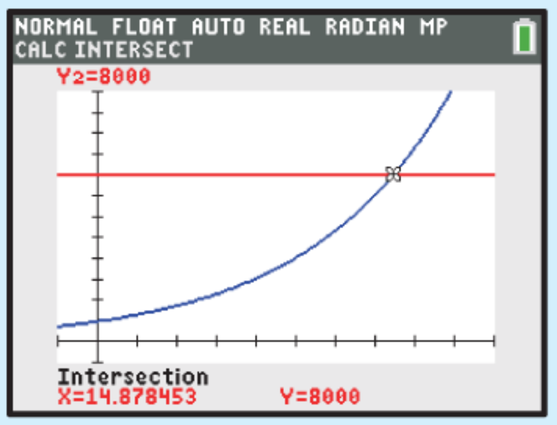
The affected area is about 4050 hectares.

c



d From the graph in **c**, it appears that it would take about 15 weeks for the affected area to reach 8000 hectares.

or Using technology, the solution is ≈ 14.9 weeks.



EXERCISE 2E.1



1 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times (1.07)^t$ grams.

a Find the initial weight.

b Find the weight after:

i 4 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of the bacteria weight over time using the results of **a** and **b** only.

Use technology to graph $Y_1 = 100 \times (1.07)^X$ and hence check your answers.



2 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times (1.23)^n$.

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of the population over time using **a** and **b** only.

d Hence estimate the time needed for the population to reach 500.

e Use technology to graph $Y_1 = 50 \times (1.23)^X$. Hence check your answer to **d**.

3 A flu virus spreads in a school. The number of people N infected after t days is given by $N = 4 \times 1.332^t, \ t \geqslant 0$.

a Find the number of people who were initially infected.

b Calculate the number of people who were infected after 16 days.

c There are 1200 people in the school. Estimate the time it will take for everybody in the school to catch the flu.

4 In 1998, 200 bears were introduced to a large island off Alaska where previously there were no bears. The population increased exponentially according to $B(t) = B_0 \times a^t$, where $a > 0$ is a constant and t is the time in years since the introduction.

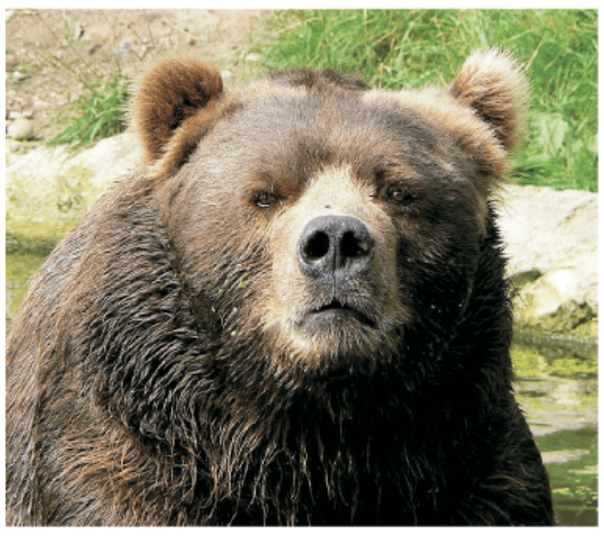
a Find B_0.

b In 2000 there were 242 bears. Find a, and interpret your answer.

c Find the expected bear population in 2018.

d Find the expected percentage increase in population from 2008 to 2018.

e How long will it take for the population to reach 2000?



5 The speed V of a chemical reaction is given by $V(t) = V_0 \times 2^{0.05t}$ where t is the temperature in °C.

a Find the reaction speed at: **i** 0°C **ii** 20°C.

b Find the percentage increase in reaction speed at 20°C compared with 0°C.

c Find $\left(\dfrac{V(50) - V(20)}{V(20)}\right) \times 100\%$ and explain what this calculation means.

6 Kayla deposited £5000 into an account. The amount in the account increases by 10% each year.

a Write a formula for the amount $A(t)$ in the account after t years.

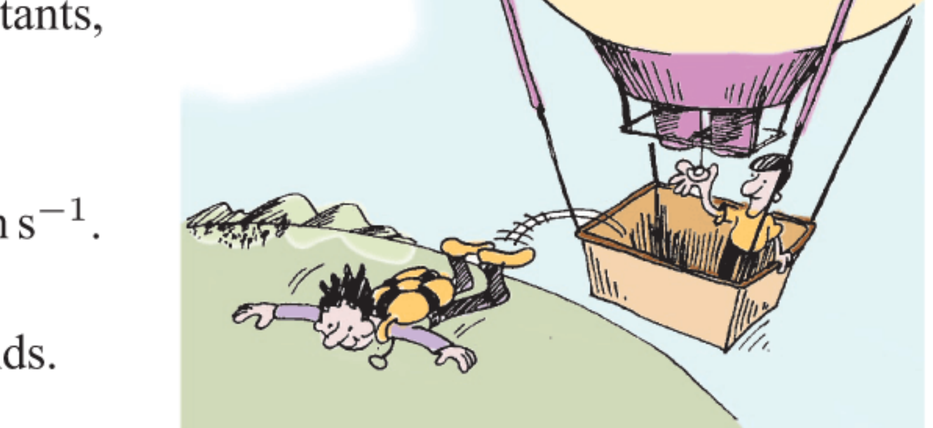
b Find the amount in the account after: **i** 2 years **ii** 5 years.

c Sketch the graph of $A(t)$.

d How long will it take for the amount in the account to reach £8000?

7 The expected value of a house in t years' time is given by the exponential function $V = k \times a^t$ dollars, where $t \geqslant 0$.

The function is graphed alongside.



a Find a and k, and interpret these values.

b How long will it take for the house's value to reach \$550 000?

8 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = c - 60 \times 2^{kt}$ m s^{-1} where c and k are constants, and t is the time in seconds.

a Explain why $c = 60$.

b After 5 seconds, the parachutist has speed 30 m s^{-1}. Find k.

c Find the speed of the parachutist after 12 seconds.

d Sketch the graph of V against t.

e Describe how the speed of the parachutist varies over time.

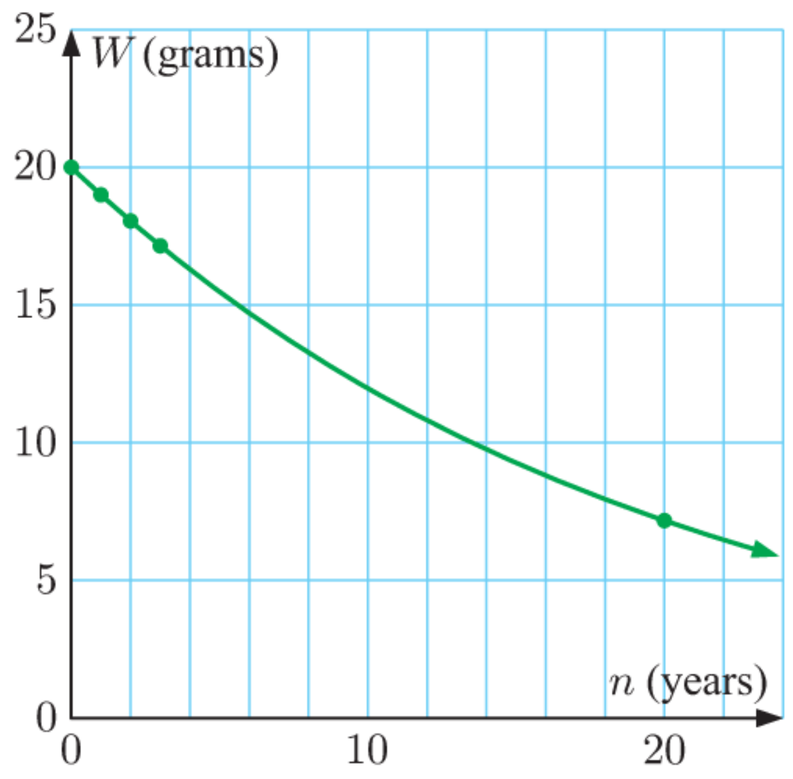
9 The number of microorganisms in a culture doubles every 6 hours. How long will it take for the number of microorganisms to increase by 30%?

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95 .

If W_n is the weight after n years, then:

$$\begin{aligned}
W_0 &= 20 \text{ grams} \\
W_1 &= W_0 \times 0.95 = 20 \times 0.95 \text{ grams} \\
W_2 &= W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams} \\
W_3 &= W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams} \\
&\vdots \\
W_{20} &= 20 \times (0.95)^{20} \approx 7.2 \text{ grams} \\
&\vdots
\end{aligned}$$



From this pattern we see that $W_n = 20 \times (0.95)^n$, $n \in \mathbb{Z}$, which is again a geometric sequence.

However, we know that radioactive decay is a continuous process, so the weight remaining will actually be given by the smooth exponential curve $W(n) = 20 \times (0.95)^n$, $n \in \mathbb{R}$.

Example 15 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a Find $I(t)$ when $t = 0$, 1, 2, and 3.

b What current flowed in the generator at the instant it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

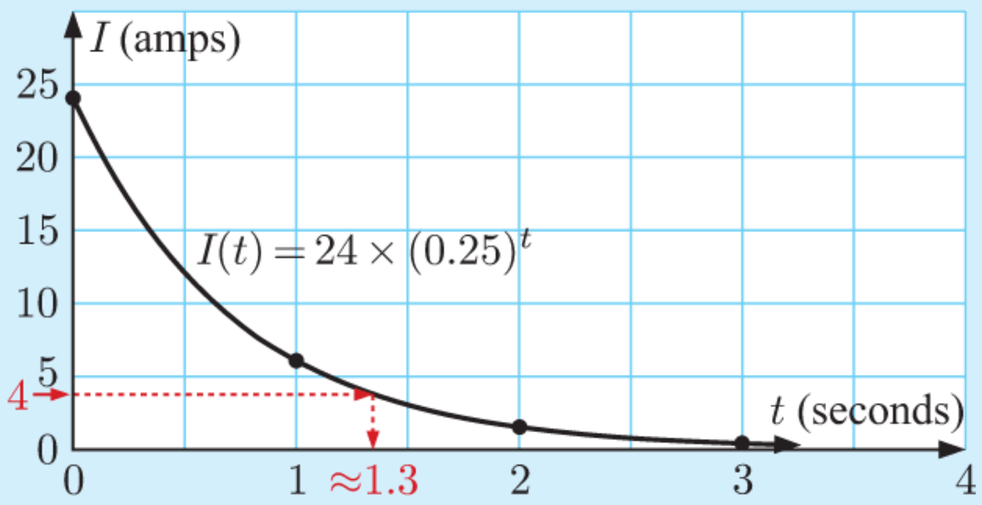
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b $I(0) = 24$

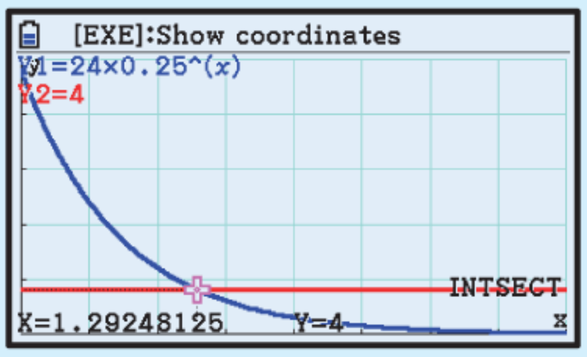
When the generator was switched off, 24 amps of current flowed in the circuit.

c



d From the graph above, the time to reach 4 amps is about 1.3 seconds.

or Using technology, the solution is ≈ 1.29 seconds.



EXERCISE 2E.2

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

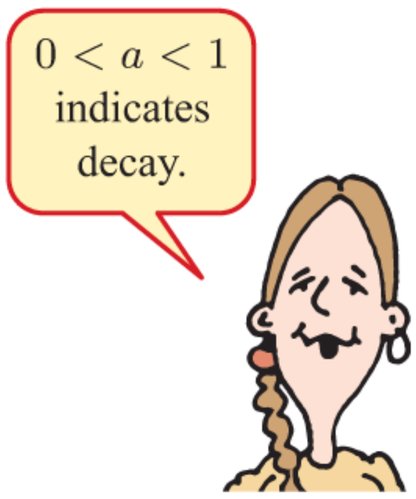
a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.



2 The temperature T of a liquid which has been placed in a refrigerator is given by $T(t) = 100 \times (0.986)^t$ °C where t is the time in minutes.

a Find the initial temperature of the liquid.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

3 The weight W of radioactive substance remaining after t years is given by $W(t) = 1000 \times 2^{-0.03t}$ grams.

a Find the initial weight of the radioactive substance.

b Find the weight remaining after:

i 10 years **ii** 100 years **iii** 1000 years.

c Graph the weight remaining over time using **a** and **b** only.

d Use your graph or graphics calculator to find the time when 10 grams of the substance remains.

e Write an expression for the amount of substance that has decayed after t years.

4 An initial count of orangutans in a forest found that the forest contained 400 orangutans. Since then, the destruction of their habitat has caused the population to fall by 8% each year.



a Write a formula for the population P of orangutans t years after the initial count.

b Find the population of orangutans after:

i 1 year **ii** 5 years.

c Sketch the graph of the population over time.

d How long will it take for the population to fall to 200?

5 The intensity of light L diminishes below the surface of the sea according to the formula $L = L_0 \times (0.95)^d$ units, where d is the depth in metres measured from the surface of the sea.

a If the intensity of light at the surface is 10 units, find the value of L_0.

b Find the intensity of light 25 m below the surface.

c A light intensity of 4 units is considered adequate for divers to be able to see clearly. Calculate the depth corresponding to this intensity of light.

d Calculate the range of depths for which the light intensity is between 1 and 3 units.

6 The value of a car after t years is $V = 24\,000 \times r^t$ dollars, $t \geqslant 0$.

a Write down the value of the car when it was first purchased.

b The value of the car after 2 years was \$17 340. Find the value of r.

c How long will it take for the value of the car to reduce to \$8000? Give your answer to the nearest year.

7 The interior of a freezer has temperature $-10°$C. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

a What was the temperature of the packet of peas:

i when placed in the freezer **ii** after 5 minutes **iii** after 10 minutes?

b Sketch the graph of $T(t)$.

c How long does it take for the temperature of the packet of peas to fall to $0°$C?

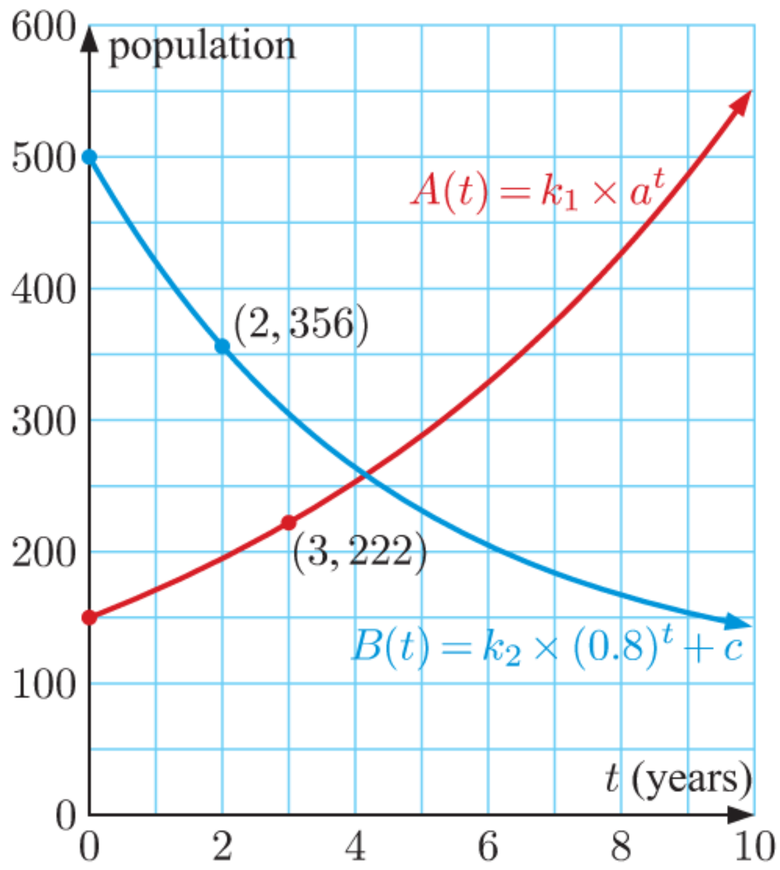
d Will the temperature of the packet of peas ever reach $-10°$C? Explain your answer.

8 The weight W_t of a radioactive uranium-235 sample remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$.

a Find the original weight. **b** Find the percentage weight loss after 1000 years.

c How long will it take until $\frac{1}{512}$ of the sample remains?

9 When scientists first observe a population of endangered marsupials, they notice two distinct groups. Group A is smaller in number, but appear to be larger and stronger individuals. Group B are more numerous, but smaller animals. The number of animals in each group over time are given by $A(t)$ and $B(t)$ respectively.



a Use the graph to find the exponential function for each animal group.

b Determine the time at which:

i there are the same number of group A and group B animals

ii there are 50 less group A than group B animals

iii there are twice as many group B animals compared to group A animals.

10 The **half-life** of a radioactive substance is the time it takes for the substance's weight to fall to half of its original value.

The radioactive isotope fermium-253 has a half-life of 3 days. The weight of fermium-253 detected t days after an explosion is $W(t) = 10 \times a^t$ mg.

a Interpret the value 10 in this model.

b Calculate the value of a, correct to 4 decimal places, and interpret this value.

c Find the weight of fermium-253 after 2 days.

d How long will it take for the weight of fermium-253 to fall to:

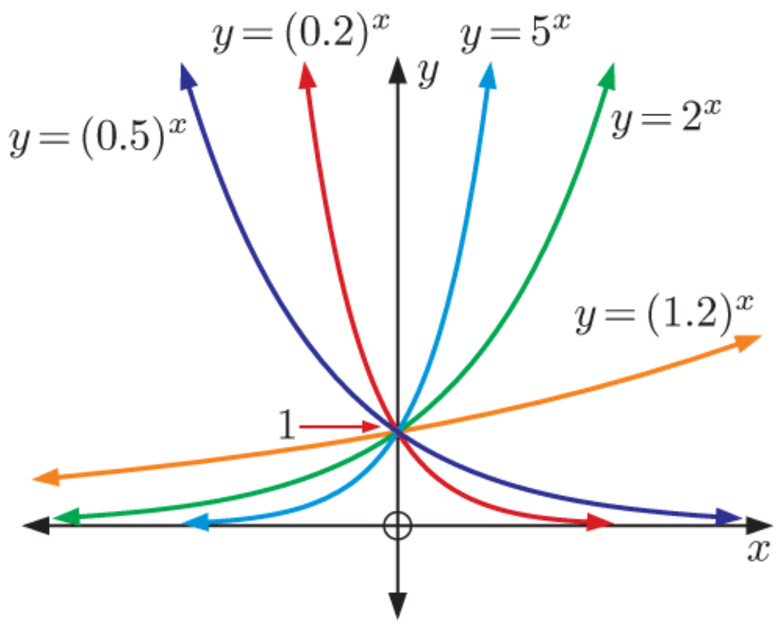
i 3 mg **ii** 1.25 mg?

11 The half-life of nitrogen-13 is 10 minutes. How long will it take for the mass of nitrogen-13 to fall to 10% of its original value?

F THE NATURAL EXPONENTIAL

We have seen that the simplest exponential functions have the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base a, the graph is always positive.

Hence $a^x > 0$ for all $a > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Investigation**.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A discrete formula for calculating the amount to which an investment grows under compound interest is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,
i is the interest rate per compounding period,
n is the number of periods, or times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to become extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$ **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$

c monthly **d** daily **e** by the second **f** by the millisecond.

Comment on your answers.

2 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

If we let $a = \frac{N}{r}$, show that the growth formula becomes $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

3 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

4 You should have found that for very large values of a, $\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$

Use the e^x key of your calculator to find the value of e^1. What do you notice?

5 For continuous growth, $u_n = u_0e^{rt}$ where u_0 is the initial amount, r is the annual percentage rate, and t is the number of years.

Use this formula to find the final amount if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is paid continuously.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0e^{rt}$, where r is the percentage rate per annum and t is the number of years.

HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748, Euler evaluated e correct to 18 decimal places.



Leonhard Euler

Euler also discovered some patterns in **continued fraction** expansions of e. He wrote that

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\ldots}}}}} \quad \text{and} \quad e-1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}}}$$

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

EXERCISE 2F

1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

GRAPHING PACKAGE

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = pe^{qx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be negative.

b Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\dfrac{1}{\sqrt{e}}$ **c** $\dfrac{1}{e^2}$ **d** $e\sqrt{e}$

7 Evaluate, to five significant figures:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$

e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

8 Expand and simplify:

a $(e^x + 1)^2$ **b** $(1 + e^x)(1 - e^x)$ **c** $e^x(e^{-x} - 3)$

9 Factorise:

a $e^{2x} + e^x$ **b** $e^{2x} - 16$ **c** $e^{2x} - 8e^x + 12$

10 **a** On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto e^{x-2}$, $h : x \mapsto e^x + 3$

b State the domain and range of each function.

11 **a** On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto -e^x$, $h : x \mapsto 10 - e^x$

b State the domain and range of each function.

c Describe the behaviour of each function as $x \to \pm\infty$.

12 Let $f(x) = e^x - 1$ and $g(x) = \dfrac{1}{x}$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

13 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture:

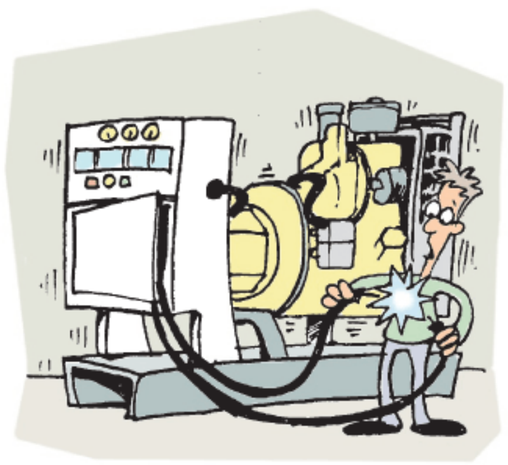
i initially **ii** after 30 minutes **iii** after $1\frac{1}{2}$ hours **iv** after 6 hours.

b Hence sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

14 Solve for x:

a $e^x = \sqrt{e}$ **b** $e^{\frac{1}{2}x} = \dfrac{1}{e^2}$ **c** $e^{2x} + e^x = 2$

15 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.



a What current is still flowing in the circuit after:

i 1 second **ii** 10 seconds?

b Use your graphics calculator to help sketch the graph of $I(t) = 75e^{-0.15t}$.

c How long will it take for the current to fall to 1 amp?

16 The population P of trout in a lake is given by $P(t) = \dfrac{800}{1 + ke^{-0.5t}}$, where t is the time in months.

a Given that there were initially 20 trout in the lake, find the value of k.

b Find the population after 6 months.

c Use technology to sketch the graph of $P(t)$.

d Describe what happens to the population as t increases.

e How long will it take for the population to reach 600?

17 Consider the function $f(x) = e^x$.

a On the same set of axes, sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

b State the domain and range of f^{-1}.

18 It can be shown that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times3}x^3 + \frac{1}{2\times3\times4}x^4 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ which is an infinite polynomial expansion.

Check this statement by using the first 20 terms of the series to find an approximation for e^1.

ACTIVITY

Click on the icon to run a card game for exponential functions.



REVIEW SET 2A

1 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$ **c** $81^{-\frac{1}{4}}$

2 Solve for x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \frac{1}{\sqrt{e}}$

3 Expand and simplify:

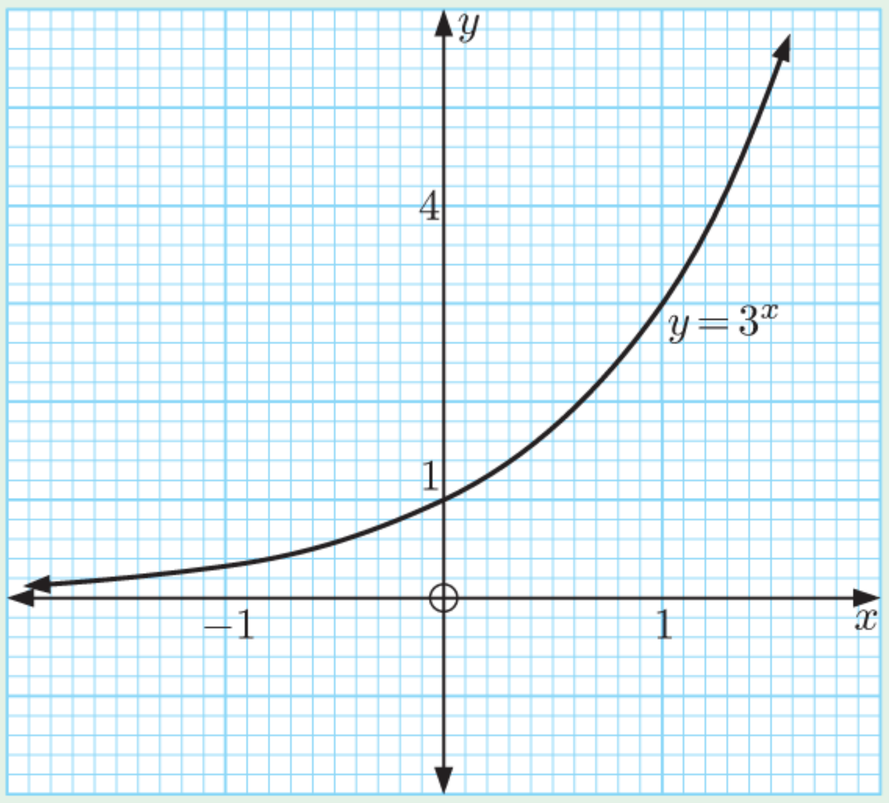
a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

4 Solve for x:

a $6 \times 2^x = 192$ **b** $9^{x-1} \times \left(\frac{1}{27}\right)^x = \sqrt{3}$ **c** $4^x - 32 = 4(2^x)$

5 The point $(1, \sqrt{8})$ lies on the graph of $y = 2^{kx}$. Find the value of k.

6 Consider the graph of $y = 3^x$ alongside.



a Use the graph to estimate the value of:

i $3^{0.7}$ **ii** $3^{-0.5}$

b Use the graph to estimate the solution to:

i $3^x = 5$ **ii** $3^x = \frac{1}{2}$

iii $6 \times 3^x = 20$

7 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

8 On the same set of axes, draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

9 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3^x - 5$.

d State the equation of any asymptote.

10 Consider $y = 3 - 2^{-x}$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3 - 2^{-x}$.

d State the equation of any asymptote.

11 Let $f(x) = 2^x$ and $g(x) = 3 - x^2$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

c Solve for x:

i $(f \circ g)(x) = 2$ **ii** $(g \circ f)(x) = -13$

12 **a** On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto e^{x-1}$, $h : x \mapsto 3 - e^x$

b State the domain and range of each function in **a**.

c Describe the behaviour of each function as $x \to \pm\infty$.

13 A plant doubles in size every 5 days. How often does it treble in size?

14 The temperature of a dish t minutes after it is removed from the microwave, is given by $T(t) = 80 \times (0.913)^t$ °C.

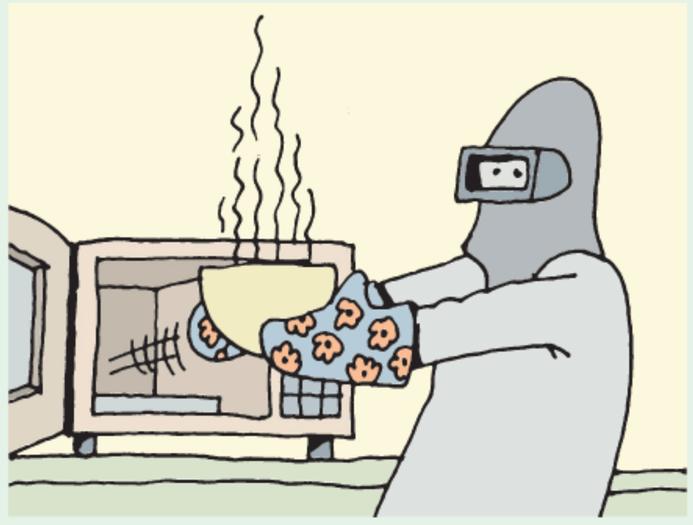
a Find the initial temperature of the dish.

b Find the temperature after:

i 12 minutes **ii** 24 minutes **iii** 36 minutes.

c Draw the graph of T against t for $t \geqslant 0$, using **a** and **b** or technology.

d Hence find the time taken for the temperature of the dish to fall to 25°C.



REVIEW SET 2B

1 Evaluate, correct to 3 significant figures:

a $3^{\frac{5}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

2 Expand and simplify:

a $(3 - e^x)^2$ **b** $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$ **c** $2^{-x}(2^{2x} + 2^x)$

3 Factorise:

a $3^{x+2} - 3^x$ **b** $4^x - 2^x - 12$ **c** $e^{2x} + 2e^x - 15$

4 Solve for x:

a $3 \times \left(\frac{1}{7}\right)^{x+1} = 1029$ **b** $9^x - 10(3^x) + 9 = 0$ **c** $2(4^{x+1}) + 1 = 6(2^x)$

5 Suppose $f(x) = 2^{-x} + 1$.

a Find $f\left(\frac{1}{2}\right)$. **b** Find a such that $f(a) = 3$.

6 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm1, \pm2$. **b** Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$. **d** State the equation of any asymptote.

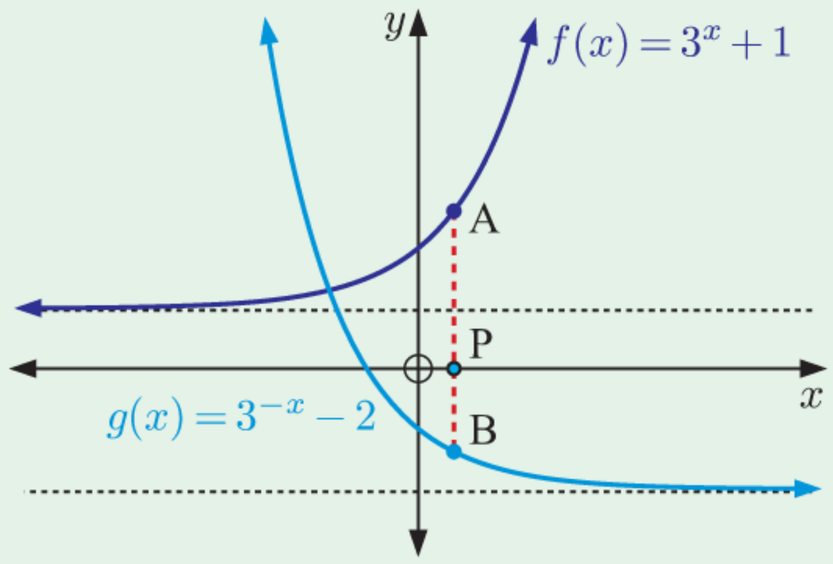
7 Answer the **Opening Problem** on page **42**.

8 Find the domain and range of $f(x) = 3^{\sqrt{x+1}}$.

9 The exponential functions $f(x) = 3^x + 1$ and $g(x) = 3^{-x} - 2$ are graphed alongside.

a Find the y-intercept of each function.

b Given that the vertical line segment [AB] has length 4 units, find the exact length of [PB]. Give your answer in the form $a + b\sqrt{5}$ units, where $a, b \in \mathbb{Q}$.



10 Let $f(x) = 3^x$.

a Write down the value of:

i $f(4)$ **ii** $f(-1)$

b Find the value of k such that $f(x + 2) = k\,f(x)$, $k \in \mathbb{Z}$.

11 Suppose $y = a^x$. Express in terms of y:

a a^{2x} **b** a^{-x} **c** $\dfrac{1}{\sqrt{a^x}}$

12 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t for $t \geqslant 0$.

d Hence find the time taken for the weight to reduce to 100 grams.

13 A phycologist investigates an algal bloom in a lake. Initially it covers 10 square metres of water. Each day after it was discovered, the area covered increases by 15%.

a Write a formula for the area $A(t)$ covered after t days.

b Find the area covered after:

i 2 days **ii** 5 days.

c Sketch the graph of $A(t)$.

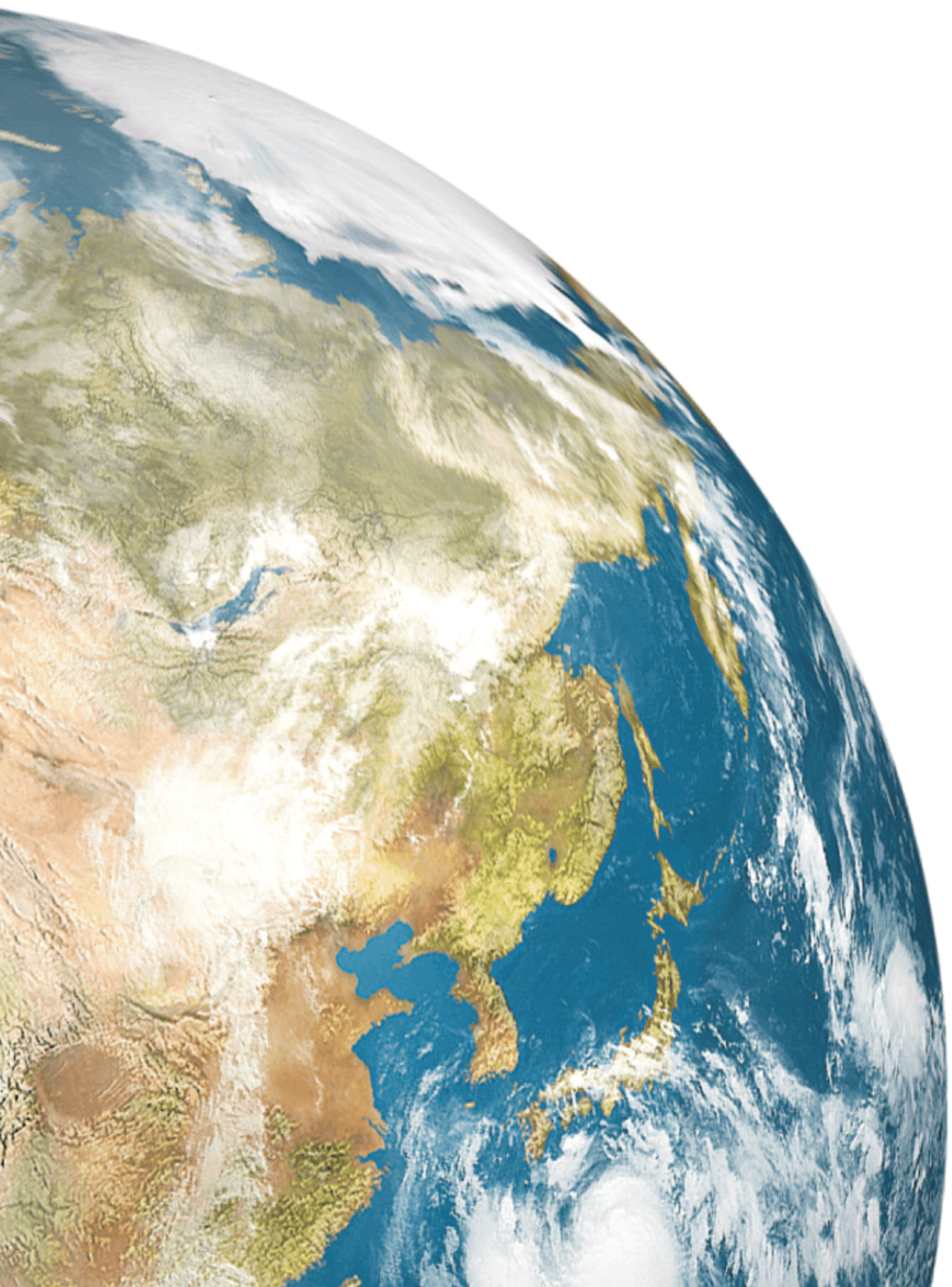
d How long will it take for the affected area to reach 300 m^2?

Chapter 3

Logarithms

Contents:

A Logarithms in base 10
B Logarithms in base a
C Laws of logarithms
D Natural logarithms
E Logarithmic equations
F The change of base rule
G Solving exponential equations using logarithms
H Logarithmic functions



OPENING PROBLEM

In a plentiful springtime, a population of 1000 mice will double every week.

The population after t weeks is given by the exponential function $P(t) = 1000 \times 2^t$ mice.



Things to think about:

a What does the graph of the population over time look like?

b How long will it take for the population to reach 20 000 mice?

c Can we write a function for t in terms of P, which determines the time at which the population P is reached?

In the previous Chapter we solved exponential equations by writing both sides with the same base, and by using graphs.

In this Chapter we study a more formal solution to exponential equations in which we use the **inverse** of the exponential function. We call this a **logarithm**.

A LOGARITHMS IN BASE 10

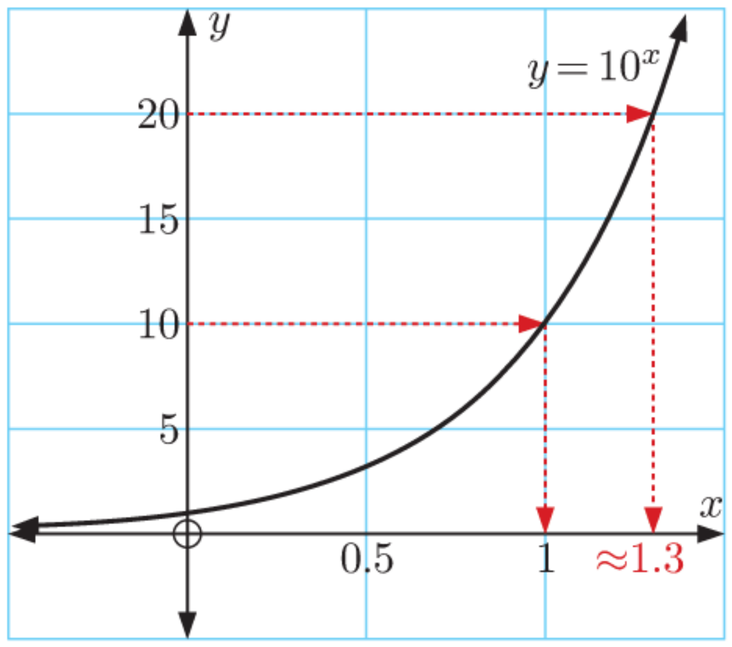
Consider the graph of $y = 10^x$ shown.

Notice that the range of the function is $\{y \mid y > 0\}$. This means that every positive number y can be written in the form 10^x.

For example:

- When $y = 10$, $x = 1$, so $10 = 10^1$.
- When $y = 20$, $x \approx 1.3$, so $20 \approx 10^{1.3}$.

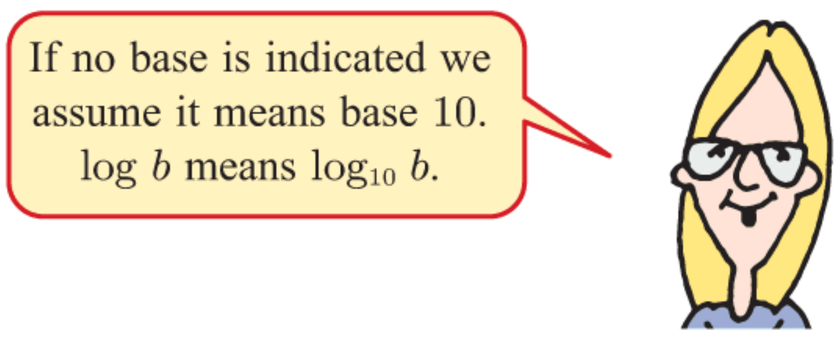
When we write a positive number y in the form 10^x, we say that x is the **logarithm in base 10**, of y.



The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

For example:

- The logarithm in base 10 of 1000 is 3, since $1000 = 10^3$.
  We write $\log_{10} 1000 = 3$ or simply $\log 1000 = 3$.
- $\log(0.01) = -2$ since $0.01 = 10^{-2}$.



By observing that $\log 1000 = \log(10^3) = 3$ and $\log(0.01) = \log(10^{-2}) = -2$, we conclude that $\log 10^x = x$ for any $x \in \mathbb{R}$.

Example 1 — Self Tutor

Find: **a** $\log 100$ **b** $\log \sqrt[4]{10}$

a $\log 100 = \log(10^2) = 2$

b $\log \sqrt[4]{10} = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

The logarithms in **Example 1** can be found by hand because it is easy to write 100 and $\sqrt[4]{10}$ as powers of 10. The logarithms of most values, however, can only be found using a calculator.

For example, $\log 34 \approx 1.53$

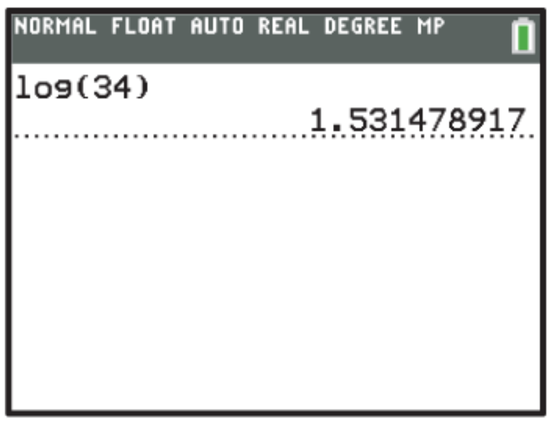
so $34 \approx 10^{1.53}$

Logarithms allow us to write any number as a power of 10. In particular:

$$x = 10^{\log x} \quad \text{for any} \quad x > 0.$$



GRAPHICS CALCULATOR INSTRUCTIONS



Example 2 — Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

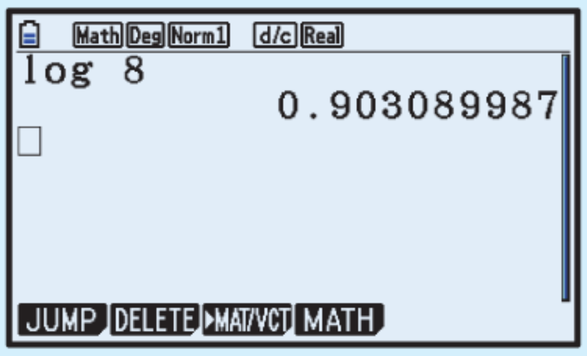
a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8} \approx 10^{0.9031}$

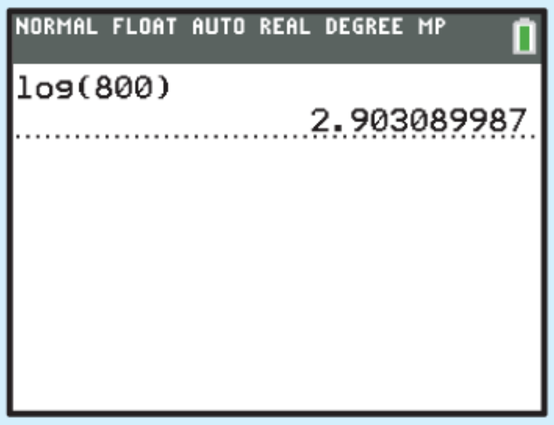
b $800 = 10^{\log 800} \approx 10^{2.9031}$

c $0.08 = 10^{\log 0.08} \approx 10^{-1.0969}$

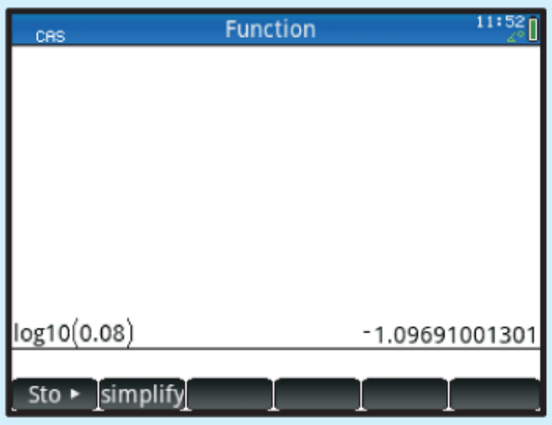
Casio fx-CG50



TI-84 Plus CE



HP Prime



EXERCISE 3A

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log 10$ **d** $\log 1$

e $\log \sqrt{10}$ **f** $\log \sqrt[3]{10}$ **g** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\log\left(10\sqrt{10}\right)$

i $\log \sqrt[3]{100}$ **j** $\log\left(\frac{100}{\sqrt{10}}\right)$ **k** $\log\left(10 \times \sqrt[3]{10}\right)$ **l** $\log\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\log(10^n)$ **b** $\log(10^a \times 100)$ **c** $\log\left(\frac{10}{10^m}\right)$ **d** $\log\left(\frac{10^a}{10^b}\right)$

3 **a** Explain why $\log 237$ must lie between 2 and 3.

b Use your calculator to evaluate $\log 237$ correct to 2 decimal places.

4 **a** Between which two consecutive whole numbers does $\log(0.6)$ lie?

b Check your answer by evaluating $\log(0.6)$ correct to 2 decimal places.

5 Use your calculator to evaluate, correct to 2 decimal places:

a $\log 76$ **b** $\log 114$ **c** $\log 3$ **d** $\log 831$

e $\log(0.4)$ **f** $\log 3247$ **g** $\log(0.008)$ **h** $\log(-7)$

6 For what values of x is $\log x$:

a positive **b** zero **c** negative **d** undefined?

7 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

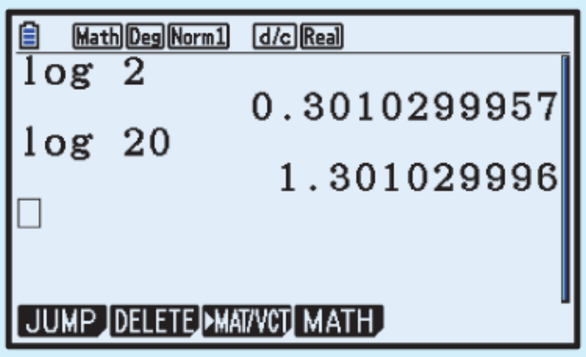
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 3 — Self Tutor

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a



i $\log 2 \approx 0.3010$

ii $\log 20 \approx 1.3010$

b
$$\begin{aligned}\log 20 &= \log(2 \times 10) \\ &= \log(10^{\log 2} \times 10^1) && \{x = 10^{\log x}\} \\ &= \log(10^{\log 2 + 1}) && \{\text{adding indices}\} \\ &= \log 2 + 1\end{aligned}$$

8 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$

b Explain why $\log 300 = \log 3 + 2$.

9 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log(0.05)$

b Explain why $\log(0.05) = \log 5 - 2$.

Example 4 — Self Tutor

Find x such that:

a $\log x = 3$ **b** $\log x \approx -0.271$

a
$$\begin{aligned}&\log x = 3 \\ \therefore\ &10^{\log x} = 10^3 \\ \therefore\ &x = 1000\end{aligned}$$

b
$$\begin{aligned}&\log x \approx -0.271 \\ \therefore\ &10^{\log x} \approx 10^{-0.271} \\ \therefore\ &x \approx 0.536\end{aligned}$$

10 Find x such that:

a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$ **d** $\log x = -1$

e $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$ **g** $\log x = 4$ **h** $\log x = -5$

i $\log x \approx 0.8351$ **j** $\log x \approx 2.1457$ **k** $\log x \approx -1.378$ **l** $\log x \approx -3.1997$

B LOGARITHMS IN BASE a

In the previous Section we defined the logarithm in base 10 of a number as the power that 10 must be raised to in order to obtain that number.

We can use the same principle to define logarithms in other bases:

The **logarithm in base a of b** is the power that a must be raised to in order to obtain b.

The logarithm in base a of b is written $\log_a b$.



For example, to find $\log_2 8$, we ask "What power must 2 be raised to in order to obtain 8?". We know that $2^3 = 8$, so $\log_2 8 = 3$.

$a^x = b$ and $x = \log_a b$ are *equivalent* statements.

For any $b > 0$, $a^x = b \Leftrightarrow x = \log_a b$

Example 5 **Self Tutor**

a Write an equivalent exponential statement for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic statement for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 3B

1 Write an equivalent exponential statement for:

a $\log_{10} 100 = 2$ b $\log_{10} 10\,000 = 4$ c $\log_{10}(0.1) = -1$

d $\log_{10} \sqrt{10} = \frac{1}{2}$ e $\log_2 8 = 3$ f $\log_3 9 = 2$

g $\log_2\left(\frac{1}{4}\right) = -2$ h $\log_3 \sqrt{27} = 1.5$ i $\log_5\left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$

2 Write an equivalent logarithmic statement for:

a $4^3 = 64$ b $5^2 = 25$ c $7^2 = 49$ d $2^6 = 64$

e $2^{-3} = \frac{1}{8}$ f $10^{-2} = 0.01$ g $2^{-1} = \frac{1}{2}$ h $3^{-3} = \frac{1}{27}$

Example 6 **Self Tutor**

Find: a $\log_2 16$ b $\log_5(0.2)$ c $\log_{10} \sqrt[5]{100}$ d $\log_2\left(\frac{1}{\sqrt{2}}\right)$

a $\log_2 16$
$= \log_2(2^4)$
$= 4$

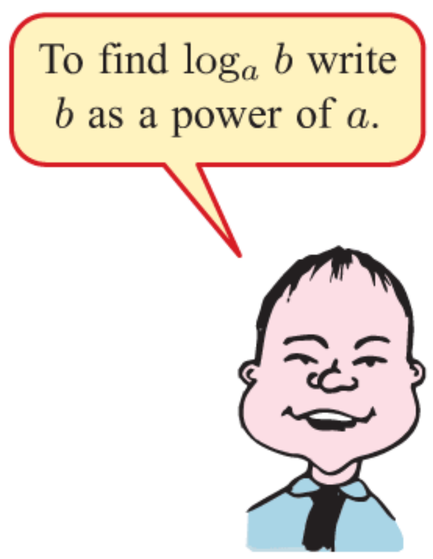
b $\log_5(0.2)$
$= \log_5\left(\frac{1}{5}\right)$
$= \log_5(5^{-1})$
$= -1$

c $\log_{10} \sqrt[5]{100}$
$= \log_{10}((10^2)^{\frac{1}{5}})$
$= \log_{10}(10^{\frac{2}{5}})$
$= \frac{2}{5}$

d $\log_2\left(\frac{1}{\sqrt{2}}\right)$
$= \log_2(2^{-\frac{1}{2}})$
$= -\frac{1}{2}$

3 Without using a calculator, find:

a $\log_{10} 100\,000$	**b** $\log_{10}(0.01)$	**c** $\log_3 \sqrt{3}$
d $\log_2 4$	**e** $\log_2 64$	**f** $\log_2 128$
g $\log_5 25$	**h** $\log_5 125$	**i** $\log_2(0.125)$
j $\log_9 3$	**k** $\log_4 16$	**l** $\log_{36} 6$
m $\log_3 243$	**n** $\log_2 \sqrt[3]{2}$	**o** $\log_8 2$
p $\log_6 \left(6\sqrt{6}\right)$	**q** $\log_4 1$	**r** $\log_9 9$
s $\log_3 \left(\frac{1}{3}\right)$	**t** $\log_{10} \sqrt[4]{1000}$	**u** $\log_7 \left(\frac{1}{\sqrt{7}}\right)$
v $\log_5 \left(25\sqrt{5}\right)$	**w** $\log_3 \left(\frac{1}{\sqrt{27}}\right)$	**x** $\log_4 \left(\frac{1}{2\sqrt{2}}\right)$



Check your answers using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

4 Simplify:

a $\log_x(x^2)$	**b** $\log_t \left(\frac{1}{t}\right)$	**c** $\log_x \sqrt{x}$
d $\log_m(m^3)$	**e** $\log_k \sqrt[4]{k}$	**f** $\log_x(x\sqrt{x})$
g $\log_a \left(\frac{1}{a^2}\right)$	**h** $\log_a \left(\frac{1}{\sqrt{a}}\right)$	**i** $\log_m \sqrt{m^5}$

Example 7 — Self Tutor

Solve for x: $\log_3 x = 5$

$\log_3 x = 5$
$\therefore \quad x = 3^5$
$\therefore \quad x = 243$

5 Solve for x:

a $\log_2 x = 3$	**b** $\log_4 x = \frac{1}{2}$	**c** $\log_5 x = -3$
d $\log_x 81 = 4$	**e** $\log_2(x-6) = 3$	**f** $\log_2(\log_3 x) = -1$

6 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_b a$.

7 If $y = \log_2 \sqrt{5x-1}$, write x in terms of y.

HISTORICAL NOTE

Acharya Virasena was an 8th century Indian mathematician. Among other areas, he worked with the concept of *ardhaccheda*, which is how many times a number of the form 2^n can be divided by 2. The result is the integer n, and is the logarithm of the number 2^n in base 2.

In 1544, the German **Michael Stifel** published *Arithmetica Integra* which contains a table expressing many other integers as powers of 2. In effect, he had created an early version of a logarithmic table.

C LAWS OF LOGARITHMS

INVESTIGATION 1 DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 **a** Use your calculator to find:

i $\log 2 + \log 3$ **ii** $\log 3 + \log 7$ **iii** $\log 4 + \log 20$

iv $\log 6$ **v** $\log 21$ **vi** $\log 80$

b From your answers, suggest a possible simplification for $\log m + \log n$.

2 **a** Use your calculator to find:

i $\log 6 - \log 2$ **ii** $\log 12 - \log 3$ **iii** $\log 3 - \log 5$

iv $\log 3$ **v** $\log 4$ **vi** $\log(0.6)$

b From your answers, suggest a possible simplification for $\log m - \log n$.

3 **a** Use your calculator to find:

i $3 \log 2$ **ii** $2 \log 5$ **iii** $-4 \log 3$

iv $\log(2^3)$ **v** $\log(5^2)$ **vi** $\log(3^{-4})$

b From your answers, suggest a possible simplification for $m \log b$.

From the **Investigation**, you should have discovered the three important **laws of logarithms**:

- $\log m + \log n = \log(mn)$ for $m, n > 0$
- $\log m - \log n = \log\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \log b = \log(b^m)$ for $b > 0$

More generally, in any base a where $a \neq 1$, $a > 0$, we have these **laws of logarithms**:

- $\log_a m + \log_a n = \log_a(mn)$ for $m, n > 0$
- $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \log_a b = \log_a(b^m)$ for $b > 0$

Proof:

- $\log_a(mn)$
  $= \log_a\left(a^{\log_a m} \times a^{\log_a n}\right)$
  $= \log_a\left(a^{\log_a m + \log_a n}\right)$
  $= \log_a m + \log_a n$
- $\log_a\left(\frac{m}{n}\right)$
  $= \log_a\left(\frac{a^{\log_a m}}{a^{\log_a n}}\right)$
  $= \log_a\left(a^{\log_a m - \log_a n}\right)$
  $= \log_a m - \log_a n$
- $\log_a(b^m)$
  $= \log_a\left(\left(a^{\log_a b}\right)^m\right)$
  $= \log_a\left(a^{m \log_a b}\right)$
  $= m \log_a b$

Example 8 — Self Tutor

Use the laws of logarithms to write as a single logarithm or as an integer:

a $\log 5 + \log 3$ **b** $\log_3 24 - \log_3 8$ **c** $\log_2 5 - 1$

a
$$\begin{aligned}&\log 5 + \log 3\\ &= \log(5 \times 3)\\ &= \log 15\end{aligned}$$

b
$$\begin{aligned}&\log_3 24 - \log_3 8\\ &= \log_3\left(\tfrac{24}{8}\right)\\ &= \log_3 3\\ &= 1\end{aligned}$$

c
$$\begin{aligned}&\log_2 5 - 1\\ &= \log_2 5 - \log_2(2^1)\\ &= \log_2\left(\tfrac{5}{2}\right)\end{aligned}$$

EXERCISE 3C

1 Write as a single logarithm or as an integer:

a $\log 8 + \log 2$ **b** $\log 4 + \log 5$ **c** $\log 40 - \log 5$

d $\log p - \log m$ **e** $\log_4 8 - \log_4 2$ **f** $\log 5 + \log(0.4)$

g $\log 250 + \log 4$ **h** $\log_5 100 - \log_5 4$ **i** $\log 2 + \log 3 + \log 4$

j $\log 5 + \log 4 - \log 2$ **k** $\log_3 6 - \log_3 2 - \log_3 3$ **l** $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

2 Write as a single logarithm:

a $\log 7 + 2$ **b** $\log 4 - 1$ **c** $1 + \log_2 3$

d $\log_3 5 - 2$ **e** $2 + \log 2$ **f** $\log 50 - 4$

g $t + \log w$ **h** $\log_m 40 - 2$ **i** $3 - \log_5 50$

Example 9 — Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2\log 7 - 3\log 2$ **b** $2\log 3 + 3$ **c** $\dfrac{\log 8}{\log 4}$

a
$$\begin{aligned}&2\log 7 - 3\log 2\\ &= \log(7^2) - \log(2^3)\\ &= \log 49 - \log 8\\ &= \log\left(\tfrac{49}{8}\right)\end{aligned}$$

b
$$\begin{aligned}&2\log 3 + 3\\ &= \log(3^2) + \log(10^3)\\ &= \log 9 + \log 1000\\ &= \log 9000\end{aligned}$$

c
$$\begin{aligned}\frac{\log 8}{\log 4} &= \frac{\log(2^3)}{\log(2^2)}\\ &= \frac{3\log 2}{2\log 2}\\ &= \tfrac{3}{2}\end{aligned}$$

3 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$

d $2\log_3 5 - 3\log_3 2$ **e** $\frac{1}{2}\log_6 4 + \log_6 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$

g $3 - \log 2 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log_n 4 - \log_n 5$

4 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log_5 27}{\log_5 9}$ **c** $\dfrac{\log 8}{\log 2}$

d $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log_3 25}{\log_3(0.2)}$ **f** $\dfrac{\log_4 8}{\log_4(0.25)}$

Example 10 **Self Tutor**

Show that:

a $\log\left(\frac{1}{9}\right) = -2\log 3$ **b** $\log 500 = 3 - \log 2$

a
$$\begin{aligned}&\log\left(\tfrac{1}{9}\right)\\ &= \log(3^{-2})\\ &= -2\log 3\end{aligned}$$

b
$$\begin{aligned}&\log 500\\ &= \log\left(\tfrac{1000}{2}\right)\\ &= \log 1000 - \log 2\\ &= \log(10^3) - \log 2\\ &= 3 - \log 2\end{aligned}$$

5 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

6 The number $a \times 10^k$ where $1 \leqslant a < 10$, $k \in \mathbb{Z}$ is written in standard form. Show that $\log(a \times 10^k) = \log a + k$.

7 Suppose $p = \log_b 2$, $q = \log_b 3$, and $r = \log_b 5$. Write in terms of p, q, and r:

a $\log_b 6$ **b** $\log_b 45$ **c** $\log_b 108$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b\left(\frac{2}{9}\right)$

8 Suppose $\log_2 P = x$, $\log_2 Q = y$, and $\log_2 R = z$. Write in terms of x, y, and z:

a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\frac{PR}{Q}\right)$

d $\log_2\left(P^2\sqrt{Q}\right)$ **e** $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$

9 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$, find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$

10 Suppose $\log_a(x+2) = \log_a x + 2$ and $a > 1$. Find x in terms of a.

11 In **factorial notation** we use $n!$ to denote the product $1 \times 2 \times 3 \times .... \times n$.

a Write $\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$ as a single logarithm.

b Write $\log_2(6!)$ in the form $a + \log_2 b$, where a, $b \in \mathbb{Z}$ and b is as small as possible.

12 Write $\log x^4 + \log\left(\frac{x^4}{y}\right) + \log\left(\frac{x^4}{y^2}\right) + ... + \log\left(\frac{x^4}{y^9}\right)$ in the form $\log\left(\frac{x^m}{y^n}\right)$.

13 Evaluate the infinite series $\log\sqrt{3} - \log\sqrt[4]{3} + \log\sqrt[8]{3} - \log\sqrt[16]{3} + ...$

14 Suppose $x^2 + y^2 = 52xy$ where $0 < y < x$. Show that $\log\left(\frac{x-y}{5}\right) = \frac{1}{2}(\log x + \log 2y)$.

D NATURAL LOGARITHMS

The logarithm in base e is called the **natural logarithm**.

We use $\ln x$ to represent $\log_e x$, and call $\ln x$ the natural logarithm of x.

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

Example 11 — Self Tutor

Find: **a** $\ln e^3$ **b** $\ln \sqrt{e}$ **c** $e^{2\ln 5}$

a $\ln e^3 = 3$

b $\ln \sqrt{e} = \ln(e^{\frac{1}{2}})$
$= \frac{1}{2}$

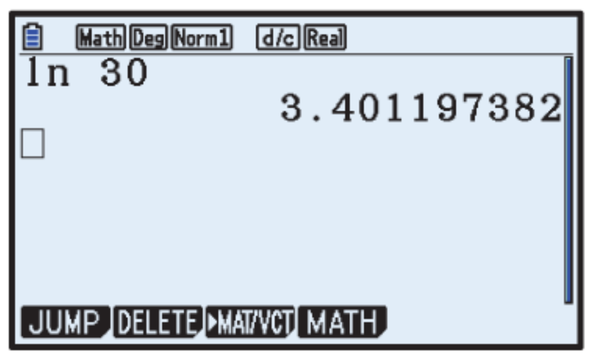
c $e^{2\ln 5} = (e^{\ln 5})^2$
$= 5^2$
$= 25$

As with base 10 logarithms, we can use our calculator to find natural logarithms.

For example, $\ln 30 \approx 3.40$, which means that $30 \approx e^{3.40}$.



GRAPHICS CALCULATOR INSTRUCTIONS



Example 12 — Self Tutor

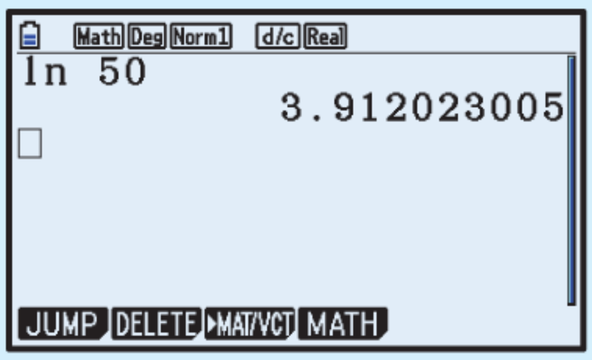
Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

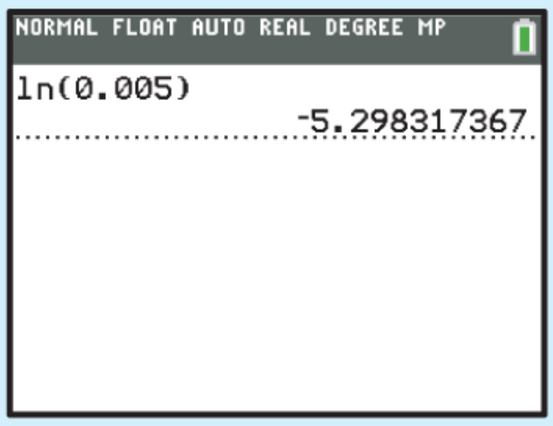
a $50 = e^{\ln 50}$ $\{x = e^{\ln x}\}$
$\approx e^{3.9120}$

b $0.005 = e^{\ln 0.005}$
$\approx e^{-5.2983}$

Casio fx-CG50



TI-84 Plus CE



LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

- $\ln m + \ln n = \ln(mn)$ for $m, n > 0$
- $\ln m - \ln n = \ln\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \ln b = \ln(b^m)$ for $b > 0$

EXERCISE 3D

1 Without using a calculator find:

a $\ln(e^2)$ **b** $\ln(e^4)$ **c** $\ln\left((\sqrt{e})^3\right)$ **d** $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ **f** $\ln \sqrt[3]{e}$ **g** $\ln\left(\frac{1}{e^2}\right)$ **h** $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ **b** $e^{2\ln 3}$ **c** $e^{-\ln 5}$ **d** $e^{-2\ln 2}$

e $\ln e^a$ **f** $\ln(e \times e^a)$ **g** $\ln(e^a \times e^b)$ **h** $\ln\left((e^a)^b\right)$

3 Use your calculator to find, correct to 3 decimal places:

a $\ln 12$ **b** $\ln 68$ **c** $\ln(1.4)$ **d** $\ln(0.7)$ **e** $\ln 500$

4 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

5 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 13 — Self Tutor

Find x if:

a $\ln x = 2.17$ **b** $\ln x = -0.384$

a $\ln x = 2.17$

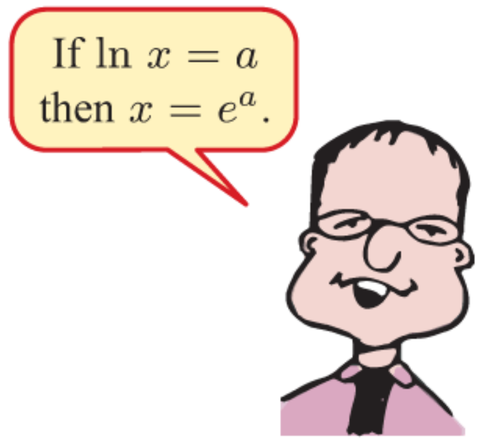
$\therefore\ x = e^{2.17}$

$\therefore\ x \approx 8.76$

b $\ln x = -0.384$

$\therefore\ x = e^{-0.384}$

$\therefore\ x \approx 0.681$



6 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$

e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

7 **a** Write in simplest form:

i $\ln(e^x)$ **ii** $e^{\ln x}$

b What does this tell us about the functions $y = e^x$ and $y = \ln x$?

Example 14 — Self Tutor

Use the laws of logarithms to write as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$

$= \ln(5 \times 3)$

$= \ln 15$

b $\ln 24 - \ln 8$

$= \ln\left(\frac{24}{8}\right)$

$= \ln 3$

c $\ln 5 - 1$

$= \ln 5 - \ln(e^1)$

$= \ln\left(\frac{5}{e}\right)$

8 Write as a single logarithm or integer:

a $\ln 15 + \ln 3$ **b** $\ln 15 - \ln 3$ **c** $\ln 20 - \ln 5$

d $\ln 4 + \ln 6$ **e** $\ln 5 + \ln(0.2)$ **f** $\ln 2 + \ln 3 + \ln 5$

g $1 + \ln 4$ **h** $\ln 6 - 1$ **i** $\ln 5 + \ln 8 - \ln 2$

j $2 + \ln 4$ **k** $\ln 20 - 2$ **l** $\ln 12 - \ln 4 - \ln 3$

Example 15 **Self Tutor**

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$ **b** $2\ln 3 + 3$

a
$$\begin{aligned} &2\ln 7 - 3\ln 2 \\ &= \ln(7^2) - \ln(2^3) \\ &= \ln 49 - \ln 8 \\ &= \ln\left(\tfrac{49}{8}\right) \end{aligned}$$

b
$$\begin{aligned} &2\ln 3 + 3 \\ &= \ln(3^2) + \ln(e^3) \\ &= \ln 9 + \ln(e^3) \\ &= \ln(9e^3) \end{aligned}$$

9 Write in the form $\ln a$, $a \in \mathbb{R}$:

a $5\ln 3 + \ln 4$ **b** $3\ln 2 + 2\ln 5$ **c** $3\ln 2 - \ln 8$

d $3\ln 4 - 2\ln 2$ **e** $\frac{1}{3}\ln 8 + \ln 3$ **f** $\frac{1}{3}\ln\left(\frac{1}{27}\right)$

g $-\ln 2$ **h** $-\ln\left(\frac{1}{2}\right)$ **i** $-2\ln\left(\frac{1}{4}\right)$

j $4\ln 2 + 2$ **k** $\frac{1}{2}\ln 9 - 1$ **l** $-3\ln 2 + \frac{1}{2}$

10 Show that:

a $\ln 27 = 3\ln 3$ **b** $\ln\sqrt{3} = \frac{1}{2}\ln 3$ **c** $\ln\left(\frac{1}{16}\right) = -4\ln 2$

d $\ln\left(\frac{1}{6}\right) = -\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

g $\ln(6e) = \ln 6 + 1$ **h** $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$

j $\ln\left(\frac{e^2}{8}\right) = 2 - 3\ln 2$ **k** $\ln\left(\frac{\sqrt{3}}{e^4}\right) = \frac{1}{2}\ln 3 - 4$ **l** $\ln\left(\frac{1}{16 \times \sqrt[3]{e}}\right) = -4\ln 2 - \frac{1}{3}$

11 Find x and y given that $\ln\left(\frac{x}{y}\right) = 6$ and $\ln(x^3y^4) = 4$.

THEORY OF KNOWLEDGE

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and techniques such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of

other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.

Logarithms were an extremely important development, and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used logarithms to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".



Johannes Kepler

1 Can anyone claim to have *invented* logarithms?

2 Can we consider the process of mathematical discovery as an *evolution* of ideas?

3 Has modern computing effectively doubled the life of a mathematician?

Many areas of mathematics have been developed over centuries as several mathematicians have worked in a particular area, or taken the knowledge from one area and applied it to another field. Sometimes the process is held up because a method for solving a particular class of problem has not yet been found. In other cases, pure mathematicians have published research papers on seemingly useless mathematical ideas, which have then become vital in applications much later.

In *Everybody Counts*: *A report to the nation on the future of Mathematical Education* by the National Academy of Sciences (National Academy Press, 1989), there is an excellent section on the Nature of Mathematics. It includes:

> "Even the most esoteric and abstract parts of mathematics - number theory and logic, for example - are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago, the leading British mathematician G.H. Hardy could boast that number theory was the most pure and least useful part of mathematics. Today, Hardy's mathematics is studied as an essential prerequisite to many applications, including control of automated systems, data transmission from remote satellites, protection of financial records, and efficient algorithms for computation."

4 Should we only study the mathematics required to enter our chosen profession?

5 Why should we explore mathematics for its own sake, rather than to address the needs of science?

E LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

Since the logarithmic function is one-to-one, we can take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

Example 16 Self Tutor

Write as a logarithmic equation (in base 10):

a $y = a^2b$ **b** $P = \dfrac{20}{\sqrt{n}}$

a
$$y = a^2b$$
$$\therefore \log y = \log(a^2b)$$
$$\therefore \log y = \log(a^2) + \log b$$
$$\therefore \log y = 2\log a + \log b$$

b
$$P = \left(\frac{20}{\sqrt{n}}\right)$$
$$\therefore \log P = \log\left(\frac{20}{n^{\frac{1}{2}}}\right)$$
$$\therefore \log P = \log 20 - \log(n^{\frac{1}{2}})$$
$$\therefore \log P = \log 20 - \tfrac{1}{2}\log n$$

Example 17 Self Tutor

Write without logarithms:

a $\log A = \log b + 2\log c$ **b** $\log_2 M = 3\log_2 a - 2$

a
$$\log A = \log b + 2\log c$$
$$\therefore \log A = \log b + \log(c^2)$$
$$\therefore \log A = \log(bc^2)$$
$$\therefore A = bc^2$$

b
$$\log_2 M = 3\log_2 a - 2$$
$$\therefore \log_2 M = \log_2(a^3) - \log_2(2^2)$$
$$\therefore \log_2 M = \log_2\left(\frac{a^3}{4}\right)$$
$$\therefore M = \frac{a^3}{4}$$

EXERCISE 3E

1 Write as a logarithmic equation (in base 10), with no powers, assuming all variables are positive:

a $y = 2^x$ **b** $y = 20b^3$ **c** $M = ad^4$ **d** $T = 5\sqrt{d}$

e $R = b\sqrt{l}$ **f** $Q = \dfrac{a}{b^n}$ **g** $y = ab^x$ **h** $F = \dfrac{20}{\sqrt{n}}$

i $L = \dfrac{ab}{c}$ **j** $N = \sqrt{\dfrac{a}{b}}$ **k** $S = 200 \times 2^t$ **l** $y = \dfrac{a^m}{b^n}$

2 Write without logarithms:

a $\log D = \log e + \log 2$ **b** $\log_a F = \log_a 5 - \log_a t$ **c** $\log P = \frac{1}{2}\log x$

d $\log_n M = 2\log_n b + \log_n c$ **e** $\log B = 3\log m - 2\log n$ **f** $\log N = -\frac{1}{3}\log p$

g $\log P = 3\log x + 1$ **h** $\log_a Q = 2 - \log_a x$

3 Write without logarithms:

a $\ln D = \ln x + 1$ **b** $\ln F = -\ln p + 2$ **c** $\ln P = \frac{1}{2}\ln x$

d $\ln M = 2\ln y + 3$ **e** $\ln B = 3\ln t - 1$ **f** $\ln N = -\frac{1}{3}\ln g$

g $\ln Q \approx 3\ln x + 2.159$ **h** $\ln D \approx 0.4\ln n - 0.6582$

4 **a** Write $y = 3 \times 2^x$ as a logarithmic equation in base 2.

b Hence write x in terms of y.

c Find the value of x when: **i** $y = 3$ **ii** $y = 12$ **iii** $y = 30$

5 Solve for x:

a $\log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x$ **b** $\log_5 x = \log_5 8 - \log_5(6 - x)$

c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$ **d** $\log_{20} x = 1 + \log_{20} 10$

e $\log x + \log(x + 1) = \log 30$ **f** $\log(x + 2) - \log(x - 2) = \log 5$

g $\log 24 = \log 3 + x \log 2$ **h** $x \log_2 3 + \log_2 36 = 2$

6 Solve simultaneously for x and y:

a $\begin{cases} \log_x y = 2 \\ \log_{y-2} x = 1 \end{cases}$ **b** $\begin{cases} \log_x y = 3 \\ \log_{y+1}(x + 1) = \frac{1}{2} \end{cases}$

7 Let $x = \log_2 7$.

a Write the equation without logarithms.

b Take the logarithm in base 10 of both sides of your equation from **a**. Hence show that $\log_2 7 = \dfrac{\log 7}{\log 2}$, and calculate this number.

8 Consider the exponential equation $a^x = b$ where $a, b > 0$.

a Explain why $x = \log_a b$.

b Take the logarithm in base 10 of both sides of $a^x = b$.

c Hence show that $x = \log_a b = \dfrac{\log b}{\log a}$.

F THE CHANGE OF BASE RULE

In the previous Exercise you should have proven the base 10 case of the **change of base rule**:

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

Proof:

If $\log_b a = x$, then $b^x = a$

$\therefore \log_c b^x = \log_c a$ {taking logarithms in base c}

$\therefore x \log_c b = \log_c a$ {power law of logarithms}

$\therefore x = \dfrac{\log_c a}{\log_c b}$

$\therefore \log_b a = \dfrac{\log_c a}{\log_c b}$

We need the change of base rule to evaluate logarithms in bases other than 10 or e.

Example 18 **Self Tutor**

Find $\log_2 9$ by:

a changing to base 10 **b** changing to base e.

a $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2}$
≈ 3.17

b $\log_2 9 = \dfrac{\ln 9}{\ln 2}$
≈ 3.17

EXERCISE 3F

1 Use the change of base rule with base 10 to calculate:

a $\log_3 7$ **b** $\log_2 40$ **c** $\log_5 180$

d $\log_{\frac{1}{2}} 1250$ **e** $\log_3(0.067)$ **f** $\log_{0.4}(0.006\,984)$

Check your results using the change of base rule with base e.

2 Simplify $\log_m n \times \log_n(m^2)$.

3 Without using technology, show that $2^{\frac{4}{\log_5 4}+\frac{3}{\log_7 8}} = 175$.

Hint: Use the change of base rule with base 2.

4 Solve for x:

a $\log_4(x^3) + \log_2 \sqrt{x} = 8$ **b** $\log_{\frac{1}{9}} x = \log_9 5$

c $\log_{16}(x^5) = \log_{64} 125 - \log_4 \sqrt{x}$ **d** $\log_3(x^3) - 4\log_9 x - 5\log_{27} \sqrt{x} = \log_9 4$

e $\log_x 4 + \log_2 x = 3$

5 Given $x = \log_3(y^2)$, express $\log_y 81$ in terms of x.

6 Given $m = \log_4 3$, express $\log_2 24$ in terms of m.

7 **a** By evaluating each expression when $x = 9$, show that $\log_9(\log_3 x)$ and $\log_3(\log_9 x)$ are not always equal.

b Find the value of x such that $\log_9(\log_3 x) = \log_3(\log_9 x)$.

c Find, in terms of a, the solution to $\log_{a^2}(\log_a x) = \log_a(\log_{a^2} x)$.

d Show that when $\log_{a^k}(\log_a x) = \log_a(\log_{a^k} x)$, $k \neq 1$, both sides of the equation have the value $\log_a k^{\frac{1}{k-1}}$.

G SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 2** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we can use **logarithms**.

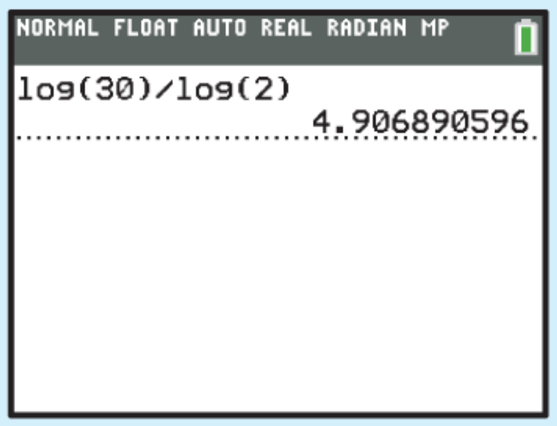
Example 19 — Self Tutor

a Solve the equation $2^x = 30$ exactly.

b Use your calculator to evaluate the solution correct to 2 decimal places.

a
$$2^x = 30$$
$$\therefore \log(2^x) = \log 30 \quad \{\text{taking the logarithm of each side}\}$$
$$\therefore x \log 2 = \log 30 \quad \{\log(b^m) = m \log b\}$$
$$\therefore x = \frac{\log 30}{\log 2}$$

b $\dfrac{\log 30}{\log 2} \approx 4.91$, so the solution is $x \approx 4.91$.



EXERCISE 3G

1 Consider the equation $3^x = 40$.

a Explain why the solution to this equation lies between $x = 3$ and $x = 4$.

b Find the solution exactly.

c Use your calculator to evaluate the solution correct to 2 decimal places.

2 Solve for x: **i** exactly **ii** correct to 2 decimal places.

a $2^x = 10$ **b** $3^x = 20$ **c** $4^x = 50$

d $\left(\frac{1}{2}\right)^x = 0.0625$ **e** $\left(\frac{3}{4}\right)^x = 0.1$ **f** $10^x = 0.000\,015$

3 Solve for x, correct to 3 significant figures:

a $5^x = 40$ **b** $3^x = 2^{x+3}$ **c** $2^{x+4} = 5^{2-x}$

Example 20 — Self Tutor

Find x exactly:

a $e^x = 30$ **b** $3e^{\frac{x}{2}} = 21$

a
$$e^x = 30$$
$$\therefore x = \ln 30$$

b
$$3e^{\frac{x}{2}} = 21$$
$$\therefore e^{\frac{x}{2}} = 7$$
$$\therefore \frac{x}{2} = \ln 7$$
$$\therefore x = 2 \ln 7$$

4 Solve for x, giving an exact answer:

a $e^x = 10$ **b** $e^x = 1000$ **c** $2e^x = 0.3$

d $e^{\frac{x}{2}} = 5$ **e** $e^{2x} = 18$ **f** $e^{-\frac{x}{2}} = 1$

5 Solve for x, giving an exact answer:

a $3 \times 2^x = 75$ **b** $7 \times (1.5)^x = 20$ **c** $5 \times (0.8)^x = 3$

d $4 \times 2^{-x} = 0.12$ **e** $300 \times 5^{0.1x} = 1000$ **f** $32 \times e^{-0.25x} = 4$

6 Solve for x exactly:

a $25^x - 3 \times 5^x = 0$ **b** $8 \times 9^x - 3^x = 0$ **c** $2^x - 2 \times 4^x = 0$

7 Solve $3^{2x} = \frac{1}{16}$, writing your answer in terms of $\ln 2$ and $\ln 3$.

8 Solve $10^{2x} = 4^{x+3}$, writing your answer in terms of $\ln 2$ and $\ln 5$.

Example 21 **Self Tutor**

Find exactly the points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.
Check your solution using technology.

The functions meet where

$$e^x - 3 = 1 - 3e^{-x}$$
$$\therefore\ e^x - 4 + 3e^{-x} = 0$$
$$\therefore\ e^{2x} - 4e^x + 3 = 0 \quad \{\text{multiplying each term by } e^x\}$$
$$\therefore\ (e^x - 1)(e^x - 3) = 0$$
$$\therefore\ e^x = 1 \text{ or } 3$$
$$\therefore\ x = \ln 1 \text{ or } \ln 3$$
$$\therefore\ x = 0 \text{ or } \ln 3$$

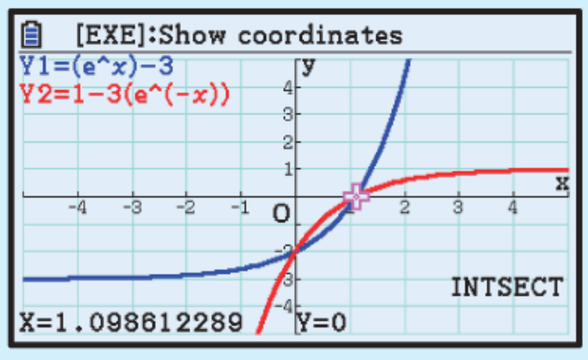
When $x = 0$, $y = e^0 - 3 = -2$

When $x = \ln 3$, $y = e^{\ln 3} - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

GRAPHING PACKAGE





9 Solve for x:

a $e^{2x} = 2e^x$ **b** $e^x = e^{-x}$ **c** $e^{2x} - 5e^x + 6 = 0$

d $e^x + 2 = 3e^{-x}$ **e** $1 + 12e^{-x} = e^x$ **f** $e^x + e^{-x} = 3$

10 Find algebraically the point(s) of intersection of:

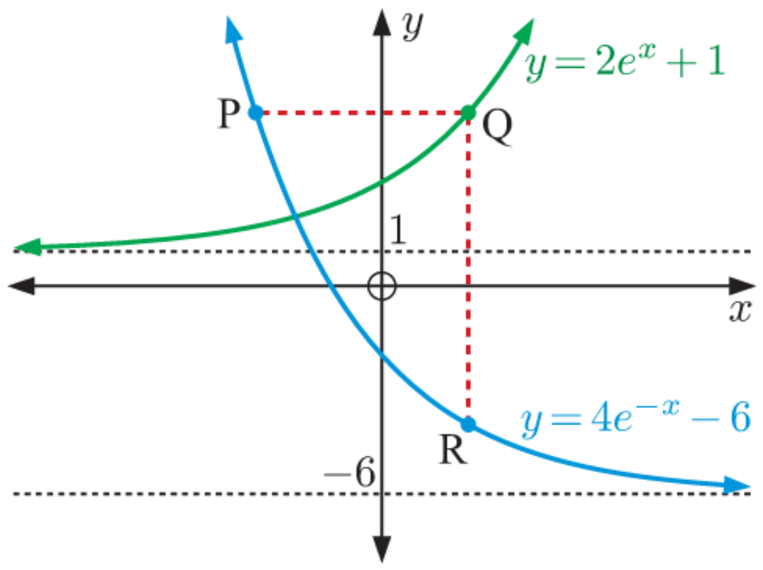
a $y = e^x$ and $y = e^{2x} - 6$ **b** $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$

Check your answers using technology.

11 **a** Find the exact coordinates of the intersection point of the graphs shown.

b Given that the vertical line segment [QR] has length 9 units, find the length of the horizontal line segment [PQ]. Give your answer in the form $\ln k$ units, where $k \in \mathbb{Q}$.



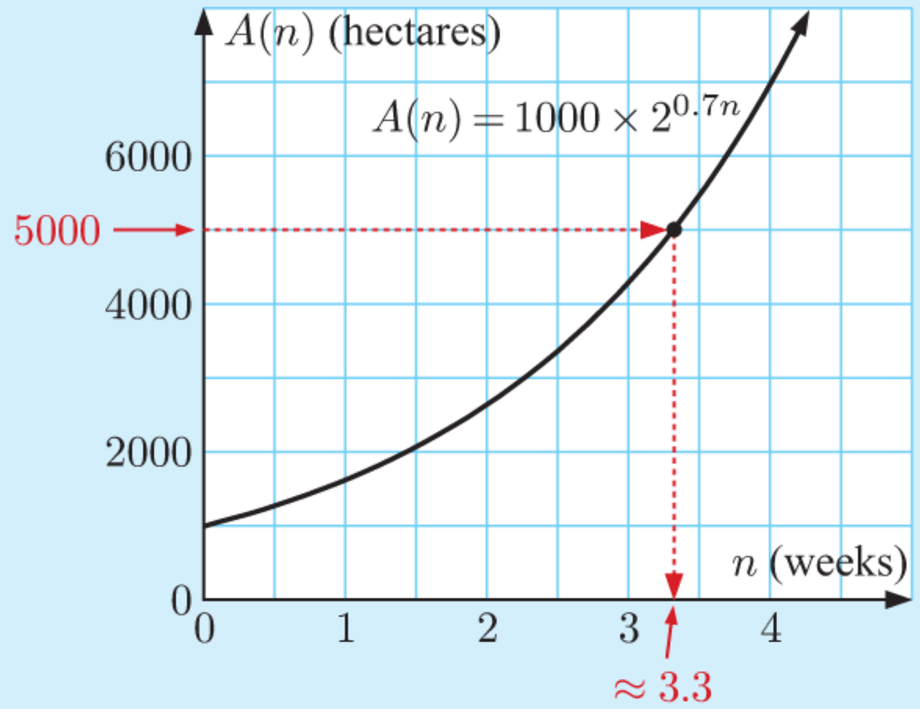
Example 22 **Self Tutor**

A farmer monitoring an insect plague finds that the area affected by the insects is given by $A(n) = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$. Hence estimate the time taken for the affected area to reach 5000 hectares.

b Check your answer to **a** using logarithms.

a From the graph, it appears that it will take about 3.3 weeks for the affected area to reach 5000 hectares.



b When $A(n) = 5000$,

$1000 \times 2^{0.7n} = 5000$

$\therefore \ 2^{0.7n} = 5$

$\therefore \ \log(2^{0.7n}) = \log 5$

$\therefore \ 0.7n \log 2 = \log 5$

$\therefore \ n = \dfrac{\log 5}{0.7 \times \log 2} \approx 3.32$

$\therefore$ it will take about 3 weeks and 2 days.

12 The population of turtles in an isolated colony is $P(t) = 852 \times (1.07)^t$, where t is the time in years after the colony was first recorded. How long will it take for the population to reach:

a 1000 turtles **b** 1500 turtles?

13 The weight of bacteria in a culture t hours after establishment is given by $W(t) = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

14 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A(n) = 2000 \times e^{0.57n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$.

b Hence estimate the time taken for the infested area to reach 10 000 hectares.

c Check your answer to **b** using logarithms.



15 A house is expected to increase in value at an average rate of 7.5% p.a. If the house is worth £360 000 now, when would you expect it to be worth £550 000?

16 Thabo has $10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to $15 000?

17 Dien invests $15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches $25 000, at which time he plans to travel. The formula $t_n = t_0 \times r^n$ can be used to model the investment, where n is the time in months.

a Explain why $r = 1.007$.

b After how many months will Dien withdraw the money?

18 The mass M_t of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

a halve **b** reach 25 grams **c** reach 1% of its original value.

19 The current I flowing in a transistor radio t seconds after it is switched off, is given by $I = I_0 \times 2^{-0.02t}$ amps. Show that it takes $\frac{50}{\log 2}$ seconds for the current to drop to 10% of its original value.

20 A sky diver jumps from an aeroplane. His speed of descent is given by $V(t) = 50(1 - e^{-0.2t})$ m s^{-1}, where t is the time in seconds.

a Show that it will take $5 \ln 5$ seconds for the sky diver's speed to reach 40 m s^{-1}.

b Write an expression for the time taken for his speed to reach v m s^{-1}.

21 Answer the **Opening Problem** on page **68**.

22 The weight of radioactive substance remaining after t years is given by $W = 1000 \times 2^{-0.04t}$ grams.

a Sketch the graph of W against t.

b Write a function for t in terms of W.

c Hence find the time required for the weight to reach:

i 20 grams **ii** 0.001 grams.

23 The temperature of a liquid t minutes after it is placed in a refrigerator, is given by $T = 4 + 96 \times e^{-0.03t}$ °C.

a Sketch the graph of T against t.

b Write a function for t in terms of T.

c Find the time required for the temperature to reach:

i 25°C **ii** 5°C.

24 A meteor hurtling through the atmosphere has speed of descent given by

$$V(t) = 650(4 + 2 \times e^{-0.1t}) \text{ m s}^{-1}$$

where t is the time in seconds after the meteor is sighted.



a Is the meteor's speed increasing or decreasing?

b Find the speed of the meteor:

i when it was first sighted

ii after 2 minutes.

c How long will it take for the meteor's speed to reach 3000 m s^{-1}?

INVESTIGATION 2 THE "RULE OF 72"

The "rule of 72" is used to estimate the time a quantity takes to double in value, given the rate at which the quantity grows.

Click on the icon to view this Investigation.



H LOGARITHMIC FUNCTIONS

We have seen that $\log_a a^x = a^{\log_a x} = x$.

Letting $f(x) = \log_a x$ and $g(x) = a^x$, we have $f \circ g = g \circ f = x$.

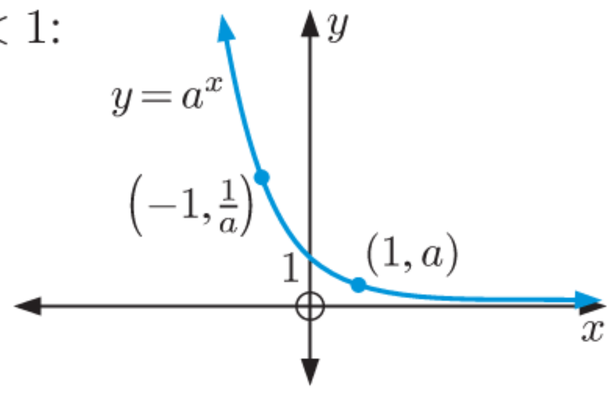
We can therefore say that the logarithmic function $\log_a x$ is the **inverse** of the exponential function a^x.

Algebraically, this has the effect that the logarithmic and exponential functions "undo" one another.

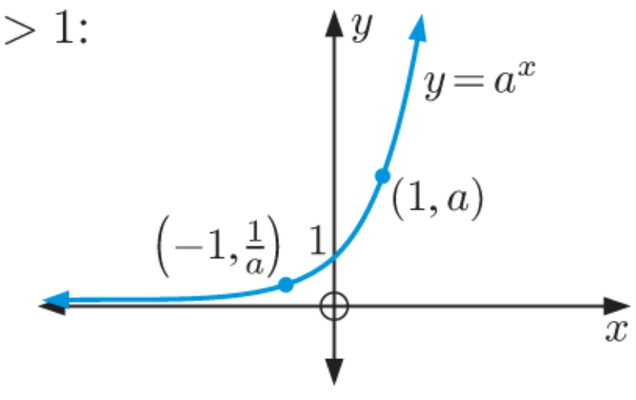
Geometrically, it means that the graph of $y = \log_a x$, $a > 0$, $a \neq 1$ is the *reflection* of the graph of $y = a^x$ in the line $y = x$.

We have seen previously the shape of the exponential function $y = a^x$ where $a > 0$, $a \neq 1$.

For $0 < a < 1$:



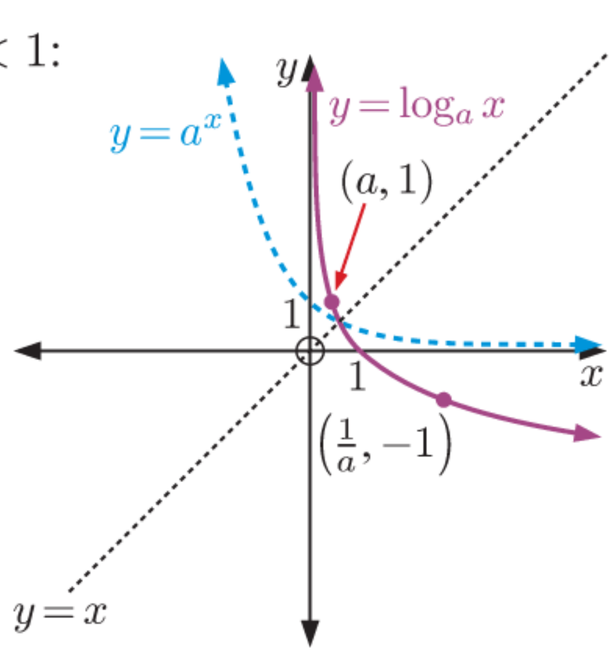
For $a > 1$:



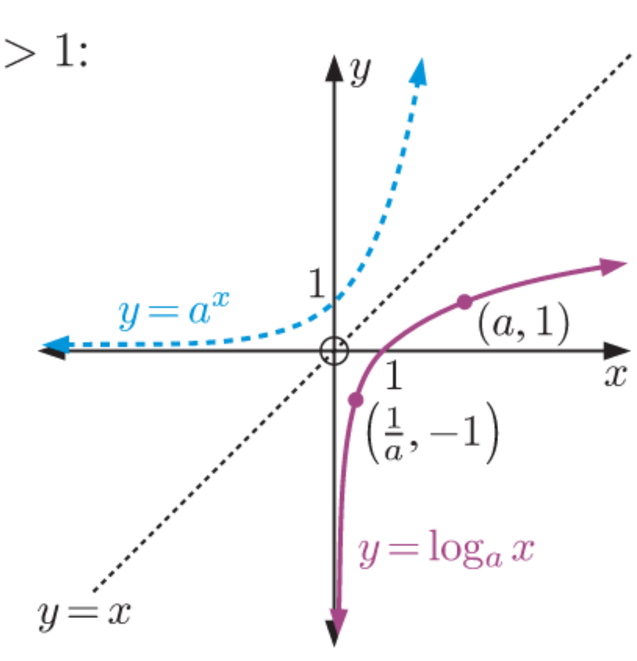
The horizontal asymptote for all of these functions is the x-axis $y = 0$.

By reflecting these graphs in the line $y = x$, we obtain the graphs for $y = \log_a x$.

For $0 < a < 1$:



For $a > 1$:



The **vertical asymptote** of $y = \log_a x$ is the y-axis $x = 0$.

For $0 < a < 1$: as $x \to \infty$, $y \to -\infty$
as $x \to 0^+$, $y \to \infty$

For $a > 1$: as $x \to \infty$, $y \to \infty$
as $x \to 0^+$, $y \to -\infty$

PROPERTIES OF $y = \log_a x$

Since we can only find logarithms of positive numbers, the domain of $y = \log_a x$ is $\{x \mid x > 0\}$.

We can compare the functions $y = a^x$ and $y = \log_a x$ as follows:

Function	$y = a^x$	$y = \log_a x$
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS

LOGARITHMIC GRAPHS

Click on the icon to explore the graphs of functions of the form $y = p\ln(x-h)+k$.

Example 23 — Self Tutor

Consider the function $f(x) = \log_2(x-1)+1$.

a State the transformation which maps $y = \log_2 x$ to $y = f(x)$.

b Find the domain and range of f.

c Find any asymptotes and axes intercepts.

d Sketch the graph of $y = f(x)$.

e Find the inverse function f^{-1}.

---

a $f(x)$ is a translation of $y = \log_2 x$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b $x - 1 > 0$ when $x > 1$

So, the domain is $x > 1$ and the range is $y \in \mathbb{R}$.

c As $x \to 1^+$, $y \to -\infty$, so the vertical asymptote is $x = 1$.

As $x \to \infty$, $y \to \infty$, so there is no horizontal asymptote.

When $x = 0$, y is undefined, so there is no y-intercept.

When $y = 0$, $\log_2(x-1) = -1$

$\therefore\ x - 1 = 2^{-1}$

$\therefore\ x = \frac{3}{2}$ So, the x-intercept is $\frac{3}{2}$.

GRAPHICS CALCULATOR INSTRUCTIONS

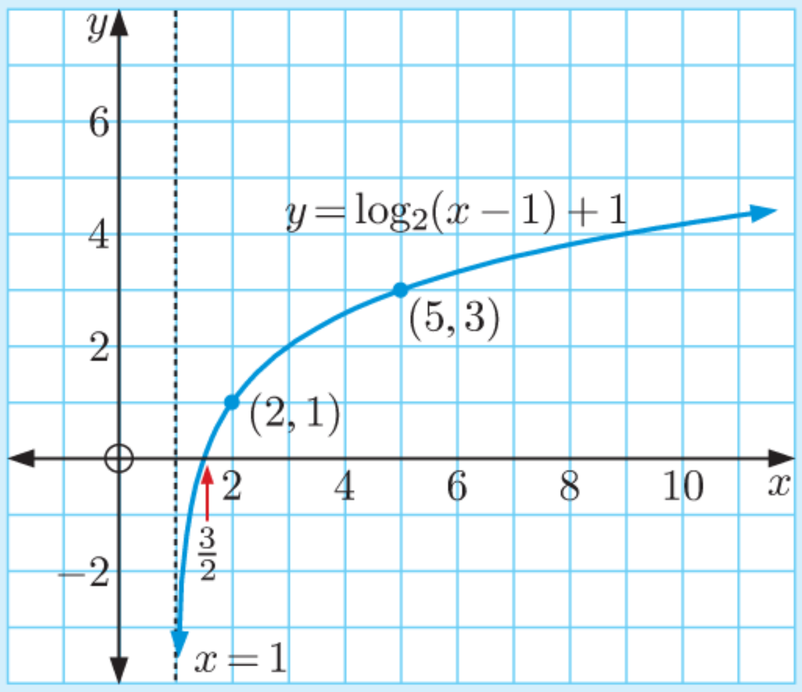
d When $x = 2$, $y = \log_2(2-1)+1 = 1$

When $x = 5$, $y = \log_2(5-1)+1$

$= \log_2 4 + 1$

$= 2 + 1$

$= 3$



e f is defined by $y = \log_2(x-1)+1$

$\therefore\ f^{-1}$ is defined by $x = \log_2(y-1)+1$

$\therefore\ x - 1 = \log_2(y-1)$

$\therefore\ y - 1 = 2^{x-1}$

$\therefore\ y = 2^{x-1} + 1$

$\therefore\ f^{-1}(x) = 2^{x-1} + 1$

EXERCISE 3H

1 For each of the following functions f:

i Find the domain and range. ii Find any asymptotes and axes intercepts.

iii Sketch the graph of $y = f(x)$, showing all important features.

iv Solve $f(x) = -1$ algebraically and check the solution on your graph.

v Find the inverse function f^{-1}.

a $f : x \mapsto \log_2 x - 2$ b $f : x \mapsto \log_3(x+1)$ c $f : x \mapsto 1 - \log_3(x+1)$

d $f : x \mapsto \log_5(x-2) - 2$ e $f : x \mapsto 1 - \log_5(x-2)$ f $f : x \mapsto 1 - 2\log_2 x$

2 For each of the functions f:

i State the transformation which maps $y = \ln x$ to $y = f(x)$.

ii State the domain and range.

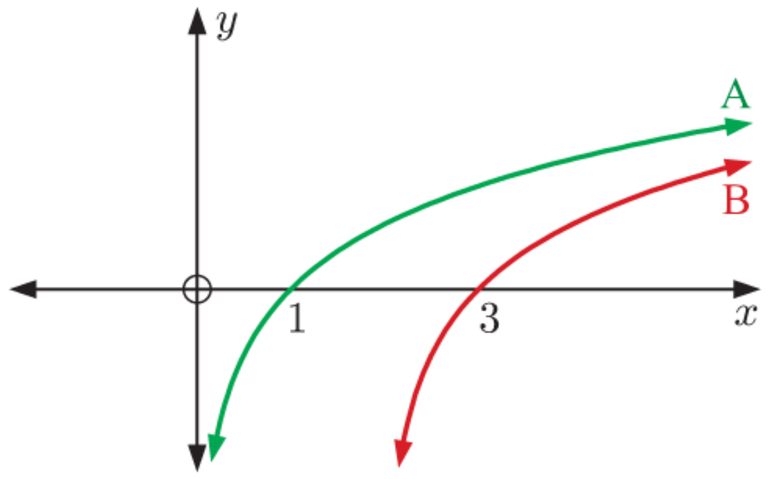
iii Find any asymptotes and intercepts.

iv Sketch the graph of $y = f(x)$, showing all important features.

v Find the inverse function f^{-1}.

a $f(x) = \ln x - 4$ **b** $f(x) = \ln(x - 1) + 2$ **c** $f(x) = 3\ln x - 1$

3 Consider the curves A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x - 2)$.

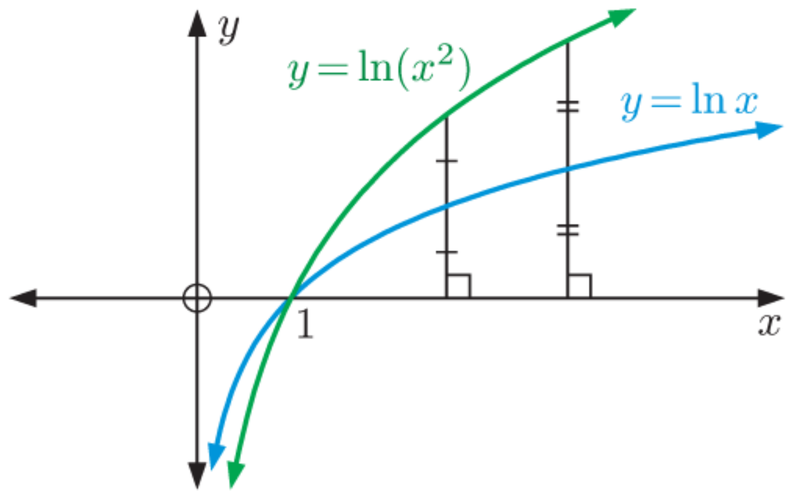


a Identify which curve is which, giving evidence for your answer.

b Copy the graphs onto a new set of axes, and then draw the graph of $y = \ln(x + 2)$.

c Find the equation of the vertical asymptote for each graph.

4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis. Is Kelly correct? Explain your answer.



5 Draw, on the same set of axes, the graphs of:

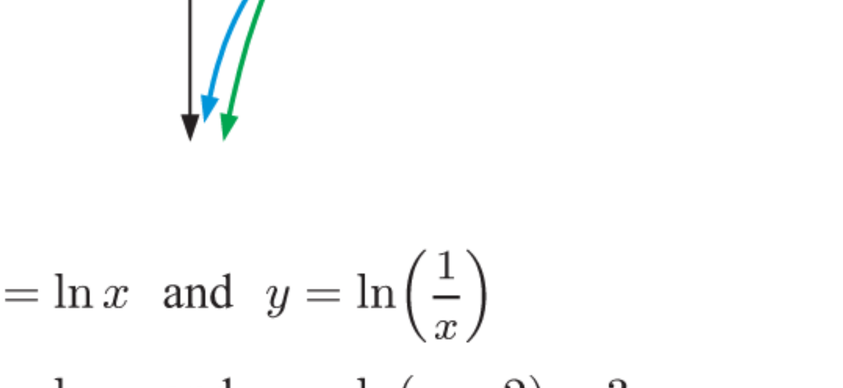
a $y = \ln x$ and $y = \ln(x^3)$ **b** $y = \ln x$ and $y = \ln\left(\frac{1}{x}\right)$

c $y = \ln x$ and $y = \ln(x + e)$ **d** $y = \ln x$ and $y = \ln(x - 2) - 3$

e $y = 2\ln x$ and $y = \ln(x^2) + 2$

6 Describe a transformation which maps the graph of $y = \log_2 x$ to the graph of $y = \log_5 x$.

7 The logarithmic functions $y = 3 - \log_2(3x + 1)$ and $y = \log_2(x + 2)$ are graphed alongside.



a Identify which curve is which, giving evidence for your answer.

b Find the axes intercepts and asymptotes of each graph.

c Find the exact coordinates of the point where the graphs intersect.

8 For each of the following functions, find the inverse function f^{-1}, and state the domain and range of f^{-1}:

a $f(x) = 3^x$ **b** $f(x) = 2^{x+1}$ **c** $f(x) = e^{2x}$ **d** $f(x) = 5^x - 3$

9 Suppose $f(x) = be^x$ and $g(x) = \ln(bx)$. Find:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

c the value of x, in terms of b, for which $(f \circ g)(x) = (g \circ f)(x)$.

10 Suppose $f(x) = e^{x+2}$ and $g(x) = \ln x - 3$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

11 Let $f(x) = \log_2(1 - 3x)$ and $g(x) = 2x + 1$.

a Find the domain and range of $f(x)$.

b Solve for x:

i $(f \circ g)(x) = 5$ **ii** $(g \circ f)(x) = 0$

c Find $f^{-1}(x)$, and state its domain and range.

12 Given $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$, find:

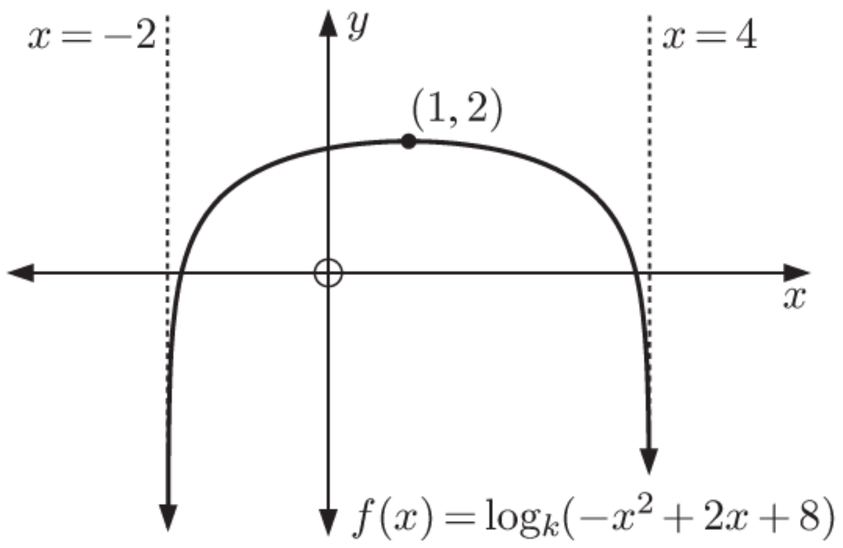
a $(f^{-1} \circ g)(x)$ **b** $(g \circ f)^{-1}(x)$

13 The function $f(x) = \log_k(-x^2 + 2x + 8)$ is graphed alongside.

a Find k.

b Find the axes intercepts of $f(x)$.

c Let $g(x) = f(x)$ on the restricted domain $-2 < x \leqslant 1$. Find $g^{-1}(x)$, and state its domain and range.



ACTIVITY

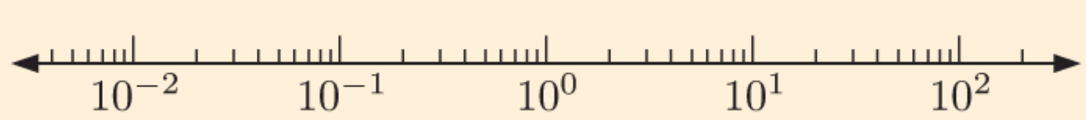
Click on the icon to obtain a card game for logarithmic functions.



INVESTIGATION 3 LOGARITHMIC SCALES

In a **logarithmic scale**, equally spaced major tick marks correspond to integer *powers* of a base number. We often call these **orders of magnitude**.

For example, in the logarithmic scale alongside, each major tick mark represents a power of 10.



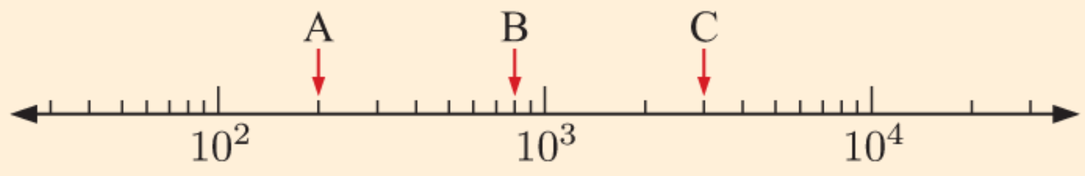
The minor tick marks correspond to integer *multiples* of each power of 10. So the minor tick marks between 10^1 and 10^2 represent 20, 30, 40, ..., and so on.

Logarithmic scales are useful when we want to represent both very large and very small numbers on the same number line. They allow us to compare real world quantities or events which are many orders of magnitude apart.

In this Investigation, we will explore the use of logarithmic scales in a variety of contexts.

What to do:

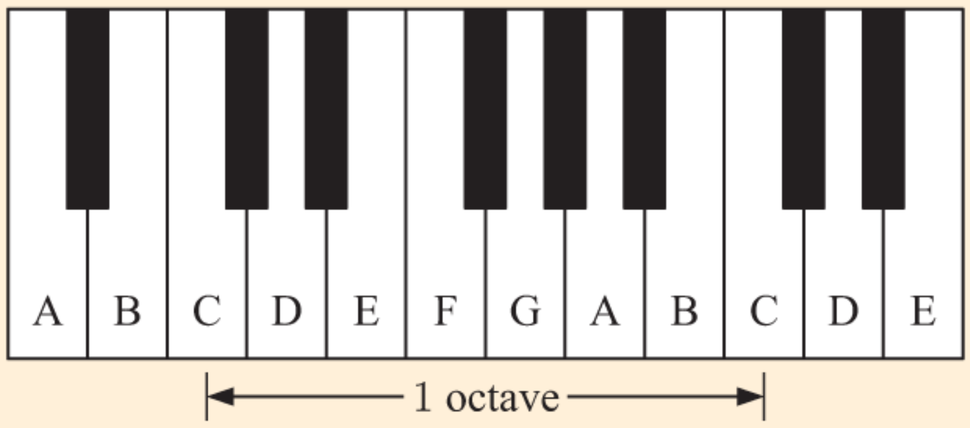
1 **a** For the logarithmic scale alongside, state the values of the points A, B, and C.



b Explain why the minor tick marks in a logarithmic scale are not equally spaced.

c Where is the value 0 on a logarithmic scale? Explain your answer.

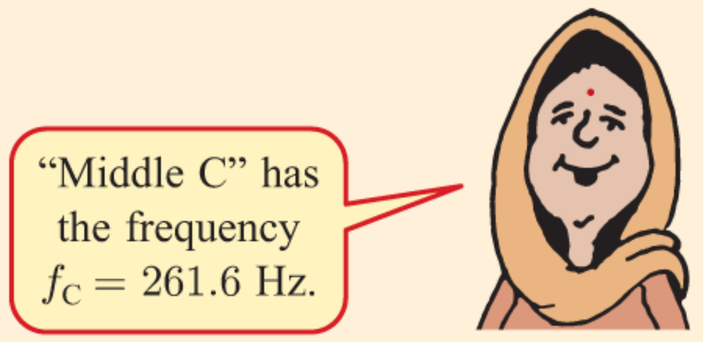
2 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet. A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.



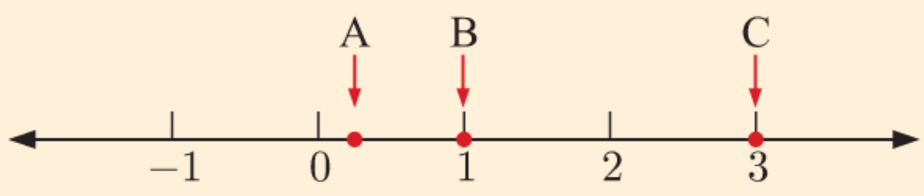
a How many orders of magnitude apart are the frequencies of two notes separated by 3 octaves?

b Write an expression for the frequency of a musical note f, in terms of the number of octaves n *above* middle C.

c There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.



3 In some situations, the logarithm is already applied to values placed on the number line. In these cases, the major tick marks represent the *exponents* rather than the numbers themselves. For example, suppose the scale alongside is logarithmic with base 10. The major tick mark "2" represents the value 10^2, the major tick mark "3" represents the value 10^3, and so on.



a How many times larger is the value at C than the value at B?

b Estimate the position on the scale representing the value:

i 10 times smaller than A **ii** twice as large as B.

4 Earthquakes can range from microscopic tremors to huge natural disasters. The magnitude of earthquakes is measured on the **Richter scale** which relates to the energy released by the earthquake. For this logarithmic scale, the logarithm is part of the formula. It is calculated as $M = \log\left(\frac{I}{I_0}\right)$, where I is the earthquake intensity and I_0 is a reference intensity level.

a What does it mean for a tremor to have magnitude:

i 0 **ii** 1?

b Explain why an earthquake of magnitude 6 is *not* twice as intense as a magnitude 3 earthquake.

c Find the magnitude of an earthquake which has half the intensity of a magnitude 4 earthquake.

5 The *acidity* of a solution is determined by the concentration of hydronium ions (H_3O^+).
The higher the concentration of H_3O^+, the more acidic it is. The opposite of acidic is *alkaline*.

a In extremely acidic solutions, the concentration of H_3O^+ is typically more than 10^{-3} units. In very alkaline solutions, it is usually less than 10^{-12} units.
Explain why a logarithmic scale would be useful in describing the acidity of a solution.

b In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by $\text{pH} = -\log C$, where C is the concentration of H_3O^+. Find:

i the pH of a solution with H_3O^+ concentration $0.000\,234$ units

ii the H_3O^+ concentration in a solution with pH 7.

c Is it possible for a solution to have a negative pH? Explain what this means in terms of the concentration of H_3O^+.

6 Research the use of **decibels** in acoustics as a unit of measurement for loudness of sound. Compare the use of decibels to the scales in questions **4** and **5**.

REVIEW SET 3A

1 Find:

a $\log\sqrt{10}$ **b** $\log\left(\frac{1}{\sqrt[3]{10}}\right)$ **c** $\log(10^a \times 10^{b+1})$

2 Find:

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_{81} 3$ **g** $\log_9\left(\frac{1}{9}\right)$ **h** $\log_k \sqrt{k}$

3 Use your calculator to evaluate, correct to 3 decimal places:

a $\log 27$ **b** $\log(0.58)$ **c** $\log 400$ **d** $\ln 40$

4 If $y = \log_3\sqrt{2-x}$, write x in terms of y.

5 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

6 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

7 Suppose $A = \log_5 2$ and $B = \log_5 3$. Write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5\left(8\sqrt{3}\right)$

d $\log_5\left(\sqrt{6}\right)$ **e** $\log_5(20.25)$ **f** $\log_5\left(\frac{8}{9}\right)$

8 Write as a logarithmic equation:

a $M = ab^n$ **b** $T = \dfrac{5}{\sqrt{l}}$ **c** $G = \dfrac{a^2b}{c}$

9 Solve for x:

a $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

10 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

11 Write without logarithms:

a $\ln P = 1.5 \ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5 \ln N$

12 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2 \ln x - 3 \ln\left(\frac{1}{x}\right) = 10$

13 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

14 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $2^x = 50$ **b** $7^x = 4$ **c** $(0.6)^x = 0.01$

15 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_a\left(\frac{1}{b}\right)$.

16 Write $\dfrac{8}{\log_5 9}$ in the form $a \log_3 b$.

17 Show that the solution to $16^x - 5 \times 8^x = 0$ is $x = \log_2 5$.

18 Solve for x, giving exact answers:

a $\ln x = 5$ **b** $3 \ln x + 2 = 0$ **c** $e^x = 400$

d $e^{2x+1} = 11$ **e** $25e^{\frac{x}{2}} = 750$

19 Consider $f(x) = e^{3x-4} + 1$.

a Show that $f^{-1}(x) = \dfrac{\ln(x-1)+4}{3}$.

b Calculate $f^{-1}(8) - f^{-1}(3)$. Give your answer in the form $a \ln b$, where $a, b \in \mathbb{Q}^+$.

20 Solve simultaneously: $x = 16y$ and $\log_y x - \log_x y = \frac{8}{3}$.

21 Consider the function $g : x \mapsto \log_3(x+2) - 2$.

a State the transformation which maps $y = \log_3 x$ to $y = g(x)$.

b Find the domain and range.

c Find any asymptotes and axes intercepts for the graph of the function.

d Find the inverse function g^{-1}.

e Sketch the graphs of g, g^{-1}, and $y = x$ on the same set of axes.

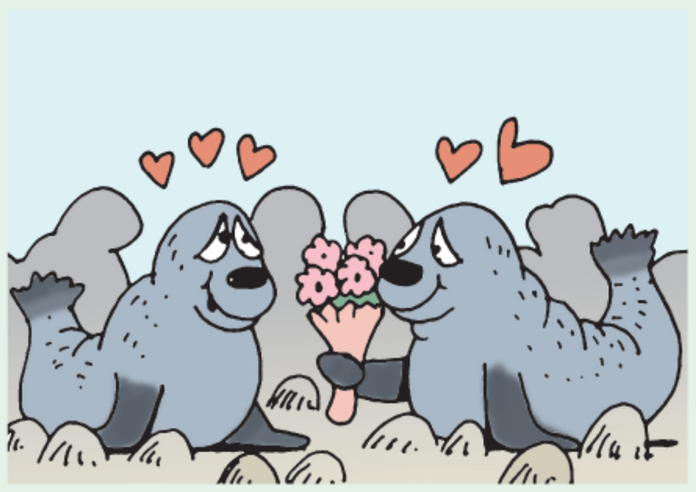
22 The weight of a radioactive isotope remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.

23 A population of seals is given by $P(t) = 80 \times (1.15)^t$ where t is the time in years, $t \geqslant 0$.

a Find the time required for the population to double in size.

b Find the percentage increase in population during the first 4 years.



24 For each of the following functions:

i State the domain and range.

ii Find any asymptotes and axes intercepts.

iii Sketch the graph of the function, showing all important features.

a $f(x) = \log_2(x+4) - 1$ **b** $f(x) = \ln x + 2$

25 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x-3)$ **b** $y = \ln x$ and $y = \frac{1}{2}\ln x$

26 Consider $f(x) = e^x$ and $g(x) = \ln(x+4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

REVIEW SET 3B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Find:

a $\log_2 128$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_5\left(\frac{1}{\sqrt{5}}\right)$

3 Write in the form 10^x, giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\frac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\frac{e}{e^x}\right)$

5 Simplify:

a $\dfrac{\log_2 25}{\log_2 125}$ **b** $\dfrac{\log 64}{\log 32}$ **c** $\dfrac{\log_5 81}{\log_5 \sqrt{3}}$

6 Simplify:

a $e^{4\ln x}$ **b** $\ln(e^5)$ **c** $\ln(\sqrt{e})$

d $10^{\log x + \log 3}$ **e** $\ln\left(\frac{1}{e^x}\right)$ **f** $\dfrac{\log(x^2)}{\log_3 9}$

7 Write in the form e^x, where x is correct to 4 decimal places:

a 20 **b** 3000 **c** 0.075

8 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $5^x = 7$ **b** $2^x = 0.1$

9 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

10 Solve for x, giving exact answers:

a $e^{2x} = 70$ **b** $3 \times (1.3)^x = 11$ **c** $5 \times 2^{0.3x} = 16$

11 What is the only value of x for which $\log x = \ln x$?

12 Show that the equation $\log_3(x-k)+\log_3(x+2)=1$ has a real solution for every real value of k.

13 Write as a logarithmic equation:

a $P=3\times b^x$ **b** $m=\dfrac{n^3}{p^2}$

14 Show that $\log_3 7\times 2\log_7 x=2\log_3 x$.

15 Solve for x:

a $\log_2(x^2)+\log_8\sqrt{x}=3$ **b** $\log_{27}\left(\dfrac{1}{x}\right)+\log_3(x^4)=\log_3 10$

16 Write the following equations without logarithms:

a $\log T=2\log x-\log y$ **b** $\log_2 K=\log_2 n+\frac{1}{2}\log_2 t$

17 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

18 Copy and complete:

	$y=\log_2 x$	$y=\ln(x+5)$
Domain		
Range		

19 **a** Factorise 4^x-2^x-20 in the form $(2^x+a)(2^x-b)$ where $a,\ b\in\mathbb{Z}^+$.

b Hence find the exact solution of $2^x(2^x-1)=20$.

c Suppose $p=\log_5 2$.

i Write the solution to **b** in terms of p.

ii Find the solution to $8^x=5^{1-x}$ in terms of p only.

20 Consider $g:x\mapsto 2e^x-5$.

a Find the inverse function g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

d State the asymptotes and intercepts of g and g^{-1}.

21 The temperature of a mug of water t minutes after it has been poured from a kettle is given by $T=60e^{-0.1t}+20$ °C.

Show that it will take $10\ln 3$ minutes for the temperature of the water to fall to 40°C.



22 The weight of a radioactive isotope after t years is given by $W(t)=2500\times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight of the isotope.

b Find the time taken for the isotope to reduce to 30% of its original weight.

c Find the percentage of weight lost after 1500 years.

23 Solve for x, giving an exact answer:

a $5^{\frac{x}{2}} = 9$ **b** $e^x = 30$ **c** $e^{1-3x} = 2$

24 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x+2)$ **b** $y = \ln x$ and $y = \ln(ex)$

25 *Hick's law* models the time taken for a person to make a selection from a number of possible options.

For a particular person, Hick's law determines that the time taken to choose between n equally probable choices is $T = 2\ln(n+1)$ seconds.

a Sketch the graph of T against n for $0 \leqslant n \leqslant 50$.

b How long will it take this person to choose between:

i 5 possible choices **ii** 15 possible choices?

c If the number of possible choices increases from 20 to 40, how much longer will the person take to make a selection?

26 Let $f(x) = \log_3 x - 2$ and $g(x) = 3 - \sqrt{x}$.

a Find the following functions, and state the domain and range of each function:

i $f^{-1}(x)$ **ii** $(f \circ g)(x)$ **iii** $(g \circ f)(x)$

b Solve for x:

i $(f \circ g)(x) = -2$ **ii** $(g \circ f)(x) = 0$

c Find $(g \circ f)^{-1}(x)$, and state its domain and range.

Chapter 4

Introduction to complex numbers

Contents:

A Complex numbers
B The sum of two squares factorisation
C Operations with complex numbers
D Equality of complex numbers
E Properties of complex conjugates

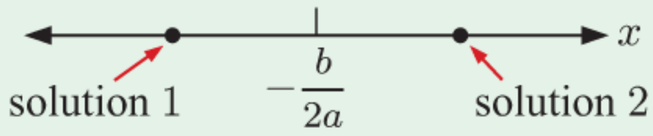


OPENING PROBLEM

The quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ is said to be *real* if $a, b, c \in \mathbb{R}$.

We have seen previously that solutions to this equation can be found using the **quadratic formula** $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$ is called the **discriminant**.

The average of these solutions is $x = -\dfrac{b}{2a}$ which corresponds to the axis of symmetry of the corresponding quadratic function.



Suppose $a > 0$ so the quadratic opens upwards, and that $\Delta > 0,$ so the quadratic equation has two real distinct solutions.

Suppose we start to increase the value of c, so Δ starts to decrease.

Things to think about:

a While Δ is still positive but getting smaller, what happens to the solutions?

b When $\Delta = 0,$ the solution to the equation is $x = -\dfrac{b}{2a}$. Is it still useful to think of this as two *identical* solutions?

c Once $\Delta < 0$ we know that the equation has no *real* solutions. However, suppose the square root of a negative number has some other meaning. If so, then the quadratic formula still gives us two solutions.

i Does the average of these solutions still correspond to the axis of symmetry of the corresponding quadratic function?

ii Do the sum and product of the solutions give us $-\dfrac{b}{a}$ and $\dfrac{c}{a}$ respectively?

d Is it *useful* to take the square root of a negative number?

HISTORICAL NOTE

Heron or **Hero of Alexandria** (10 - 70) was a mathematician and engineer in Roman Egypt. He is most commonly known for **Heron's formula** for finding the area of a triangle, but he also developed an iterative process called **Heron's method** or the **Babylonian method** for computing square roots.

Heron is the first known person to consider whether it might make sense to take the square root of a negative number. In 50 AD, while finding the height of a pyramid using a known formula, he obtained the following result which was published in the book *Stereometrica*:

$$\begin{aligned} h &= \sqrt{(15)^2 - 2\left(\frac{28-4}{2}\right)^2} \\ &= \sqrt{225 - 2(12)^2} \\ &= \sqrt{225 - 144 - 144} \\ &= \sqrt{81 - 144} \end{aligned}$$

Since this result had no use in the context of a real pyramid, Heron did not consider the concept further.

Nearly 1500 years later, the square root of a negative number was considered again by **Gerolamo Cardano** (1501 - 1576). In his book *Ars Magna* published in 1545, Cardano gives the solutions to an equation as $5 \pm \sqrt{-15}$. Rather than dismissing the result, he went on to find the product of the two solutions:

$$(5 + \sqrt{-15}) \times (5 - \sqrt{-15}) = 25 - (-15) = 40$$

In 1572, **Rafael Bombelli** defined the **imaginary number** $i = \sqrt{-1}$. It is called "imaginary" because we cannot place it on the real number line.



Rafael Bombelli

Having defined imaginary i, we can now write down solutions for quadratic equations with $\Delta < 0$. These solutions are said to be *complex* because they include a real and an imaginary part.

For example, consider the equation $x^2 + 1 = 0$. The solutions $x = \pm\sqrt{-1}$ cannot be placed on a number line, and so are not real.

Using Bombelli's idea, we can write the solutions conveniently as $x = \pm i$.

For other quadratic equations with $\Delta < 0$, we obtain solutions of the form $x = a + bi$ where a and b are real and $i = \sqrt{-1}$.

A COMPLEX NUMBERS

Any number of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a **complex number**.

Notice that:

- All real numbers can be written as complex numbers by simply choosing $b = 0$.
- In the special case $a = 0$, we have a number of the form bi, $b \in \mathbb{R}$. We say this number is **purely imaginary**.

REAL AND IMAGINARY PARTS

If $z = a + bi$ where $a, b \in \mathbb{R}$, then:

- a is the **real part** of z, written $\mathcal{Re}(z)$
- b is the **imaginary part** of z, written $\mathcal{Im}(z)$.

For example:

- If $z = 2 + 3i$, then $\mathcal{Re}(z) = 2$ and $\mathcal{Im}(z) = 3$.
- If $z = -\sqrt{2}i$, then $\mathcal{Re}(z) = 0$ and $\mathcal{Im}(z) = -\sqrt{2}$.

COMPLEX CONJUGATES

Suppose $z = a + bi$ where $a, b \in \mathbb{R}$.

The **complex conjugate** of z is $z^* = a - bi$.

$\mathcal{Re}(z^*) = \mathcal{Re}(z)$ and $\mathcal{Im}(z^*) = -\mathcal{Im}(z)$

For example, if $z = 2 + 3i$ then $z^* = 2 - 3i$.

WORKING WITH IMAGINARY i

With $i = \sqrt{-1}$ defined, we can expand our rules for square roots to include negative numbers.

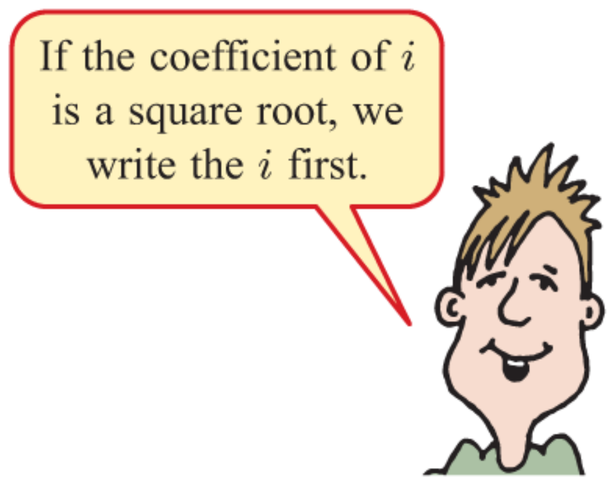
Example 1 **Self Tutor**

Write in terms of i:

a $\sqrt{-81}$ **b** $-\sqrt{-3}$

a
$$\sqrt{-81}$$
$$= \sqrt{81} \times \sqrt{-1}$$
$$= 9i$$

b
$$-\sqrt{-3}$$
$$= -\sqrt{3}\sqrt{-1}$$
$$= -i\sqrt{3}$$



EXERCISE 4A

1 Copy and complete:

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3+2i$			
$5-i$			
3			
0			
$-3+4i$			
$-7-2i$			
$-11i$			
$i\sqrt{3}$			

2 Write in terms of i:

a $\sqrt{-9}$ **b** $\sqrt{-64}$ **c** $\sqrt{-\frac{1}{4}}$ **d** $\sqrt{-5}$ **e** $\sqrt{-8}$

Example 2 **Self Tutor**

Solve the quadratic equations:

a $x^2 = -4$ **b** $x^2 + x + 2 = 0$

a
$$x^2 = -4$$
$$\therefore\ x = \pm\sqrt{-4}$$
$$\therefore\ x = \pm\sqrt{4}\sqrt{-1}$$
$$\therefore\ x = \pm 2i$$

b Using the quadratic formula,
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$$
$$\therefore\ x = \frac{-1 \pm \sqrt{-7}}{2}$$
$$\therefore\ x = \frac{-1 \pm i\sqrt{7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

3 Solve for x:

a $x^2 = 25$ **b** $x^2 = -25$ **c** $x^2 = 5$

d $x^2 = -5$ **e** $4x^2 = 9$ **f** $4x^2 = -9$

4 Solve for x using the quadratic formula:

a $x^2 - 10x + 29 = 0$ **b** $x^2 + 6x + 25 = 0$ **c** $x^2 + 14x + 50 = 0$

d $2x^2 + 5 = 6x$ **e** $x^2 - 2\sqrt{3}x + 4 = 0$ **f** $2x + \frac{1}{x} = 1$

B THE SUM OF TWO SQUARES FACTORISATION

We have previously used the **difference between two squares factorisation** $a^2 - b^2 = (a+b)(a-b)$ extensively.

However, without $i = \sqrt{-1}$ we have been unable to factorise the sum of two squares.

Notice that $a^2 + b^2 = a^2 - b^2 i^2$ {as $i^2 = -1$}
$= (a+bi)(a-bi)$ {the difference between two squares}

The **sum of two squares factorisation** is: $\boxed{a^2 + b^2 = (a+bi)(a-bi)}$

Example 3 Self Tutor

Write as a product of linear factors:

a $x^2 + 4$ **b** $x^2 + 11$

a
$x^2 + 4$
$= x^2 - 4i^2$
$= (x+2i)(x-2i)$

b
$x^2 + 11$
$= x^2 - 11i^2$
$= (x + i\sqrt{11})(x - i\sqrt{11})$

EXERCISE 4B

1 Write as a product of linear factors:

a $x^2 - 9$ **b** $x^2 + 9$ **c** $x^2 - 7$ **d** $x^2 + 7$

e $4x^2 - 1$ **f** $4x^2 + 1$ **g** $2x^2 - 9$ **h** $2x^2 + 9$

2 Explain why the product of the complex conjugates $a + bi$ and $a - bi$ is always real.

Example 4 Self Tutor

Solve for x by factorisation:

a $x^2 + 9 = 0$ **b** $2x^2 + 3 = 0$

a
$x^2 + 9 = 0$
$\therefore\ x^2 - 9i^2 = 0$
$\therefore\ (x+3i)(x-3i) = 0$
$\therefore\ x = \pm 3i$

b
$2x^2 + 3 = 0$
$\therefore\ (\sqrt{2}x)^2 - (i\sqrt{3})^2 = 0$
$\therefore\ (\sqrt{2}x + i\sqrt{3})(\sqrt{2}x - i\sqrt{3}) = 0$
$\therefore\ \sqrt{2}x = \pm i\sqrt{3}$
$\therefore\ x = \pm i\frac{\sqrt{3}}{\sqrt{2}}$

3 Solve for x by factorisation:

a $x^2 + 16 = 0$ **b** $36 + x^2 = 0$ **c** $x^2 + 5 = 0$

d $9x^2 + 1 = 0$ **e** $4x^2 + 25 = 0$ **f** $3x^2 + 2 = 0$

Example 5 **Self Tutor**

Solve for x:

a $x^3 + 2x = 0$ **b** $x^4 + x^2 = 6$

a
$$x^3 + 2x = 0$$
$$\therefore\ x(x^2 + 2) = 0$$
$$\therefore\ x(x^2 - 2i^2) = 0$$
$$\therefore\ x(x + i\sqrt{2})(x - i\sqrt{2}) = 0$$
$$\therefore\ x = 0 \text{ or } \pm i\sqrt{2}$$

b
$$x^4 + x^2 = 6$$
$$\therefore\ x^4 + x^2 - 6 = 0$$
$$\therefore\ (x^2 + 3)(x^2 - 2) = 0$$
$$\therefore\ (x + i\sqrt{3})(x - i\sqrt{3})(x + \sqrt{2})(x - \sqrt{2}) = 0$$
$$\therefore\ x = \pm i\sqrt{3} \text{ or } \pm\sqrt{2}$$

4 Write as a product of linear factors:

a $x^3 - x$ **b** $x^3 + x$ **c** $x^4 - 1$ **d** $x^4 - 16$

5 Solve for x:

a $x^3 - 4x = 0$ **b** $x^3 + 4x = 0$ **c** $x^3 - 3x = 0$

d $x^3 + 3x = 0$ **e** $x^4 - 1 = 0$ **f** $x^4 = 81$

6 Solve for x:

a $x^4 + 2x^2 = 3$ **b** $x^4 = x^2 + 6$ **c** $x^4 + 5x^2 = 36$

d $x^4 + 9x^2 + 14 = 0$ **e** $x^4 + 1 = 2x^2$ **f** $x^4 + 2x^2 + 1 = 0$

C OPERATIONS WITH COMPLEX NUMBERS

Operations with complex numbers are identical to those with radicals, but with $i^2 = -1$ rather than $(\sqrt{2})^2 = 2$ or $(\sqrt{3})^2 = 3$.

For example:

- addition:
$$(2 + \sqrt{3}) + (4 + 2\sqrt{3}) = (2 + 4) + (1 + 2)\sqrt{3} = 6 + 3\sqrt{3}$$
$$(2 + i) \quad + \quad (4 + 2i) = (2 + 4) + (1 + 2)i \quad = 6 + 3i$$

- multiplication:
$$(2 + \sqrt{3})(4 + 2\sqrt{3}) = 8 + 4\sqrt{3} + 4\sqrt{3} + 2(\sqrt{3})^2 = 8 + 8\sqrt{3} + 6 = 14 + 8\sqrt{3}$$
$$(2 + i)(4 + 2i) = 8 + \ 4i \ + \ 4i \ + \ 2i^2 \ = 8 + \ 8i \ - 2 = 6 + 8i$$

$(a + bi) + (c + di) = (a + c) + (b + d)i$	**addition**
$(a + bi) - (c + di) = (a - c) + (b - d)i$	**subtraction**
$(a + bi)(c + di) = ac + adi + bci + bdi^2$	**multiplication**

When we perform division with radicals, we use the radical conjugate of the denominator to achieve a rational denominator.

For example, $\dfrac{1+2\sqrt{3}}{4-\sqrt{3}} = \left(\dfrac{1+2\sqrt{3}}{4-\sqrt{3}}\right)\left(\dfrac{4+\sqrt{3}}{4+\sqrt{3}}\right) = \dfrac{4+\sqrt{3}+8\sqrt{3}+6}{16-3} = \dfrac{10+9\sqrt{3}}{13}$

In the previous Exercise we saw that multiplying a complex number by its complex conjugate also produces a real number. We can use this to achieve a real denominator.

For example, $\dfrac{1+2i}{4-i} = \left(\dfrac{1+2i}{4-i}\right)\left(\dfrac{4+i}{4+i}\right) = \dfrac{4+i+8i+2i^2}{16-i^2} = \dfrac{2+9i}{17}$

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) = \frac{ac-adi+bci-bdi^2}{c^2+d^2} \qquad \textbf{division}$$

Example 6 — Self Tutor

If $z = 3+2i$ and $w = 4-i$ find:

a $z+w$ **b** $z-w$ **c** zw **d** $\dfrac{z}{w}$

a
$$\begin{aligned} & z+w \\ &= (3+2i)+(4-i) \\ &= 7+i \end{aligned}$$

b
$$\begin{aligned} & z-w \\ &= (3+2i)-(4-i) \\ &= 3+2i-4+i \\ &= -1+3i \end{aligned}$$

c
$$\begin{aligned} & zw \\ &= (3+2i)(4-i) \\ &= 12-3i+8i-2i^2 \\ &= 12+5i+2 \\ &= 14+5i \end{aligned}$$

d
$$\begin{aligned} \frac{z}{w} &= \frac{3+2i}{4-i} \\ &= \left(\frac{3+2i}{4-i}\right)\left(\frac{4+i}{4+i}\right) \\ &= \frac{12+3i+8i+2i^2}{16-i^2} \\ &= \frac{10+11i}{17} \\ &= \tfrac{10}{17}+\tfrac{11}{17}i \end{aligned}$$

You can use your calculator to perform operations with complex numbers.

After solving the questions in the following Exercise by hand, check your answers using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 4C

1 If $z = 5-2i$ and $w = 2+i$, find in simplest form:

a $z+w$ **b** $2z$ **c** iw **d** $z-w$

e $2z-3w$ **f** zw **g** w^2 **h** z^2

2 For $z = 1+i$ and $w = -2+3i$, find in simplest form:

a $z+2w$ **b** z^2 **c** z^3 **d** iz

e w^2 **f** zw **g** z^2w **h** izw

3 Prove that the sum of complex conjugates is always real.

4 **a** Simplify i^n for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and also for $n = -1, -2, -3, -4, -5$.

b Hence simplify i^{4n+3} where n is any integer.

5 Write $(1+i)^4$ in simplest form. Hence find $(1+i)^{101}$ in simplest form.

6 **a** Use complex number notation to find $\sqrt{-4} \times \sqrt{-9}$.

b We have previously seen that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for any $a, b \geqslant 0$.
Is this rule also valid for negative a, b? Justify your answer.

7 Suppose $z = 2 - i$ and $w = 1 + 3i$. Write in the form $a + bi$ where $a, b \in \mathbb{R}$ are given exactly:

a $\dfrac{z}{w}$ **b** $\dfrac{i}{z}$ **c** $\dfrac{w}{iz}$ **d** z^{-2}

8 Simplify:

a $\dfrac{i}{1-2i}$ **b** $\dfrac{i(2-i)}{3-2i}$ **c** $\dfrac{1}{2-i} - \dfrac{2}{2+i}$ **d** $\dfrac{i}{x+i} - \dfrac{i}{x-i}$

9 If $z = 2 + i$ and $w = -1 + 2i$, find:

a $\mathcal{Im}(4z - 3w)$ **b** $\mathcal{Re}(zw)$ **c** $\mathcal{Im}(iz^2)$ **d** $\mathcal{Re}\left(\dfrac{z}{w}\right)$

10 Express $z = \left(\dfrac{4}{1+i} + 7 - 2i\right)^2$ in the form $a + bi$ where $a, b \in \mathbb{Z}$.

11 Express $z = \dfrac{3i}{\sqrt{2} - i} + 1$ in the form $a + bi$ where $a, b \in \mathbb{R}$ are given exactly.

12 Suppose $w = \dfrac{z-1}{z^* + 1}$ where $z = a + bi$ for $a, b \in \mathbb{R}$, and z^* is the complex conjugate of z.

a Write w in the form $x + yi$.

b Hence determine the conditions under which w is purely imaginary.

13 Solve for z: $z^2 - (2+i)z + 3 + i = 0$

D EQUALITY OF COMPLEX NUMBERS

Two complex numbers are **equal** when their **real parts** are equal and their **imaginary parts** are equal.

$$a + bi = c + di \Leftrightarrow a = c \text{ and } b = d.$$

Proof:

Suppose $b \neq d$. Now if $a + bi = c + di$ where a, b, c, and d are real,

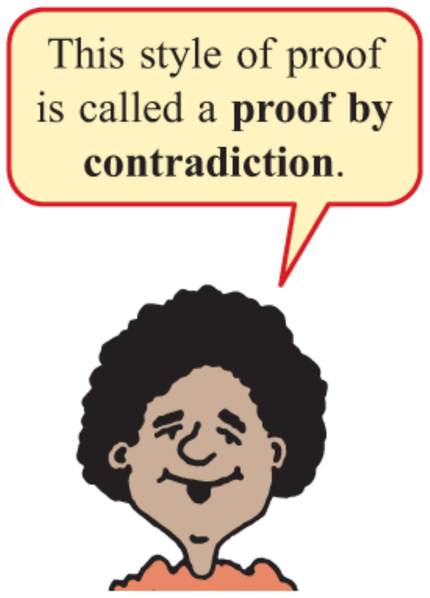
then $bi - di = c - a$

$\therefore\ i(b-d) = c - a$

$\therefore\ i = \dfrac{c-a}{b-d}$ $\{\text{as } b \neq d\}$

This is false as the RHS is real but the LHS is imaginary.

Thus, the supposition is false. Hence $b = d$ and furthermore $a = c$.



For the complex number $a + bi$, where a and b are real, $a + bi = 0 \Leftrightarrow a = 0$ and $b = 0$.

Example 7 **Self Tutor**

Find real numbers x and y such that:

a $(x+yi)(2-i) = -i$

b $(x+2i)(1-i) = 5+yi$

a
$$(x+yi)(2-i) = -i$$
$$\therefore \quad x+yi = \frac{-i}{2-i}$$
$$= \left(\frac{-i}{2-i}\right)\left(\frac{2+i}{2+i}\right)$$
$$= \frac{-2i-i^2}{4+1}$$
$$= \frac{1-2i}{5}$$
$$= \tfrac{1}{5} - \tfrac{2}{5}i$$

Equating real and imaginary parts, $x = \frac{1}{5}$ and $y = -\frac{2}{5}$.

b
$$(x+2i)(1-i) = 5+yi$$
$$\therefore \quad x - xi + 2i + 2 = 5 + yi$$
$$\therefore \quad (x+2) + (2-x)i = 5 + yi$$

Equating real and imaginary parts,
$$\therefore \quad x+2 = 5 \text{ and } 2-x = y$$
$$\therefore \quad x = 3 \text{ and } y = -1$$

EXERCISE 4D

1 Find exact real numbers x and y such that:

a $2x + 3yi = -x - 6i$

b $x^2 + xi = 4 - 2i$

c $(x+yi)(2-i) = 8+i$

d $(3+2i)(x+yi) = -i$

2 Find exact $x, y \in \mathbb{R}$ such that:

a $2(x+yi) = x - yi$

b $(x+2i)(y-i) = -4-7i$

c $(x+i)(3-iy) = 1+13i$

d $(x+yi)(2+i) = 2x - (y+1)i$

3 The complex number z satisfies the equation $3z + 17i = iz + 11$. Write z in the form $a+bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

4 Find the real values of m and n for which $3(m+ni) = n - 2mi - (1-2i)$.

5 Suppose $(a+bi)^2 = -16 - 30i$ where $a, b \in \mathbb{R}$. Find the possible values of a and b.

6 Given $z^2 = 1 + i + \dfrac{58}{9(3-7i)}$, find z in the form $a+bi$.

HISTORICAL NOTE

18th century mathematicians enjoyed playing with these new "imaginary" numbers, but they were regarded as little more than interesting curiosities until the work of **Carl Friedrich Gauss** (1777 - 1855), the German mathematician, astronomer, and physicist.

For centuries mathematicians had attempted to find a method of trisecting an angle using a compass and straight edge. Gauss put an end to this when he used complex numbers to prove the impossibility of such a construction. By his systematic use of complex numbers, he was able to convince mathematicians of their usefulness.

Early last century, the American engineer **Steinmetz** used complex numbers to solve electrical problems, illustrating that complex numbers did have a practical application.

Complex numbers are now used extensively in electronics, engineering, and physics.

E PROPERTIES OF COMPLEX CONJUGATES

INVESTIGATION PROPERTIES OF CONJUGATES

The purpose of this Investigation is to discover any properties that complex conjugates might have.

What to do:

1 Given $z_1 = 1 - i$ and $z_2 = 2 + i$ find:

a z_1^*	**b** z_2^*	**c** $(z_1^*)^*$	**d** $(z_2^*)^*$
e $z_1 + z_1^*$	**f** $z_2 + z_2^*$	**g** $z_1 z_1^*$	**h** $z_2 z_2^*$
i $(z_1 + z_2)^*$	**j** $z_1^* + z_2^*$	**k** $(z_1 - z_2)^*$	**l** $z_1^* - z_2^*$
m $(z_1 z_2)^*$	**n** $z_1^* z_2^*$	**o** $\left(\frac{z_1}{z_2}\right)^*$	**p** $\frac{z_1^*}{z_2^*}$
q $(z_1^2)^*$	**r** $(z_1^*)^2$	**s** $(z_2^3)^*$	**t** $(z_2^*)^3$

2 Repeat **1** with z_1 and z_2 of your choice.

3 Examine your results from **1** and **2**, and hence suggest some rules for complex conjugates.

From the **Investigation** you should have discovered the following rules for complex conjugates:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}, \quad z_2 \neq 0$
- $(z^n)^* = (z^*)^n$ for positive integers n
- $z + z^*$ and zz^* are real.

Example 8 Self Tutor

Show that for all complex numbers z_1 and z_2:

a $(z_1 + z_2)^* = z_1^* + z_2^*$ **b** $(z_1 z_2)^* = z_1^* \times z_2^*$

a Let $z_1 = a + bi$ and $z_2 = c + di$

$\therefore z_1^* = a - bi$ and $z_2^* = c - di$

Now $z_1 + z_2 = (a + c) + (b + d)i$

$\therefore (z_1 + z_2)^* = (a + c) - (b + d)i$

$= a + c - bi - di$

$= a - bi + c - di$

$= z_1^* + z_2^*$

b Let $z_1 = a + bi$ and $z_2 = c + di$

$\therefore z_1 z_2 = (a + bi)(c + di)$

$= ac + adi + bci + bdi^2$

$= (ac - bd) + i(ad + bc)$

$\therefore (z_1 z_2)^* = (ac - bd) - i(ad + bc)$ .... (1)

Also, $z_1^* \times z_2^*$

$= (a - bi)(c - di)$

$= ac - adi - bci + bdi^2$

$= (ac - bd) - i(ad + bc)$ .... (2)

From (1) and (2), $(z_1 z_2)^* = z_1^* \times z_2^*$

EXERCISE 4E

1 Show that $(z_1 - z_2)^* = z_1^* - z_2^*$ for all complex numbers z_1 and z_2.

2 Simplify the expression $(w^* - z)^* - (w - 2z^*)$ using the properties of conjugates.

3 Suppose $z_1 = a + bi$ and $z_2 = c + di$ are complex numbers.

a Find $\frac{z_1}{z_2}$ in the form $X + iY$. **b** Show that $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$ for all z_1 and $z_2 \neq 0$.

4 Prove that $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$ by starting with $\left(\frac{z_1}{z_2}\right)^* \times z_2^*$ and using the result "the conjugate of a product is the product of the conjugates" from **Example 8 b**.

5 Let $z = a + bi$ be a complex number. Prove:

a If $z = z^*$, then z is real. **b** If $z^* = -z$, then z is purely imaginary or zero.

6 Prove that for all complex numbers z and w:

a $zw^* + z^*w$ is always real **b** $zw^* - z^*w$ is purely imaginary or zero

c $\frac{z}{w} + \frac{z^*}{w^*}$ is always real.

7 **a** Given $z = a + bi$, write z^2 in the form $X + iY$.
Hence show that $(z^2)^* = (z^*)^2$ for all complex numbers z.

b Repeat **a** with z^3 instead of z^2.

8 Suppose $w = \frac{z - 1}{z^* + 1}$ where $z = a + bi$. Find the conditions under which w is:

a real **b** purely imaginary.

REVIEW SET 4A

1 Solve for x using the quadratic formula:

a $x^2 - 6x + 11 = 0$ **b** $x^2 + 3 = 2\sqrt{2}x$

2 Solve for x by factorisation:

a $3x^2 + 16 = 0$ **b** $x^4 + 2x^2 = 15$

3 If $z = 3 + i$ and $w = -2 - i$, find in simplest form:

a $2z - 3w$ **b** $\frac{z^*}{w}$ **c** z^3

4 Find exactly the real and imaginary parts of z if $z = \frac{3}{i + \sqrt{3}} + \sqrt{3}$.

5 If $z = \left(\frac{5}{2 - i} - 3 - 2i\right)^3$, express z in the form $z = x + yi$, $x, y \in \mathbb{Z}$.

6 Find real numbers a and b such that:

a $a + ib = 4$ **b** $(1 - 2i)(a + bi) = -5 - 10i$ **c** $(a + 2i)(1 + bi) = 17 - 19i$

7 Find a complex number z such that $2z - 1 = iz - i$. Write your answer in the form $z = a + bi$ where $a, b \in \mathbb{R}$.

8 Find rationals a and b such that $\frac{2 - 3i}{2a + bi} = 3 + 2i$.

9 Prove that if z is a complex number then both $z + z^*$ and zz^* are real.

10 Suppose $z = x + 2i$ and $u = 3 + iy$ where $x, y \in \mathbb{R}$. Find the smallest positive value of x for which $\dfrac{z+u}{z-u}$ is purely imaginary.

REVIEW SET 4B

1 Write as a product of linear factors:

a $x^2 + 121$ **b** $3x^2 + 7$

2 Solve for x by factorisation: $4x^3 + x = 0$

3 If $z = 4 + i$ and $w = 3 - 2i$, find $2w^* - iz$.

4 z and w are two complex numbers such that $2z + w = i$ and $z - 3w = 7 - 10i$. Find $z + w$ in the form $a + bi$.

5 Find z if $\sqrt{z} = \dfrac{2}{3-2i} + 2 + 5i$.

6 Given $w = \dfrac{z+1}{z^*+1}$ where $z = a + bi$, $a, b \in \mathbb{R}$, write w in the form $x + yi$, $x, y \in \mathbb{R}$. Hence determine the conditions under which w is purely imaginary.

7 Find real x and y such that:

a $x + iy = 0$ **b** $(3 - 2i)(x + i) = 17 + yi$

c $(x + iy)^2 = x - iy$ **d** $(3x + 2yi)(1 - i) = (3y + 1)i - x$

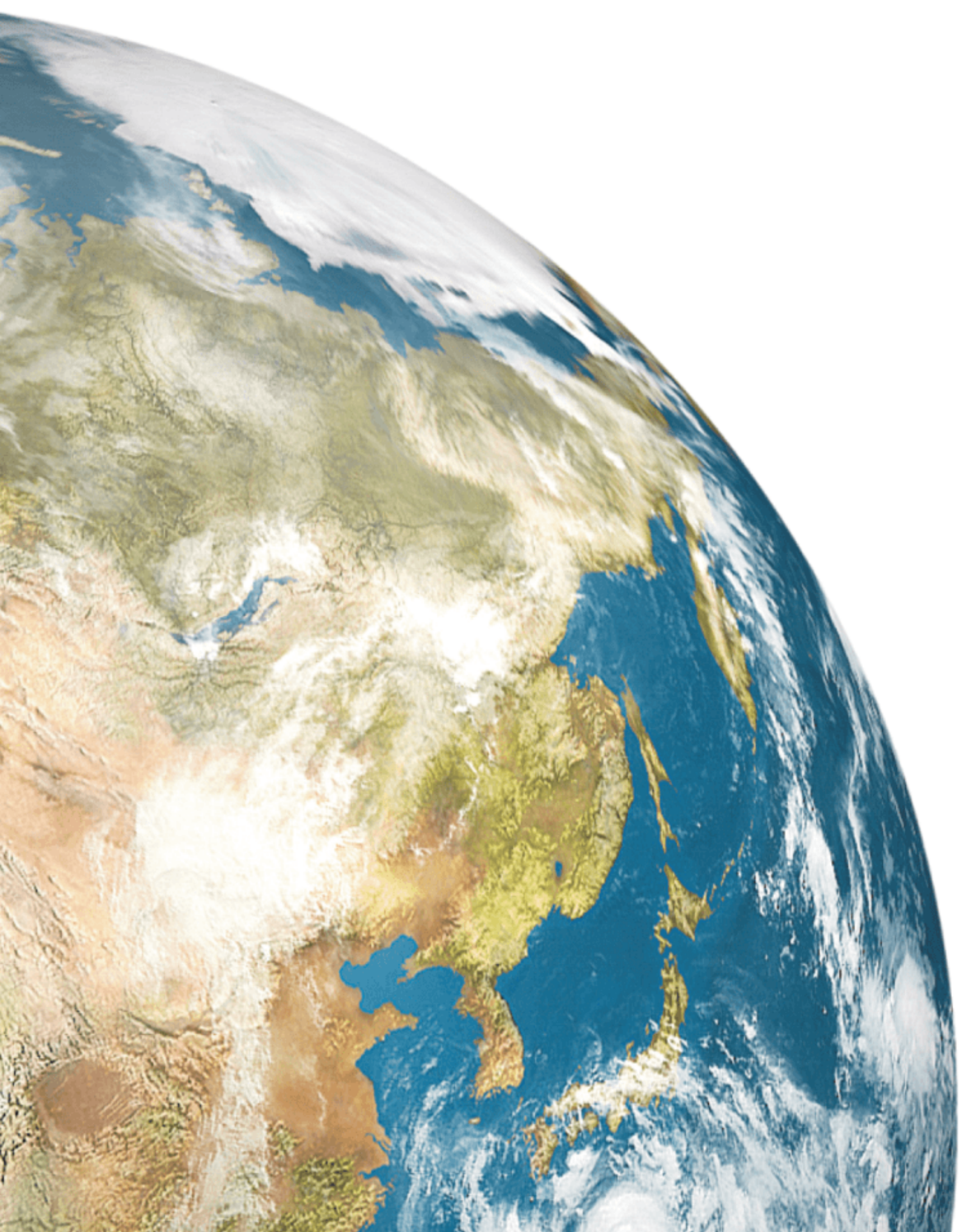
8 Without using a calculator, find z if $z^2 = 5 - 12i$. Check your answer using a calculator.

9 Let $z = a + bi$ be a complex number and k be a real constant. Prove that $(kz)^* = kz^*$.

10 z and w are non-real complex numbers with the property that both $z + w$ and zw are real. Prove that $z^* = w$.

Chapter 5

Real polynomials



Contents:

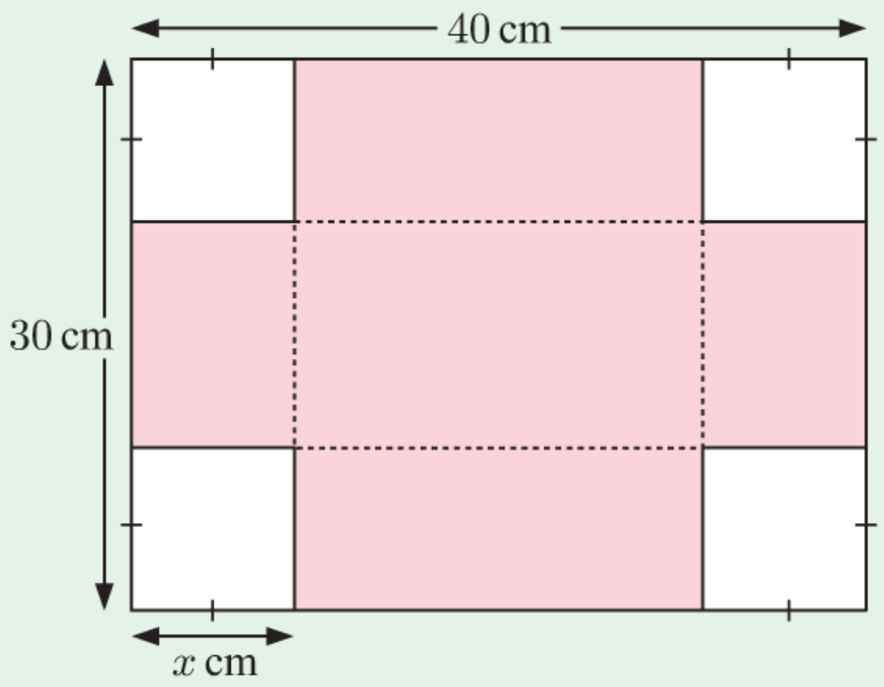
A Polynomials
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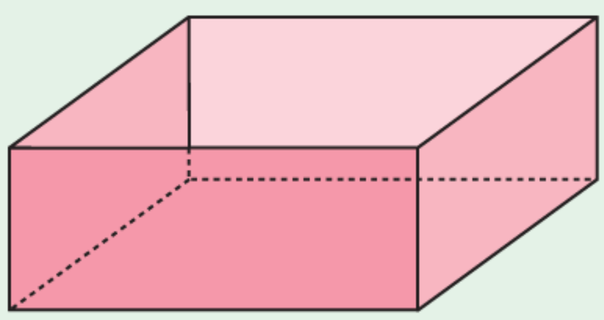
OPENING PROBLEM

For his Technical Studies project, Marcus is using a 40 cm by 30 cm sheet of tinplate to make a container.

Squares are cut from the corners of the sheet, and the metal is then folded upwards along the dashed lines. The edges are then welded together.

The volume of the container will depend on the size of the squares cut from the corners. When considering what this size should be, Marcus supposes that squares with side length x cm are cut from the corners.





Things to think about:

a Can you explain why the volume of the container is given by the function $V = x(40 - 2x)(30 - 2x)$ cm^3 ?

b What type of function is this?

c For what values of x is it reasonable to use this function?

d What does the graph of this function look like?

e What size squares should be cut from the corners to make a container with volume 2000 cm^3 ?

To model the world around us, we need to be able to select and apply appropriate mathematical functions. Usually, we will be fitting a curve to a set of data points, and in this case we use functions whose values are always real numbers.

The linear and quadratic functions we have studied are the first two members of the general class of functions called **polynomials**.

A POLYNOMIALS

A **polynomial** is a function which can be written in the form

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 = \sum_{r=0}^{n} a_r x^r$ where $a_0, a_1, ..., a_n$ are constants, $a_n \neq 0$.

We say that: x is the **variable**
a_0 is the **constant term**
a_n is the **leading coefficient**
a_r is the **coefficient of x^r** for $r = 0, 1, 2, ..., n$
n is the **degree** of the polynomial, being the highest power of the variable
$a_r x^r$ is the **term** of the polynomial with degree r.

A **real polynomial** $P(x)$ is a polynomial for which $a_r \in \mathbb{R}$, $r = 0, 1, 2, ..., n$.

The low degree members of the polynomial family have special names, some of which you are already familiar with. For these polynomials, we commonly write their coefficients as a, b, c, ...

Polynomial function	*Degree*	*Name*
$ax + b, \quad a \neq 0$	1	linear
$ax^2 + bx + c, \quad a \neq 0$	2	quadratic
$ax^3 + bx^2 + cx + d, \quad a \neq 0$	3	cubic
$ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	4	quartic

EXERCISE 5A

1 For each of the following polynomials, state the:

i degree **ii** leading coefficient **iii** constant term.

a $P(x) = 2x^3 + x^2 - 2x + 5$ **b** $f(x) = x^4 + 5x^2 + 3x - 2$

c $Q(x) = -3x^5 - x^3 + 4x^2 + 1$ **d** $g(x) = 5x^3 - 3x^2 + 4x$

e $P(x) = 7 + 5x - x^4$ **f** $h(x) = 2x^2 - 4x^3 + 7 - 3x$

2 State the coefficient of:

a x^2 in $P(x) = 4x^3 + 7x^2 - 5x + 2$ **b** x in $f(x) = x^4 + 2x^3 - 4x + 1$

c x^3 in $Q(x) = x^5 - 3x^4 + 2x^2 - 7$ **d** x^2 in $g(x) = \frac{1}{3}x^3 - 2x + \frac{1}{4}x^2 + 3$.

B OPERATIONS WITH POLYNOMIALS

ADDITION AND SUBTRACTION

To **add** or **subtract** two polynomials, we collect "like" terms.

To make it easier to collect "like" terms, we often write them one above the other.

Example 1 Self Tutor

If $P(x) = x^3 - 2x^2 + 3x - 5$ and $Q(x) = 2x^3 + x^2 - 11$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

a
$$\begin{aligned} P(x) + Q(x) &= x^3 - 2x^2 + 3x - 5 \\ &\quad +2x^3 + x^2 - 11 \\ &= 3x^3 - x^2 + 3x - 16 \end{aligned}$$

b
$$\begin{aligned} P(x) - Q(x) &= x^3 - 2x^2 + 3x - 5 - (2x^3 + x^2 - 11) \\ &= x^3 - 2x^2 + 3x - 5 \\ &\quad -2x^3 - x^2 + 11 \\ &= -x^3 - 3x^2 + 3x + 6 \end{aligned}$$



The brackets around $Q(x)$ help us get the signs correct in the next line.

SCALAR MULTIPLICATION

To **multiply** a polynomial by a **scalar** (constant) we multiply each term by the scalar.

Example 2 — Self Tutor

If $P(x) = x^4 - 2x^3 + 4x + 7$, find:

a $3P(x)$ **b** $-2P(x)$

a
$$\begin{aligned} &3P(x) \\ &= 3(x^4 - 2x^3 + 4x + 7) \\ &= 3x^4 - 6x^3 + 12x + 21 \end{aligned}$$

b
$$\begin{aligned} &-2P(x) \\ &= -2(x^4 - 2x^3 + 4x + 7) \\ &= -2x^4 + 4x^3 - 8x - 14 \end{aligned}$$

POLYNOMIAL MULTIPLICATION

To **multiply** two polynomials, we multiply each term of the first polynomial by each term of the second polynomial, and then collect "like" terms.

Example 3 — Self Tutor

If $P(x) = x^3 - 2x + 4$ and $Q(x) = 2x^2 + 3x - 5$, find $P(x)Q(x)$.

$$\begin{aligned} &P(x)Q(x) \\ &= (x^3 - 2x + 4)(2x^2 + 3x - 5) \\ &= x^3(2x^2 + 3x - 5) - 2x(2x^2 + 3x - 5) + 4(2x^2 + 3x - 5) \\ &= 2x^5 + 3x^4 - 5x^3 \\ &\qquad - 4x^3 - 6x^2 + 10x \\ &\qquad\qquad + 8x^2 + 12x - 20 \\ &= 2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20 \end{aligned}$$



We line up the "like" terms so we can add them in columns.

We can also perform polynomial multiplication using:

- **synthetic multiplication** where we just use the coefficients

		1	0	-2	4 ← coefficients of $x^3 - 2x + 4$
	$\times$		2	3	-5 ← coefficients of $2x^2 + 3x - 5$
		-5	0	10	-20
	3	0	-6	12	
2	0	-4	8		
2	3	-9	2	22	-20
x^5	x^4	x^3	x^2	x	constant term

$$\begin{aligned} \therefore\ &(x^3 - 2x + 4)(2x^2 + 3x - 5) \\ &= 2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20 \end{aligned}$$

Unlike long multiplication with numbers, we do not "carry" tens, and we can have negative coefficients.



- a **grid** where we can add "like" terms along diagonals.



EXERCISE 5B

1 If $P(x) = x^2 + 2x + 3$ and $Q(x) = 4x^2 + 5x + 6$, find in simplest form:

a $3P(x)$ **b** $P(x) + Q(x)$ **c** $P(x) - 2Q(x)$ **d** $P(x)Q(x)$

2 If $f(x) = x^2 - x + 2$ and $g(x) = x^3 - 3x + 5$, find in simplest form:

a $f(x) + g(x)$ **b** $g(x) - f(x)$ **c** $2f(x) + 3g(x)$

d $g(x) + xf(x)$ **e** $f(x)\,g(x)$ **f** $[f(x)]^2$

3 Expand and simplify:

a $(x^2 - 2x + 3)(2x + 1)$ **b** $(x - 1)^2(x^2 + 3x - 2)$ **c** $(x + 2)^3$

d $(2x^2 - x + 3)^2$ **e** $(2x - 1)^4$ **f** $(3x - 2)^2(2x + 1)(x - 4)$

4 Find the following products:

a $(2x^2 - 3x + 5)(3x - 1)$ **b** $(4x^2 - x + 2)(2x + 5)$

c $(2x^2 + 3x + 2)(5 - x)$ **d** $(x - 2)^2(2x + 1)$

e $(x^2 - 3x + 2)(2x^2 + 4x - 1)$ **f** $(3x^2 - x + 2)(5x^2 + 2x - 3)$

g $(x^2 - x + 3)^2$ **h** $(2x^2 + x - 4)^2$

i $(2x + 5)^3$ **j** $(x^3 + x^2 - 2)^2$

5 Suppose the polynomial $P(x)$ has degree 3. Find the degree of $Q(x) = (x - 1)P(x)$. Explain your answer.

6 Suppose $P(x) = (x^2 - 3x + 4)Q(x)$ has degree 5. Find the degree of $Q(x)$.

7 $P(x)$ has degree 2, leading coefficient 3, and constant term 5. $Q(x)$ has degree 4, leading coefficient -5, and constant term -2. Find the degree, leading coefficient, and constant term of $P(x)Q(x)$.

8 If $P(x)$ and $Q(x)$ are polynomials with degrees p and q respectively, state possibilities for the degree of:

a $P(x)+Q(x),\ P(x) \neq -Q(x)$ **b** $kP(x),\ k \neq 0$ **c** $P(x)Q(x)$ **d** $(P(x))^2$

Explain each answer.

ACTIVITY 1 POLYNOMIALS IN POWERS OF $x \pm k$



Instead of writing a polynomial in powers of x, it is often useful to write it in terms of a simple linear expression.

For example, we might wish to write $x^3 + 2x^2 - 3x + 7$ in the form $a(x + 1)^3 + b(x + 1)^2 + c(x + 1) + d$.

Click on the icon to explore the **shift-expand-shift** process for achieving this.

C ZEROS, ROOTS, AND FACTORS

A **zero** of a polynomial is a value of the variable which makes the polynomial equal to zero.

$$\alpha \text{ is a } \textbf{zero} \text{ of polynomial } P(x) \Leftrightarrow P(\alpha) = 0.$$

The **roots** of a polynomial **equation** are the solutions to the equation.

$$\alpha \text{ is a } \textbf{root} \text{ (or } \textbf{solution}\text{) of } P(x) = 0 \Leftrightarrow P(\alpha) = 0.$$

The **roots** of $P(x) = 0$ are the **zeros** of $P(x)$ and the x-intercepts of the graph of $y = P(x)$.



Consider $P(x) = x^3 - 2x^2 + x - 12$

$\therefore\ P(3) = 3^3 - 2(3)^2 + 3 - 12 = 27 - 18 + 3 - 12 = 0$

This tells us that:

- 3 is a zero of $x^3 - 2x^2 + x - 12$
- 3 is a root of $x^3 - 2x^2 + x - 12 = 0$
- the graph of $y = x^3 - 2x^2 + x - 12$ has the x-intercept 3.

DISCUSSION

Will a zero of a polynomial always correspond to an x-intercept of its graph?

You might like to consider the function $P(x) = x^2 + 1$.

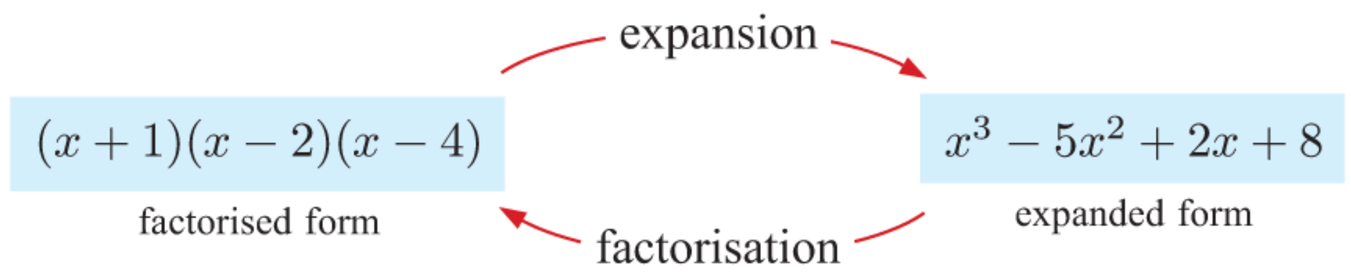
FACTORISATION AND EXPANSION

Polynomials can be written in **factorised form** or **expanded form**.

For example, the **expansion** of $(x+1)(x-2)(x-4)$

$$= (x^2 - x - 2)(x - 4)$$
$$= x^3 - 5x^2 + 2x + 8$$

In reverse, the process of writing $x^3 - 5x^2 + 2x + 8$ as $(x+1)(x-2)(x-4)$ is called **factorisation**.



If $P(x) = (2x+1)(x-3)(x-1)$, then $(2x+1)$, $(x-3)$, and $(x-1)$ are its **linear factors**.

Likewise $P(x) = (x+2)^2(x-1)$ has been factorised into 3 linear factors, one of which is repeated.

$(x-\alpha)$ is a **linear factor** of the polynomial $P(x) \Leftrightarrow$ there exists a polynomial $Q(x)$ such that $P(x) = (x-\alpha)Q(x)$.

If $\alpha \in \mathbb{R}$ then $(x-\alpha)$ is a **real linear factor**.

Example 4 — Self Tutor

Find the zeros of: **a** $x^2 - 4x + 53$ **b** $z^3 + 3z$

a We wish to find x such that $x^2 - 4x + 53 = 0$

$$\therefore \quad x = \frac{4 \pm \sqrt{16 - 4(1)(53)}}{2}$$

$$\therefore \quad x = \frac{4 \pm \sqrt{-196}}{2} = \frac{4 \pm 14i}{2}$$

$$\therefore \quad x = 2 \pm 7i$$

The zeros are $2 - 7i$ and $2 + 7i$.

b We wish to find z such that

$$z^3 + 3z = 0$$

$$\therefore \quad z(z^2 + 3) = 0$$

$$\therefore \quad z(z + i\sqrt{3})(z - i\sqrt{3}) = 0$$

$$\therefore \quad z = 0 \text{ or } \pm i\sqrt{3}$$

The zeros are $-i\sqrt{3}$, 0, and $i\sqrt{3}$.

EXERCISE 5C

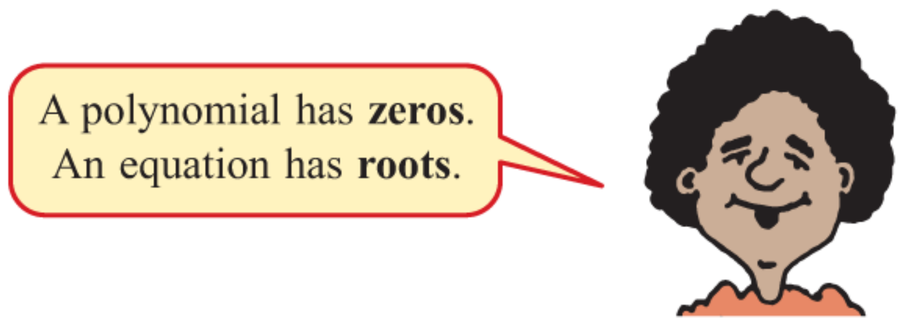
1 Decide whether:

a 3 is a zero of $x^2 - 5x + 6$

b -1 is a zero of $x^3 + 2x^2 - 4x + 3$

c 2 is a root of $3x^2 - 5x - 4 = 0$

d -2 is a root of $x^3 + 3x^2 + 5x + 6 = 0$.



2 Find the zeros of:

a $3x^2 - 2x - 5$ **b** $x^2 + 6x + 10$ **c** $z^2 - 6z + 6$

d $x^3 - 4x$ **e** $z^3 + 2z$ **f** $z^4 + 4z^2 - 5$

3 Find the roots of:

a $5x^2 = 3x + 2$ **b** $(2x + 1)(x^2 + 3) = 0$ **c** $-2z(z^2 - 2z + 2) = 0$

d $x^3 = 5x$ **e** $z^3 + 5z = 0$ **f** $z^4 = 3z^2 + 10$

4 Use expansion to verify that:

a $(x - 4)(x - 2)(x + 3)$ is the factorised form of $x^3 - 3x^2 - 10x + 24$

b $(3x + 1)(x + 3)(x - 4)$ is the factorised form of $3x^3 - 2x^2 - 37x - 12$

c $(2x + 1)(2x - 3)(x + 2)$ is the factorised form of $4x^3 + 4x^2 - 11x - 6$.

Example 5 — Self Tutor

Factorise: **a** $2x^3 + 5x^2 - 3x$ **b** $z^2 + 4z + 9$

a $2x^3 + 5x^2 - 3x$
$= x(2x^2 + 5x - 3)$
$= x(2x - 1)(x + 3)$

b $z^2 + 4z + 9$ is zero when $z = \dfrac{-4 \pm \sqrt{16 - 4(1)(9)}}{2}$

$$\therefore \quad z = \frac{-4 \pm \sqrt{-20}}{2}$$

$$\therefore \quad z = \frac{-4 \pm 2i\sqrt{5}}{2}$$

$$\therefore \quad z = -2 \pm i\sqrt{5}$$

$$\therefore \quad z^2 + 4z + 9 = (z - [-2 + i\sqrt{5}])(z - [-2 - i\sqrt{5}])$$
$$= (z + 2 - i\sqrt{5})(z + 2 + i\sqrt{5})$$

5 Factorise into the product of linear factors:

a $2x^2 - 7x - 15$ **b** $z^2 - 6z + 16$ **c** $x^3 + 2x^2 - 4x$

d $6z^3 - z^2 - 2z$ **e** $z^4 - 6z^2 + 5$ **f** $z^4 - z^2 - 2$

6 Find the *real* linear factors and corresponding zeros of:

a $x^2 - 6x - 16$ **b** $2x^2 - 7x - 15$ **c** $x^2 + 3x + 4$

d $x^3 + 2x^2 - 4x$ **e** $6z^3 - z^2 - 2z$ **f** $z^4 + 3z^2 - 18$

7 If $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$ then α, β, and γ are its zeros. Verify this statement by finding $P(\alpha)$, $P(\beta)$, and $P(\gamma)$.

8 The graph of the polynomial $P(x)$ has three x-intercepts, α, β, and γ.

a State the values of $P(\alpha)$, $P(\beta)$, and $P(\gamma)$.

b Explain why $P(x)$ cannot have a factor $(x-a)$ such that $a \neq \alpha$, β, or γ.

Example 6 — Self Tutor

Find all real cubic polynomials with zeros $\frac{1}{2}$ and $-3 \pm 2i$.

The zeros $-3 \pm 2i$ have sum $= -3 + 2i - 3 - 2i = -6$ and product $= (-3+2i)(-3-2i) = 13$

$\therefore$ they come from the quadratic factor $x^2 + 6x + 13$

$\frac{1}{2}$ comes from the linear factor $2x - 1$.

$\therefore \quad P(x) = a(2x-1)(x^2+6x+13), \quad a \neq 0.$

Example 7 — Self Tutor

Find all real quartic polynomials with zeros 2, $-\frac{1}{3}$, and $-1 \pm \sqrt{5}$.

The zeros $-1 \pm \sqrt{5}$ have sum $= -1 + \sqrt{5} - 1 - \sqrt{5} = -2$
and product $= (-1+\sqrt{5})(-1-\sqrt{5}) = -4$

$\therefore$ they come from the quadratic factor $x^2 + 2x - 4$.

The zeros 2 and $-\frac{1}{3}$ come from the linear factors $(x-2)$ and $(3x+1)$.

$\therefore \quad P(x) = a(x-2)(3x+1)(x^2+2x-4), \quad a \neq 0.$

9 Find all real quadratic functions with zeros:

a 0 and $-\frac{2}{3}$ **b** $2 \pm \sqrt{3}$ **c** -1 and $1 - \sqrt{2}$

d $3 \pm i$ **e** $1 \pm 3i$ **f** $-2 \pm 5i$

g $-3 \pm \sqrt{2}$ **h** $\pm i\sqrt{2}$ **i** $\sqrt{2} \pm i$

10 Show that if the zeros of the quadratic function $P(x)$ are $p \pm qi$, then the function has the form $P(x) = a(x^2 - 2px + (p^2 + q^2))$ for some constant $a \neq 0$.

11 Find all real cubic polynomials with zeros:

a $\pm 2,\ 3$ **b** $-2,\ \pm 4i$ **c** $3,\ -1 \pm i$

d $-5,\ -3 \pm 2i$ **e** $-1,\ 3 \pm \sqrt{2}$ **f** $\sqrt{2},\ -2 \pm i\sqrt{2}$

12 Find all real quartic polynomials with zeros:

a $-2,\ 0,\ 1,\ 5$ **b** $\pm 1,\ \pm\sqrt{2}$ **c** $2,\ -1,\ \pm i\sqrt{3}$

d $2 \pm \sqrt{3},\ 4,\ 0$ **e** $\pm\sqrt{3},\ 1 \pm i$ **f** $2 \pm \sqrt{5},\ -2 \pm 3i$

13 **a** Suppose a quadratic function with *rational* coefficients has the *irrational* zero $p + q\sqrt{n}$. Prove that the other zero must be the radical conjugate $p - q\sqrt{n}$.

b $1 - \sqrt{2}$ is a zero of $x^2 + ax + b$ where $a,\ b \in \mathbb{Q}$. Find a and b.

c Find all real quadratic functions with rational coefficients and one zero $2 + \sqrt{5}$.

14 **a** Prove that if a *real* quadratic function has one zero $p + qi$, then the other zero must be the complex conjugate $p - qi$.

b $3 + i$ is a zero of $x^2 + ax + b$ where $a,\ b \in \mathbb{R}$. Find a and b.

c Find all real quadratic functions with one zero $\sqrt{2} + i$.

d $a + ai$ is a zero of $x^2 + 4x + b$ where $a,\ b \in \mathbb{R}$. Find a and b.

D POLYNOMIAL EQUALITY

Two polynomials are **equal** if and only if they have the **same degree** (order) and corresponding terms have equal coefficients.

If we know two polynomials are **equal**, we can **equate coefficients** to find unknown coefficients.

For example, if $x^3 - 2x^2 + 3x + 7 = ax^3 + bx^2 + cx + d$, where $a,\ b,\ c,\ d \in \mathbb{R}$, then $a = 1,\ b = -2,\ c = 3,$ and $d = 7$.

Example 8 **Self Tutor**

Find constants a, b, and c given that: $4x^3 + 10x^2 - x + 15 = (x + 3)(ax^2 + bx + c)$ for all x.

$$\begin{aligned} 4x^3 + 10x^2 - x + 15 &= (x + 3)(ax^2 + bx + c) \\ &= ax^3 + bx^2 + cx + 3ax^2 + 3bx + 3c \\ &= ax^3 + (3a + b)x^2 + (3b + c)x + 3c \end{aligned}$$

Since this is true for all x, we equate coefficients:

$\therefore$ $\underbrace{a = 4}_{x^3 \text{ s}}$ $\underbrace{3a + b = 10}_{x^2 \text{ s}}$ $\underbrace{3b + c = -1}_{x \text{ s}}$ and $\underbrace{3c = 15}_{\text{constant terms}}$

$\therefore$ $a = 4$ and $c = 5$

$12 + b = 10$ and $3b + 5 = -1$ $\Rightarrow$ $b = -2$ in both equations.

So, $a = 4,\ b = -2,$ and $c = 5$.

Example 9 **Self Tutor**

Find constants a and b if $z^4 + 9 = (z^2 + az + 3)(z^2 + bz + 3)$ for all z.

$$\begin{aligned} z^4 + 9 &= (z^2 + az + 3)(z^2 + bz + 3) \quad \text{for all } z \\ &= z^4 + bz^3 + \ 3z^2 \\ &\qquad + az^3 + abz^2 + 3az \\ &\qquad\quad + \ 3z^2 + 3bz + 9 \\ &= z^4 + (a+b)z^3 + (ab+6)z^2 + (3a+3b)z + 9 \quad \text{for all } z \end{aligned}$$

Equating coefficients gives
$$\begin{cases} a + b = 0 & \ldots. \ (1) \quad \{z^3 \text{ s}\} \\ ab + 6 = 0 & \ldots. \ (2) \quad \{z^2 \text{ s}\} \\ 3a + 3b = 0 & \ldots. \ (3) \quad \{z \text{ s}\} \end{cases}$$

From (1) and (3), $b = -a$

$\therefore$ in (2), $a(-a) + 6 = 0$

$\therefore \ a^2 = 6$

$\therefore \ a = \pm\sqrt{6}$ and so $b = \mp\sqrt{6}$

$\therefore \ a = \sqrt{6}, \ b = -\sqrt{6}$ *or* $a = -\sqrt{6}, \ b = \sqrt{6}$

When simultaneously solving more equations than there are unknowns, we must check that any solutions fit **all** equations. If they do not, there are **no solutions**.



EXERCISE 5D

1 Find constants a, b, and c given that:

a $2x^2 + 4x + 5 = ax^2 + (2b - 6)x + c$ for all x

b $x^3 + 2x^2 - 3x + 4 = (x - 2)(x^2 + ax + b) + c$ for all x

c $2x^3 - x^2 + 6 = (x - 1)^2(2x + a) + bx + c$ for all x.

2 Find constants a and b if:

a $z^4 + 4 = (z^2 + az + 2)(z^2 + bz + 2)$ for all z

b $z^4 + z^3 + z^2 - 9z - 10 = (z^2 + az + 5)(z^2 + bz - 2)$ for all z

c $2z^4 + 5z^3 + 4z^2 + 7z + 6 = (z^2 + az + 2)(2z^2 + bz + 3)$ for all z.

3 Show that $z^4 + 64$ can be factorised into two real quadratic factors of the form $z^2 + az + 8$ and $z^2 + bz + 8$, but cannot be factorised into two real quadratic factors of the form $z^2 + az + 16$ and $z^2 + bz + 4$.

4 Find real numbers a and b such that $x^4 - 4x^2 + 8x - 4 = (x^2 + ax + 2)(x^2 + bx - 2)$.
Hence find $x \in \mathbb{R}$ such that $x^4 + 8x = 4x^2 + 4$.

5 Find real numbers c and d such that $x^4 - 10x^2 + 1 = (x^2 + cx + 1)(x^2 + dx + 1)$.
Hence find $x \in \mathbb{R}$ such that $x^4 + 1 = 10x^2$.

Example 10 **Self Tutor**

$(2x+3)$ and $(x-1)$ are factors of $2x^4+ax^3-3x^2+bx+3$.

Find constants a and b, and all zeros of the polynomial.

Since $(2x+3)$ and $(x-1)$ are factors,

The coefficient of x^4 is $2\times1\times1=2$ ↓ This must be -1 so the constant term is $3\times-1\times-1=3$ ↓

$$2x^4+ax^3-3x^2+bx+3=(2x+3)(x-1)(x^2+cx-1) \quad \text{for some } c$$

$$\begin{aligned}&=(2x^2+x-3)(x^2+cx-1)\\&=2x^4+2cx^3-2x^2\\&\quad+\ x^3+cx^2-\ x\\&\quad\quad-3x^2-3cx+3\\&=2x^4+(2c+1)x^3+(c-5)x^2+(-1-3c)x+3\end{aligned}$$

Equating coefficients gives $2c+1=a$, $c-5=-3$, and $-1-3c=b$

$\therefore\ c=2$

$\therefore\ a=5$ and $b=-7$

$$\begin{aligned}\therefore\ P(x)&=2x^4+5x^3-3x^2-7x+3\\&=(2x+3)(x-1)(x^2+2x-1)\end{aligned}$$

Now x^2+2x-1 has zeros $\dfrac{-2\pm\sqrt{4-4(1)(-1)}}{2}=\dfrac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$

$\therefore\ P(x)$ has zeros $-\frac{3}{2}$, 1, and $-1\pm\sqrt{2}$.

6 **a** $(2z-3)$ is a factor of $2z^3+az^2-17z+12$. Find $a\in\mathbb{R}$ and all zeros of the cubic.

b $(3z+2)$ is a factor of $3z^3-z^2+(a-4)z+a$. Find $a\in\mathbb{R}$ and all the zeros of the cubic.

7 **a** $(2x+1)$ and $(x-2)$ are factors of $P(x)=2x^4+ax^3+bx^2+18x+8$.
Find constants a and b, and all zeros of $P(x)$.

b $(x+3)$ and $(2x-1)$ are factors of $2x^4+ax^3+bx^2+ax+3$.
Find constants a and b, and hence determine all zeros of the quartic.

8 Find k given that $4x^4-4x^3+kx^2-6x+9$ is a perfect square.

9 Write $x^4-2x^3-3x^2+4x+3$ in the form $a(x^2-x)^2+b(x^2-x)+c$ where a, b, c are constants.
Hence find all zeros of the polynomial.

10 **a** $x^3+x^2-16x+k$, $k\in\mathbb{R}$, has two identical linear factors. Find k, and hence factorise the cubic into linear factors.

b x^3+3x^2-9x+c, $c\in\mathbb{R}$, has two identical linear factors. Prove that c is either 5 or -27, and factorise the cubic into linear factors in each case.

c $3x^3+4x^2-x+m$, $m\in\mathbb{R}$, has two identical linear factors. Find the possible values of m, and find the zeros of the polynomial in each case.

E POLYNOMIAL DIVISION

We can use a division process to find the result when one polynomial is divided by another. This is usually only useful if the degree of the denominator is less than or equal to the degree of the numerator.

DIVISION BY LINEARS

Consider $(2x^2 + 3x + 4)(x + 2) + 7$.

If we expand this expression we obtain $(2x^2 + 3x + 4)(x + 2) + 7 = 2x^3 + 7x^2 + 10x + 15$.

Dividing both sides by $(x + 2)$, we obtain

$$\frac{2x^3 + 7x^2 + 10x + 15}{x + 2} = \frac{(2x^2 + 3x + 4)(x + 2) + 7}{x + 2}$$

$$= \frac{(2x^2 + 3x + 4)(x + 2)}{x + 2} + \frac{7}{x + 2}$$

$$= 2x^2 + 3x + 4 + \frac{7}{x + 2}$$

where $x + 2$ is the divisor,
$2x^2 + 3x + 4$ is the quotient,
and 7 is the remainder.

If $P(x)$ is divided by $ax + b$ until a constant remainder R is obtained, then

$$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$$ where $ax + b$ is the **divisor**, $D(x)$,
$Q(x)$ is the **quotient**,
and R is the **remainder**.

Notice that $P(x) = Q(x) \times (ax + b) + R$.

DIVISION ALGORITHM

We can divide a polynomial by another polynomial using an algorithm similar to that used for division of whole numbers:

Step 1: What do we multiply x by to get $2x^3$?
The answer is $2x^2$, so we expand $2x^2(x + 2) = 2x^3 + 4x^2$ and write it underneath.

Step 2: We then subtract $2x^3 + 4x^2$ from $2x^3 + 7x^2$. The answer is $3x^2$.

Step 3: Bring down the $10x$ to obtain $3x^2 + 10x$.

Return to *Step 1* with the question:

What must we multiply x by to get $3x^2$?
The answer is $3x$, and $3x(x + 2) = 3x^2 + 6x$.

We continue the process until we are left with a constant.

$$\begin{array}{r|l} & 2x^2 + 3x + 4 \\ \hline x + 2 & 2x^3 + 7x^2 + 10x + 15 \\ & -(2x^3 + 4x^2) \\ \hline & 3x^2 + 10x \\ & -(3x^2 + 6x) \\ \hline & 4x + 15 \\ & -(4x + 8) \\ \hline & 7 \end{array}$$

We find that $\frac{2x^3 + 7x^2 + 10x + 15}{x + 2} = 2x^2 + 3x + 4 + \frac{7}{x + 2}$.

The division algorithm can also be performed by leaving out the variable, as shown alongside.

$$
\begin{array}{rr|rrrr}
 & & & 2 & 3 & 4 \\
\hline
1 & 2 & 2 & 7 & 10 & 15 \\
 & & -(2 & 4) & \downarrow & \\
\hline
 & & & 3 & 10 & \\
 & & -(3 & 6) & & \downarrow \\
\hline
 & & & & 4 & 15 \\
 & & & -(4 & 8) & \\
\hline
 & & & & & 7
\end{array}
$$

Example 11 **Self Tutor**

Perform the division $\dfrac{x^4+2x^2-1}{x+3}$. Hence write x^4+2x^2-1 in the form $Q(x)\times(x+3)+R$.

$$
\begin{array}{rr|rrrrr}
 & & & x^3 & -3x^2 & +11x & -33 \\
\hline
x+3 & & x^4 & +0x^3 & +2x^2 & +0x & -1 \\
 & & -(x^4 & +3x^3) & \downarrow & & \\
\hline
 & & & -3x^3 & +2x^2 & & \\
 & & & -(-3x^3 & -9x^2) & \downarrow & \\
\hline
 & & & & 11x^2 & +0x & \\
 & & & & -(11x^2 & +33x) & \downarrow \\
\hline
 & & & & & -33x & -1 \\
 & & & & & -(-33x & -99) \\
\hline
 & & & & & & 98
\end{array}
$$



$\therefore \dfrac{x^4+2x^2-1}{x+3} = x^3-3x^2+11x-33+\dfrac{98}{x+3}$

$\therefore x^4+2x^2-1 = (x^3-3x^2+11x-33)(x+3)+98$

EXERCISE 5E.1

1 Find the quotient and remainder for the following, and hence write the division in the form $P(x) = Q(x)\,D(x)+R$, where $D(x)$ is the divisor.

a $\dfrac{x^2+2x-3}{x+2}$ **b** $\dfrac{x^2-5x+1}{x-1}$ **c** $\dfrac{2x^3+6x^2-4x+3}{x-2}$

2 Perform the following divisions, and hence write the division in the form $P(x) = Q(x)\,D(x)+R$.

a $\dfrac{x^2-3x+6}{x-4}$ **b** $\dfrac{x^2+4x-11}{x+3}$ **c** $\dfrac{2x^2-7x+2}{x-2}$

d $\dfrac{2x^3+3x^2-3x-2}{2x+1}$ **e** $\dfrac{3x^3+11x^2+8x+7}{3x-1}$ **f** $\dfrac{2x^4-x^3-x^2+7x+4}{2x+3}$

3 Perform the divisions:

a $\dfrac{x^2+5}{x-2}$ **b** $\dfrac{2x^2+3x}{x+1}$ **c** $\dfrac{3x^2+2x-5}{x+2}$

d $\dfrac{x^3+2x^2-5x+2}{x-1}$ **e** $\dfrac{2x^3-x}{x+4}$ **f** $\dfrac{x^3+x^2-5}{x-2}$

Example 12 **Self Tutor**

Carry out the division, and hence factorise the numerator into linear factors:

$$\frac{2x^3 - x^2 - 13x - 6}{x - 3}$$

$$\begin{array}{r} 2x^2 + 5x + 2 \\ x-3 \;\overline{)\; 2x^3 - x^2 - 13x - 6} \\ -(2x^3 - 6x^2) \\ \hline 5x^2 - 13x \\ -(5x^2 - 15x) \\ \hline 2x - 6 \\ -(2x - 6) \\ \hline 0 \end{array}$$

$\therefore \quad \dfrac{2x^3 - x^2 - 13x - 6}{x - 3} = 2x^2 + 5x + 2 \qquad \{\text{the remainder is } 0\}$

$\therefore \quad 2x^3 - x^2 - 13x - 6 = (x - 3)(2x^2 + 5x + 2)$

$= (x - 3)(2x + 1)(x + 2)$

4 Carry out the division, and hence factorise the numerator into linear factors:

a $\dfrac{3x^3 - 5x^2 - 4x + 4}{x + 1}$ **b** $\dfrac{2x^3 + 7x^2 - 10x - 24}{x - 2}$ **c** $\dfrac{4x^3 + 21x^2 + 2x - 15}{x + 5}$

5 **a** Use the division process to divide $\dfrac{x^3 + a^3}{x + a}$.

Check your answer by expanding and simplifying $(x^2 - ax + a^2)(x + a)$.

b Hence factorise into two factors:

i $x^3 + 1$ **ii** $x^3 + 8$ **iii** $x^3 + 125$

6 **a** Carry out the following divisions:

i $\dfrac{x^2 - a^2}{x - a}$ **ii** $\dfrac{x^3 - a^3}{x - a}$ **iii** $\dfrac{x^4 - a^4}{x - a}$ **iv** $\dfrac{x^5 - a^5}{x - a}$

b Hence deduce the factorisation of $x^n - a^n$ into two factors, one of which is linear.

c Write as a product of two factors, one of which is linear:

i $x^3 - 8$ **ii** $x^4 - 1$ **iii** $x^4 - 16$ **iv** $x^5 - 243$

ACTIVITY 2 SYNTHETIC DIVISION

Click on the icon to learn an alternative procedure for performing division of a polynomial by a linear factor.

DIVISION BY QUADRATICS

As with division by linears, we can use the **division algorithm** to divide polynomials by quadratics. The division process stops when the remainder has degree less than that of the divisor, so

If $P(x)$ is divided by ax^2+bx+c then

$$\frac{P(x)}{ax^2+bx+c} = Q(x) + \frac{ex+f}{ax^2+bx+c}$$

where ax^2+bx+c is the **divisor**,
$Q(x)$ is the **quotient**,
and $ex+f$ is the **remainder**.

The remainder will be linear if $e \neq 0$, and constant if $e = 0$.

Example 13 **Self Tutor**

Find the quotient and remainder for $\dfrac{x^4+4x^3-x+1}{x^2-x+1}$.

Hence write x^4+4x^3-x+1 in the form $Q(x) \times (x^2-x+1) + R(x)$.

$$\begin{array}{r} x^2+5x+4 \\ x^2-x+1 \overline{\big)\; x^4+4x^3+0x^2-\ x+1} \\ \underline{-(x^4-\ x^3+\ x^2)\quad\downarrow} \\ 5x^3-\ x^2-\ x \\ \underline{-(5x^3-5x^2+5x)\quad\downarrow} \\ 4x^2-6x+1 \\ \underline{-(4x^2-4x+4)} \\ -2x-3 \end{array}$$

The quotient is x^2+5x+4 and the remainder is $-2x-3$.

$\therefore\ x^4+4x^3-x+1 = (x^2+5x+4)(x^2-x+1) - 2x - 3$

EXERCISE 5E.2

1 Find the quotient and remainder for:

a $\dfrac{x^3+2x^2+x-3}{x^2+x+1}$ **b** $\dfrac{3x^2-x}{x^2-1}$ **c** $\dfrac{3x^3+x-1}{x^2+1}$ **d** $\dfrac{x-4}{x^2+2x-1}$

2 Carry out the following divisions and write each in the form $P(x) = Q(x)\,D(x) + R(x)$:

a $\dfrac{x^2-x+1}{x^2+x+1}$ **b** $\dfrac{x^3}{x^2+2}$ **c** $\dfrac{x^4+3x^2+x-1}{x^2-x+1}$

d $\dfrac{2x^3-x+6}{(x-1)^2}$ **e** $\dfrac{x^4}{(x+1)^2}$ **f** $\dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}$

3 Suppose $P(x) = (x-2)(x^2+2x+3)+7$. Find the quotient and remainder when $P(x)$ is divided by $x-2$.

4 Suppose $f(x) = (x-1)(x+2)(x^2-3x+5)+15-10x$. Find the quotient and remainder when $f(x)$ is divided by x^2+x-2.

ACTIVITY 3 POLYNOMIAL DIVISION USING A GRID

We can perform polynomial division using the same grid structure that we used for multiplication.

This method is very fast when dividing by quadratics.



F THE REMAINDER THEOREM

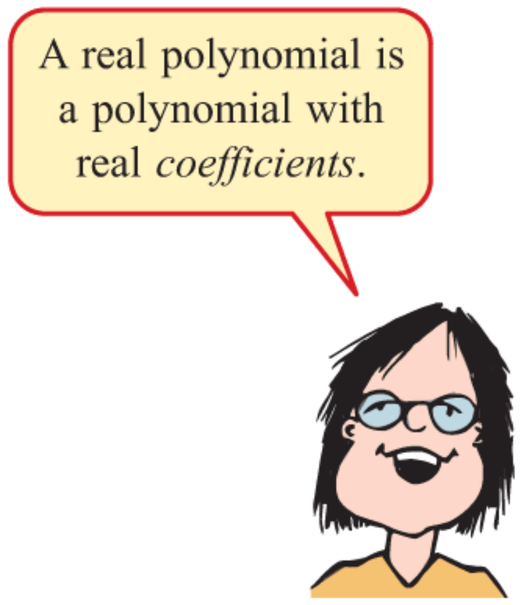
Consider the real cubic polynomial $P(x) = x^3 + 5x^2 - 11x + 3$.

If we divide $P(x)$ by $x - 2$, we find that

$$\frac{x^3 + 5x^2 - 11x + 3}{x - 2} = x^2 + 7x + 3 + \frac{9}{x - 2}$$ ← remainder

So, when $P(x)$ is divided by $x - 2$, the remainder is 9.

Notice that $P(2) = 8 + 20 - 22 + 3$
$= 9$, which is the remainder.



By considering other examples like the one above, we formulate the **Remainder theorem**:

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

Proof:

By the division algorithm, $P(x) = Q(x)(x - k) + R$
Letting $x = k$, $P(k) = Q(k) \times 0 + R$
$\therefore$ $P(k) = R$

When using the Remainder theorem, it is important to realise that the following statements are equivalent:

- $P(x) = (x - k)Q(x) + R$
- $P(k) = R$
- $P(x)$ divided by $x - k$ leaves a remainder of R.

Example 14 Self Tutor

Use the Remainder theorem to find the remainder when $x^4 - 3x^3 + x - 4$ is divided by $x + 2$.

If $P(x) = x^4 - 3x^3 + x - 4$, then
$P(-2) = (-2)^4 - 3(-2)^3 + (-2) - 4$
$= 16 + 24 - 2 - 4$
$= 34$

$\therefore$ when $x^4 - 3x^3 + x - 4$ is divided by $x + 2$, the remainder is 34. {Remainder theorem}

EXERCISE 5F

1 For $P(x)$ a real polynomial, write two equivalent statements for each of:

a If $P(2) = 7$, then .....

b If $P(x) = Q(x)(x+3) - 8$, then .....

c If $P(x)$ divided by $x-5$ has a remainder of 11 then .....

2 Without performing division, find the remainder when:

a $x^3 + 2x^2 - 7x + 5$ is divided by $x - 1$

b $2x^3 - 8x + 11$ is divided by $x + 3$

c $x^4 - 2x^2 + 3x - 1$ is divided by $x + 2$.

3 Find $a \in \mathbb{R}$ given that:

a when $x^3 - 2x + a$ is divided by $x - 2$, the remainder is 7

b when $2x^3 + x^2 + ax - 5$ is divided by $x + 1$, the remainder is -8.

Example 15 **Self Tutor**

When $x^3 + 4x^2 + ax + b$ is divided by $x - 2$ the remainder is 20, and when divided by $x + 5$ the remainder is 6. Find a and b.

Let $P(x) = x^3 + 4x^2 + ax + b$

$\therefore P(2) = 20$ and $P(-5) = 6$ {Remainder theorem}

So, $(2)^3 + 4(2)^2 + a(2) + b = 20$

$\therefore 8 + 16 + 2a + b = 20$

$\therefore 2a + b = -4$ .... (1)

and $(-5)^3 + 4(-5)^2 + a(-5) + b = 6$

$\therefore -125 + 100 - 5a + b = 6$

$\therefore -5a + b = 31$ .... (2)

Solving simultaneously:

$$\begin{array}{rll} & 2a + b = -4 & \{(1)\} \\ & 5a - b = -31 & \{-(2)\} \\ \hline \text{Adding,} & 7a = -35 & \end{array}$$

$\therefore a = -5$

Substituting $a = -5$ into (1), $2(-5) + b = -4$

$\therefore b = 6$

4 When $x^3 + 2x^2 + ax + b$ is divided by $x - 1$ the remainder is 4, and when divided by $x + 2$ the remainder is 16. Find constants a and b.

5 $2x^n + ax^2 - 6$ leaves a remainder of -7 when divided by $x - 1$, and 129 when divided by $x + 3$. Find a and n given that $n \in \mathbb{Z}^+$.

6 Consider $f(x) = 2x^3 + ax^2 - 3x + b$. When $f(x)$ is divided by $x + 1$, the remainder is 7. When $f(x)$ is divided by $x - 2$, the remainder is 28. Find the remainder when $f(x)$ is divided by $x + 3$.

7 **a** Suppose a polynomial $P(x)$ is divided by $2x - 1$ until a constant remainder R is obtained. Show that $R = P(\frac{1}{2})$.

Hint: $P(x) = Q(x)(2x - 1) + R$.

b Hence find the remainder when:

i $4x^2 - 10x + 1$ is divided by $2x - 1$ **ii** $2x^3 - 5x^2 + 8$ is divided by $2x - 1$.

8 Find the remainder when $4x^3 + 7x - 3$ is divided by $2x + 1$.

9 When $2x^3 + ax^2 + bx + 4$ is divided by $x + 1$ the remainder is -5, and when divided by $2x - 1$ the remainder is 10. Find a and b.

Example 16 **Self Tutor**

When $P(x)$ is divided by $x^2 - 3x + 7$, the quotient is $x^2 + x - 1$ and the remainder $R(x)$ is unknown.

When $P(x)$ is divided by $x - 2$ the remainder is 29. When $P(x)$ is divided by $x + 1$ the remainder is -16.

Find $R(x)$ in the form $ax + b$.

When the divisor is $x^2 - 3x + 7$, $P(x) = \underbrace{(x^2 + x - 1)}_{Q(x)}\underbrace{(x^2 - 3x + 7)}_{D(x)} + \underbrace{ax + b}_{R(x)}$.

Now $P(2) = 29$ {Remainder theorem}

$$\therefore (2^2 + 2 - 1)(2^2 - 6 + 7) + 2a + b = 29$$
$$\therefore (5)(5) + 2a + b = 29$$
$$\therefore 2a + b = 4 \quad \ldots. (1)$$

Also, $P(-1) = -16$ {Remainder theorem}

$$\therefore ((-1)^2 + (-1) - 1)((-1)^2 - 3(-1) + 7) + (-a + b) = -16$$
$$\therefore (-1)(11) - a + b = -16$$
$$\therefore -a + b = -5 \quad \ldots. (2)$$

Solving (1) and (2) simultaneously gives $a = 3$ and $b = -2$.

$\therefore R(x) = 3x - 2$.

10 When $P(z)$ is divided by $z^2 - 3z + 2$ the remainder is $4z - 7$.
Find the remainder when $P(z)$ is divided by:

a $z - 1$ **b** $z - 2$.

11 When $P(x)$ is divided by $x^2 - x + 3$, the quotient is $x^2 + 4x - 1$ and the remainder $R(x)$ is unknown.
When $P(x)$ is divided by $x - 1$, the remainder is 10. When $P(x)$ is divided by $x + 3$, the remainder is -42.
Find $R(x)$ in the form $ax + b$.

12 When $P(z)$ is divided by $z + 1$ the remainder is -8, and when divided by $z - 3$ the remainder is 4. Find the remainder when $P(z)$ is divided by $(z - 3)(z + 1)$.

13 If $P(x)$ is divided by $(x - a)(x - b)$, where $a \neq b$, $a, b \in \mathbb{R}$, prove that the remainder is:

$$\left(\frac{P(b) - P(a)}{b - a}\right) \times (x - a) + P(a).$$

G THE FACTOR THEOREM

For any polynomial $P(x)$, k is a zero of $P(x) \Leftrightarrow (x-k)$ is a factor of $P(x)$.

Proof:

k is a zero of $P(x) \Leftrightarrow P(k) = 0$ {definition of a zero}
$\Leftrightarrow R = 0$ {Remainder theorem}
$\Leftrightarrow P(x) = Q(x)(x-k)$ {division algorithm}
$\Leftrightarrow (x-k)$ is a factor of $P(x)$ {definition of a factor}

The **Factor theorem** says that if 2 is a zero of $P(x)$ then $(x-2)$ is a factor of $P(x)$, and vice versa.

Example 17 **Self Tutor**

Find k given that $(x-2)$ is a factor of $x^3 + kx^2 - 3x + 6$.
Hence fully factorise $x^3 + kx^2 - 3x + 6$.

Let $P(x) = x^3 + kx^2 - 3x + 6$

Since $(x-2)$ is a factor, $P(2) = 0$ {Factor theorem}

$\therefore (2)^3 + k(2)^2 - 3(2) + 6 = 0$
$\therefore 8 + 4k = 0$
$\therefore k = -2$

We now use division to find the other factors of $P(x)$:

Division algorithm

$$\begin{array}{r} x^2 \quad -3 \\ x-2 \;\overline{)\; x^3 - 2x^2 - 3x + 6} \\ \underline{-(x^3 - 2x^2)} \\ 0 \quad -3x + 6 \\ \underline{-(-3x + 6)} \\ 0 \end{array}$$

or

Synthetic division

$$\begin{array}{r|cccc} 2 & 1 & -2 & -3 & 6 \\ & 0 & 2 & 0 & -6 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

$\therefore P(x) = (x-2)(x^2 - 3)$
$= (x-2)(x+\sqrt{3})(x-\sqrt{3})$

EXERCISE 5G

1 Use the Factor theorem to determine whether:

a $(x-2)$ is a factor of $x^3 - 3x^2 + 3x - 2$

b $(x+1)$ is a factor of $x^4 - 2x^3 + 3x^2 - 4$

c $(x+3)$ is a factor of $2x^3 + 5x^2 + x + 12$.

2 Find a given that:

a $(x-1)$ is a factor of $2x^3 - 3x^2 + ax - 4$

b $(x+2)$ is a factor of $3x^5 + ax^3 - 7x + 3$.

3 Find the constant k and hence factorise the polynomial if:

a $2x^3 + x^2 + kx - 4$ has the factor $(x + 2)$

b $x^4 - 3x^3 - kx^2 + 6x$ has the factor $(x - 3)$.

4 $(x - 4)$ is a factor of $P(x) = 3x^3 - 17x^2 + kx + 8$.

a Find k.

b Write $P(x)$ in the form $P(x) = (x - 4)(ax^2 + bx + c)$.

c Hence find the roots of $P(x) = 0$.

5 Find constants a and b given that $2x^3 + ax^2 + bx + 5$ has factors $(x - 1)$ and $(x + 5)$.

6 Suppose 3 is a zero of $P(z) = z^3 - z^2 + (k - 5)z + (k^2 - 1)$.
Find the possible values of $k \in \mathbb{R}$ and all the corresponding zeros of $P(z)$.

7 Show that $(z - 2)$ is a factor of $P(z) = z^3 + mz^2 + (3m - 2)z - 10m - 4$ for all values of $m \in \mathbb{R}$. For what value of m is $(z - 2)^2$ a factor of $P(z)$?

8 $(x + 3)$ is a factor of $P(x) = 2x^3 + 9x^2 + ax + b$.
When $P(x)$ is divided by $x + 4$, the remainder is -18.

a Find a and b.

b Find the remainder when $P(x)$ is divided by $x - 2$.

c Write $P(x)$ in the form $P(x) = (x + 3)(px^2 + qx + r)$.

d Find the zeros of $P(x)$.

9 $(2x - 1)$ is a factor of $P(x) = 2x^3 + ax^2 - 8x + b$.
When $P(x)$ is divided by $x - 1$, the remainder is 3.

a Find a and b.

b Find the irrational roots of $P(x) = 0$, giving your answer in the form $x = p \pm \sqrt{q}$ where $p, q \in \mathbb{Z}$.

10 Prove that $(x + 1)$ is a factor of $x^n + 1$, $n \in \mathbb{Z} \Leftrightarrow n$ is odd.

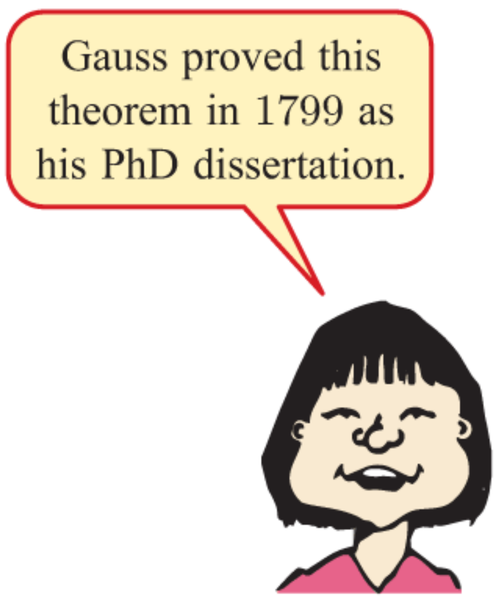
11 Find the real number a such that $(x - 1 - a)$ is a factor of $P(x) = x^3 - 3ax - 9$.

H THE FUNDAMENTAL THEOREM OF ALGEBRA

The theorems we have just seen for real polynomials can be generalised in the **Fundamental Theorem of Algebra**:

> If $P(x)$ is a polynomial of degree n, then $P(x)$ has exactly n zeros, each of which may be written in the form $a + bi$ where $a, b \in \mathbb{R}$, and some of which may be repeated.

Using the Fundamental Theorem of Algebra, the following properties of **real polynomials** can be established.



- Every **real** polynomial of degree n can be factorised into n complex linear factors, some of which may be repeated.
- Every real polynomial can be expressed as a product of **real** linear and **real** irreducible quadratic factors (where $\Delta < 0$).
- Every **real** polynomial of degree n has exactly n zeros, some of which may be repeated.
- If $p+qi$ $(q \neq 0)$ is a zero of a **real** polynomial then its complex conjugate $p-qi$ is also a zero.
- Every **real** polynomial of odd degree has at least one real zero.

Example 18 **Self Tutor**

Suppose $-3+i$ is a zero of $P(x) = ax^3 + 9x^2 + ax - 30$ where a is real.

Find a and hence find all zeros of the cubic.

Since $P(x)$ is real, both $-3+i$ and $-3-i$ are zeros.

These have sum $= -6$ and product $= (-3+i)(-3-i) = 10$.

$\therefore$ the zeros $-3 \pm i$ come from the quadratic $x^2 + 6x + 10$.

The coefficient of x^3 is a. The constant term is $10 \times (-3) = -30$.

Consequently, $ax^3 + 9x^2 + ax - 30 = (x^2 + 6x + 10)(ax - 3)$

$= ax^3 + (6a-3)x^2 + (10a - 18)x - 30$

Equating coefficients of x^2 gives: $6a - 3 = 9$ $\therefore$ $a = 2$

Equating coefficients of x gives: $10a - 18 = a$ $\therefore$ $a = 2$

$\therefore$ $a = 2$ and the linear factor is $(2x - 3)$

$\therefore$ $P(x)$ has zeros $-3 \pm i$ and $\frac{3}{2}$.

EXERCISE 5H

1 $2 - i\sqrt{3}$ is a root of $x^2 + ax + b = 0$, where $a, b \in \mathbb{R}$. Find a and b.

2 Find all real polynomials of degree 3 with zeros $-\frac{1}{2}$ and $1 - 3i$.

3 $p(x)$ is a real cubic polynomial for which $p(1) = p(2+i) = 0$ and $p(0) = -20$.
Find $p(x)$ in expanded form.

4 $2 - 3i$ is a zero of $P(z) = z^3 + pz + q$ where p and q are real.

a Using conjugate pairs, find p and q and the other two zeros.

b Check your answer by solving for p and q using $P(2-3i) = 0$.

5 $3+i$ is a root of $z^4 - 2z^3 + az^2 + bz + 10 = 0$, where a and b are real. Find a and b and the other roots of the equation.

6 $2+i$ is a root of $x^4 + ax^3 + 8x^2 + 3ax + b = 0$, $a, b \in \mathbb{R}$. Find a and b and the other roots of the equation.

Example 19 **Self Tutor**

One zero of $ax^3 + (a+1)x^2 + 10x + 15$, $a \in \mathbb{R}$, is purely imaginary.

Find a and the zeros of the polynomial.

Let the purely imaginary zero be bi, $b \neq 0$.

Since $P(x)$ is real, both bi and $-bi$ are zeros.

These have sum $= 0$ and product $= -b^2i^2 = b^2$

$\therefore$ the zeros $\pm bi$ come from the quadratic $x^2 + b^2$.

The coefficient of x^3 is a.

The constant term is $b^2 \times \frac{15}{b^2} = 15$.

Consequently, $ax^3 + (a+1)x^2 + 10x + 15 = (x^2 + b^2)(ax + \frac{15}{b^2})$

$$= ax^3 + \left(\frac{15}{b^2}\right)x^2 + ab^2x + 15$$

Equating coefficients of x^2 gives: $a + 1 = \frac{15}{b^2}$ .... (1)

Equating coefficients of x gives: $ab^2 = 10$ .... (2)

$\therefore$ $ab^2 + b^2 = 15$ {using (1)}

$\therefore$ $10 + b^2 = 15$ {using (2)}

$\therefore$ $b^2 = 5$

$\therefore$ $b = \pm\sqrt{5}$

In (2), since $b^2 = 5$, $5a = 10$ $\therefore$ $a = 2$.

The linear factor $ax + \frac{15}{b^2}$ becomes $2x + 3$

$\therefore$ $a = 2$ and the zeros are $\pm i\sqrt{5}$, $-\frac{3}{2}$.

7 One zero of $P(z) = z^3 + az^2 + 3z + 9$, $a \in \mathbb{R}$, is purely imaginary. Find a, and hence factorise $P(z)$ into the product of linear factors.

8 At least one zero of $P(x) = 3x^3 + kx^2 + 15x + 10$, $k \in \mathbb{R}$, is purely imaginary. Find k, and hence factorise $P(x)$ into the product of linear factors.

9 A real polynomial $P(x) = x^4 + ax^3 + bx^2 + cx - 10$ has two integer zeros p and q, and a complex zero $1 + ki$, $k \in \mathbb{Z}$, $k \neq 0$.

a Use the zero $1 + ki$ to find a real quadratic factor of $P(x)$.

b State all possible values of k.

c Use p and q to write another real quadratic factor of $P(x)$. Hence list all possible values of pq.

d Given that $p + q = -1$, show that there is only one possible value for pq. Hence find all zeros of $P(x)$.

I SUM AND PRODUCT OF ROOTS THEOREM

INVESTIGATION 1 SUM AND PRODUCT OF ROOTS

We have seen that for the quadratic equation $ax^2 + bx + c = 0, \ a \neq 0,$ the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. In this Investigation we seek a rule for the sum and product of roots of higher order polynomial equations.

What to do:

1 Expand and collect like terms in x:

a $P_2(x) = a(x-\alpha)(x-\beta)$

b $P_3(x) = a(x-\alpha)(x-\beta)(x-\gamma)$
$= P_2(x)(x-\gamma)$

c $P_4(x) = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
$= P_3(x)(x-\delta)$

2 For each case in **1**, compare:

a the constant term with the product of the roots of $P_n(x) = 0$

b the coefficient of x^{n-1} with the sum of the roots of $P_n(x) = 0$.

3 Generalise your observations in **2** for the polynomial equation $\sum_{r=0}^{n} a_r x^r = 0, \ a_n \neq 0.$

From the **Investigation** you should have established that:

For the polynomial equation $\sum_{r=0}^{n} a_r x^r = 0, \ a_n \neq 0$

the sum of the roots is $-\frac{a_{n-1}}{a_n}$, and the product of the roots is $\frac{(-1)^n a_0}{a_n}$.

We can explain this result as follows:

Consider the polynomial equation $a_n(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)....(x-\alpha_n) = 0$ with roots $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$.

Expanding the LHS we have

$$\begin{aligned}
&a_n(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)....(x-\alpha_n) \\
&= a_n(x^2 - [\alpha_1+\alpha_2]x + (-1)^2\alpha_1\alpha_2)(x-\alpha_3)(x-\alpha_4)....(x-\alpha_n) \\
&= a_n(x^3 - [\alpha_1+\alpha_2+\alpha_3]x^2 + ... + (-1)^3\alpha_1\alpha_2\alpha_3)(x-\alpha_4)....(x-\alpha_n) \\
&\quad \{\text{the term of order } x \text{ is no longer important as it will not contribute to} \\
&\quad \ \text{either the } x^{n-1} \text{ term or the constant term}\} \\
&\quad \ \vdots \\
&= a_n(x^n - [\alpha_1+\alpha_2+\alpha_3+....+\alpha_n]x^{n-1} + ... + (-1)^n\alpha_1\alpha_2\alpha_3....\alpha_n) \\
&= a_n x^n - a_n[\alpha_1+\alpha_2+\alpha_3+....+\alpha_n]x^{n-1} + ... + (-1)^n\alpha_1\alpha_2\alpha_3....\alpha_n a_n
\end{aligned}$$

Equating coefficients,

$$a_{n-1} = -a_n[\alpha_1+\alpha_2+\alpha_3+....+\alpha_n] \quad \text{and} \quad a_0 = (-1)^n\alpha_1\alpha_2\alpha_3....\alpha_n a_n$$

$$\therefore \ -\frac{a_{n-1}}{a_n} = \alpha_1+\alpha_2+\alpha_3+....+\alpha_n \quad \text{and} \quad \frac{(-1)^n a_0}{a_n} = \alpha_1\alpha_2\alpha_3....\alpha_n$$

Example 20 **Self Tutor**

Find the sum and product of the roots of $2x^3 - 7x^2 + 8x - 1 = 0$.

The sum of the roots $= -\dfrac{(-7)}{2}$

$= \frac{7}{2}$

The polynomial equation has degree 3.

$\therefore$ the product of the roots $= \dfrac{(-1)^3(-1)}{2}$

$= \frac{1}{2}$

EXERCISE 5I

1 Find the sum and product of the roots of:

a $2x^2 - 3x + 4 = 0$ **b** $3x^3 - 4x^2 + 8x - 5 = 0$

c $x^4 - x^3 + 2x^2 + 3x - 4 = 0$ **d** $2x^5 - 3x^4 + x^2 - 8 = 0$

e $x^7 - x^5 + 2x - 9 = 0$ **f** $x^6 - 1 = 0$

2 A real cubic polynomial $P(x)$ has zeros $3 \pm \sqrt{2}$ and $\frac{2}{3}$, and its leading coefficient is 6. Find:

a the sum and product of its zeros **b** the coefficient of x^2

c the constant term.

3 A real cubic polynomial $P(x)$ has zeros $1 \pm i\sqrt{2}$ and 1, and its leading coefficient is $\frac{2}{3}$. Find:

a the sum and product of its zeros **b** the coefficient of x^2

c the constant term.

4 A real polynomial of degree 4 has leading coefficient -1. Its zeros are -2, 3, and $\sqrt{k} \pm 1$, and its graph has y-intercept 18. Find:

a k **b** the coefficient of x^3.

Example 21 **Self Tutor**

A real polynomial has the form $P(x) = 3x^4 - 12x^3 + cx^2 + dx + e$. The graph of $y = P(x)$ has y-intercept 180, cuts the x-axis at 2 and 6, and does not meet the x-axis anywhere else.

Suppose the other two zeros are $m \pm ni$, $n > 0$. Use the sum and product formulae to find m and n.

If the other two zeros are $m \pm ni$, the sum of the zeros is

$$2 + 6 + (m + ni) + (m - ni) = -\frac{-12}{3}$$

$$\therefore \quad 2m + 8 = 4$$

$$\therefore \quad m = -2$$

The constant term e is the y-intercept.

$$\therefore \quad e = 180$$

The product of the zeros is $2 \times 6 \times (-2+ni)(-2-ni) = \dfrac{(-1)^4 180}{3} = 60$

$\therefore \ 12(4+n^2) = 60$

$\therefore \ 4+n^2 = 5$

$\therefore \ n^2 = 1$

$\therefore \ n = 1 \quad \{\text{as } n > 0\}$

So, $m = -2$ and $n = 1$.

5 A real polynomial of degree 5 has leading coefficient 2, and the coefficient of x^4 is 3. Its graph has y-intercept 5, and it cuts the x-axis at $\frac{1}{2}$ and $1 \pm \sqrt{2}$ only.
Suppose the other two zeros are $m \pm ni$, $n > 0$. Use the sum and product formulae to find m and n.

6 A real quartic polynomial has leading coefficient 1 and constant term 25. Its zeros have the form $a \pm i$ and $3 \pm a$, where $a \in \mathbb{R}$. Find the possible values that a may take.

7 $x^3 - px^2 + qx - r = 0$ has non-zero roots p, q, and r, where $p, q, r \in \mathbb{R}$.

a Show that $q = -r$ and $p = -\dfrac{1}{r}$.

b Hence find p, q, and r.

8 $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ where $a \neq 0$, $b \neq 0$, $c \neq 0$, have a single common root. Prove that the other roots satisfy $x^2 + cx + ab = 0$.

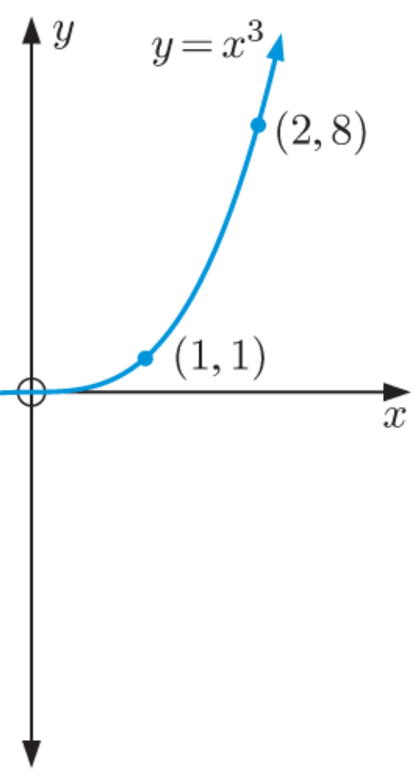
9 $f(x)$ is a polynomial of degree n with leading coefficient 1. The sum of its roots is 5, and the product of its roots is -3. Suppose $g(x) = [f(x)]^2$. Find:

a the degree of $g(x)$ **b** the sum of the roots of $g(x)$

c the product of the roots of $g(x)$ **d** the y-intercept of $y = g(x)$.

J GRAPHING CUBIC FUNCTIONS

The most basic cubic function is $y = x^3$. Its graph is shown alongside.



Notice that:

- The graph is horizontal at $(0, 0)$.
- As x gets very large and positive, y also gets very large and positive. We write:
  As $x \to \infty$, $y \to \infty$.
- As x gets very large and negative, y also gets very large and negative. We write:
  As $x \to -\infty$, $y \to -\infty$.



INVESTIGATION 2 FAMILIES OF CUBICS

In this Investigation we will graph some different types of cubic functions, and observe the geometrical significance of each type.

What to do:

1 **a** Use technology to help you sketch the graphs of:

GRAPHING PACKAGE



i $y = x^3$, $y = 2x^3$, $y = \frac{1}{2}x^3$, and $y = \frac{1}{3}x^3$

ii $y = x^3$ and $y = -x^3$

iii $y = -x^3$, $y = -2x^3$, $y = -\frac{1}{2}x^3$, and $y = -\frac{1}{10}x^3$

b For the curve $y = ax^3$, what is the geometrical significance of a? You should comment on both the sign and the size of a.

2 **a** Use technology to help you sketch the graphs of:

i $y = x^3$ and $y = (x-2)^3 + 3$

ii $y = 2x^3$ and $y = 2(x+1)^3 - 4$

iii $y = -\frac{1}{2}x^3$ and $y = -\frac{1}{2}(x-3)^3 + 2$

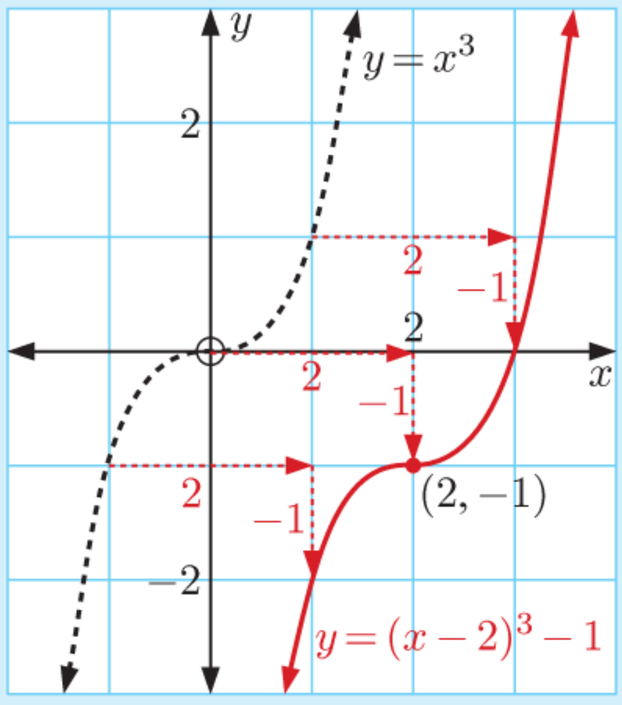
b Describe how the graph of $y = a(x-b)^3 + c$ is obtained from the graph of $y = ax^3$.

From **Investigation 2**, you should have found that:

- The graph of $y = ax^3$ has shape:
  -  if $a > 0$
  -  if $a < 0$.
- As the size of a increases, the graph of $y = ax^3$ becomes steeper.
- The graph of $y = a(x-b)^3 + c$ is obtained by translating the graph of $y = ax^3$ by the vector $\begin{pmatrix} b \\ c \end{pmatrix}$.

Example 22 **Self Tutor**

On the same set of axes, sketch the graph of $y = x^3$ and $y = (x-2)^3 - 1$.



CUBICS IN FACTORISED FORM

The factorised form of a cubic tells us the number and nature of the x-intercepts of its graph.

INVESTIGATION 3 GRAPHS OF CUBIC POLYNOMIALS

Every real cubic polynomial can be categorised into one of four types. In each case $a \in \mathbb{R}$, $a \neq 0$, and the constants α, β, and γ correspond to zeros.

Type 1: Three real, distinct zeros: $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$

Type 2: Two real zeros, one repeated: $P(x) = a(x-\alpha)^2(x-\beta)$

Type 3: One real zero repeated three times: $P(x) = a(x-\alpha)^3$

Type 4: One real zero: $P(x) = (x-\alpha)(ax^2+bx+c)$, $\Delta = b^2-4ac < 0$.

What to do:

GRAPHING PACKAGE



1 Experiment with the graphs of *Type 1* cubics. State the effect of changing both the size and sign of a. What is the geometrical significance of α, β, and γ?

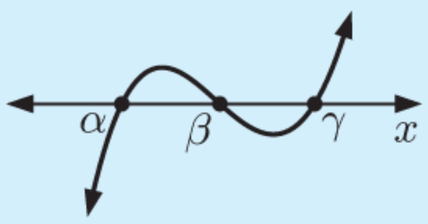
2 Experiment with the graphs of *Type 2* cubics. What is the geometrical significance of the squared factor?

3 Experiment with the graphs of *Type 3* cubics. What is the geometrical significance of α?

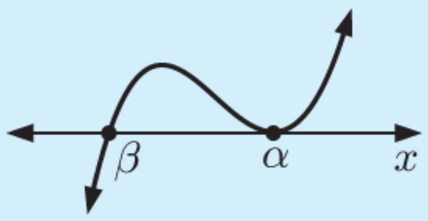
4 Experiment with the graphs of *Type 4* cubics. What is the geometrical significance of α? What is the significance of the quadratic factor which has no real zeros?

From **Investigation 3** you should have discovered that:

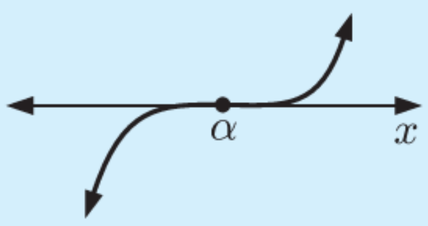
- If $a > 0$, the graph has shape  or  . If $a < 0$ it is  or  .
- Every cubic polynomial must cut the x-axis at least once, and so has at least one real zero.
- For a cubic of the form $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$ where $\alpha, \beta, \gamma \in \mathbb{R}$, the graph has three distinct x-intercepts corresponding to the three distinct zeros α, β, and γ. The graph crosses over or **cuts** the x-axis at these points.



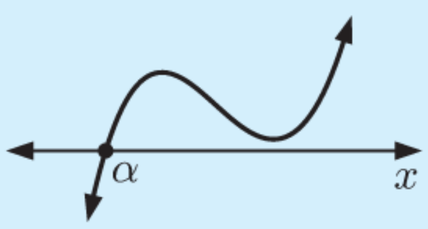
- For a cubic of the form $P(x) = a(x-\alpha)^2(x-\beta)$ where $\alpha, \beta \in \mathbb{R}$, the graph **touches** the x-axis at α and **cuts** it at β.



- For a cubic of the form $P(x) = a(x-\alpha)^3$, $x \in \mathbb{R}$, the graph has only one x-intercept, α. The graph is horizontal at this point.



- For a cubic of the form $P(x) = (x-\alpha)(ax^2+bx+c)$ where $\Delta < 0$, there is only one x-intercept, α. The graph cuts the x-axis at this point.

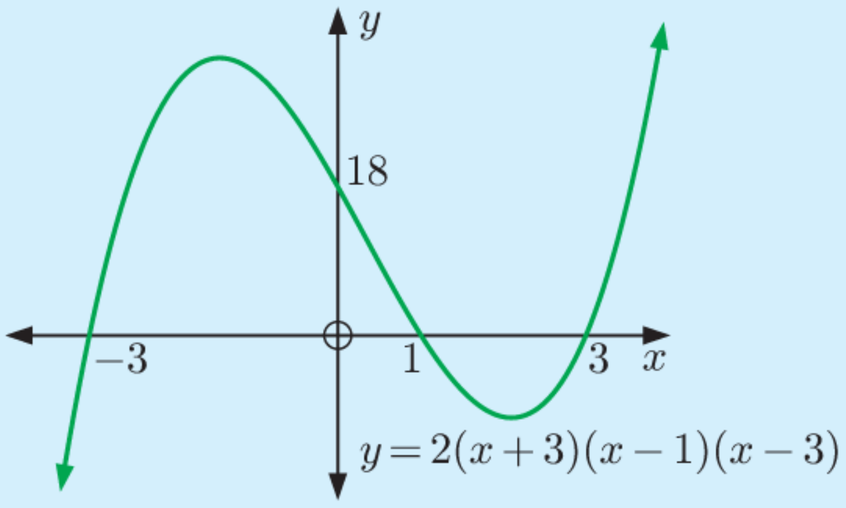


Example 23 — Self Tutor

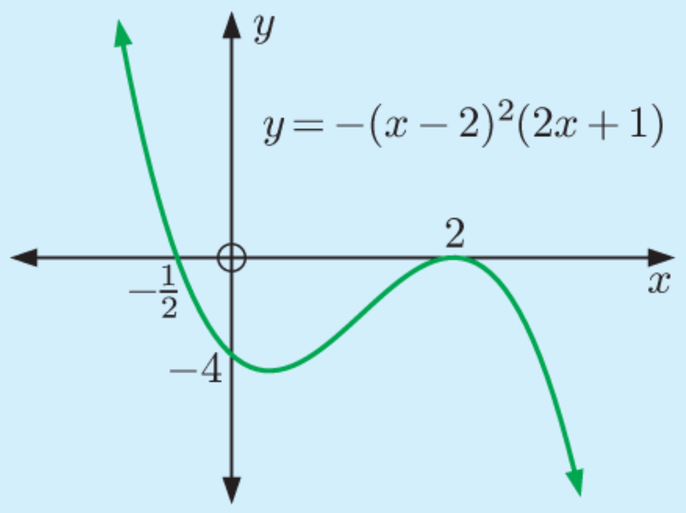
Use the axes intercepts to sketch the graph of:

a $y = 2(x+3)(x-1)(x-3)$ **b** $y = -(x-2)^2(2x+1)$

a The graph cuts the x-axis at -3, 1, and 3.
When $x = 0$,
$y = 2(3)(-1)(-3) = 18$
$\therefore$ the y-intercept is 18.



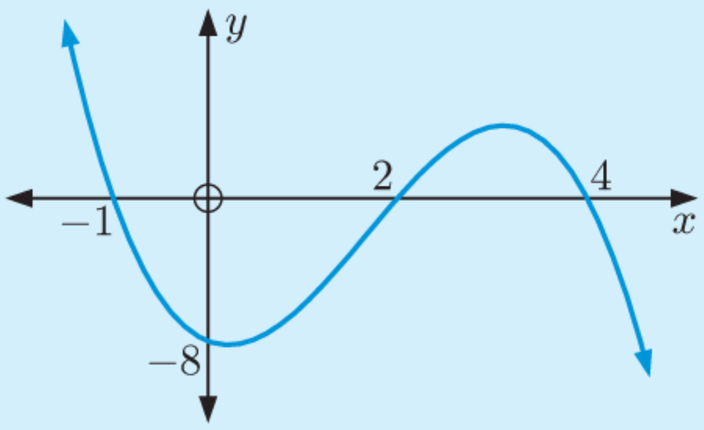
b The graph touches the x-axis at 2 and cuts the x-axis at $-\frac{1}{2}$.
When $x = 0$,
$y = -(-2)^2(1) = -4$
$\therefore$ the y-intercept is -4.



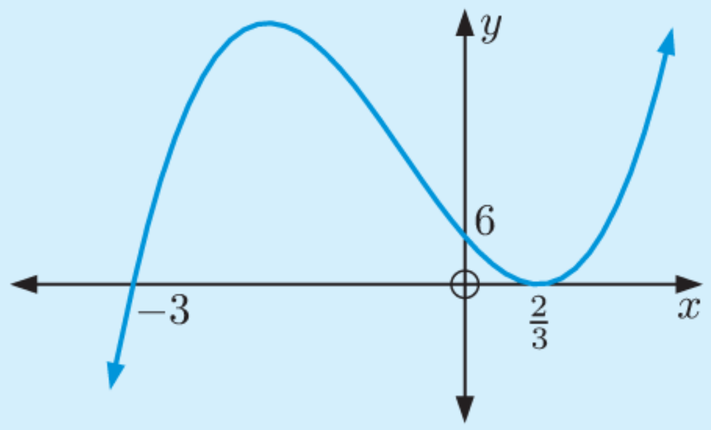
Example 24 — Self Tutor

Find the equation of the cubic with graph:

a



b



a The x-intercepts are -1, 2, 4.
$\therefore\ y = a(x+1)(x-2)(x-4)$
But when $x = 0$, $y = -8$
$\therefore\ a(1)(-2)(-4) = -8$
$\therefore\ a = -1$
So, $y = -(x+1)(x-2)(x-4)$

b The graph touches the x-axis at $\frac{2}{3}$, indicating a squared factor $(3x-2)^2$.
The other x-intercept is -3,
so $y = a(3x-2)^2(x+3)$.
But when $x = 0$, $y = 6$
$\therefore\ a(-2)^2(3) = 6$
$\therefore\ a = \frac{1}{2}$
So, $y = \frac{1}{2}(3x-2)^2(x+3)$

EXERCISE 5J

1 Sketch the following graphs. Use a separate set of axes for each part.

a $y = x^3$ and $y = (x+3)^3 + 2$ **b** $y = -x^3$ and $y = -(x-1)^3 - 4$

c $y = 3x^3$ and $y = 3(x-2)^3 + 5$ **d** $y = -\frac{1}{2}x^3$ and $y = -\frac{1}{2}(x+4)^3 - 3$

2 Use axes intercepts to sketch the graph of:

a $y = (x+4)(x+1)(x-2)$ **b** $y = (x-1)^2(x+3)$

c $y = (x+2)^3$ **d** $f(x) = -(x+2)^2(2x-3)$

e $y = 3x(x+4)(x-2)$ **f** $f(x) = -2(x-4)^3$

g $y = \frac{1}{2}(x+4)^2(3x-2)$ **h** $f(x) = \frac{1}{3}(2x-1)(x+3)(x-2)$

i $f(x) = -4(2x+5)^2(3x-1)$ **j** $y = (x-3)(x^2-4x+5)$

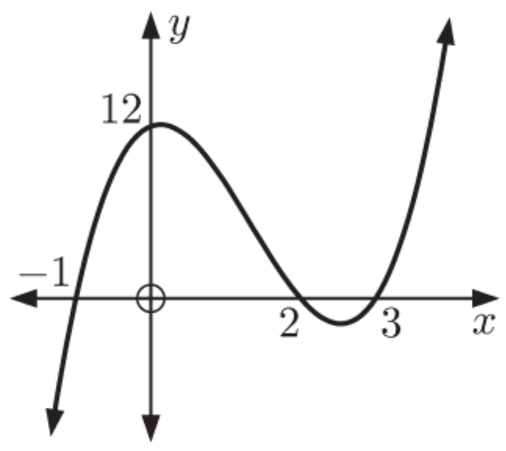
k $y = (x-3)(x^2+2x+3)$ **l** $y = (x+2)(x^2+6x+9)$



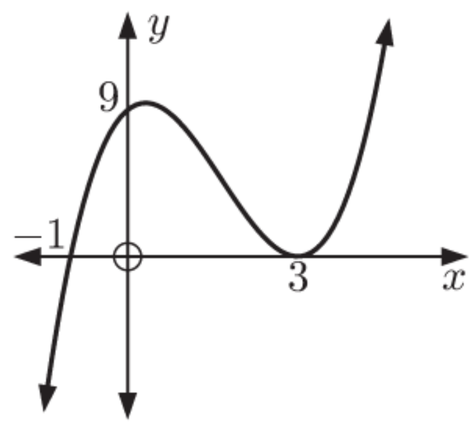
Check your answers using technology.

3 Find the equation of the cubic with graph:

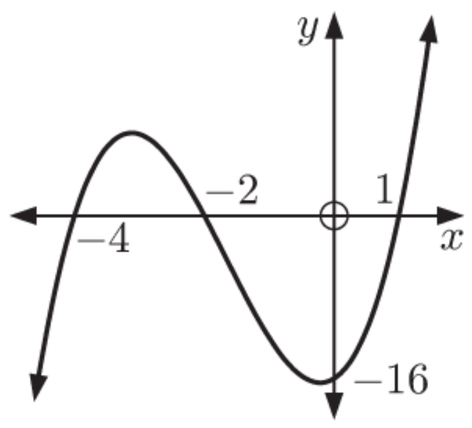
a



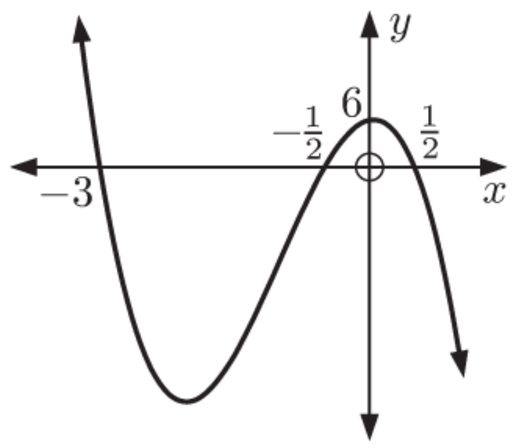
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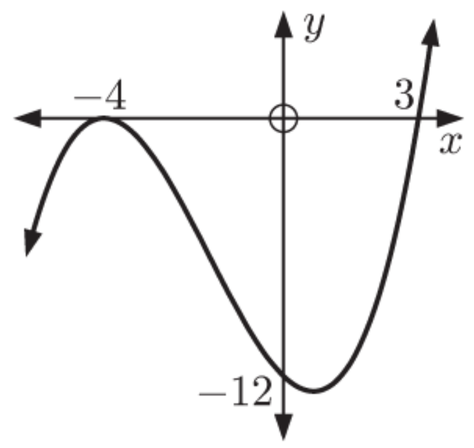
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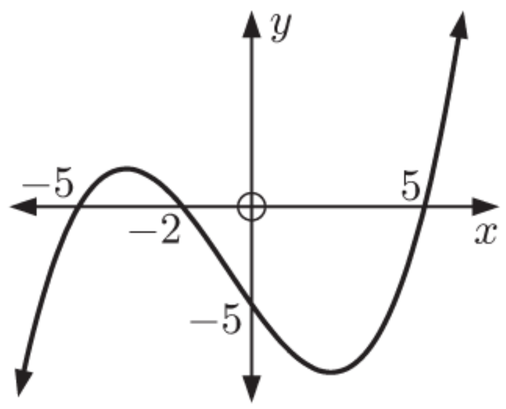
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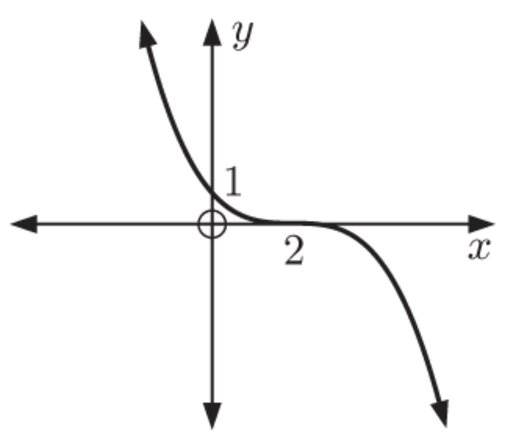
e



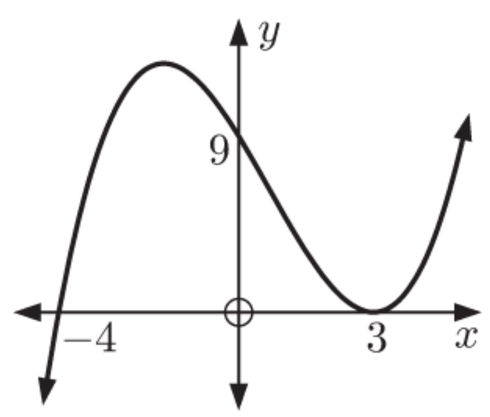
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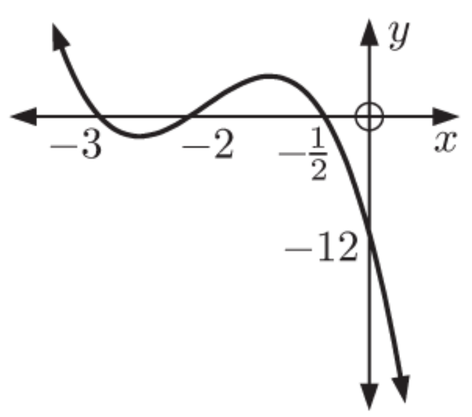
g



h



i



4 **a** Show that -2 is an x-intercept of the graph of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

b Fully factorise $2x^3 - 3x^2 - 11x + 6$.

c Hence sketch the graph of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

d State the domain and range of the function.

e Describe the behaviour of the graph as $x \to \infty$ and $x \to -\infty$.

5 Find the equation of the cubic whose graph:

a cuts the x-axis at 3, 1, and -2, and passes through $(2, -4)$

b cuts the x-axis at -2, 0, and $\frac{1}{2}$, and passes through $(-1, 9)$

c touches the x-axis at 1, cuts the x-axis at -2, and passes through $(4, 54)$

d touches the x-axis at $-\frac{2}{3}$, cuts the x-axis at 4, and passes through $(1, 25)$.

6 **a** Given that 1 is a zero of $-2x^3 - 2x^2 + 10x - 6$, fully factorise the polynomial.

b Hence sketch the graph of $y = -2x^3 - 2x^2 + 10x - 6$.

c Describe the behaviour of the graph as $x \to \infty$ and $x \to -\infty$.

Example 25 **Self Tutor**

Find the equation of the cubic which cuts the x-axis at 2 and -3, cuts the y-axis at -48, and which passes through the point $(1, -40)$.

The zeros are 2 and -3, so $y = (x-2)(x+3)(ax+b)$, $a \neq 0$.

When $x = 0$, $y = -48$

$\therefore (-2)(3)b = -48$

$\therefore b = 8$

When $x = 1$, $y = -40$

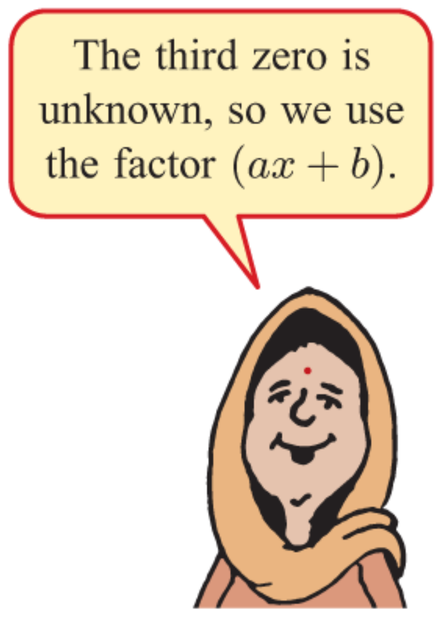
$\therefore (-1)(4)(a+8) = -40$

$\therefore a + 8 = 10$

$\therefore a = 2$

So, the equation is $y = (x-2)(x+3)(2x+8)$

which we can write as $y = 2(x-2)(x+3)(x+4)$



7 Find the equation of a real cubic polynomial which:

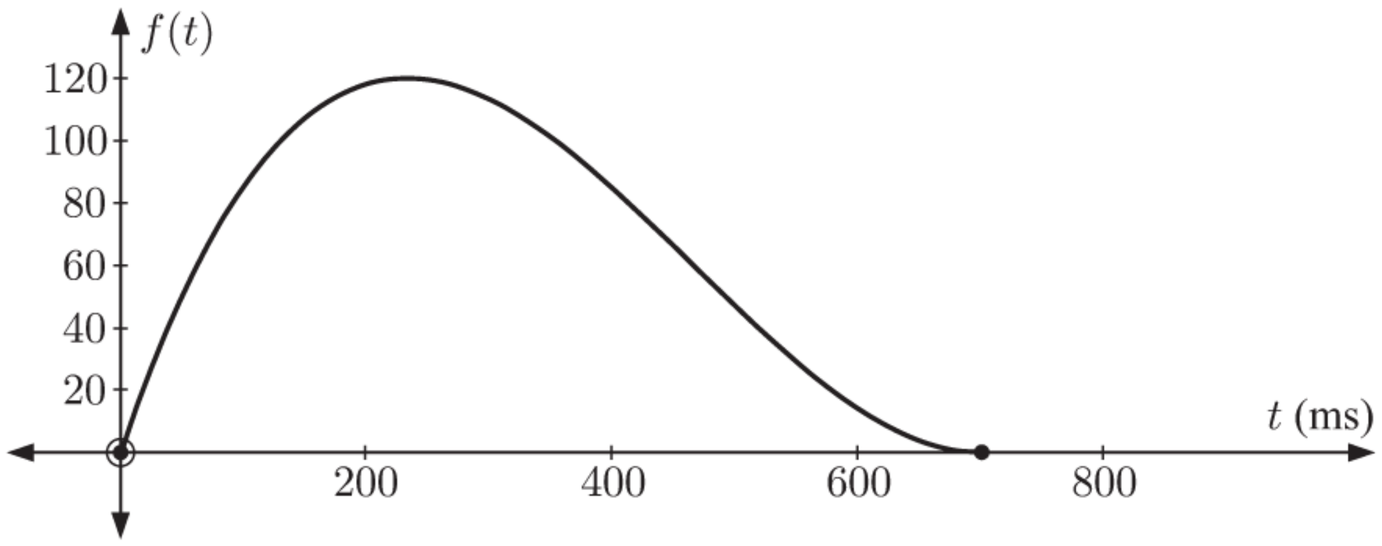
a cuts the x-axis at $\frac{1}{2}$ and -3, cuts the y-axis at 30, and passes through $(1, -20)$

b touches the x-axis at -2, cuts the y-axis at 8, and passes through $(-1, 4)$

c cuts the x-axis at 2, cuts the y-axis at -4, and passes through $(1, -1)$ and $(-1, -21)$.



8 A scientist is trying to design a crash test barrier with the characteristics shown in the graph below. The independent variable t is the time after impact, measured in milliseconds with $0 \leqslant t \leqslant 700$. The dependent variable is the distance the barrier is depressed during the impact, measured in millimetres.



a The equation for this graph has the form $f(t) = kt(t-a)^2$, $0 \leqslant t \leqslant 700$.
Use the graph to find a. Discuss the significance of this value.

b The ideal crash barrier is depressed by 85 mm after 100 milliseconds. Find the value of k which gives this result, and hence find the equation of the graph given.

9 Consider the **Opening Problem** on page **110**.

a Explain why the volume of the container is $V = x(40 - 2x)(30 - 2x)$ cm^3.

b For what values of x is it reasonable to use this function?

c Use technology to help sketch the function.

d What size squares should be cut from the corners to make a container with volume 2000 cm^3 ?

ACTIVITY 4

Click on the icon to practise working with cubic functions.

CARD GAME



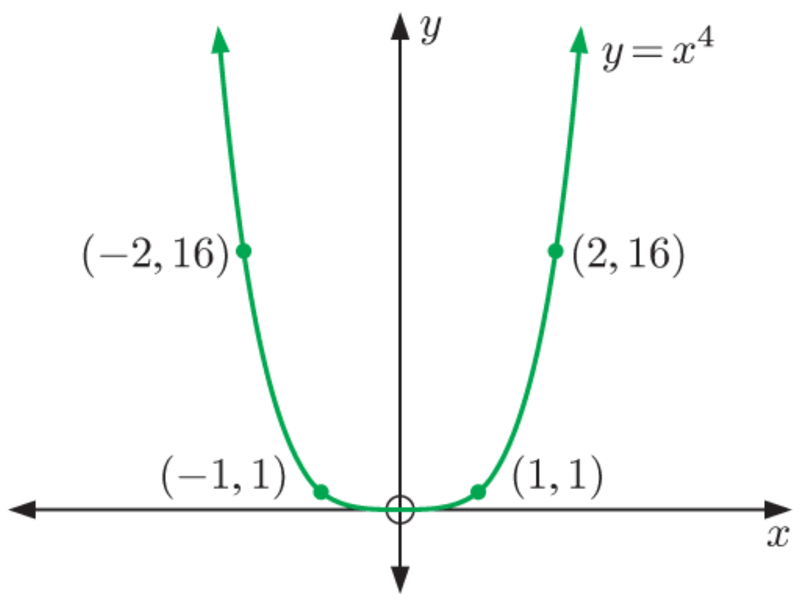
K GRAPHING QUARTIC FUNCTIONS

A **quartic function** is a function of the form $y = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d, and e are constants, $a \neq 0$.

The simplest quartic function is $y = x^4$.

Notice that:

- the graph opens upwards
- the graph is "flat" at $(0, 0)$
- as $x \to \infty$, $y \to \infty$
- as $x \to -\infty$, $y \to \infty$.



As with quadratic and cubic functions, quartic functions may be written in **expanded form** such as $y = x^4 - 4x^3 - x^2 + 16x - 12$, or in **factorised form** such as $y = (x + 2)(x - 1)(x - 2)(x - 3)$. The factorised form of a quartic function gives us the most information about the function's graph.

INVESTIGATION 4 GRAPHS OF QUARTIC FUNCTIONS

What to do:

1 Experiment with the graphs of quartics which have:

a four distinct real linear factors

b a squared real linear factor and two distinct real linear factors

c two squared real linear factors

d a cubed real linear factor and one distinct real linear factor

e a real linear factor raised to the fourth power

f one real quadratic factor with $\Delta < 0$ and two real linear factors

g two real quadratic factors each with $\Delta < 0$.

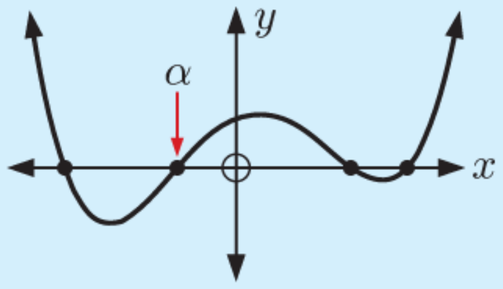
GRAPHING PACKAGE



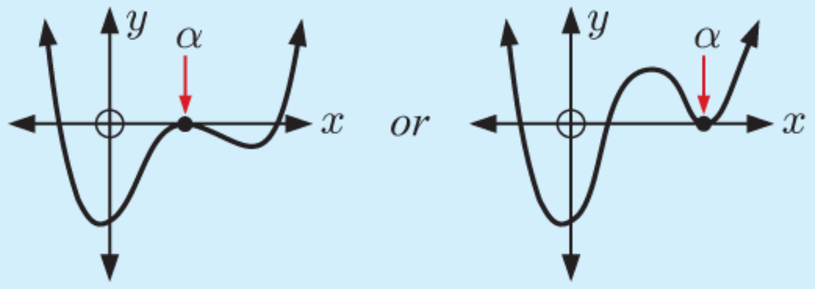
2 Summarise your observations.

From **Investigation 4** you should have discovered that:

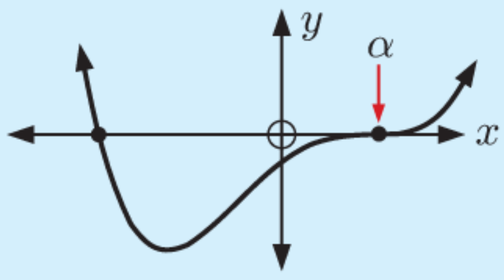
- For a quartic polynomial in which a is the coefficient of x^4:
  - If $a > 0$ the graph opens upwards.
  - If $a < 0$ the graph opens downwards.
- If a quartic with $a > 0$ is fully factorised into real linear factors, then:
  - for a **single factor** $(x - \alpha)$, the graph **cuts** the x-axis at α



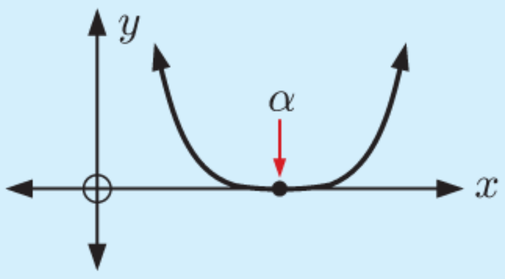
  - for a **square factor** $(x-\alpha)^2$, the graph **touches** the x-axis at α



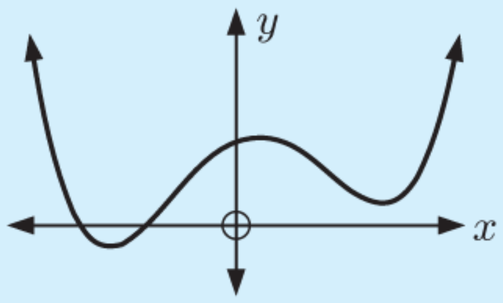
  - for a **cubed factor** $(x - \alpha)^3$, the graph **cuts** the x-axis at α and is "flat" at α



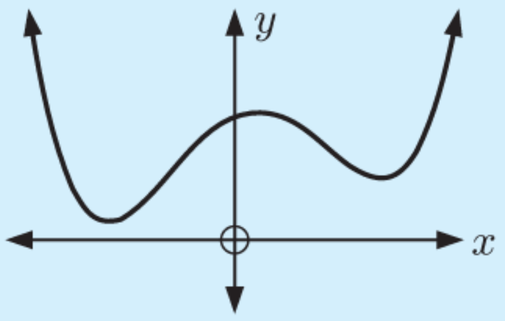
  - for a **quadruple factor** $(x - \alpha)^4$, the graph **touches** the x-axis and is "flat" at that point.



  - If a quartic with $a > 0$ has one real quadratic factor with $\Delta < 0$ we could have:



  - If a quartic with $a > 0$ has two real quadratic factors both with $\Delta < 0$, the graph does not meet the x-axis at all.



Example 26 **Self Tutor**

Sketch the graph of:

a $y = (3x - 2)(x + 3)(x + 1)^2$ **b** $y = -(x - 3)^2(x + 2)^2$

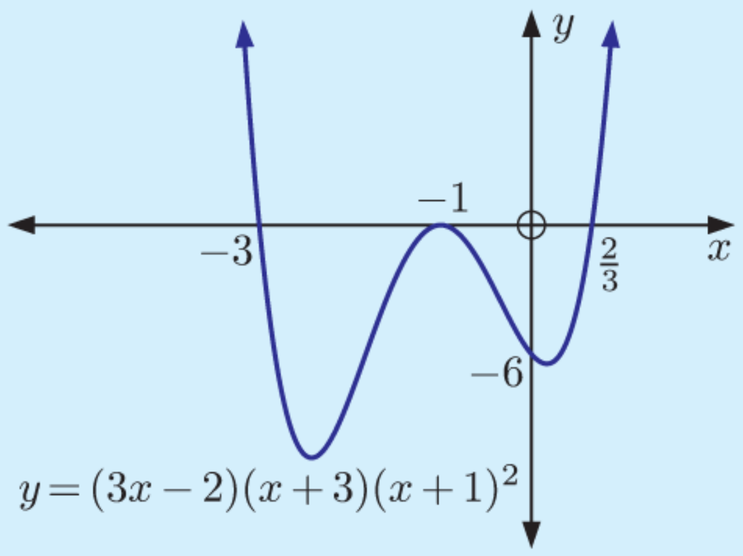
a $a > 0$, so the graph opens upwards.

The graph cuts the x-axis at $\frac{2}{3}$ and -3, and touches the x-axis at -1.

When $x = 0$,

$y = (-2)(3)(1)^2 = -6$

$\therefore$ the y-intercept is -6.



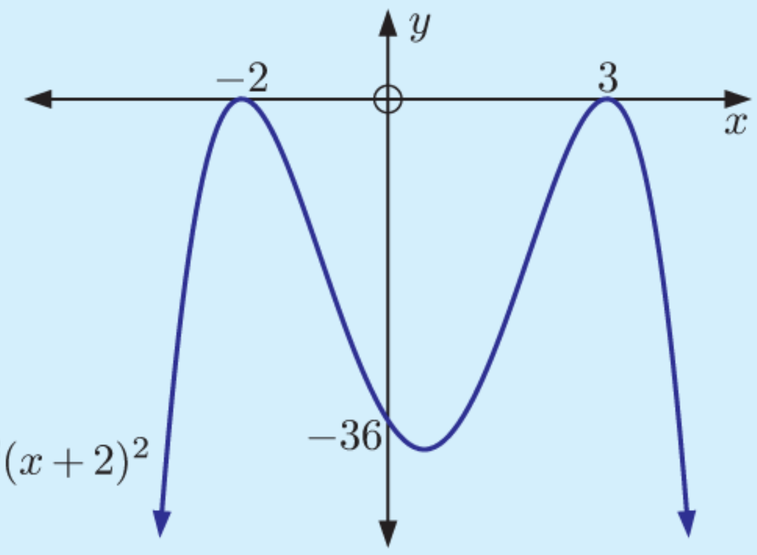
b $a < 0$, so the graph opens downwards.

The graph touches the x-axis at 3 and -2.

When $x = 0$,

$$y = -(-3)^2(2)^2 = -36$$

$\therefore$ the y-intercept is -36.



$y = -(x-3)^2(x+2)^2$

EXERCISE 5K

1 Sketch the graph of:

a $y = x^2(x-1)(2x+3)$

b $y = 3x^2(x-2)^2$

c $f(x) = 2(x+3)(x+1)(x-1)(x-2)$

d $y = -2x(x-1)^2(x+3)$

e $f(x) = -(x-1)(x+1)^3$

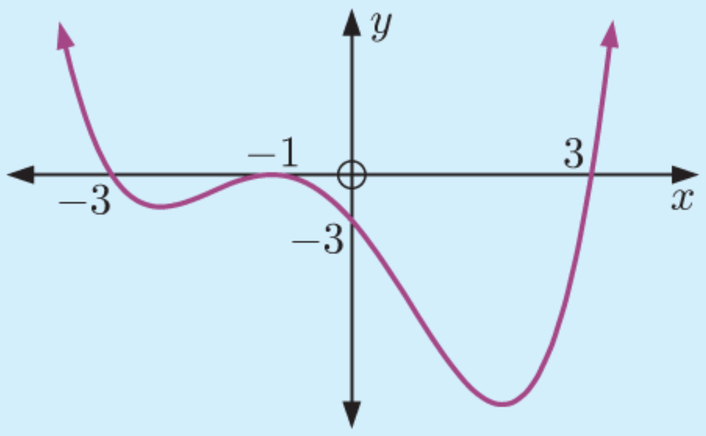
f $f(x) = -2(x+2)^2(2x-1)^2$

2 **a** Fully factorise $x^4 - x^3 - 2x^2$.

b Hence sketch the graph of $f(x) = x^4 - x^3 - 2x^2$.

Example 27 — Self Tutor

Find the equation of the quartic with graph:



The graph touches the x-axis at -1 and cuts it at -3 and 3.

$\therefore \quad y = a(x+1)^2(x+3)(x-3), \quad a \neq 0$

But when $x = 0, \quad y = -3$

$\therefore \quad -3 = a(1)^2(3)(-3)$

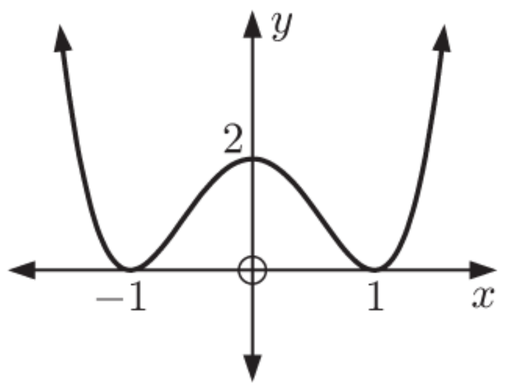
$\therefore \quad -3 = -9a$

$\therefore \quad a = \frac{1}{3}$

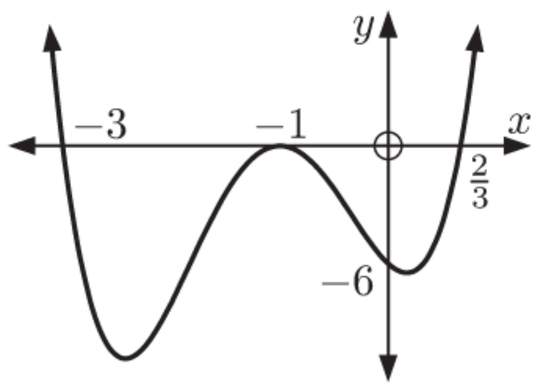
$\therefore \quad y = \frac{1}{3}(x+1)^2(x+3)(x-3)$

3 Find the equation of the quartic with graph:

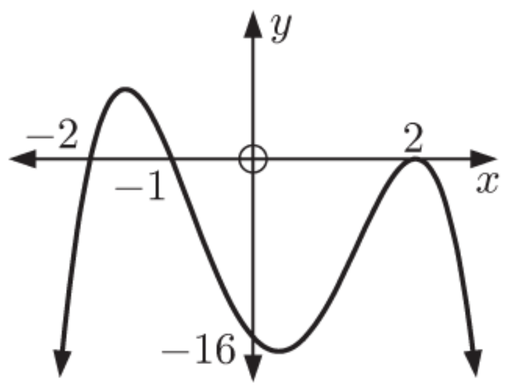
a



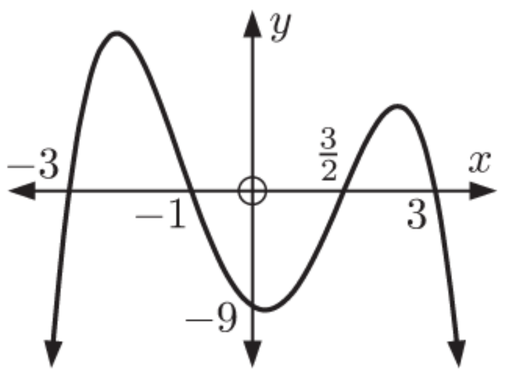
b



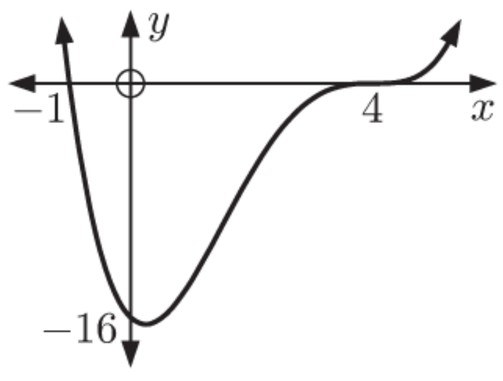
c



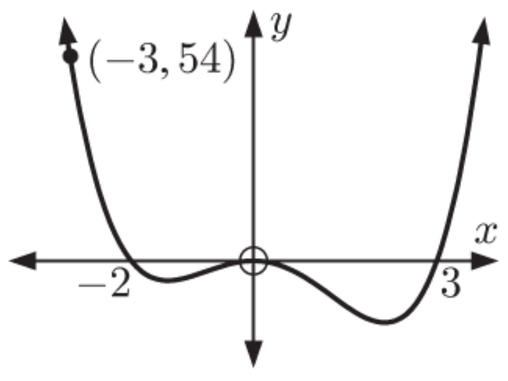
d



e



f



Example 28 **Self Tutor**

Find the quartic function $P(x)$ which touches the x-axis at 2, cuts the x-axis at -3, and passes through $(1, -12)$ and $(3, 6)$.

The graph *touches* the x-axis at 2, so $(x-2)^2$ is a factor.

The graph *cuts* the x-axis at -3, so $(x+3)$ is a factor.

$\therefore \quad P(x) = (x-2)^2(x+3)(ax+b)$ where a and b are constants, $a \neq 0$.

Now $P(1) = -12$ and $P(3) = 6$

$\therefore \quad (-1)^2(4)(a+b) = -12$ $\qquad$ $\therefore \quad (1)^2(6)(3a+b) = 6$

$\therefore \quad a+b = -3 \quad \ldots. (1)$ $\qquad$ $\therefore \quad 3a+b = 1 \quad \ldots. (2)$

Solving (1) and (2) simultaneously gives $a = 2, \ b = -5$.

$\therefore \quad P(x) = (x-2)^2(x+3)(2x-5)$

4 Find the equation of the quartic whose graph:

a cuts the x-axis at -4 and $\frac{1}{2}$, touches it at 2, and passes through the point $(1, 5)$

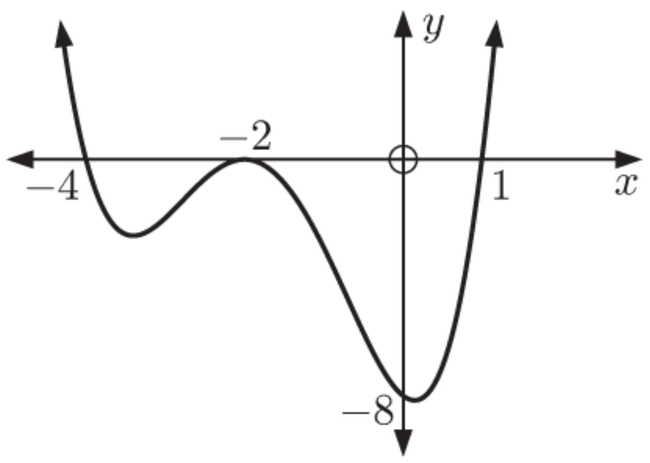
b touches the x-axis at $\frac{2}{3}$ and -3, and passes through the point $(-4, 49)$

c cuts the x-axis at $\pm\frac{1}{2}$ and ± 2, and passes through the point $(1, -18)$

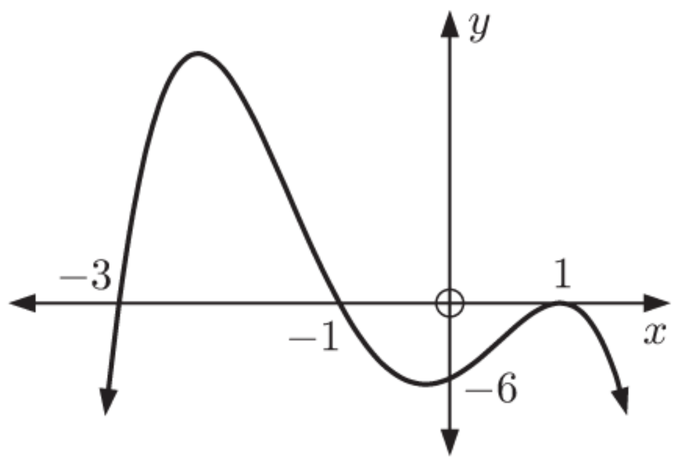
d touches the x-axis at 1, cuts the y-axis at -1, and passes through $(-1, -4)$ and $(2, 15)$.

5 Find the equation of the following quartics. Give your answers in expanded form:

a



b



6 The graph of $f(x) = x^4 - x^3 - 5x^2 + 12$ touches the x-axis at 2.

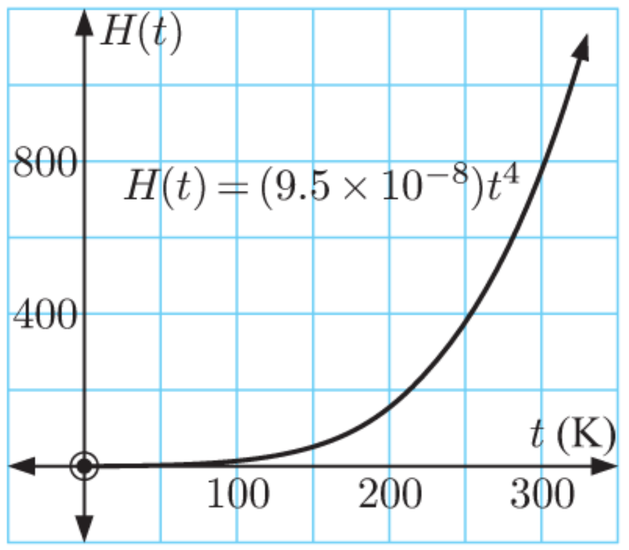
a By fully factorising $x^4 - x^3 - 5x^2 + 12$, show that the graph does not meet the x-axis anywhere else.

b Use technology to help sketch the graph of $y = f(x)$.

7 The heat radiated from a person is given by the quartic function $H(t) \approx (9.5 \times 10^{-8})t^4$ joules per second, where t is the person's body temperature in Kelvins (K).
Find the heat radiated by:

a a person with a normal body temperature of 37°C or 310 K

b a person who has a fever, with body temperature 39°C or 312 K.

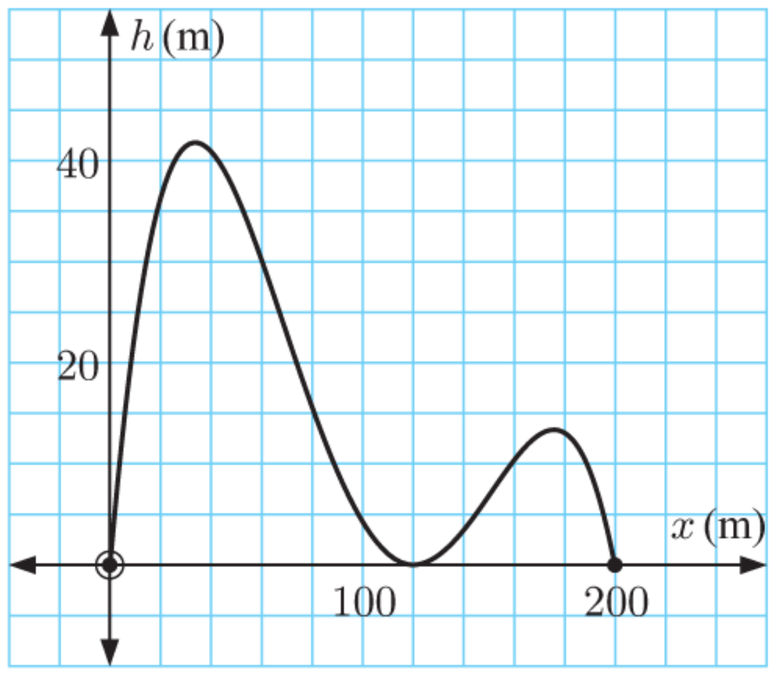


8 **a** Given that x^2+x+1 is a factor of $x^4-4x^3+3x^2+2x+7$, write $x^4-4x^3+3x^2+2x+7$ as a product of two quadratic factors.

b Hence show that the graph of $f(x)=x^4-4x^3+3x^2+2x+7$ does not meet the x-axis at all.

c Use technology to help sketch the graph of $f(x)=x^4-4x^3+3x^2+2x+7$.

9 The graph alongside shows the height (h metres) of a roller coaster above the ground against the horizontal distance travelled (x metres), for $0 \leqslant x \leqslant 200$.



The equation for this graph has the form

$$h(x)=\frac{-x(x-a)^2(x-b)}{k}, \quad 0 \leqslant x \leqslant 200.$$

a Use the graph to find a and b.

b Given that the roller coaster is 4 m above the ground after travelling 100 m, find the value of k.

c Find the height of the roller coaster above the ground, after it has travelled 150 m.

ACTIVITY 5

Click on the icon to practise working with quartic functions.

CARD GAME

DISCUSSION GENERAL POLYNOMIALS

Consider the general polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+....+a_1x+a_0, \quad a_n \neq 0$.

- Discuss the behaviour of the graph as $x \to -\infty$ and $x \to \infty$ depending on
  - the sign of a_n
  - whether n is odd or even.
- Under what circumstances is $P(x)$ guaranteed to have at least one real zero?

L POLYNOMIAL EQUATIONS

We have already seen that every real cubic polynomial must cut the x-axis at least once, and so has at least one real zero.

If the exact value of the zero is difficult to find, we can use technology to help us. We can then factorise the cubic as a linear factor times a quadratic, and if necessary use the quadratic formula to find the other zeros.

This method is particularly useful if we have one rational zero and two irrational zeros that are radical conjugates.

Example 29 — Self Tutor

Find exactly the zeros of $P(x) = 3x^3 - 14x^2 + 5x + 2$.

Using technology $x \approx 0.666\,667 = \frac{2}{3}$ is a zero.

$\therefore$ $(3x - 2)$ is a factor.

$\therefore$ $3x^3 - 14x^2 + 5x + 2 = (3x - 2)(x^2 + ax - 1)$ for some a.

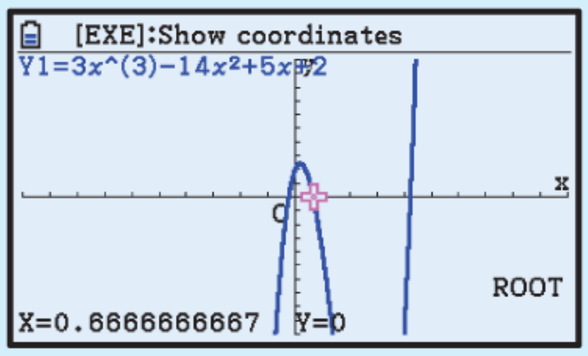
Equating coefficients of x^2: $3a - 2 = -14$

$\therefore$ $3a = -12$

$\therefore$ $a = -4$

Equating coefficients of x: $-3 - 2a = 5$ ✓

GRAPHING PACKAGE



$\therefore$ $P(x) = (3x - 2)(x^2 - 4x - 1)$ which has zeros $\frac{2}{3}$ and $2 \pm \sqrt{5}$ {quadratic formula}

Example 30 — Self Tutor

Find exactly the roots of $6x^3 + 13x^2 + 20x + 3 = 0$.

Using technology, $x \approx -0.166\,666\,67 = -\frac{1}{6}$ is a root, so $(6x + 1)$ is a factor of the cubic.

$\therefore$ $6x^3 + 13x^2 + 20x + 3 = (6x + 1)(x^2 + ax + 3)$ for some a.

Equating coefficients of x^2: $1 + 6a = 13$

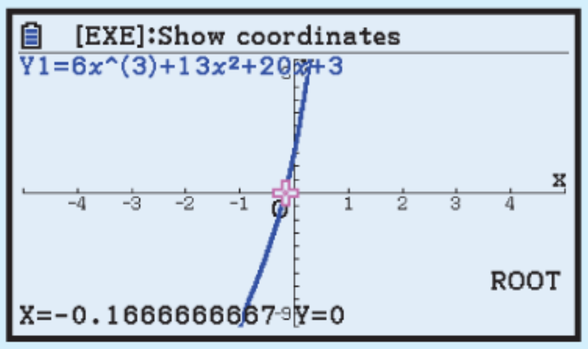
$\therefore$ $6a = 12$

$\therefore$ $a = 2$

Equating coefficients of x: $a + 18 = 20$ ✓

$\therefore$ $(6x + 1)(x^2 + 2x + 3) = 0$

$\therefore$ $x = -\frac{1}{6}$ or $-1 \pm i\sqrt{2}$ {quadratic formula}



For a quartic polynomial $P(x)$ we first need to establish if there are any x-intercepts at all. If there are not then the polynomial must have four complex zeros. If there *are* x-intercepts then we can try to identify linear or quadratic factors.

EXERCISE 5L

1 Find exactly all zeros of:

a $x^3 - 3x^2 - 3x + 1$ **b** $x^3 - 3x^2 + 4x - 2$

c $2x^3 - 3x^2 - 4x - 35$ **d** $2x^3 - x^2 + 20x - 10$

e $4x^4 - 4x^3 - 25x^2 + x + 6$ **f** $x^4 - 6x^3 + 22x^2 - 48x + 40$

2 Find exactly the roots of:

a $x^3 + 2x^2 + 3x + 6 = 0$ **b** $2x^3 + 3x^2 - 3x - 2 = 0$

c $x^3 - 6x^2 + 12x - 8 = 0$ **d** $2x^3 + 18 = 5x^2 + 9x$

e $x^4 - x^3 - 9x^2 + 11x + 6 = 0$ **f** $2x^4 - 13x^3 + 27x^2 = 13x + 15$

3 Solve for x exactly:

a $x^3 - 3x^2 + 4x - 4 = 0$ **b** $x^3 + 3x^2 + 4x + 12 = 0$

c $2x^3 - 9x^2 + 6x - 1 = 0$ **d** $x^3 - 4x^2 + 9x - 10 = 0$

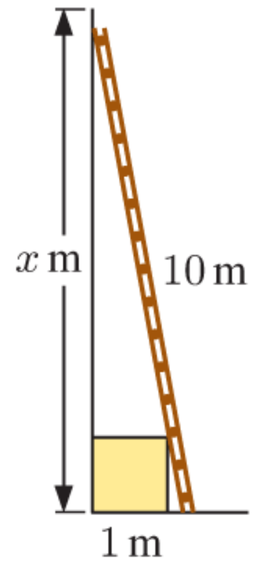
e $4x^3 - 8x^2 + x + 3 = 0$ **f** $3x^4 + 4x^3 + 5x^2 + 12x - 12 = 0$

g $2x^4 - 3x^3 + 5x^2 + 6x - 4 = 0$ **h** $2x^3 + 5x^2 + 8x + 20 = 0$

4 The following cubics will not factorise neatly. Solve for x using only technology:

a $x^3 + 2x^2 - 6x - 6 = 0$ **b** $x^3 + x^2 - 7x - 8 = 0$

5 A ladder of length 10 metres is leaning against a wall so that it is just touching a cube of edge length one metre.
What height up the wall does the ladder reach?



M CUBIC INEQUALITIES

A **cubic inequality** can be written in either the form $ax^3 + bx^2 + cx + d \geqslant 0$ or $ax^3 + bx^2 + cx + d > 0$ where $a \neq 0$.

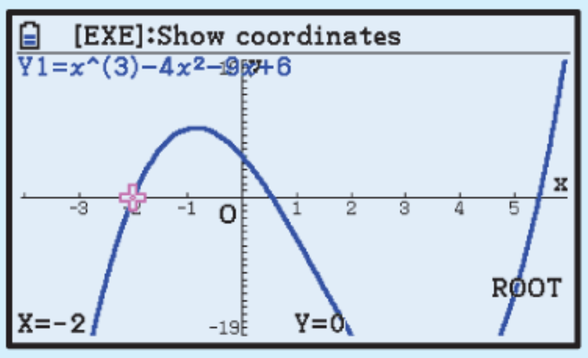
Cubic inequalities are solved in the same way as quadratic inequalities. We shift all terms to the LHS so we can factorise the cubic and draw its sign diagram.

Example 31 Self Tutor

Solve exactly for x: $x^3 + 6 \leqslant 4x^2 + 9x$

$$x^3 + 6 \leqslant 4x^2 + 9x$$
$$\therefore \quad x^3 - 4x^2 - 9x + 6 \leqslant 0$$

Using technology, $x = -2$ is a root, so $(x + 2)$ is a factor of the cubic.



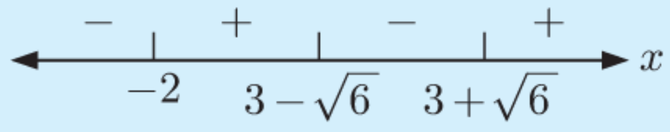
$\therefore \quad x^3 - 4x^2 - 9x + 6 = (x + 2)(x^2 + ax + 3)$ for some a.

Equating coefficients of x^2: $a + 2 = -4$

$\therefore \quad a = -6$

$\therefore$ the LHS of the inequality is $(x + 2)(x^2 - 6x + 3)$, which has zeros -2 and $3 \pm \sqrt{6}$ {quadratic formula}

Sign diagram of LHS is:



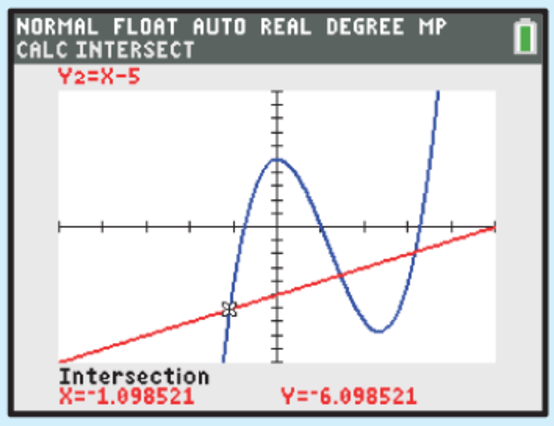
$\therefore \quad x \leqslant -2$ or $3 - \sqrt{6} \leqslant x \leqslant 3 + \sqrt{6}$

Alternatively, we can solve the inequality graphically. This will give us approximate solutions to the inequality.

Example 32 **Self Tutor**

Solve graphically: $2x^3 - 7x^2 + 5 > x - 5$

We graph $y = 2x^3 - 7x^2 + 5$ and $y = x - 5$ on the same set of axes.



Using technology, the graphs intersect at $x \approx -1.10$, $x \approx 1.44$, and $x \approx 3.16$.

$2x^3 - 7x^2 + 5 > x - 5$ whenever the graph of $y = 2x^3 - 7x^2 + 5$ is above the graph of $y = x - 5$.

This occurs when $-1.10 < x < 1.44$ or $x > 3.16$.

EXERCISE 5M

1 **a** Given that 2 is a zero of $x^3 + 3x^2 - 4x - 12$, fully factorise $x^3 + 3x^2 - 4x - 12$.

b Hence solve $x^3 + 3x^2 - 4x - 12 \geqslant 0$ exactly.

2 **a** Show that $-\frac{2}{3}$ is a zero of $3x^3 - 10x^2 - 17x - 6$.

b Hence find the remaining zeros of $3x^3 - 10x^2 - 17x - 6$.

c Hence solve $3x^3 - 6 < 10x^2 + 17x$ exactly.

3 Solve for x exactly:

a $x^3 + x^2 - 10x + 8 < 0$ **b** $x^3 + x^2 - 11x - 15 \geqslant 0$ **c** $2x^3 - 9x^2 + 2x + 1 \leqslant 0$

d $3x^3 + 10x^2 > 22x + 40$ **e** $4x^3 - 7 \leqslant 28x - x^2$ **f** $5x^3 - 26x > x^2 - 24$

4 Solve graphically for x:

a $x^3 - 5x > x^2 - 3$ **b** $2x^3 - 8x - 5 < x^2 - 2x - 7$

c $1 - 3.5x + 0.6x^3 \geqslant 0$ **d** $3x^3 - 5x \leqslant 6 - x^3$

e $x^3 - \sqrt{2}x + 1 > 0.2x + 1.2$ **f** $x^3 + 4.5x^2 - 3 < 3 - 0.2x - 1.5x^3$

5 Last year, the volume of water in a particular reservoir could be described by the model $V(t) = -t^3 + 30t^2 - 131t + 250$ ML, where t is the time in months.
The local government rules state that if the volume falls below 100 ML, irrigation is prohibited. During which months, if any, was irrigation prohibited in the last twelve months? Include in your answer a neat sketch of any graphs you may have used.

REVIEW SET 5A

1 Let $P(z) = 2z^3 - z^2 + 4z$ and $Q(z) = 3 - z$. Find, in simplest form:

a $P(z) + 2Q(z)$ **b** $P(z)\,Q(z)$ **c** $\dfrac{P(z)}{Q(z)}$

2 Expand and simplify:

a $(3x^3 + 2x - 5)(4x - 3)$ **b** $(2x^2 - x + 3)^2$

3 Carry out the divisions:

a $\dfrac{x^3}{x+2}$ **b** $\dfrac{2x^3-3x+4}{x-1}$ **c** $\dfrac{x^3}{(x+2)(x+3)}$

4 Decide whether:

a 4 is a zero of x^3-3x^2-5x+4

b -3 is a root of $2x^3+5x^2-x-6=0$.

5 Factorise z^4+4 into a product of real quadratics.

6 Find all real quartic polynomials with rational coefficients, having $2-i\sqrt{3}$ and $\sqrt{2}+1$ as two of the zeros.

7 $-2+bi$ is a solution to $z^2+az+(3+a)=0$. Find constants a and b given that they are real.

8 Find constants a and b given that:

a $4x^2-7x+5=4x^2+ax+(a+b)$ for all x

b $6x^3+11x^2-9x-4=(2x^2+3x-4)(ax+b)$ for all x.

9 $(2x-3)$ is a factor of $2x^3+3x^2-29x+30$. Fully factorise the polynomial.

10 $(3x+2)$ and $(x-2)$ are factors of $6x^3+ax^2-4ax+b$. Find a and b.

11 State and prove the Remainder theorem.

12 When a polynomial $P(x)$ is divided by x^2-3x+2 the remainder is $2x+3$.
Find the remainder when $P(x)$ is divided by $x-2$.

13 Find the remainder when $2x^{17}+5x^{10}-7x^3+6$ is divided by $x-2$.

14 Find constants a and b given x^3+x^2+ax+b has factors $(x+2)$ and $(x-4)$.

15 If $f(x)=x^3-3x^2-9x+b$ has $(x-k)^2$ as a factor, show that there are two possible values of k. For each of these two values of k, find the corresponding value for b, and hence solve $f(x)=0$.

16 $3-2i$ is a zero of $z^4+kz^3+32z+3k-1$, where k is real.
Find k and all zeros of the quartic.

17 Find all the zeros of the polynomial $z^4+2z^3+6z^2+8z+8$ given that one of the zeros is purely imaginary.

18 Find the sum and product of the zeros of:

a $3x^4-4x^3+3x^2+8$ **b** $2x^6+2x^4-x^3+7x-10$

19 $P(x)=2x^4-8x^3+ax^2+bx-110$, $a, b\in\mathbb{R}$, has zeros $m\pm 2i$ and $1\pm n\sqrt{3}$ where $m, n\in\mathbb{R}$.

a Find m and n. **b** Sketch the graph of $y=P(x)$.

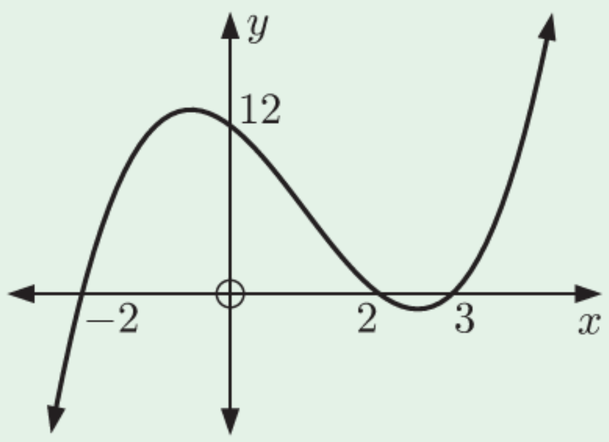
20 If α and β are two of the roots of $x^3-x+1=0$, show that $\alpha\beta$ is a root of $x^3+x^2-1=0$.
Hint: Let $x^3-x+1=(x-\alpha)(x-\beta)(x-\gamma)$.

21 $a+ai$ is a root of $x^2-6x+b=0$ where $a, b\in\mathbb{R}$.
Explain why b has two possible values. Find a in each case.

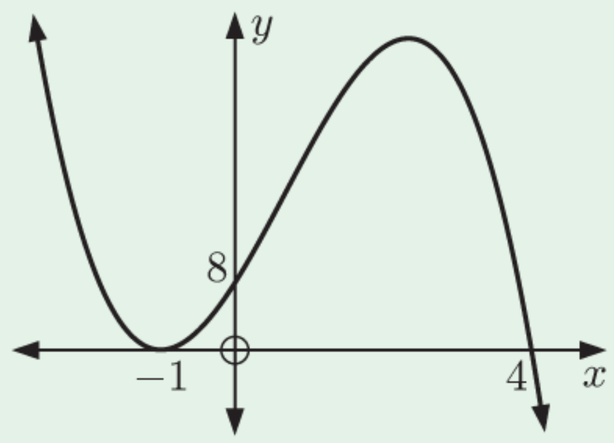
22 On the same set of axes, sketch the graphs of $y = \frac{1}{2}x^3$ and $y = \frac{1}{2}(x+2)^3 - 3$.

23 Find the equation of the cubic with graph:

a



b



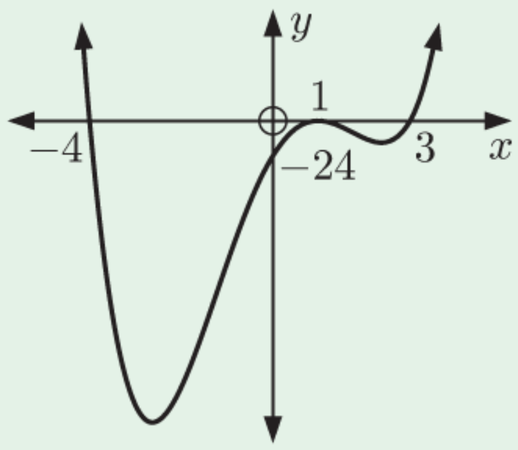
24 Sketch the graph of:

a $y = x(x+2)(x-1)(x-4)$ **b** $y = -2(x+3)^2(x-2)(x+1)$

c $y = (3x-1)^2(x+2)^2$

25 A quartic polynomial $P(x)$ has graph $y = P(x)$ which touches the x-axis at $(-2, 0)$, cuts the x-axis at $(1, 0)$, cuts the y-axis at $(0, 12)$, and passes through $(2, 80)$.
Find an expression for $P(x)$ in factored form, and hence sketch the graph of $y = P(x)$.

26 Find the quartic polynomial $P(x)$ with the graph shown.



27 **a** Fully factorise $P(x) = 2x^4 + 7x^3 + 4x^2 - 4x$.

b Hence sketch the graph of $y = P(x)$.

c For what values of x is $P(x) \geqslant 0$?

28 Solve for x:

a $6x^3 + 17x^2 = 5x + 6$ **b** $3x^4 + x^3 + 18x^2 - 54x - 20 = 0$

29 Solve for x exactly:

a $x^3 + x^2 - 12x - 18 < 0$ **b** $5x^3 - 20 \geqslant 25x - 4x^2$

REVIEW SET 5B

1 Let $f(x) = x^4 - 5x^3 + 2x - 1$ and $g(x) = -2x^3 + 3x^2 + 7$.

a Find $f(x) - g(x)$, giving your answer in simplest form.

b For the product $f(x)\,g(x)$, state the:

i degree **ii** leading coefficient **iii** constant term.

2 Expand and simplify:

a $(3x^2 - 2x + 1)(x^3 + 5x^2 - 2)$ **b** $(x^3 - 2x^2 + 1)^2$

3 Find the linear factors and corresponding zeros of:

a $3x^2 - 11x + 6$ **b** $2z^4 - 5z^3 + 13z^2 - 4z - 6$

4 Use expansion to verify that $(3x-2)(x+2)(x-3)$ is the factorisation of $3x^3-5x^2-16x+12$.

5 Find all real polynomials of least degree which have zeros:

a -4, 1, and 6 **b** $\frac{1}{3}$ and $3\pm\sqrt{5}$ **c** $i\sqrt{2}$ and $\frac{1}{2}$ **d** $1-i$ and $-3-i$.

6 Find a, b, and c such that $2x^4+3x^3+11x^2+ax+15=(2x^2+bx+c)(x^2+2x+5)$ for all x.

7 Carry out the divisions:

a $\dfrac{2x^3+9x^2+5x-1}{2x+3}$ **b** $\dfrac{x^4+x^2+2}{x-1}$ **c** $\dfrac{x^4+x^2}{(x-1)(x+2)}$

8 $(3x-2)$ is a factor of $6x^3+23x^2-33x+10$. Fully factorise the polynomial.

9 Given that $(2x+1)$ is a factor of $4x^3-8x^2-11x-3$, find all of the solutions of $4x^3-8x^2-11x-3=0$.

10 Show that $x=\frac{1}{4}$ is the only real solution of $4x^3-13x^2+15x-3=0$.

11 Suppose a and k are real. For what values of k does $z^3+az^2+kz+ka=0$ have:

a one real root **b** 3 real roots?

12 Find the remainder when $x^{47}-3x^{26}+5x^3+11$ is divided by $x+1$.

13 $P(x)=x^m+6x^2+50$ leaves a remainder of -54 when divided by $x+2$. Find m.

14 Find the quotient and remainder when $x^4+3x^3-7x^2+11x-1$ is divided by x^2+2.
Hence find constants a and b for which $x^4+3x^3-7x^2+(2+a)x+b$ is exactly divisible by x^2+2.

15 **a** Show that x^2+1 is a factor of $x^{10}+1$. **Hint:** $x^2+1=(x+i)(x-i)$

b Hence find the remainder when $x^{11}+1$ is divided by x^2+1.

16 Factorise $2z^3+z^2+10z+5$ as a product of linear factors with exact terms.

17 $4+i$ is a root of $x^2+ax+b=0$, where a and b are real. Find a and b.

18 $5-i$ is a zero of $2z^3+az^2+62z+(a-5)$, where a is real. Find a and the other two zeros.

19 Find the sum and product of the roots of:

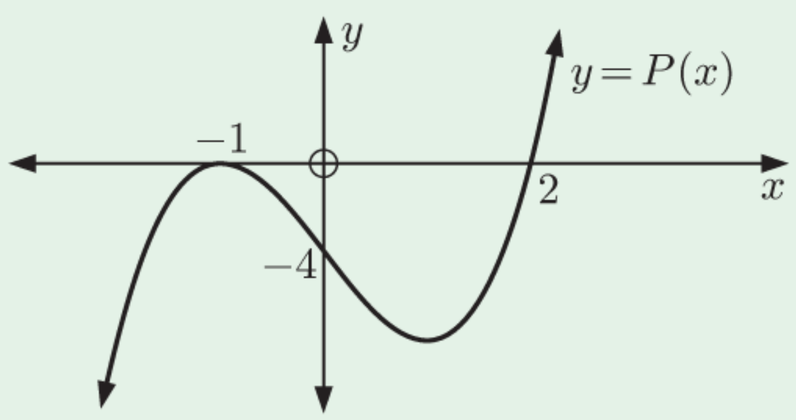
a $2x^3+3x^2-4x+6=0$ **b** $4x^4=x^2+2x-6$

20 $P(x)$ is a real polynomial of degree 5 with leading coefficient 1. It has zeros $m\pm2i$, $1\pm mi$, and 2, where $m\in\mathbb{R}$. The graph of $y=P(x)$ cuts the y-axis at -56.

a Show that $m=\pm\sqrt{3}$. **b** Find the coefficient of x^4.

21 α and β are two of the roots of $x^3+ax^2+bx+c=0$.
Prove that $\alpha\beta$ is a root of $x^3-bx^2+acx-c^2=0$.

22



Find the cubic polynomial $P(x)$ with the graph shown.

23 Use axes intercepts to sketch the graphs of:

a $y = (x+5)(x+1)(x-3)$ **b** $y = -(x+2)^3$

c $f(x) = 2(x-2)^2(2x+3)$

24 **a** Show that -5 is an x-intercept of the graph of $f(x) = x^3 + 3x^2 - 14x - 20$.

b Find the other x-intercepts.

c Hence sketch the graph of $f(x) = x^3 + 3x^2 - 14x - 20$.

d Describe the behaviour of the graph as $x \to \infty$ and $x \to -\infty$.

25 The power generated by a wind turbine is given by $P = 20v^3$ watts, where v is the wind speed in m s^{-1}.

a For what values of v is it reasonable to use this function?

b Find the power generated at a wind speed of $6\ \text{m s}^{-1}$.

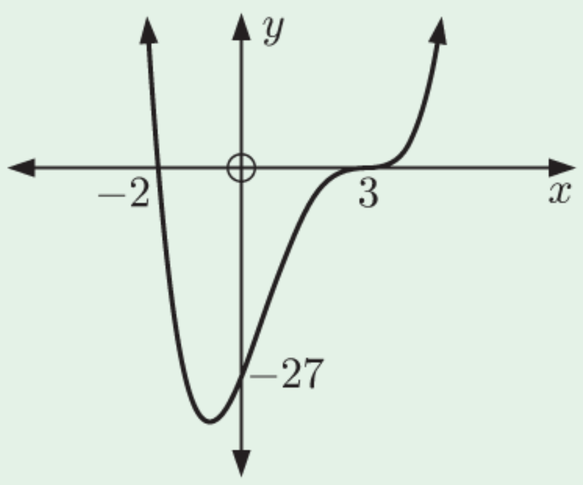
c Use technology to help sketch the graph of the function for $v \geqslant 0$.

d Find, rounded to 2 decimal places, the wind speed required to generate 10 000 watts of power.

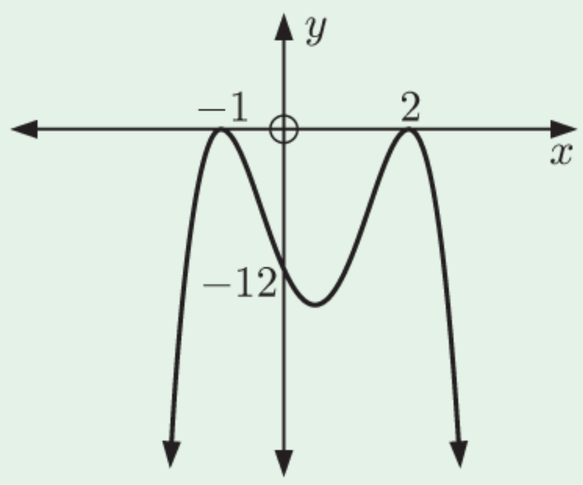


26 Find the equation of the quartic with graph:

a



b



27 The graph of $f(x) = 2x^4 - 12x^3 + 15x^2 + 9x - 20$ cuts the x-axis at -1 and 4. By fully factorising $2x^4 - 12x^3 + 15x^2 + 9x - 20$, show that the graph does not meet the x-axis anywhere else. Hence sketch the graph of $y = f(x)$.

28 Find all zeros of:

a $2z^3 + z^2 + 10z + 5$ **b** $x^3 + 6x^2 - 9x - 14$

29 Solve for x graphically:

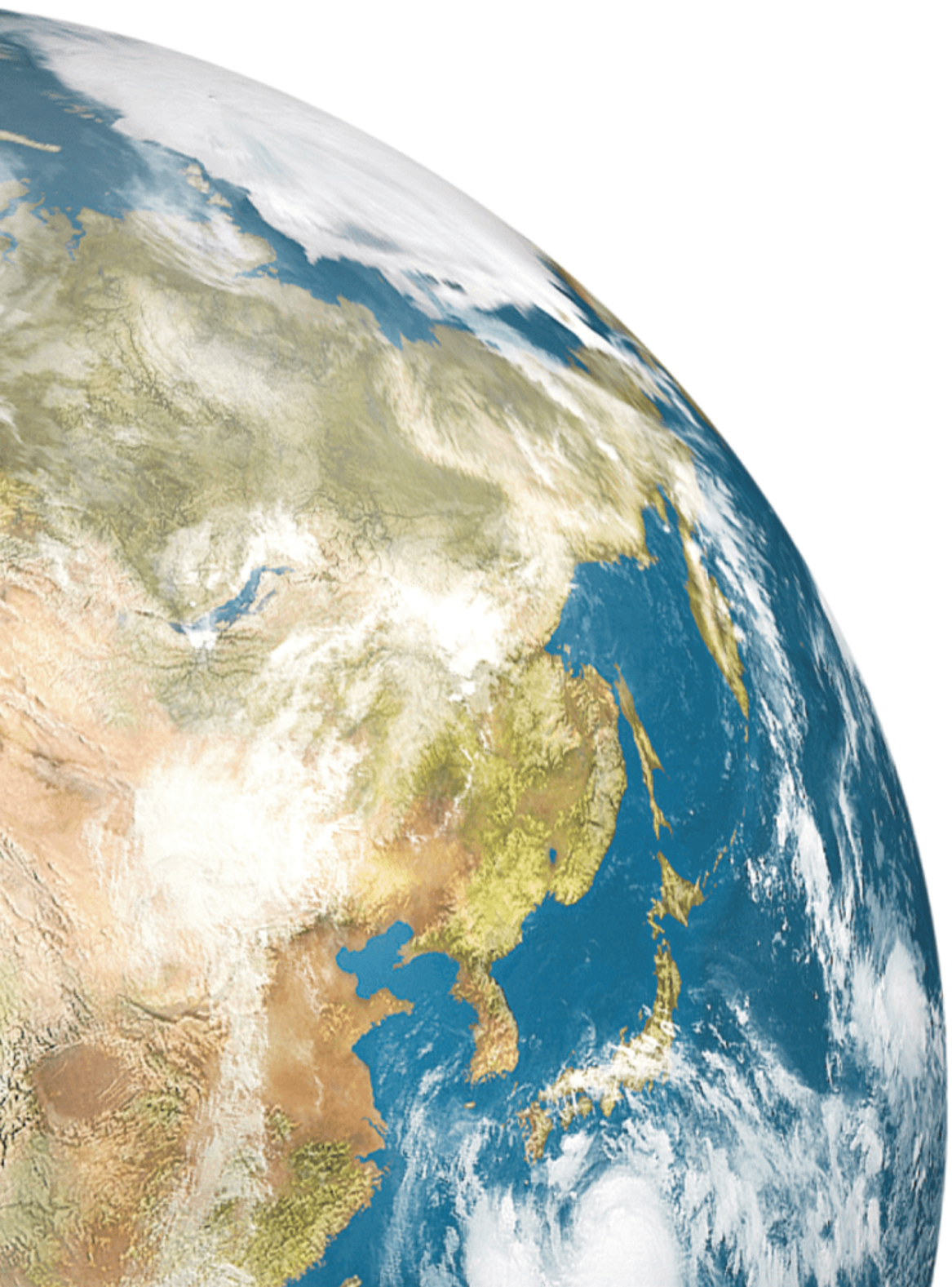
a $x^3 - 4x^2 + 1 \leqslant 2x - 3$ **b** $x^3 - 5 > 6x - 0.5x^3$

Chapter 6

Further functions

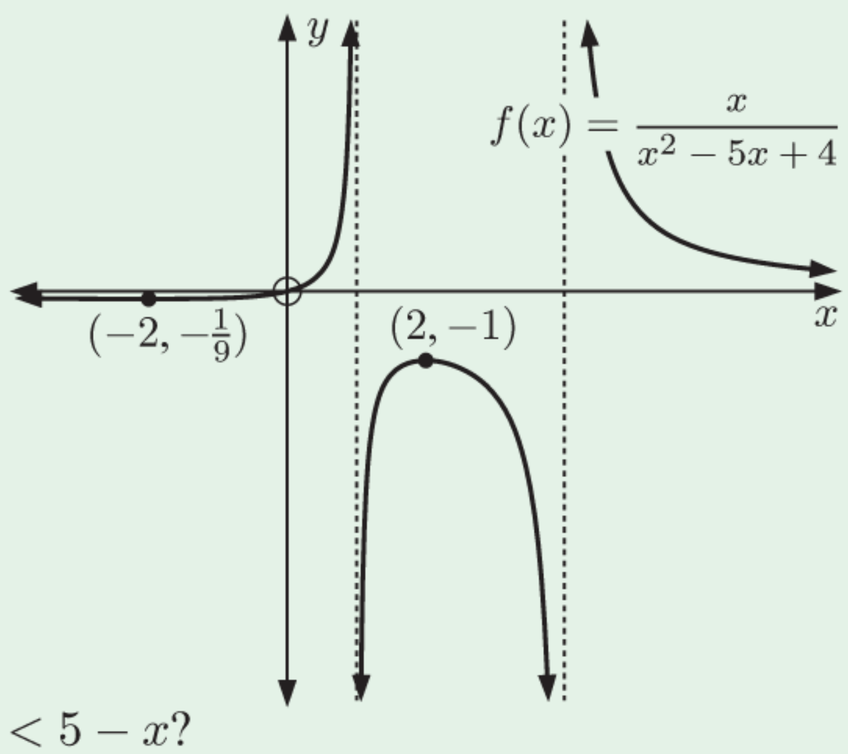
Contents:

A Even and odd functions
B The graph of $y = [f(x)]^2$
C Absolute value functions
D Rational functions
E Partial fractions



OPENING PROBLEM

The graph of $f(x) = \dfrac{x}{x^2 - 5x + 4}$ is shown alongside.



Things to think about:

a What are the asymptotes of the function?

b What would these graphs look like?

i $y = |f(x)|$ **ii** $y = f(|x|)$

iii $y = [f(x)]^2$

c Can you write the function in the form $f(x) = \dfrac{A}{x-1} + \dfrac{B}{x-4}$?

d How could you find the values of x such that $f(x) < 5 - x$?

In this Chapter we expand our knowledge of functions and their properties.

A EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS

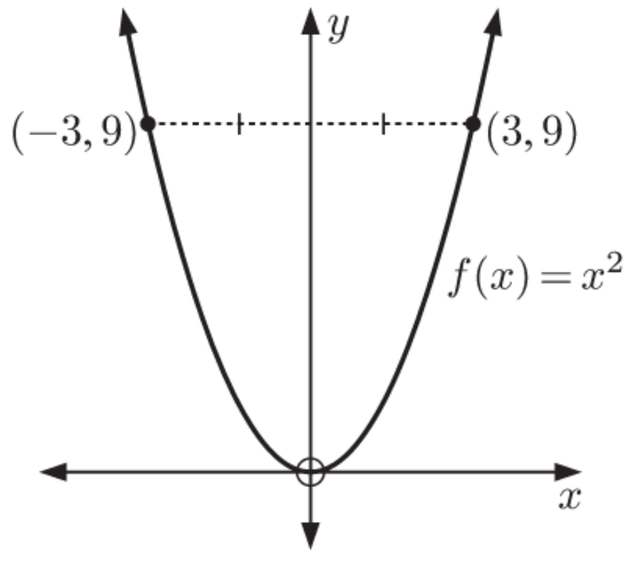
Consider the function $f(x) = x^2$.

Notice that $f(-3) = f(3) = 9$.

In fact, $f(-x) = f(x)$ for all x.

We know this is true since
$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x). \end{aligned}$$

We say that $f(x) = x^2$ is an **even function**.



A function $f(x)$ is **even** if $f(-x) = f(x)$ for all x in the domain of f.

The graph of an even function is symmetrical about the y-axis.

ODD FUNCTIONS

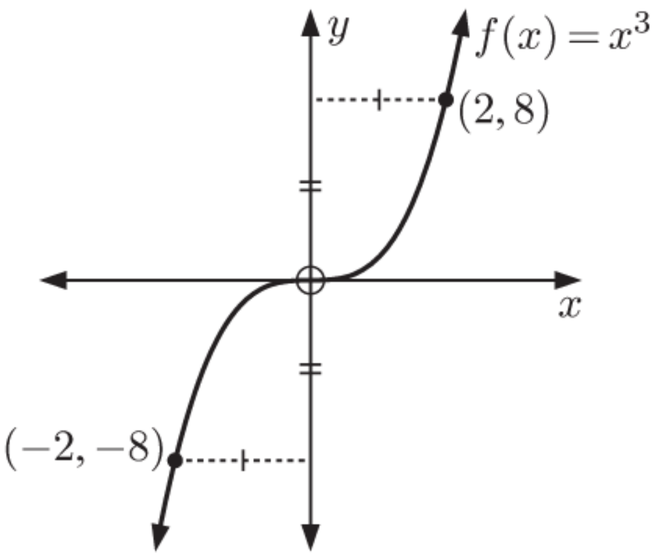
Consider the function $f(x) = x^3$.

Notice that $f(-2) = -8$ and $f(2) = 8$, so $f(-2) = -f(2)$.

In fact, $f(-x) = -f(x)$ for all x.

We know this is true since
$$\begin{aligned} f(-x) &= (-x)^3 \\ &= (-1)^3 x^3 \\ &= -x^3 \\ &= -f(x). \end{aligned}$$

We say that $f(x) = x^3$ is an **odd function**.



A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
The graph of an odd function has rotational symmetry about the origin.

Example 1 Self Tutor

Determine whether each function is even, odd, or neither:

a $f(x) = \dfrac{1}{x}$ b $g(x) = x^4 + x^2$ c $h(x) = x^2 - 2x + 3$

a $f(x) = \dfrac{1}{x}$

$\therefore\ f(-x) = \dfrac{1}{-x}$

$= -\dfrac{1}{x}$

$= -f(x)$

$\therefore\ f(x)$ is an odd function.

b $g(x) = x^4 + x^2$

$\therefore\ g(-x) = (-x)^4 + (-x)^2$

$= x^4 + x^2$

$= g(x)$

$\therefore\ g(x)$ is an even function.

c $h(x) = x^2 - 2x + 3$

$\therefore\ h(-x) = (-x)^2 - 2(-x) + 3$

$= x^2 + 2x + 3$

which is neither $h(x)$ nor $-h(x)$.

$\therefore\ h(x)$ is neither even nor odd.

Example 2 Self Tutor

Prove that the sum of two odd functions is also an odd function.

Suppose $h(x) = f(x) + g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Now $h(-x) = f(-x) + g(-x)$

$= -f(x) - g(x)$

$= -(f(x) + g(x))$

$= -h(x)$ for all x

$\therefore\ h(x)$ is odd.

Thus the sum of two odd functions is also an odd function.

EXERCISE 6A

1 Show that $f(x) = \dfrac{1}{x^2} + 2$ is an even function.

2 Show that $f(x) = x^3 - 3x$ is an odd function.

3 Explain why $f(x) = 4x^3$ is an odd function, even though 4 is an even number.

4 Determine whether each function is even, odd, or neither:

a $f(x) = 5x$ b $f(x) = -4x + 3$ c $f(x) = \dfrac{3}{x^2 - 4}$

d $f(x) = 2x^3 - \dfrac{5}{x}$ e $f(x) = x^2 + \dfrac{7}{x^2} - 3$ f $f(x) = \sqrt{x}$

5 Suppose $f(x)$ is an even function, and $(1, 3)$ and $(-5, -2)$ lie on the graph of $y = f(x)$. Write down two other points which lie on the graph of $y = f(x)$.

6 Suppose $g(x)$ is an odd function, and $(4, 6)$ and $(-1, 2)$ lie on the graph of $y = g(x)$. Write down two other points which lie on the graph of $y = g(x)$.

7 Suppose $f(x) = (2x + 3)(x + a)$, $a \in \mathbb{R}$ is an even function. Find the value of a.

8 Suppose $g(x) = (x + 1)\left(\frac{1}{x} + b\right)$, $b \in \mathbb{R}$ is an odd function. Find the value of b.

9 **a** If the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$ is an even function, prove that $b = 0$.

b Under what conditions is the cubic function $g(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, an odd function?

c Under what conditions is the quartic function $h(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$, an even function?

10 Decide whether each of the following is even, odd, or neither. Explain how this is observed in the graph of the function.

a $f(x) = \cos x$ **b** $f(x) = \sin x$ **c** $f(x) = \tan x$

11 For what values of k is $f(x) = \cos(x - k)$:

a even **b** odd **c** neither?

12 Prove that:

a the sum of two even functions is an even function

b the difference between two odd functions is an odd function

c the product of two odd functions is an even function.

13 Suppose $f(x)$ is an even function and $g(x)$ is an odd function. Determine whether the function $(f \circ g)(x)$ is even, odd, or neither.

14 $f(x)$ is an odd, invertible function. Determine whether the function $f^{-1}(x)$ is even, odd, or neither.

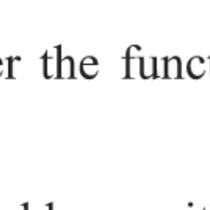
B THE GRAPH OF $y = [f(x)]^2$

INVESTIGATION 1 THE GRAPH OF $y = [f(x)]^2$

What to do:

1 Sketch each pair of functions on the same set of axes. Include all axes intercepts and asymptotes.

GRAPHING PACKAGE



a $y = x + 1$ and $y = (x + 1)^2$ **b** $y = x^2 - 4$ and $y = (x^2 - 4)^2$

c $y = \frac{2}{x}$ and $y = \left(\frac{2}{x}\right)^2$ **d** $y = \frac{1}{x - 3}$ and $y = \frac{1}{(x - 3)^2}$

e $y = 2^x - 1$ and $y = (2^x - 1)^2$ **f** $y = \log x$ and $y = (\log x)^2$

2 Determine the points where each pair of functions intersect. What do you notice?

3 What other observations can you make about the graph of $y = [f(x)]^2$?

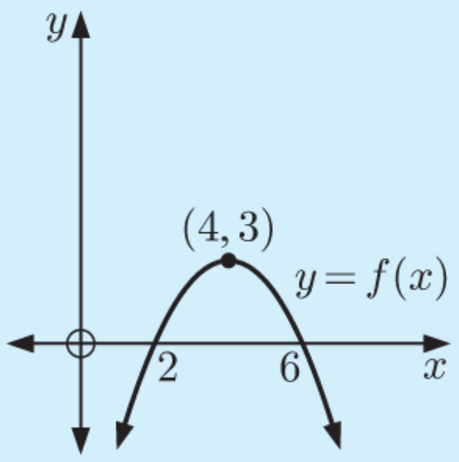
From the **Investigation**, you should have found that:

When $y = [f(x)]^2$ is graphed from $y = f(x)$:

- The points with y-coordinate 0 or 1 are invariant.
- The graph of $y = [f(x)]^2$ *touches* the x-axis at its x-intercepts.
- The graph of $y = [f(x)]^2$ lies above or on the x-axis for all x.
- The vertical asymptotes of $y = f(x)$ are also vertical asymptotes of $y = [f(x)]^2$.
- When $-1 < f(x) < 1$, $y = [f(x)]^2$ is closer to the x-axis than $y = f(x)$.
- When $|f(x)| > 1$, $y = [f(x)]^2$ is further from the x-axis than $y = f(x)$.

Example 3 **Self Tutor**

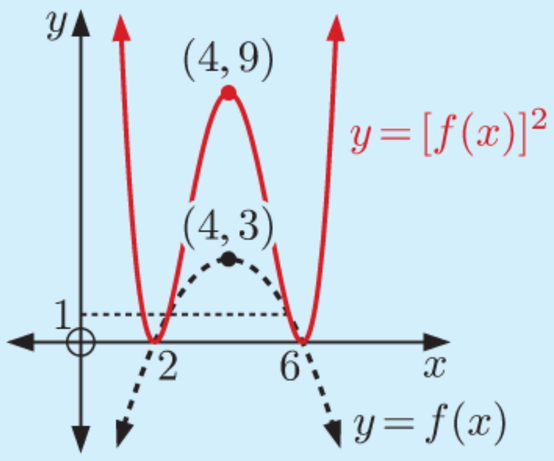
For the graph of $y = f(x)$ alongside, draw the graph of $y = [f(x)]^2$.



$y = f(x)$ cuts the x-axis at 2 and 6, so $y = [f(x)]^2$ touches the x-axis at 2 and 6.

$y = f(x)$ and $y = [f(x)]^2$ also intersect when $y = 1$.

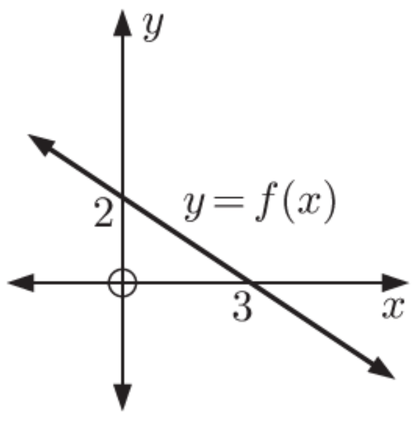
$y = f(x)$ has a turning point at $(4, 3)$, so $y = [f(x)]^2$ has a turning point at $(4, 9)$.



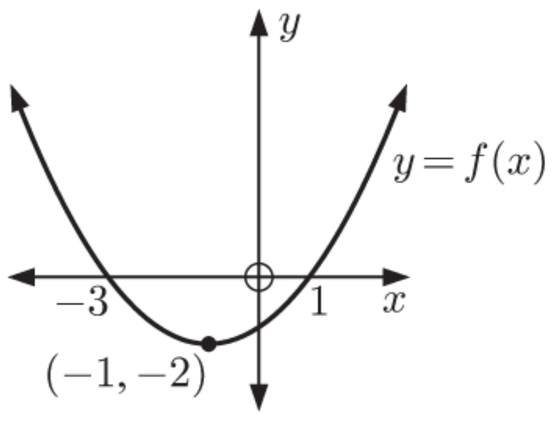
EXERCISE 6B

1 For each graph of $y = f(x)$, sketch the graph of $y = [f(x)]^2$.

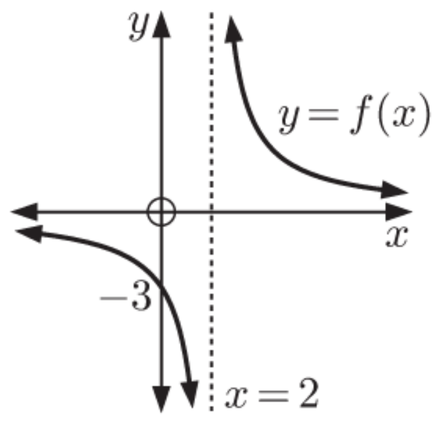
a



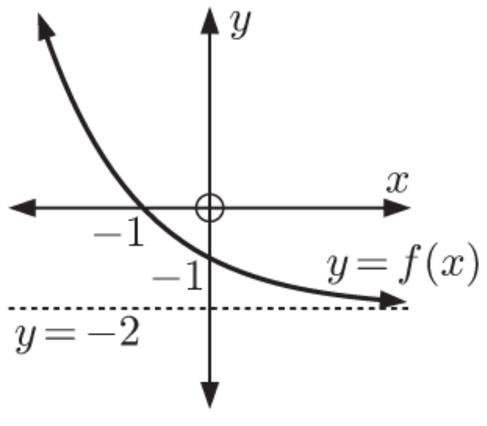
b



c



2 Copy the graph alongside, and on the same set of axes, sketch the graph of $y = [f(x)]^2 + 1$.



PRINTABLE GRAPHS

3 **a** Draw the graph of $y = f(x) = 3^x$.

b On the same set of axes, sketch the graph of $y = [f(x)]^2$.

c Describe a stretch which transforms $y = f(x)$ to $y = [f(x)]^2$.

4 **a** Draw the graph of $y = f(x) = \log_3 x$.

b On the same set of axes, sketch the graph of $y = [f(x)]^2$.

c Find the invariant points when $y = f(x)$ is transformed to $y = [f(x)]^2$.

5 Show that if $f(x)$ is odd, then $[f(x)]^2$ is even.

6 Suppose $f(x)$ has domain $0 \leqslant x \leqslant 5$ and range $-4 \leqslant y \leqslant 3$. Find the domain and range of $[f(x)]^2$.

7 Consider the function $f(x) = \dfrac{2x+4}{x-1}$

a Find the axes intercepts and asymptotes of the function.

b Hence find the axes intercepts and asymptotes of $y = [f(x)]^2$.

c Which points are invariant when $y = f(x)$ is transformed to $y = [f(x)]^2$?

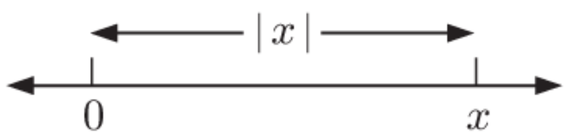
d Sketch $y = f(x)$ and $y = [f(x)]^2$ on the same set of axes.

C ABSOLUTE VALUE FUNCTIONS

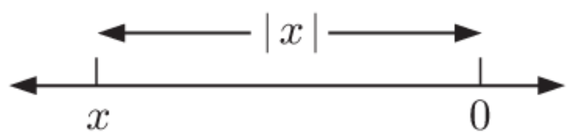
The **absolute value** or **modulus** of a real number x is its distance from 0 on the number line. We write the absolute value of x as $|x|$.

Because the absolute value is a distance, it cannot be negative.

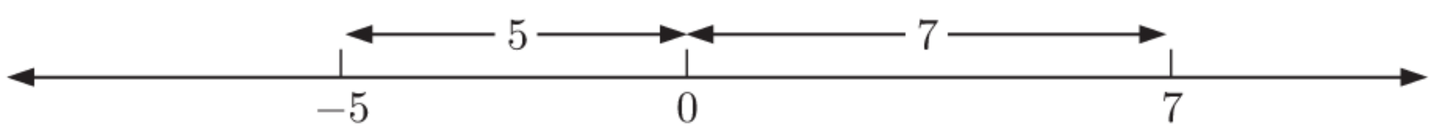
- If $x > 0$, $|x| = x$.



- If $x < 0$, $|x| = -x$.



For example: $|7| = 7$ and $|-5| = 5$.

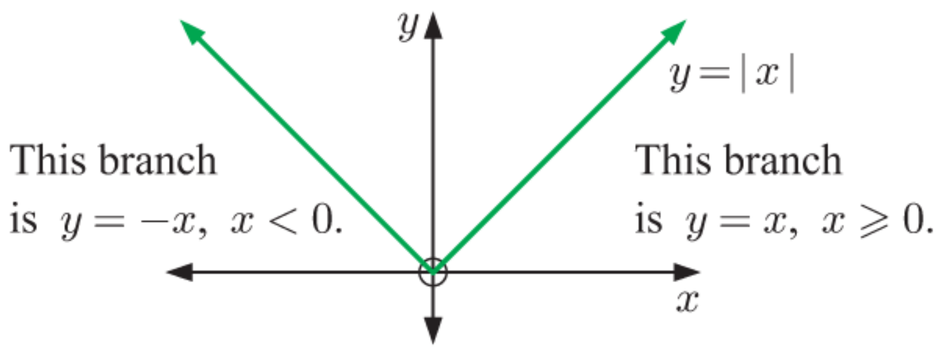


This leads us to the **algebraic definition**:

The **absolute value of** x is $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$

or alternatively $|x| = \sqrt{x^2}$.

The relation $y = |x|$ is in fact a function. We call it the **absolute value function**, and it has the graph shown.



THE GRAPH $y = |f(x)|$

The absolute value of the function $f(x)$ is $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$.



To obtain the graph of $y = |f(x)|$ from the graph of $y = f(x)$:

- keep the graph for $f(x) \geqslant 0$
- reflect the graph in the x-axis for $f(x) < 0$, discarding what was there
- points on the x-axis are invariant.

THE GRAPH $y = f(|x|)$

We know that $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$, so $f(|x|) = \begin{cases} f(x) & \text{if } x \geqslant 0 \\ f(-x) & \text{if } x < 0 \end{cases}$.

To obtain the graph of $y = f(|x|)$ from the graph of $y = f(x)$:

- discard the graph for $x < 0$
- reflect the graph for $x \geqslant 0$ in the y-axis, keeping what was there
- points on the y-axis are invariant.

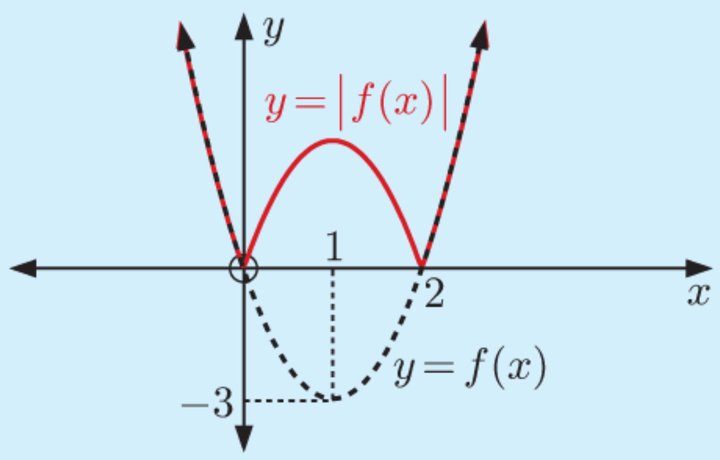
Example 4 **Self Tutor**

Draw the graph of $f(x) = 3x(x-2)$, and on the same set of axes draw the graph of:

a $y = |f(x)|$ **b** $y = f(|x|)$

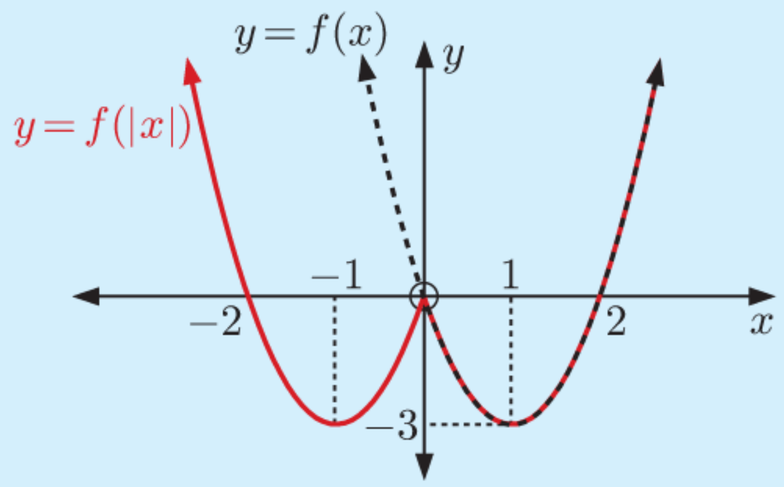
a $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for $f(x) \geqslant 0$ and reflected in the x-axis for $f(x) < 0$.



b $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geqslant 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for $x \geqslant 0$ and reflected in the y-axis for $x < 0$.



EXERCISE 6C.1

1 Draw $y = f(x) = 2x - 4$, and on the same set of axes draw the graph of:

a $y = |f(x)|$ **b** $y = f(|x|)$

2 Draw $y = f(x) = x(x+2)$, and on the same set of axes draw the graph of:

a $y = |f(x)|$ **b** $y = f(|x|)$

3 Draw $y = f(x) = -x^2 + 6x - 8$, and on the same set of axes draw the graph of:

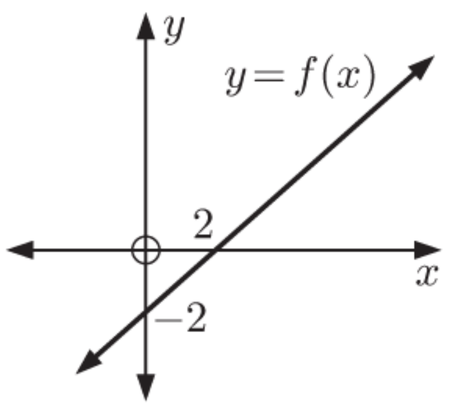
a $y = |f(x)|$ **b** $y = f(|x|)$

4 Draw $y = f(x) = 3^x - 3$, and on the same set of axes draw the graph of:

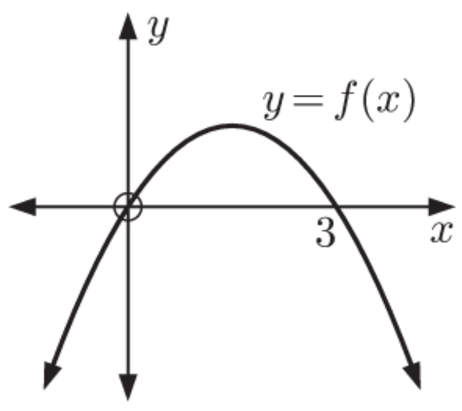
a $y = |f(x)|$ **b** $y = f(|x|)$

5 **a** Copy the following graphs for $y = f(x)$ and on the same set of axes draw the graph of $y = |f(x)|$:

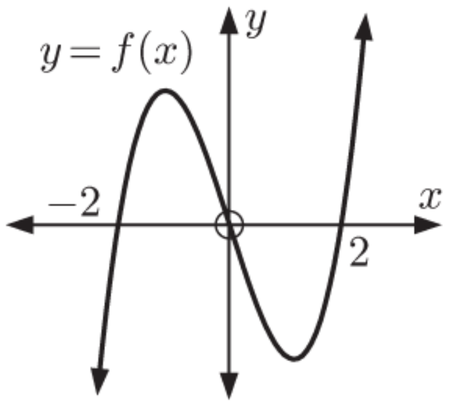
i



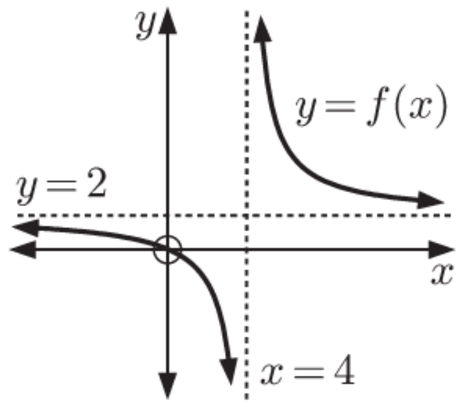
ii



iii



iv



b Repeat **a**, but this time draw the graphs of $y = f(x)$ and $y = f(|x|)$ on the same set of axes.

6 Show that if $f(x)$ is an odd function, then $|f(x)|$ is an even function.

7 Let $f(x) = \sqrt{4 - x}$.

a Draw the graph of $y = f(x)$ and state its domain and range.

b Draw the graph of $y = f(|x|)$ and state its domain and range.

8 Suppose $f(x)$ has domain $-2 \leqslant x \leqslant 6$ and range $-7 \leqslant y \leqslant 5$.

a Find:

i the domain of $f(|x|)$ **ii** the range of $|f(x)|$.

b Can we determine the range of $f(|x|)$? Explain your answer.

9 Suppose $f(x)$ has x-intercepts -3 and 4, and y-intercept -2. Find the axes intercepts of:

a $|f(x)|$ **b** $f(|x|)$

10 Suppose $f(x) = \log_2(x+4)$. Sketch the graphs of $y = f(x)$ and $y = |f(|x|)|$ on the same set of axes.

11 Use the definition $|x| = \sqrt{x^2}$ to prove that:

a $|-x| = |x|$ for all x

b $|x|^2 = x^2$ for all x

c $|xy| = |x||y|$ for all x, y

d $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for all x and y, $y \neq 0$

e $|x - y| = |y - x|$ for all x, y

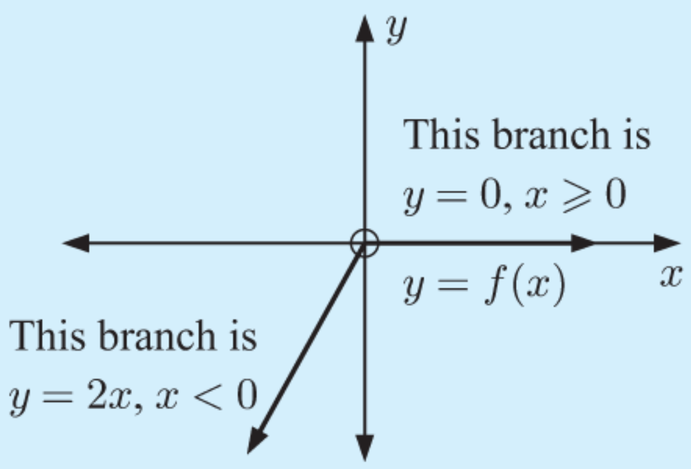
Example 5 **Self Tutor**

Use the definition $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$ to write the following functions without the modulus sign. Hence graph each function:

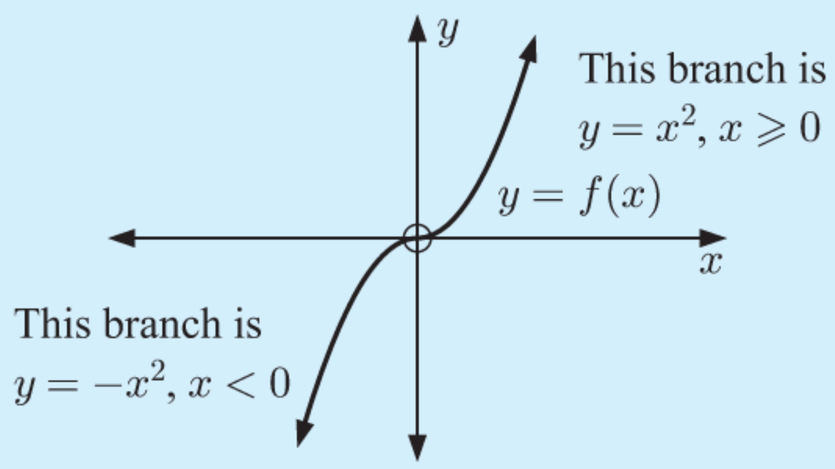
a $f(x) = x - |x|$

b $f(x) = x|x|$

a If $x < 0$, $f(x) = x - (-x) = 2x$.
If $x \geqslant 0$, $f(x) = x - x = 0$.



b If $x \geqslant 0$, $f(x) = x(x) = x^2$.
If $x < 0$, $f(x) = x(-x) = -x^2$.



12 Use $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$ to write the following functions without modulus signs.

Hence graph each function.

a $y = |x - 2|$

b $y = |x + 1|$

c $y = -|x|$

d $y = |x| + x$

e $y = \frac{|x|}{x}$

f $y = x - 2|x|$

g $y = |x| + |x - 2|$

h $y = |x| - |x - 1|$

MODULUS EQUATIONS

A modulus equation is an equation which involves the absolute value function.

To solve modulus equations we use the definitions of absolute value and the properties found in the previous Exercise:

- $|x| \geqslant 0$ for all x
- $|x|^2 = x^2$ for all x
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for all x and y, $y \neq 0$
- $|-x| = |x|$ for all x
- $|xy| = |x||y|$ for all x and y
- $|x - y| = |y - x|$ for all x and y.

We notice in particular that if $|x| = a, \ a > 0$

then $|x|^2 = a^2$

$\therefore \ x^2 = a^2 \qquad \{|x|^2 = x^2\}$

$\therefore \ x = \pm\sqrt{a^2}$

$\therefore \ x = \pm a$

If $|x| = a$ where $a > 0$, then $x = \pm a$.

Example 6 **Self Tutor**

Solve for x:

a $|2x + 3| = 7$ **b** $|3 - 2x| = -1$

a $|2x + 3| = 7$

$\therefore \ 2x + 3 = \pm 7$

$\therefore \ 2x + 3 = 7$ or $2x + 3 = -7$

$\therefore \ 2x = 4$ or $2x = -10$

$\therefore \ x = 2$ or $x = -5$

So, $x = 2$ or -5.

b $|3 - 2x| = -1$

has no solution as LHS is never negative.

Example 7 **Self Tutor**

Solve for x:

a $\left|\dfrac{3x+2}{1-x}\right| = 4$ **b** $|x + 1| = |2x - 3|$

a If $\left|\dfrac{3x+2}{1-x}\right| = 4$ then $\dfrac{3x+2}{1-x} = \pm 4$.

$\therefore \ \dfrac{3x+2}{1-x} = 4$ or $\dfrac{3x+2}{1-x} = -4$

$\therefore \ 3x + 2 = 4(1 - x)$ or $3x + 2 = -4(1 - x)$

$\therefore \ 3x + 2 = 4 - 4x$ or $3x + 2 = -4 + 4x$

$\therefore \ 7x = 2$ or $6 = x$

$\therefore \ x = \frac{2}{7}$ or $x = 6$

So, $x = \frac{2}{7}$ or 6.

b If $|x + 1| = |2x - 3|$, then $x + 1 = \pm(2x - 3)$

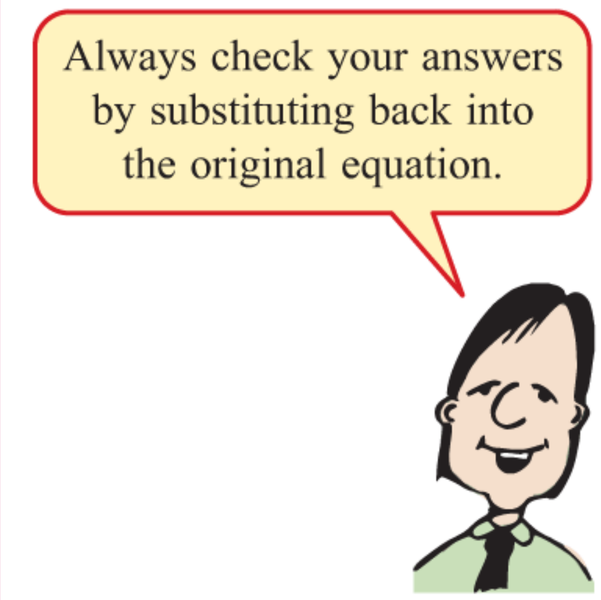
$\therefore \ x + 1 = 2x - 3$ or $x + 1 = -(2x - 3)$

$\therefore \ 4 = x$ or $x + 1 = -2x + 3$

$\therefore \ x = 4$ or $3x = 2$

$\therefore \ x = 4$ or $x = \frac{2}{3}$

So, $x = \frac{2}{3}$ or 4.



EXERCISE 6C.2

1 Solve for x:

a $|x| = 3$ **b** $|x| = -5$ **c** $|x| = 0$

d $|x - 1| = 3$ **e** $|3 - x| = 4$ **f** $|x + 5| = -1$

g $|3x - 2| = 1$ **h** $|3 - 2x| = 3$ **i** $|2 - 5x| = 12$

2 Solve for x:

a $\left|\dfrac{x}{x-1}\right| = 3$ **b** $\left|\dfrac{2x-1}{x+1}\right| = 5$ **c** $\left|\dfrac{x+3}{1-3x}\right| = \dfrac{1}{2}$

3 **a** Explain why the equation $\left|\dfrac{3x+1}{x-1}\right| = 3$ has only one solution.

b Find the solution.

4 Prove that if $|x| = |a|$ then $x = \pm a$.

5 Solve for x:

a $|3x - 1| = |x + 2|$ **b** $|2x + 5| = |1 - x|$ **c** $|x + 1| = |2 - x|$

d $|x| = |5 - x|$ **e** $|1 - 4x| = 2|x - 1|$ **f** $|3x + 2| = 2|2 - x|$

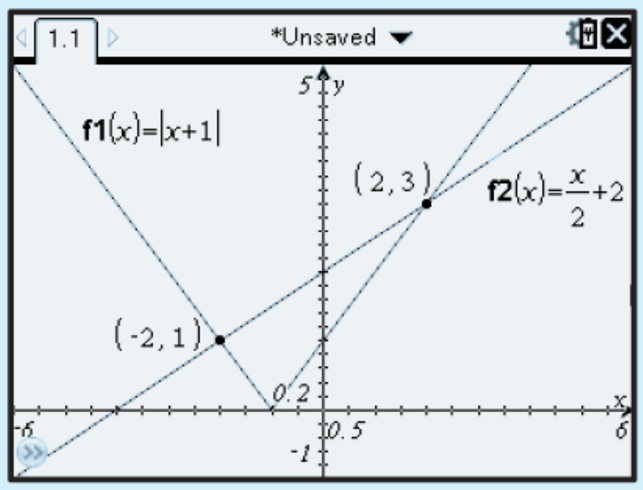
Example 8 **Self Tutor**

Solve graphically: $|x + 1| = \dfrac{x}{2} + 2$

We graph $y = |x + 1|$ and $y = \dfrac{x}{2} + 2$ on the same set of axes.

The graphs intersect at $(-2, 1)$ and $(2, 3)$

$\therefore$ the solutions are $x = -2$ or 2.





GRAPHICS CALCULATOR INSTRUCTIONS

6 **a** Sketch the graphs of $y = |x + 3|$ and $y = 7 - \dfrac{x}{3}$ on the same set of axes.

b Solve the equation $|x + 3| = 7 - \dfrac{x}{3}$.

7 Solve graphically:

a $|x + 2| = 2x + 1$ **b** $|2x + 3| = 3|x| - 1$ **c** $|x - 2| = \frac{2}{5}x + 1$

d $x^2 - 1 = |5x - x^2|$ **e** $|e^x - 6| = 2^x$ **f** $x^3 - 4x = \ln|x|$

MODULUS INEQUALITIES

There are several methods we can use to solve modulus inequalities, depending on the situation:

- For some modulus inequalities it is easiest to separate the inequality into two.

Example 9 — Self Tutor

Solve for x:

a $|2x-1| < 3$ **b** $|4-x| \geqslant 5$

a
$$|2x-1| < 3$$
$$\therefore\ -3 < 2x-1 < 3$$
$$\therefore\ -2 < 2x < 4$$
$$\therefore\ -1 < x < 2$$

b
$$|4-x| \geqslant 5$$
$$\therefore\ 4-x \leqslant -5 \quad \text{or} \quad 4-x \geqslant 5$$
$$\therefore\ x \geqslant 9 \quad \text{or} \quad x \leqslant -1$$

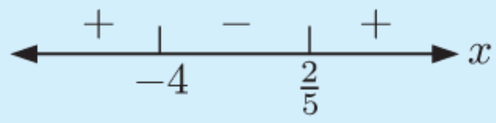
- For modulus inequalities where we know each side of the inequality is positive, we can square both sides.

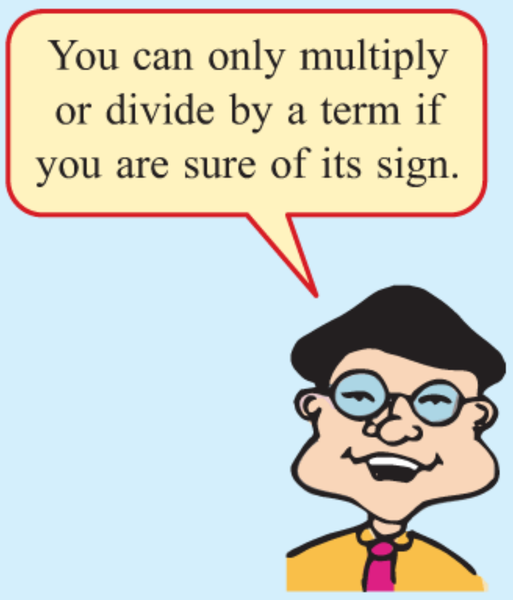
Example 10 — Self Tutor

Solve for x:

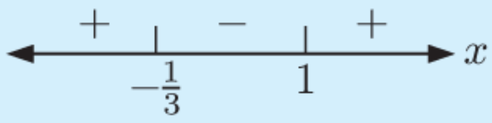
a $|3x+1| \geqslant |2x-3|$ **b** $\left|\dfrac{x}{x+1}\right| \geqslant \dfrac{1}{2}$

a
$$|3x+1| \geqslant |2x-3|$$
$$\therefore\ |3x+1|^2 \geqslant |2x-3|^2 \qquad \{\text{squaring both sides}\}$$
$$\therefore\ (3x+1)^2 \geqslant (2x-3)^2 \qquad \{|a|^2 = a^2\}$$
$$\therefore\ 9x^2+6x+1 \geqslant 4x^2-12x+9$$
$$\therefore\ 5x^2+18x-8 \geqslant 0$$
$$\therefore\ (5x-2)(x+4) \geqslant 0$$
$$\therefore\ x \leqslant -4 \quad \text{or} \quad x \geqslant \tfrac{2}{5}$$





b
$$\left|\frac{x}{x+1}\right| \geqslant \frac{1}{2}$$
$$\therefore\ \left|\frac{x}{x+1}\right|^2 \geqslant \left(\frac{1}{2}\right)^2 \qquad \{\text{squaring both sides}\}$$
$$\therefore\ \left(\frac{x}{x+1}\right)^2 \geqslant \frac{1}{4} \qquad \{|a|^2 = a^2\}$$
$$\therefore\ \frac{x^2}{(x+1)^2} \geqslant \frac{1}{4}$$
$$\therefore\ 4x^2 \geqslant (x+1)^2 \qquad \text{provided } x \neq -1 \quad \{\text{since } (x+1)^2 \geqslant 0\}$$
$$\therefore\ 4x^2 \geqslant x^2+2x+1 \qquad \text{provided } x \neq -1$$
$$\therefore\ 3x^2-2x-1 \geqslant 0 \qquad \text{provided } x \neq -1$$
$$\therefore\ (3x+1)(x-1) \geqslant 0 \qquad \text{provided } x \neq -1$$
$$\therefore\ x \leqslant -\tfrac{1}{3},\ x \neq -1 \quad \text{or} \quad x \geqslant 1$$



EXERCISE 6C.3

1 Solve for x:

a $|x-1| \leqslant 4$ **b** $|x+2| > 7$ **c** $|2x-3| < 1$

d $|3x+5| \geqslant 2$ **e** $|1-2x| < 4$ **f** $|-5x-4| > \frac{1}{2}$

2 Solve for x:

a $|x+5| < |x-1|$ **b** $|2x+1| > |x+2|$ **c** $|x-3| \leqslant |2x-6|$

d $|3x+4| \geqslant |x+3|$ **e** $2|x-4| < |3x+2|$ **f** $\frac{1}{3}|2x+5| \geqslant |4-x|$

3 Solve for x:

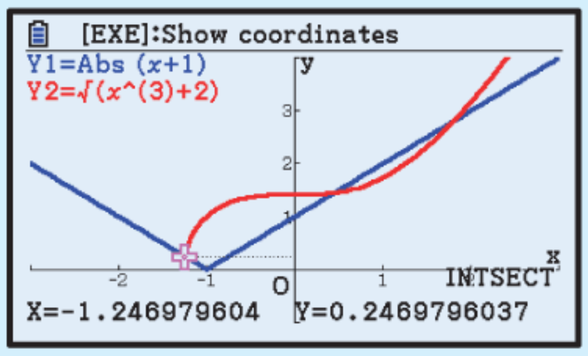
a $\left|\dfrac{x}{x-2}\right| \geqslant 3$ **b** $\left|\dfrac{2x+3}{x-1}\right| \geqslant 2$ **c** $\left|\dfrac{x-4}{1-2x}\right| < \dfrac{2}{3}$

Example 11 — Self Tutor

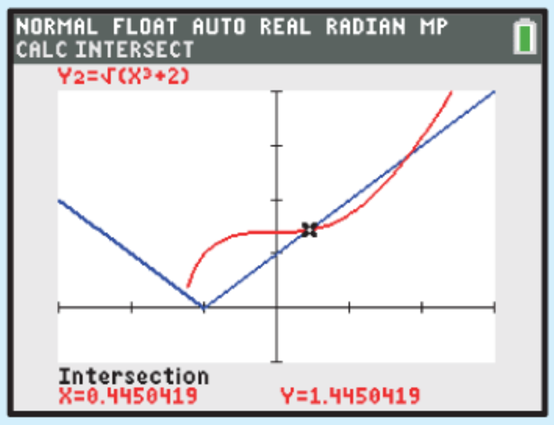
Solve graphically: $|x+1| > \sqrt{x^3+2}$.

We draw graphs of $y = |x+1|$ and $y = \sqrt{x^3+2}$ on the same set of axes.

Casio fx-CG50



TI-84 Plus CE

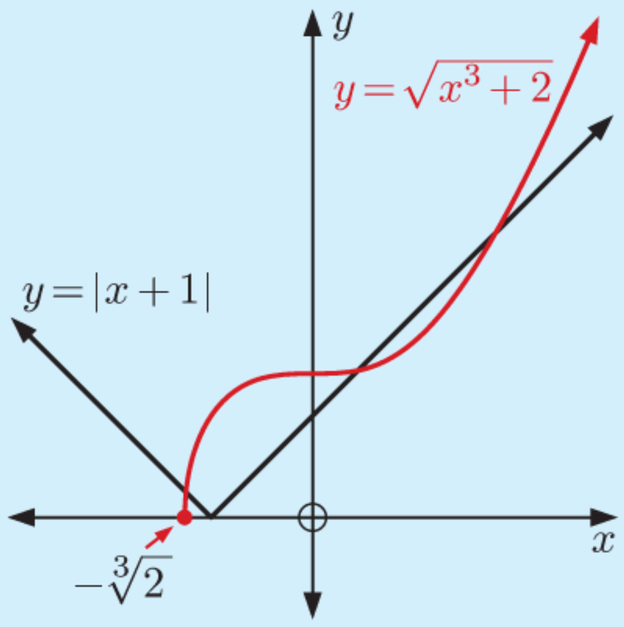


HP Prime



The graphs intersect at $x \approx -1.2470$, $x \approx 0.4450$, and $x \approx 1.8019$.

GRAPHICS CALCULATOR INSTRUCTIONS



$\therefore$ $|x+1| > \sqrt{x^3+2}$ when $-\sqrt[3]{2} \leqslant x < -1.247$ and when $0.445 < x < 1.802$.

Make sure you consider the domain of each function you graph.



4 Solve graphically:

a $|2x-3| < x$ **b** $|x| - 2 \geqslant |4-x|$

c $x^2 - 4 > |x-1|$ **d** $|2x-1| + |x-4| \leqslant 10 - x$

e $\ln\left|x^2 - x + 5\right| \geqslant 0.1x^4$ **f** $7|x| - e^{|x|} \geqslant \log_2(x^2 - x - 2)$

g $|3x \arccos x| > 1$ **h** $0.2\ln(7-2x) < \left|\sqrt{15-x^2} - 3\right|$

D RATIONAL FUNCTIONS

In **Chapter 15** of the **Core Topics HL** book we studied **rational functions** of the form $y = \dfrac{ax+b}{cx+d}$.

In this Section we consider rational functions of the form $y = \dfrac{ax+b}{cx^2+dx+e}$ and $y = \dfrac{ax^2+bx+c}{dx+e}$.

INVESTIGATION 2 RATIONAL FUNCTIONS OF THE FORM $y = \dfrac{ax+b}{cx^2+dx+e}$, $c \neq 0$

What to do:

1 Use technology to examine the graphs of the following functions. For each graph:

i State the domain.

ii Write down the equations of the asymptotes.

a $y = \dfrac{x+5}{(x+1)(x-2)}$ **b** $y = \dfrac{2x-6}{(x+3)(x-2)}$ **c** $y = \dfrac{1-3x}{x(x-4)}$

d $y = \dfrac{2x-5}{x^2-2x+3}$ **e** $y = \dfrac{1-x}{x^2+7x+10}$ **f** $y = \dfrac{3x+6}{x^2-6x+9}$

GRAPHING PACKAGE

2 For a function of the form $y = \dfrac{ax+b}{cx^2+dx+e}$, $c \neq 0$:

a What is the horizontal asymptote?

b How can we determine the vertical asymptotes?

From the **Investigation**, you should have found that:

For a function of the form $y = \dfrac{ax+b}{cx^2+dx+e}$, $c \neq 0$:

- the horizontal asymptote is $y = 0$
- the vertical asymptotes are the zeros of the denominator cx^2+dx+e.

Example 12 Self Tutor

Consider the function $y = \dfrac{2x-6}{x^2-3x-4}$.

a Find the asymptotes of the function. **b** Find the axes intercepts.

c Draw a sign diagram of the function.

d Hence discuss the behaviour of the function near the asymptotes.

e Sketch the function, showing the features you have found.

a The horizontal asymptote is $y = 0$.

The function is undefined when $x^2-3x-4 = 0$

$\therefore\ (x+1)(x-4) = 0$

$\therefore\ x = -1$ or 4

$\therefore$ the vertical asymptotes are $x = -1$ and $x = 4$.

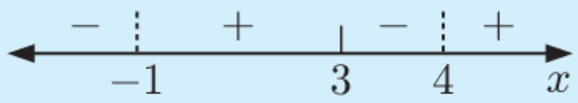
b When $x = 0$, $y = \dfrac{-6}{-4} = \frac{3}{2}$, so the y-intercept is $\frac{3}{2}$.

When $y = 0$, $2x - 6 = 0$

$\therefore \quad x = 3$

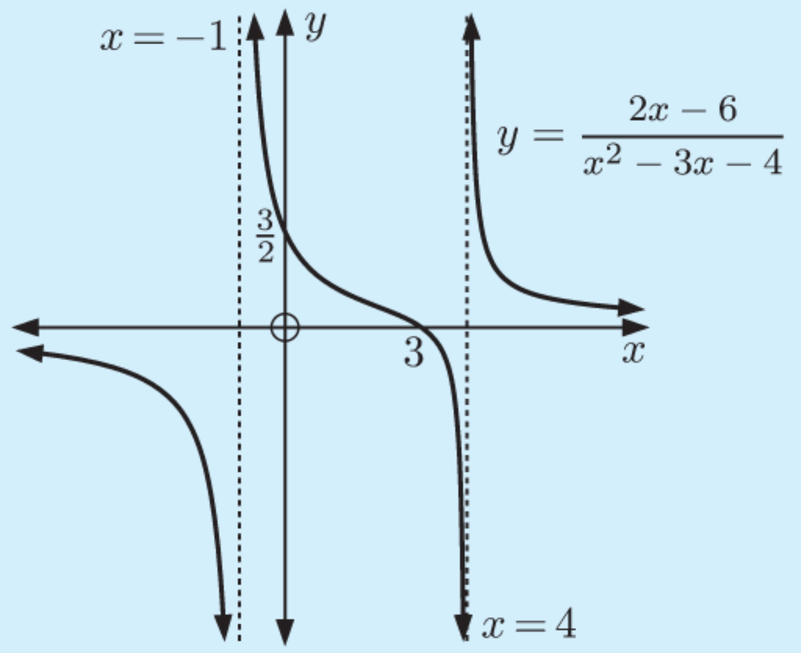
$\therefore$ the x-intercept is 3.

c $y = \dfrac{2(x-3)}{(x+1)(x-4)}$ has sign diagram



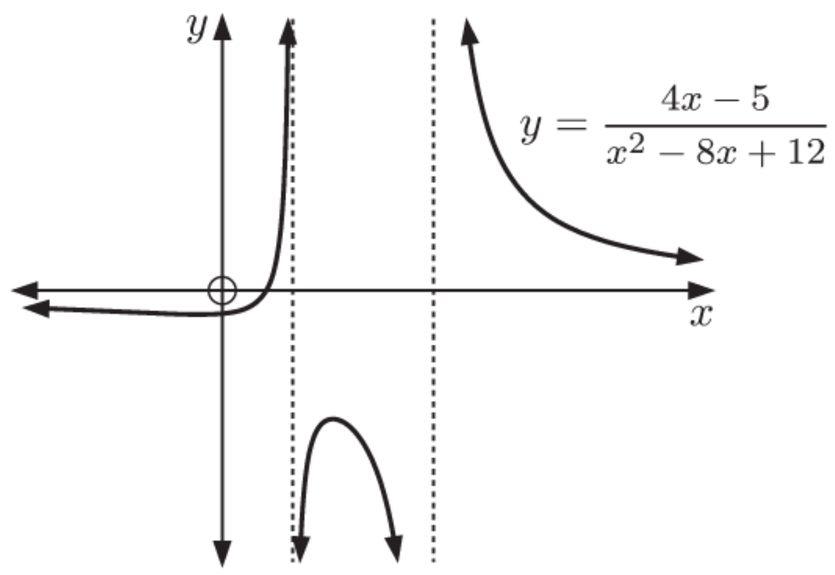
d As $x \to -1^-$, $y \to -\infty$
As $x \to -1^+$, $y \to \infty$
As $x \to 4^-$, $y \to -\infty$
As $x \to 4^+$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

e



EXERCISE 6D.1

1 The graph of $y = \dfrac{4x-5}{x^2-8x+12}$ is shown alongside.



a Find the asymptotes of the function.

b Find the axes intercepts.

2 For each of the following functions:

i Find the equations of the asymptotes.

ii Find the axes intercepts.

iii Draw a sign diagram of the function.

iv Hence discuss the behaviour of the function near its asymptotes.

v Sketch the graph of the function.

a $y = \dfrac{x-1}{(x+3)(x-2)}$ **b** $y = \dfrac{2x-8}{(x-1)(x-3)}$ **c** $f(x) = \dfrac{5-3x}{(x+2)(x-4)}$

d $f(x) = \dfrac{-x-4}{x^2+3x}$ **e** $y = \dfrac{2x+5}{x^2-4x-5}$ **f** $y = \dfrac{12-4x}{x^2-3x-18}$

g $f(x) = \dfrac{5x-9}{2x^2+10x-12}$ **h** $y = \dfrac{3x+6}{x^2+2x+1}$ **i** $f(x) = \dfrac{4-x}{2x^2-9x-5}$

j $y = \dfrac{6x+15}{3x^2+2x-8}$ **k** $f(x) = \dfrac{7-4x}{21+8x-4x^2}$ **l** $y = \dfrac{1-3x}{9x^2-12x+4}$

3 Consider the function $y = \dfrac{4x - 6}{x^2 - 5x + 7}$.

a State the equation of the horizontal asymptote.

b Explain why the function has no vertical asymptotes.

c Find the axes intercepts.

d Use technology to help sketch the function.

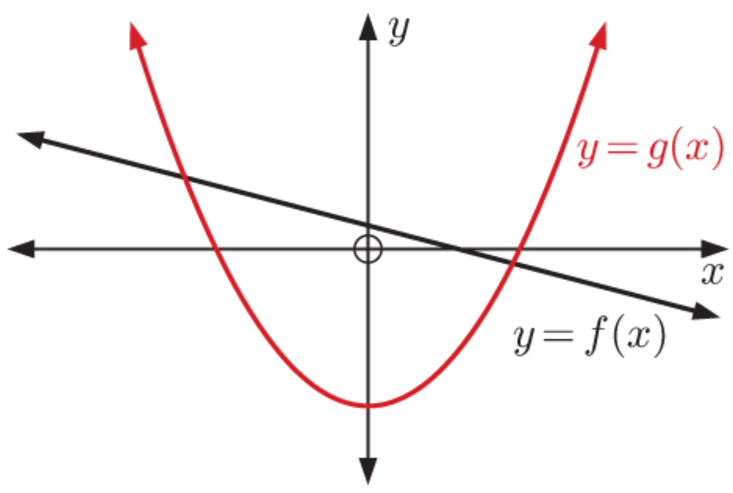
4 The function $f(x) = \dfrac{8x - 3}{cx^2 + dx + 9}$ has one vertical asymptote, and passes through the point $(1, 5)$.

a Show that $f(x)$ can take two possible forms. Find the vertical asymptote of $f(x)$ in each case.

b Sketch the possible graphs of $y = f(x)$ on the same set of axes.

5 The graph alongside shows a linear function $f(x)$ and a quadratic function $g(x)$.

Copy the graph, and on the same set of axes, graph $y = \dfrac{f(x)}{g(x)}$. Indicate clearly where the x-intercept(s) and asymptote(s) occur.



RATIONAL FUNCTIONS OF THE FORM $y = \dfrac{ax^2 + bx + c}{dx + e}$

INVESTIGATION 3 RATIONAL FUNCTIONS OF THE FORM $y = \dfrac{ax^2 + bx + c}{dx + e}$, $d \neq 0$

What to do:

1 Consider the function $y = \dfrac{x^2 - x - 5}{x - 3}$.

GRAPHING PACKAGE

a State the equation of the vertical asymptote.

b Show that $\dfrac{x^2 - x - 5}{x - 3} = x + 2 + \dfrac{1}{x - 3}$.

c Use technology to sketch $y = \dfrac{x^2 - x - 5}{x - 3}$ and $y = x + 2$ on the same set of axes.

d Explain why the graph of $y = \dfrac{x^2 - x - 5}{x - 3}$ approaches the line $y = x + 2$ as $|x| \to \infty$.

2 Consider the function $y = \dfrac{6x^2 + x - 3}{2x + 1}$.

a State the equation of the vertical asymptote.

b Show that $\dfrac{6x^2 + x - 3}{2x + 1} = 3x - 1 - \dfrac{2}{2x + 1}$.

c Use technology to sketch $y = \dfrac{6x^2 + x - 3}{2x + 1}$ and $y = 3x - 1$ on the same set of axes.

d Describe the behaviour of $y = \dfrac{6x^2 + x - 3}{2x + 1}$ as $|x| \to \infty$.

To help us describe graphs of the form $y = \dfrac{ax^2 + bx + c}{dx + e}$, we use polynomial division to write them in the form $y = px + q + \dfrac{r}{dx + e}$.

- The function is undefined when $dx + e = 0$, so the graph has the vertical asymptote $x = -\dfrac{e}{d}$.
- As $|x| \to \infty$, the term $\dfrac{r}{dx + e} \to 0$, and so the graph gets closer to the line $y = px + q$. The graph has the **oblique asymptote** $y = px + q$.

An **oblique asymptote** is an asymptote which is neither vertical nor horizontal.



Example 13 **Self Tutor**

Consider the function $f(x) = \dfrac{2x^2 + 3x - 9}{x + 2}$.

a Find the vertical asymptote of the function.
b Find the axes intercepts.
c Find the oblique asymptote.
d Draw a sign diagram of the function.
e Discuss the behaviour of the function near its asymptotes.
f Sketch the function, showing the features you have found.

a The vertical asymptote is $x = -2$.

b $f(0) = \dfrac{-9}{2}$, so the y-intercept is $-\frac{9}{2}$.

$f(x) = 0$ when $2x^2 + 3x - 9 = 0$

$\therefore\ (2x - 3)(x + 3) = 0$

$\therefore\ x = \frac{3}{2}$ or -3

$\therefore$ the x-intercepts are $\frac{3}{2}$ and -3.

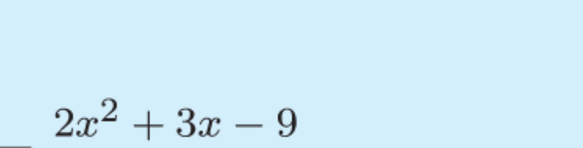
c $f(x) = \dfrac{2x^2 + 3x - 9}{x + 2}$

$= 2x - 1 - \dfrac{7}{x + 2}$

$$\begin{array}{r|l} & 2x - 1 \\ \hline x + 2 & 2x^2 + 3x - 9 \\ & \underline{-(2x^2 + 4x)\ \downarrow} \\ & -\ x - 9 \\ & \underline{-(-x - 2)} \\ & -\ 7 \end{array}$$

$\therefore$ the oblique asymptote is $y = 2x - 1$.

d $f(x) = \dfrac{(2x - 3)(x + 3)}{x + 2}$ has sign diagram



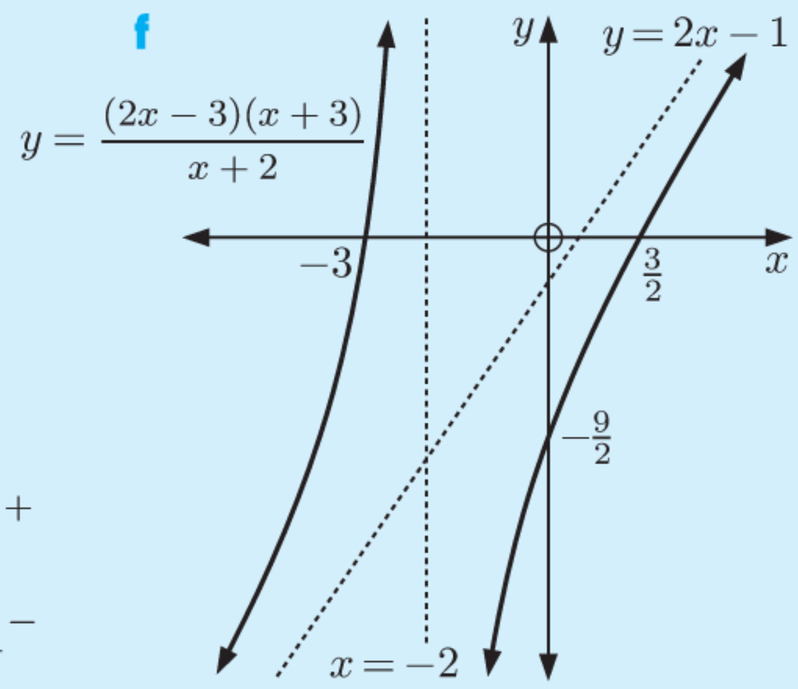
e As $x \to -2^-$, $f(x) \to \infty$

As $x \to -2^+$, $f(x) \to -\infty$

As $x \to -\infty$, $-\dfrac{7}{x + 2} \to 0^+$, so $f(x) \to 2x - 1^+$

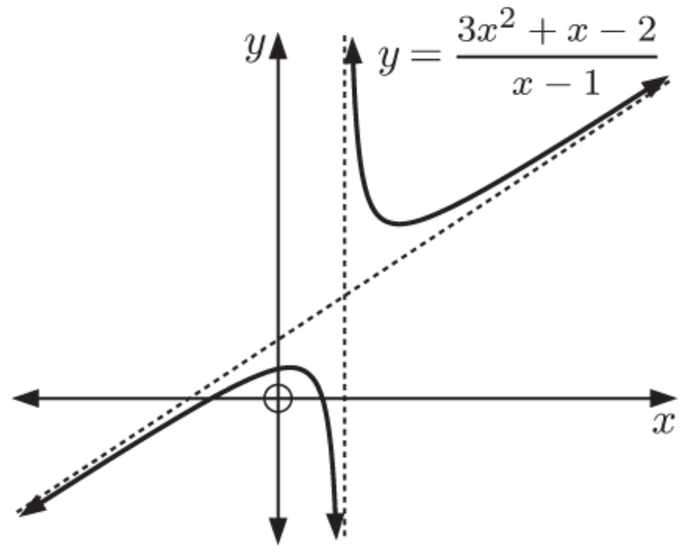
As $x \to \infty$, $-\dfrac{7}{x + 2} \to 0^-$, so $f(x) \to 2x - 1^-$

f



EXERCISE 6D.2

1 The graph of $y = \dfrac{3x^2 + x - 2}{x - 1}$ is shown alongside.



a Find the equation of the vertical asymptote.

b Find the axes intercepts.

c Write the function in the form $y = px + q + \dfrac{r}{x - 1}$, and hence find the equation of the oblique asymptote.

2 For each of the following functions:

i Find the equation of the vertical asymptote.

ii Find the axes intercepts.

iii Find the oblique asymptote.

iv Draw a sign diagram of the function.

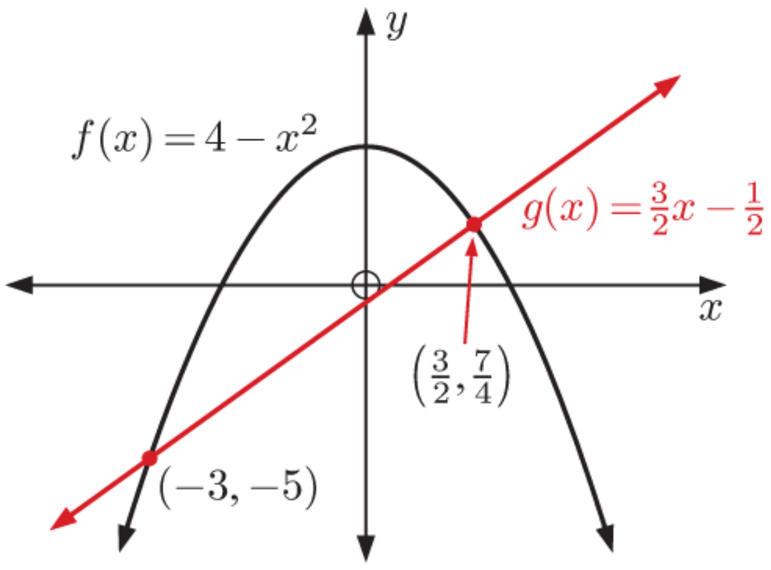
v Hence discuss the behaviour of the function near its asymptotes.

vi Sketch the graph of the function.

a $y = \dfrac{x^2 + x - 6}{x - 1}$ **b** $f(x) = \dfrac{-x^2 + 3x - 2}{x}$ **c** $y = \dfrac{2x^2 - 8x + 8}{x - 3}$

d $f(x) = \dfrac{-6x^2 + 5x}{2x + 1}$ **e** $y = \dfrac{-6x^2 - 4x - 1}{3x + 2}$ **f** $f(x) = \dfrac{8x^2 - 19x - 15}{1 - 2x}$

3



Consider the graphs of $y = f(x)$ and $y = g(x)$ shown alongside.

a For $y = \dfrac{f(x)}{g(x)}$ find the equation of the:

i vertical asymptote **ii** oblique asymptote.

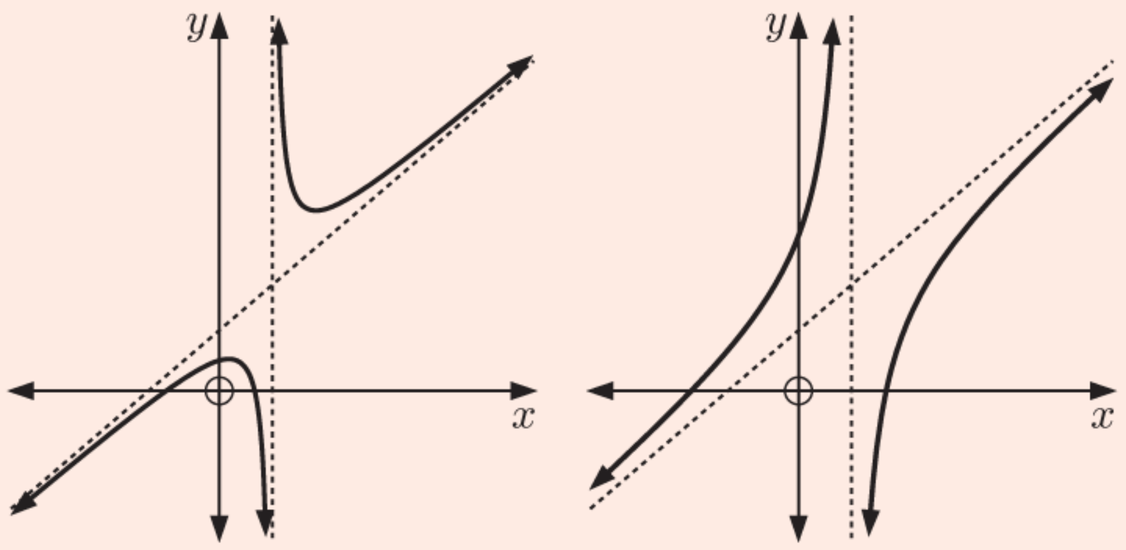
b Copy the graph, and on the same set of axes, graph $y = \dfrac{f(x)}{g(x)}$. Indicate clearly where the x-intercept(s) and asymptote(s) occur.

4 Find the equation of the oblique asymptote of $y = \dfrac{ax^2 + bx + c}{dx + e}$.

DISCUSSION

You should have noticed that the graph of $y = \dfrac{ax^2 + bx + c}{dx + e}$ takes one of the two shapes shown.

What determines which shape the graph takes?



E PARTIAL FRACTIONS

Just like fractions involving numbers, fractions involving algebra can be added by writing them with a common denominator.

For example:
$$\frac{3}{x-2} + \frac{4}{x+3}$$
$$= \frac{3}{x-2} \times \frac{x+3}{x+3} + \frac{4}{x+3} \times \frac{x-2}{x-2}$$
$$= \frac{3(x+3)+4(x-2)}{(x-2)(x+3)}$$
$$= \frac{3x+9+4x-8}{x^2+3x-2x-6}$$
$$= \frac{7x+1}{x^2+x-6} \quad \text{which is a rational function.}$$

When we study calculus later in the course, we are more likely to want to *reverse* this process. We need to **decompose** a function such as $\frac{7x+1}{x^2+x-6}$ into the sum of **partial fractions** $\frac{3}{x-2} + \frac{4}{x+3}$ so it can be integrated.

To decompose a rational function into the sum of partial fractions, we follow the following procedure:

Step 1: Factorise the denominator.

Step 2: Write a sum of partial fractions with the factors as denominators and unknown numerators.

Step 3: Multiply both sides by the denominator.

Step 4: Find the unknown numerators by substituting the zeros of the denominators.

Example 14 **Self Tutor**

Write as a sum of partial fractions: $\frac{7x+1}{x^2+x-6}$

Step 1: $x^2+x-6 = (x-2)(x+3)$

Step 2: Let $\frac{7x+1}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3}$.

Step 3: $\therefore \quad 7x+1 = A(x+3) + B(x-2)$

Step 4: Substituting $x = -3$, $7(-3)+1 = B(-3-2)$
$\therefore \quad -5B = -20$
$\therefore \quad B = 4$

Substituting $x = 2$, $7(2)+1 = A(2+3)$
$\therefore \quad 5A = 15$
$\therefore \quad A = 3$

$\therefore \quad \frac{7x+1}{x^2+x-6} = \frac{3}{x-2} + \frac{4}{x+3}$

By substituting the zeros of the denominators, we are able to solve for A and B independently. From *Step 3* we could expand the RHS and equate coefficients of x, but this requires solving simultaneous equations.



EXERCISE 6E

1 Show that:

a $\dfrac{3}{x^2+5x+4}$ can be written as the sum of partial fractions $\dfrac{-1}{x+4}+\dfrac{1}{x+1}$

b $\dfrac{9-x}{x^2+3x-10}$ can be written as the sum of partial fractions $\dfrac{-2}{x+5}+\dfrac{1}{x-2}$

c $\dfrac{3x-23}{2x^2-5x-12}$ can be written as the sum of partial fractions $\dfrac{5}{2x+3}-\dfrac{1}{x-4}$.

2 Write as the sum of partial fractions:

a $\dfrac{1}{x^2+3x+2}$ **b** $\dfrac{x}{x^2-5x+6}$ **c** $\dfrac{3x}{x^2-2x-8}$

d $\dfrac{3x+2}{x^2-4}$ **e** $\dfrac{2x-1}{x^2-2x-3}$ **f** $\dfrac{2-5x}{x^2+4x-12}$

g $\dfrac{7x+10}{2x^2+5x+3}$ **h** $\dfrac{x+9}{3x^2-5x-2}$ **i** $\dfrac{2x+27}{8x^2+18x-5}$

Example 15 — Self Tutor

Write $\dfrac{33-12x}{(x+1)(x-2)^2}$ as the sum of partial fractions in the form $\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$.

Let $\dfrac{33-12x}{(x+1)(x-2)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$\therefore\ 33-12x=A(x-2)^2+B(x+1)(x-2)+C(x+1)$

Substituting $x=-1$, $33-12(-1)=A(-1-2)^2$

$\therefore\ 45=9A$

$\therefore\ A=5$

Substituting $x=2$, $33-12(2)=C(2+1)$

$\therefore\ 9=3C$

$\therefore\ C=3$

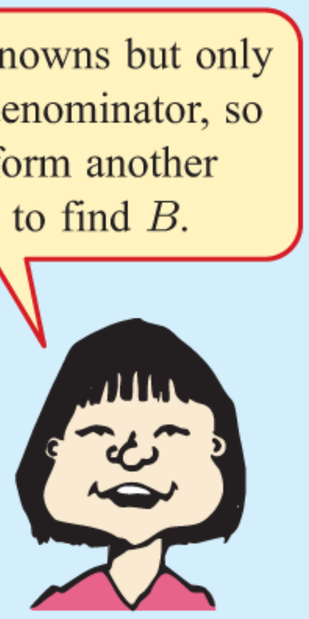
Substituting $x=0$, $33-12(0)=5(-2)^2+B(1)(-2)+3(1)$

$\therefore\ 33=20-2B+3$

$\therefore\ 2B=-10$

$\therefore\ B=-5$

$\therefore\ \dfrac{33-12x}{(x+1)(x-2)^2}=\dfrac{5}{x+1}-\dfrac{5}{x-2}+\dfrac{3}{(x-2)^2}$



3 Show that $\dfrac{5x^2+11x+16}{(x-5)(x+2)^2}$ can be written as the sum of partial fractions $\dfrac{4}{x-5}+\dfrac{1}{x+2}-\dfrac{2}{(x+2)^2}$.

4 Write $\dfrac{-3x-4}{(x+2)(x+3)^2}$ as the sum of partial fractions in the form $\dfrac{A}{x+2}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$.

5 **a** Show that $(x+3)(x-1)^2 = x^3 + x^2 - 5x + 3$.

b Write $\dfrac{19x+9}{x^3+x^2-5x+3}$ as the sum of partial fractions.

6 **a** Show that $(2x-1)(x^2+x+4) = 2x^3 + x^2 + 7x - 4$.

b Write $\dfrac{10x^2-3x+18}{2x^3+x^2+7x-4}$ in the form $\dfrac{A}{2x-1} + \dfrac{Bx+C}{x^2+x+4}$.

REVIEW SET 6A

1 Determine whether each function is even, odd, or neither:

a $f(x) = -\dfrac{4}{x}$ **b** $f(x) = \dfrac{2x-3}{x+1}$ **c** $f(x) = \sqrt{x^2-5}$

2 For what values of k is $f(x) = \sin(x-k)$:

a even **b** odd **c** neither?

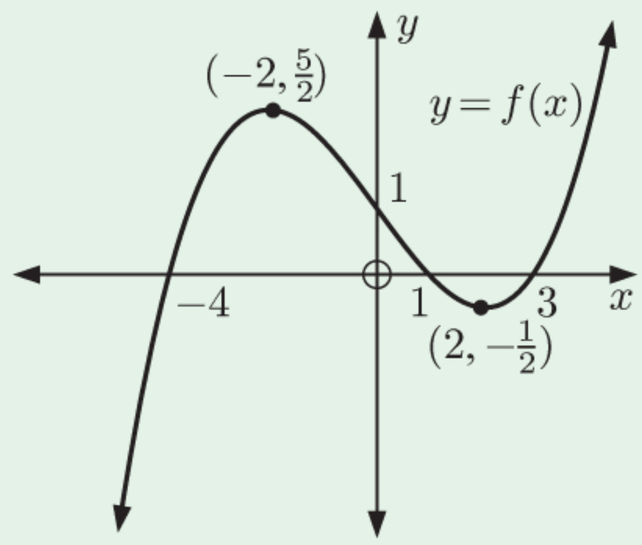
3 Given that $f(x)$ is an odd function, determine whether $(f \circ f)(x)$ is even, odd, or neither.

4 Suppose $f(x) = (x^2-3x)(x+b)$, $b \in \mathbb{R}$ is an odd function. Find the value of b.

5 The graph of $y = f(x)$ is given. On the same set of axes, graph the functions:

a $y = f(x)$ and $y = f(|x|)$

b $y = f(x)$ and $y = [f(x)]^2$.



6 Suppose $f(x)$ has domain $0 \leqslant x < 5$ and range $-5 \leqslant y \leqslant 2$. Find:

a the domain of $f(|x|)$ **b** the range of $|f(x)|$.

7 Suppose $f(x)$ is an odd function. Determine whether the following are even, odd, or neither:

a $f(2x)$ **b** $f(x^2)$ **c** $f(x) + 2$

d $[f(x)]^2$ **e** $|f(x)|$ **f** $\dfrac{1}{f(x)}$

8 Suppose $f(x) = e^x - 1$.

a On the same set of axes, sketch $y = f(x)$ and $y = [f(x)]^2$.

b Find the invariant points when $y = f(x)$ is transformed to $y = [f(x)]^2$.

9 Draw the graph of:

a $y = |(x-1)(x-4)|$ **b** $y = 2x - |x-2|$

10 Solve for x:

a $\left|\dfrac{2x+1}{x-2}\right| = 3$ **b** $|3x+5| = |x-7|$

11 Solve for x:

a $|x+3| \leqslant 6$ **b** $|3-2x| > 1$ **c** $|3x+2| < |x-5|$

12 Solve graphically:

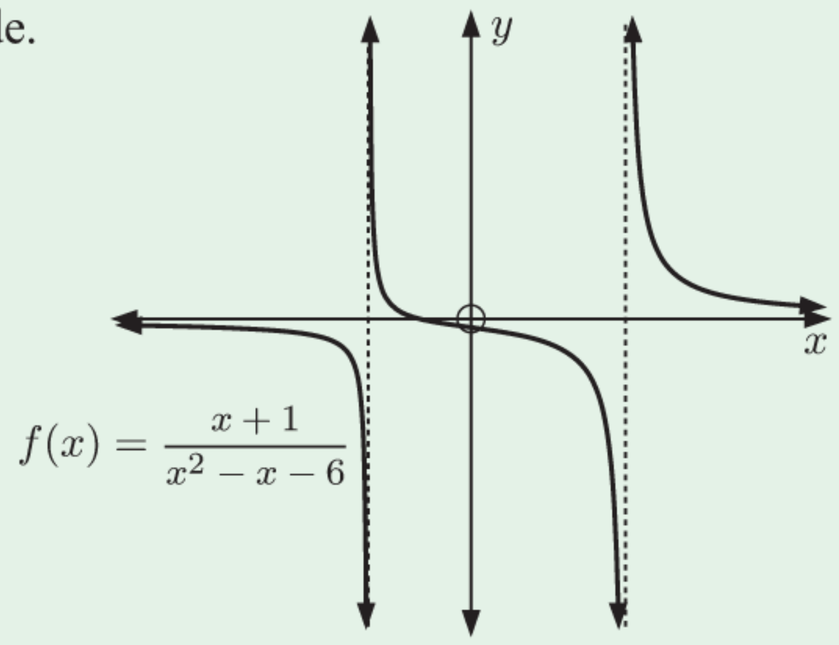
a $|2x-6| > x+3$ **b** $\sqrt{x^2+x+5} \leqslant |7-2^x|$

13 The graph of $f(x) = \dfrac{x+1}{x^2-x-6}$ is shown alongside.

a Find the asymptotes of the function.

b Find the axes intercepts.

c Sketch the graph of $y = |f(x)|$.



14 Consider the function $f(x) = \dfrac{2x^2+3x-5}{2x-3}$.

a Find the equation of the vertical asymptote.

b Find the axes intercepts.

c Find the oblique asymptote.

d Draw a sign diagram of the function.

e Hence discuss the behaviour of the function near its asymptotes.

f Sketch the graph of the function.

15 Write as the sum of partial fractions:

a $\dfrac{x+1}{x^2+4x-5}$ **b** $\dfrac{2x-4}{2x^2+7x+3}$

16 Write $\dfrac{13x+2}{(x+4)(x-1)^2}$ in the form $\dfrac{A}{x+4} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

REVIEW SET 6B

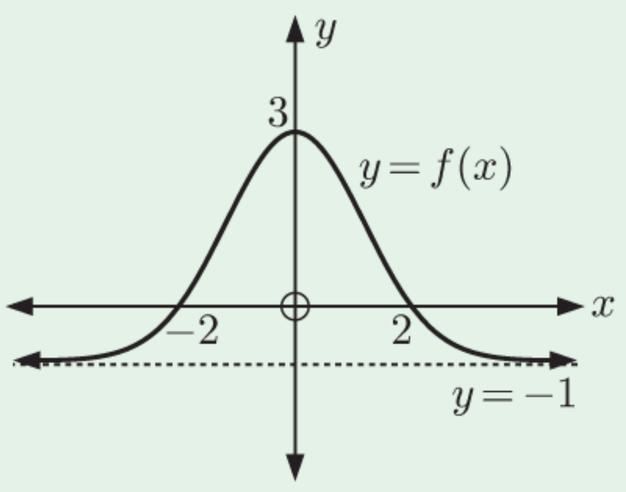
1 Suppose $f(x)$ is an even function, and $(-2, 6)$ and $(1, -4)$ lie on the graph of $y = f(x)$. Write down two other points which lie on the graph of $y = f(x)$.

2 **a** Prove that the product of two even functions is an even function.

b Give an example of two functions, both of which are neither odd nor even, whose product is an even function.

3 $f(x) = \cos(2(x-a)) + b$ is an odd function. Find the possible values of a and b.

4 The graph of $y = f(x)$ is given. On the same set of axes, graph the functions $y = f(x)$ and $y = [f(x)]^2$.



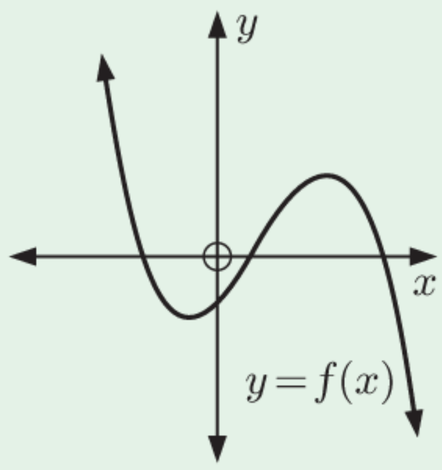
5 For any function $f(x)$, prove that $f(|x|)$ is an even function.

6 The graph of $y = f(x)$ has 3 turning points and 2 x-intercepts. If possible, state how many turning points and x-intercepts each of the following graphs has:

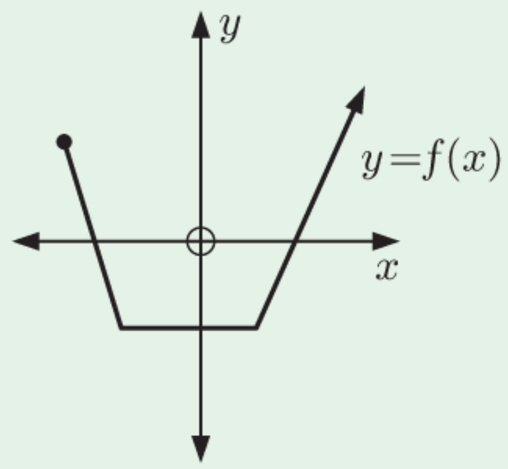
a $y = |f(x)|$ **b** $y = f(|x|)$ **c** $y = [f(x)]^2$

7 Draw the graph of $y = |f(x)|$ and $y = f(|x|)$ for:

a



b



8 Solve for x:

a $|2 - 5x| = 6$ **b** $|4x - 2| = |x + 7|$

9 Solve for x:

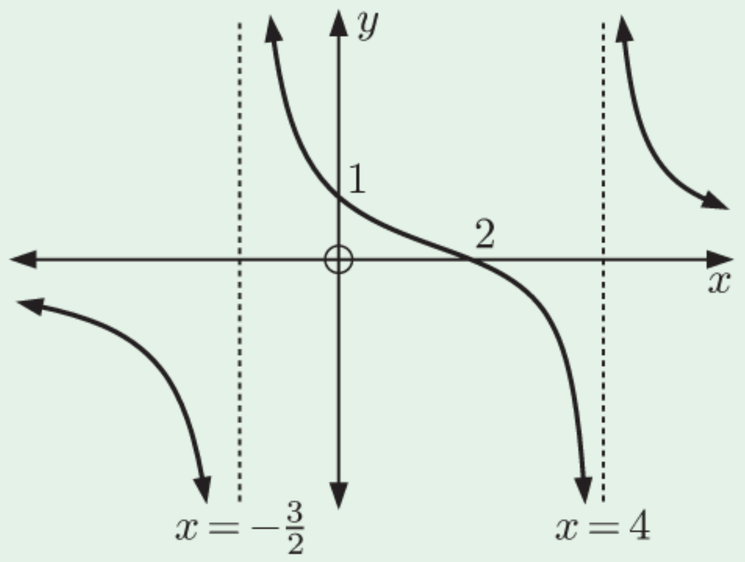
a $|3x + 2| > 3$ **b** $\frac{1}{4}|2x + 5| \geqslant |x - 1|$ **c** $\left|\frac{x}{x+4}\right| \leqslant 2$

10 Solve graphically:

a $\left|\frac{4}{x-1}\right| = 6 - \frac{x}{2}$ **b** $\frac{x}{|x|+1} \geqslant \frac{1}{3}$

11 The graph of $y = \dfrac{ax + b}{2x^2 + dx + e}$ is shown alongside.

Find the value of a, b, d, and e.



12 Consider the function $f(x) = \dfrac{2x - 5}{2x^2 - x - 6}$.

a Find the equations of the asymptotes. **b** Find the axes intercepts.

c Draw a sign diagram of the function.

d Hence discuss the behaviour of the function near its asymptotes.

e Sketch the graph of the function.

13 Consider the function $g(x) = \dfrac{3x^2 - 11x - 4}{3x - 2}$.

a Find the equation of the vertical asymptote.

b Find the axes intercepts. **c** Find the oblique asymptote.

d Draw a sign diagram of the function.

e Discuss the behaviour of the function near its asymptotes.

f Sketch the graph of the function.

14 Write as the sum of partial fractions:

a $\dfrac{2x}{-x^2+4x-3}$ **b** $\dfrac{x-7}{x^2+7x+12}$

15 **a** Show that $(2x-3)(x+2)^2 = 2x^3+5x^2-4x-12$.

b Hence write $\dfrac{9x^2-2x-5}{2x^3+5x^2-4x-12}$ as the sum of partial fractions.

16 Consider a function $f(x)$ with domain $x \in \mathbb{R}$.

a Determine whether each of the following is even, odd, or neither:

i $\dfrac{f(x)+f(-x)}{2}$ **ii** $\dfrac{f(x)-f(-x)}{2}$

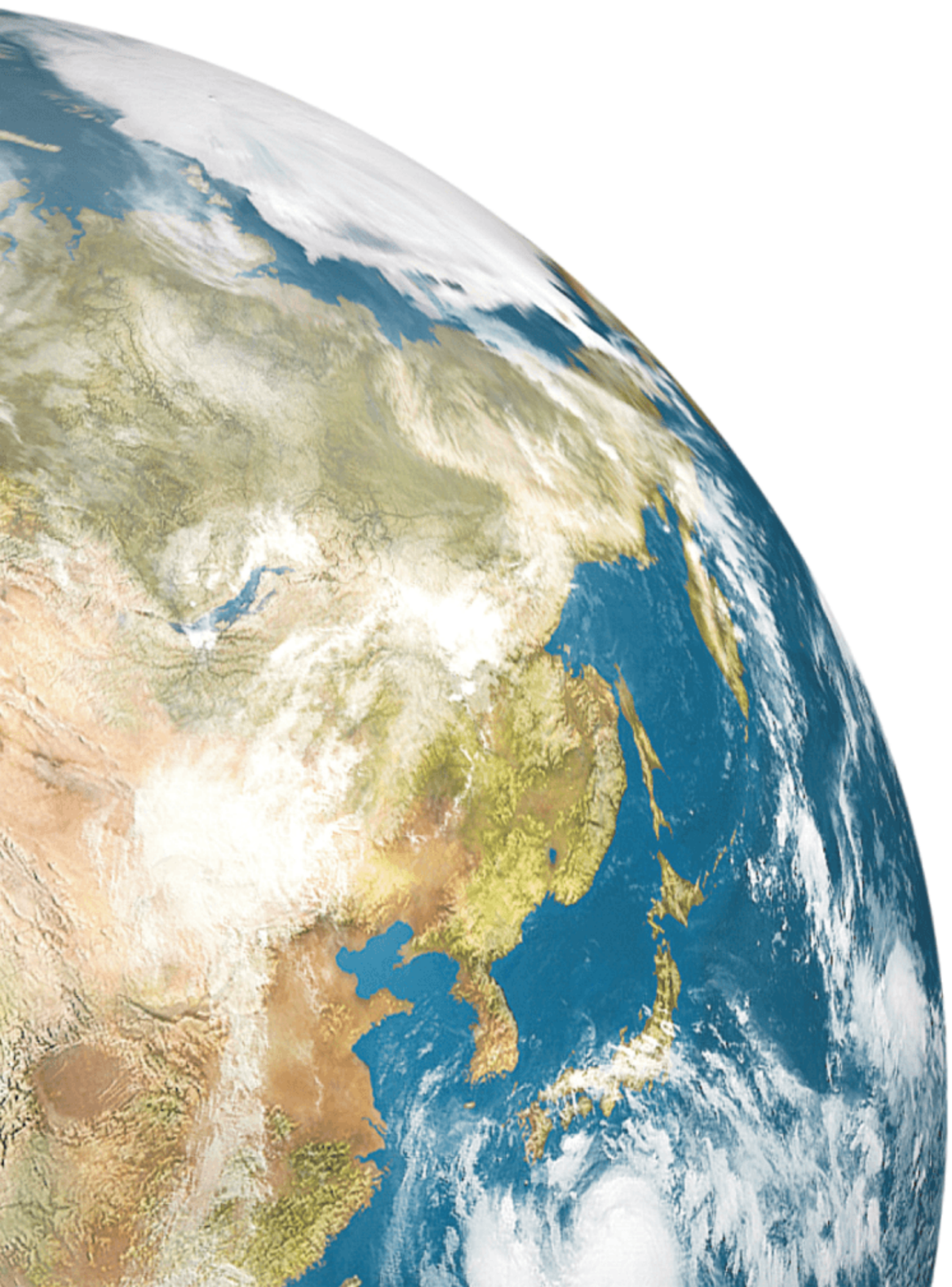
b Prove that $f(x)$ can be written as the sum of an even function and an odd function.

Chapter 7

Counting

Contents:

A The product principle
B The sum principle
C Factorial notation
D Permutations
E Combinations



OPENING PROBLEM

Students arriving for a week long camp are asked to fill out this form:

1 Select which activities you would most like to take part in (number your preferences from 1 to 3):

- ☐ Fishing
- ☐ Bicycle riding
- ☐ Swimming
- ☐ Hiking
- ☐ Orienteering
- ☐ Horse riding
- ☐ Kayaking

2 Select which days you would like to help with the cooking (tick any 3 days):

- ☐ Monday
- ☐ Tuesday
- ☐ Wednesday
- ☐ Thursday
- ☐ Friday
- ☐ Saturday
- ☐ Sunday

Things to think about:

a In both questions, students must make 3 selections from 7 options. What *differences* are there in the way students should answer the questions?

b Do you think there are more possible ways to answer question **1** or question **2**?

c Can you find the number of possible ways to answer:

i question **1** **ii** question **2**?

d How many possible ways are there to fill out the form?

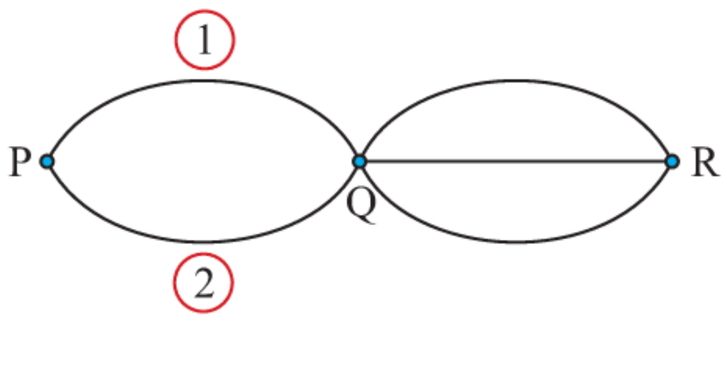
The **Opening Problem** is an example of a **counting** problem.

The following Exercises will help us to solve counting problems without having to list and count the possibilities one by one. To do this we will examine:

- the product and sum principles
- permutations
- combinations.

A THE PRODUCT PRINCIPLE

Suppose there are three towns P, Q, and R linked by roads as shown. How many different ways are there to get from P to R via Q?



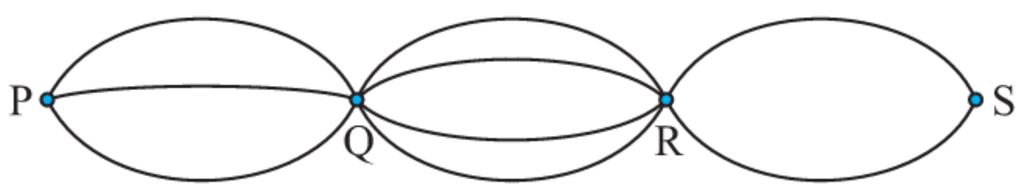
If we take road (1) from P to Q, then there are 3 possible roads to get from Q to R.

Likewise, if we take road (2) from P to Q, there are the same 3 choices to get from Q to R.

So, there are $3 + 3 = 2 \times 3$ ways of getting from P to R via Q.

number of roads from P to Q — number of roads from Q to R

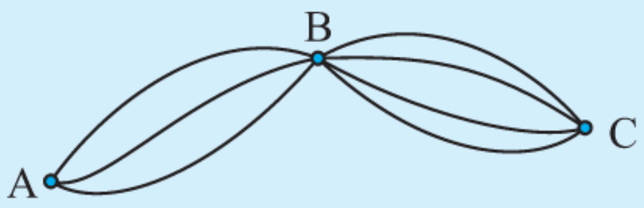
Using the same argument for the road connections alongside, there are $3 \times 4 \times 2 = 24$ ways to get from P to S via Q and R.



More formally, the **product principle** is:

If there are m different ways of performing an operation, and for each of these there are n different ways of performing a second **independent** operation, then there are mn different ways of performing the two operations in succession.

Example 1 **Self Tutor**



Three towns A, B, and C are connected by the road network shown. How many different ways are there to get from A to C via B?

For each of the 3 roads from A to B, there are 4 ways of getting from B to C.

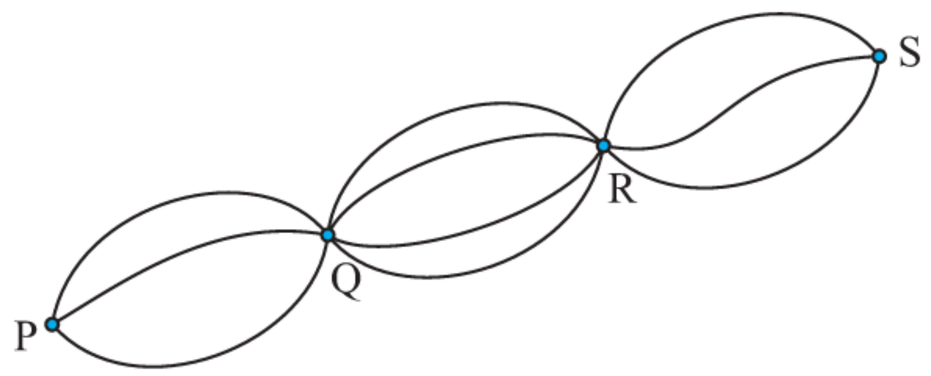
Using the product principle, there are $3 \times 4 = 12$ different ways to get from A to C via B.

EXERCISE 7A

1 The diagram shows the possible pathways for a bus service between four towns P, Q, R, and S. How many different routes are possible from:

a Q to S via R

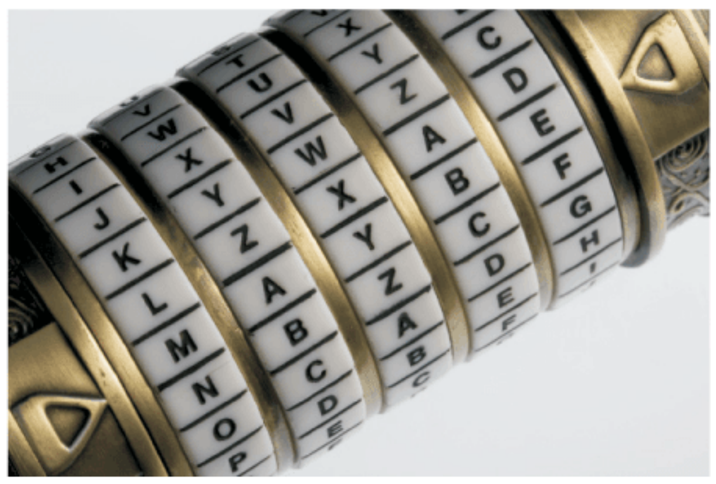
b S to P via R and Q?



2 Look at the menu below.
How many different combinations are there for a person to order an entrée, a main course, and a dessert?



3 The code for this lock consists of five letters of the alphabet.
How many different codes are possible?

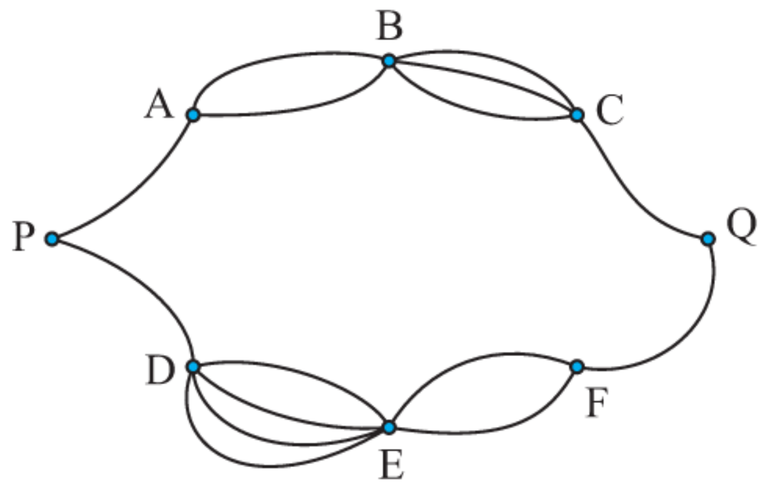


4 How many different alpha-numeric number plates for motor vehicle registration can be made if the first 3 places are letters from the English alphabet and the remaining 4 places are digits from 0 to 9?



B THE SUM PRINCIPLE

Consider the road system from P to Q.



From A to C there are $2 \times 3 = 6$ pathways.

From D to F there are $4 \times 2 = 8$ pathways.

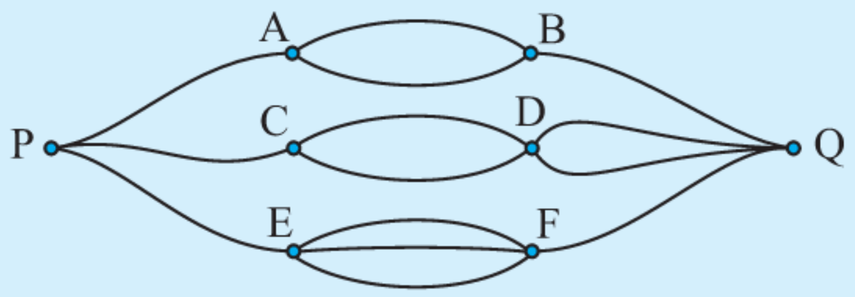
So, in total there are $6 + 8 = 14$ different pathways from P to Q.

We added the pathways because we had the choice of going from P to either A **or** D. This is the **sum principle**.

The word **and** suggests **multiplying** the possibilities.
The word **or** suggests **adding** the possibilities.

Example 2 Self Tutor

How many different paths lead from P to Q?



From P we could go to A **or** C **or** E.

From A there are 2 paths to Q.

From C there are $2 \times 2 = 4$ paths to Q.

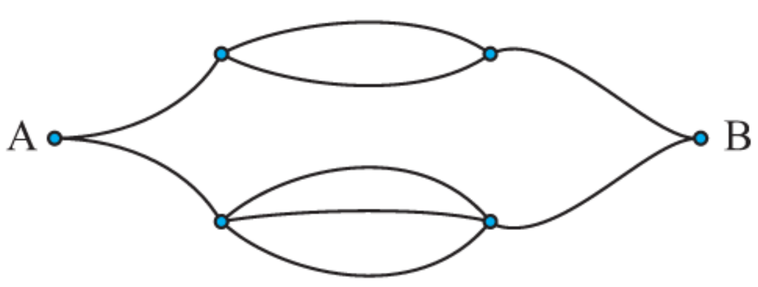
From E there are 3 paths to Q.

$\therefore$ in total there are $2 + 4 + 3 = 9$ different paths from P to Q.

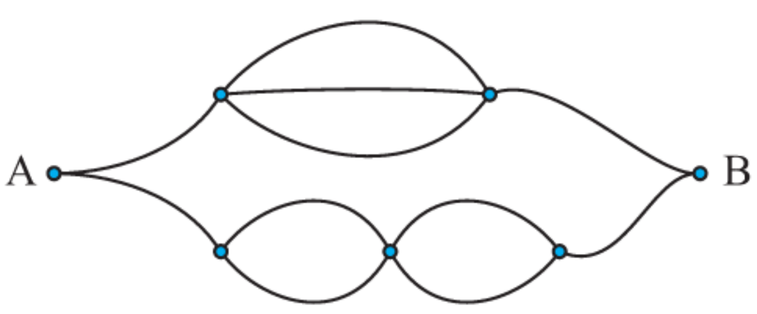
EXERCISE 7B

1 How many different paths lead from A to B?

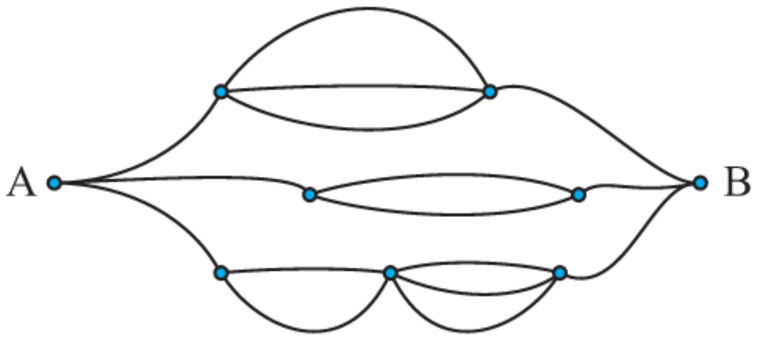
a



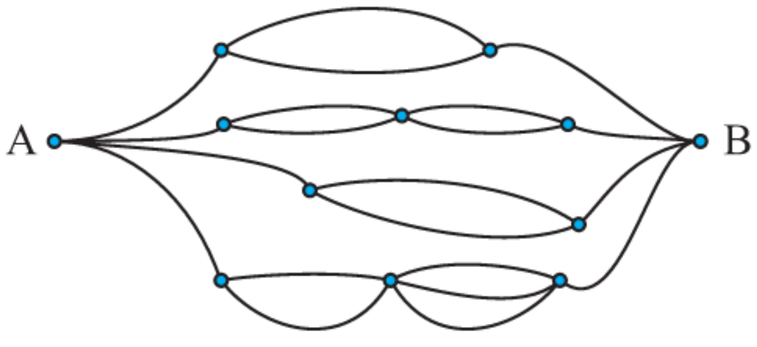
b



c



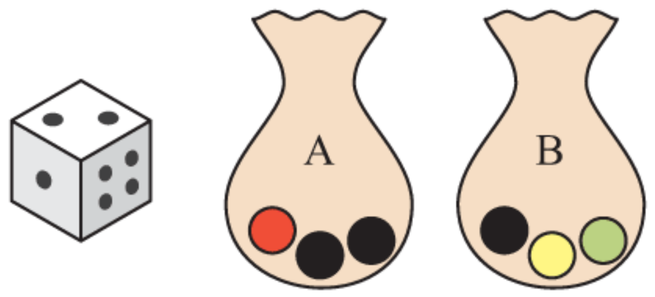
d



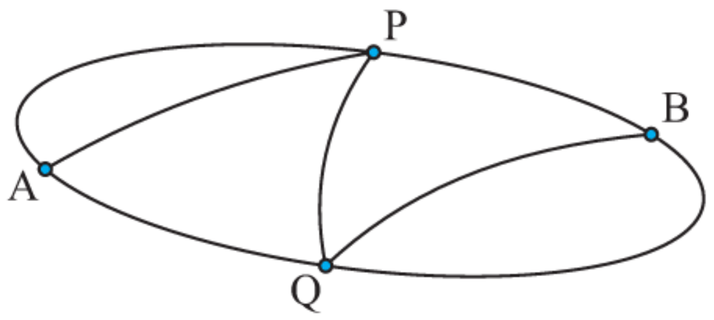
2 Michelle rolls a die. If the result is even, she picks a ball from bag A. Otherwise, she picks a ball from bag B.
How many ways lead to:

a a yellow ball being picked

b a black ball being picked?



3 Deanne is driving from A to visit her sister, who lives in a different town B. She has a choice of routes which could take her through other towns on the way.
How many different routes could Deanne take? Assume that she will not return to a town already visited.



C FACTORIAL NOTATION

When we count the number of ways in which objects can be **chosen** or **ordered**, we often need to calculate the products of consecutive integers, for example $5 \times 4 \times 3 \times 2 \times 1$ or $11 \times 10 \times 9 \times 8$.

For convenience we can use **factorial numbers** to write such products.

For $n \geqslant 1$, $n!$ is the product of the first n positive integers.

$$\boldsymbol{n! = n(n-1)(n-2) \ldots . 3 \times 2 \times 1}$$

$n!$ is read "*n factorial*".

For example, $5 \times 4 \times 3 \times 2 \times 1 = 5!$

Notice that $11 \times 10 \times 9 \times 8$ can be written using factorial numbers only as

$$11 \times 10 \times 9 \times 8 = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{11!}{7!}.$$

An alternative definition of factorial numbers is that $n! = n \times (n-1)!$ for $n \geqslant 1$.

For example, $6! = 6 \times 5!$

Under this rule we notice that $1! = 1 \times 0!$

We therefore define $0! = 1$

Example 3 — Self Tutor

Express in factorial form:

a $9 \times 8 \times 7$ **b** $\dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1}$

a $9 \times 8 \times 7 = \dfrac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \dfrac{9!}{6!}$

b $\dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \dfrac{11!}{4!7!}$

EXERCISE 7C

1 Evaluate:

a $2!$ **b** $3!$ **c** $4!$ **d** $5!$ **e** $6!$ **f** $10!$

2 Express in factorial form:

a $4 \times 3 \times 2 \times 1$ **b** $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ **c** 6×5

d $8 \times 7 \times 6$ **e** $10 \times 9 \times 8 \times 7$ **f** $15 \times 14 \times 13 \times 12$

g $\dfrac{9 \times 8 \times 7}{3 \times 2 \times 1}$ **h** $\dfrac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1}$ **i** $\dfrac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1}$

3 Simplify without using a calculator:

a $\dfrac{7!}{6!}$ **b** $\dfrac{8!}{6!}$ **c** $\dfrac{12!}{10!}$ **d** $\dfrac{120!}{119!}$ **e** $\dfrac{10!}{8! \times 2!}$ **f** $\dfrac{100!}{98! \times 2!}$

4 Simplify:

a $\dfrac{n!}{(n-1)!}$ **b** $\dfrac{(n+2)!}{n!}$ **c** $\dfrac{(n+1)!}{(n-1)!}$

Example 4 — Self Tutor

Write as a product by factorising:

a $8! + 6!$ **b** $10! - 9! + 8!$

a
$$\begin{aligned} & 8! + 6! \\ &= 8 \times 7 \times 6! \ + \ 6! \\ &= 6!(8 \times 7 + 1) \\ &= 6! \times 57 \end{aligned}$$

b
$$\begin{aligned} & 10! - 9! + 8! \\ &= 10 \times 9 \times 8! \ - \ 9 \times 8! \ + \ 8! \\ &= 8!(90 - 9 + 1) \\ &= 8! \times 82 \end{aligned}$$

5 Write as a product by factorising:

a $5! + 4!$ **b** $11! - 10!$ **c** $5! + 7!$ **d** $12! - 10!$

e $9! + 8! + 7!$ **f** $7! - 6! + 8!$ **g** $12! \ - \ 2 \times 11!$ **h** $3 \times 9! \ + \ 5 \times 8!$

Example 5 — Self Tutor

Simplify $\dfrac{7! - 6!}{6}$ by factorising.

$$\begin{aligned} & \frac{7! - 6!}{6} \\ &= \frac{7 \times 6! - 6!}{6} \\ &= \frac{6!(7 - 1)^{1}}{\cancel{6}_{1}} \\ &= 6! \end{aligned}$$

6 Simplify by factorising:

a $\dfrac{12! - 11!}{11}$ **b** $\dfrac{10! + 9!}{11}$ **c** $\dfrac{10! - 8!}{89}$ **d** $\dfrac{10! - 9!}{9!}$

e $\dfrac{6! + 5! - 4!}{4!}$ **f** $\dfrac{n! + (n-1)!}{(n-1)!}$ **g** $\dfrac{n! - (n-1)!}{n-1}$ **h** $\dfrac{(n+2)! + (n+1)!}{n+3}$

DISCUSSION

- Is the definition $0! = 1$ "arbitrary"?
- Is it logical to extend the definition of factorials to include 0!, even though it is not meaningful in the context of the original definition "product of the first n positive integers"? Can you think of other areas of mathematics in which a definition is similarly expanded?

D PERMUTATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a *definite order*.

For example, consider the three symbols P, Q, and R.

A permutation which uses all of these symbols is RQP. We say they have been taken "three at a time".

There are in fact six different permutations on the symbols P, Q, and R taken three at a time. They are PQR, PRQ, QPR, QRP, RPQ, and RQP.

Example 6 Self Tutor

List the permutations on the symbols X, Y, and Z when they are taken:

a one at a time **b** two at a time **c** three at a time.

a There are 3 permutations: X, Y, Z

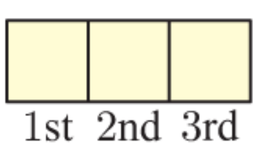
b There are 6 permutations: XY, XZ, YX, YZ, ZX, ZY

c There are 6 permutations: XYZ, XZY, YXZ, YZX, ZXY, ZYX

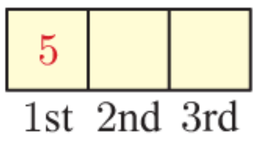
When we have lots of symbols, counting permutations by listing all of them becomes very difficult. For example, we would not want to list and then count all of the permutations on the alphabet letters A to Z taken 15 at a time. We therefore need an alternative method for counting permutations.

Consider the permutations on the symbols V, W, X, Y, and Z taken 3 at a time.

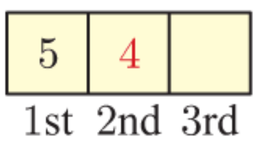
We see that there are 3 positions to fill:



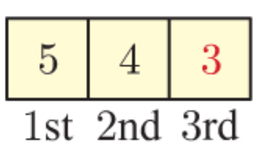
In the 1st position we can put any of the 5 original symbols.



In the 2nd position we can put any of the 4 remaining symbols.



In the 3rd position we can put any of the 3 remaining symbols.



Using the product principle, the total number of different permutations is $5 \times 4 \times 3 = \frac{5!}{2!} = 60$.

The number of permutations on n different symbols taken r at a time is $\dfrac{n!}{(n-r)!}$.

Example 7 **Self Tutor**

A chess association runs a tournament with 16 teams. In how many different ways could the top 8 positions be filled on the competition ladder?

Any of the 16 teams could fill the "top" position.
Any of the remaining 15 teams could fill the 2nd position.
Any of the remaining 14 teams could fill the 3rd position.
⋮
Any of the remaining 9 teams could fill the 8th position.

16	15	14	13	12	11	10	9
1st	2nd	3rd	4th	5th	6th	7th	8th

$$\begin{aligned}\text{The total number of permutations} &= 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \\ &= \frac{16!}{8!} \\ &= 518\,918\,400\end{aligned}$$

EXERCISE 7D.1

1 List the permutations on the symbols W, X, Y, and Z taken:

a 1 at a time **b** two at a time **c** three at a time.

2 List the permutations on the symbols A, B, C, D, and E taken:

a 2 at a time **b** 3 at a time.

3 In how many ways can 5 different books be ordered on a shelf?

4 Find the number of permutations there are on the symbols:

a A, B, C, D, E, and F taken 3 at a time **b** P, Q, R, S, T, U, and V taken 5 at a time

c A to Z taken 15 at a time.

5 A badminton competition consists of 9 teams. In how many different ways can the top two positions be filled?

6 A football competition is organised between 8 teams. In how many ways is it possible to fill the top four places on the competition ladder?

7 There are 8 paintings in a collection. In how many ways can 3 be chosen and hung in a row?

8 How many 3-digit numbers can be formed using the digits 2, 3, 4, and 5:

a as often as desired **b** at most once each?

9 Suppose you have 4 different coloured flags. How many different signals could you make using:

a 2 flags in a row **b** 3 flags in a row **c** 2 or 3 flags in a row?

Example 8 **Self Tutor**

The alphabet blocks A, B, C, D, and E are placed in a row in front of you.

a How many different permutations are there?

b How many permutations end in C?

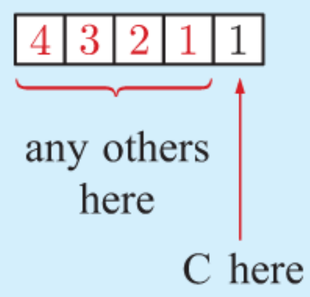
c How many permutations have the form |...|A|...|B|...| ?

d How many permutations begin and end with a vowel (A or E)?

a There are 5 letters taken 5 at a time.

$\therefore$ the total number of permutations $= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$.

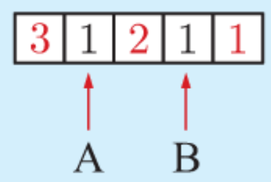
b



C must be in the last position. The other 4 letters can be placed in 4! ways.

$\therefore$ the number of permutations $= 1 \times 4! = 24$.

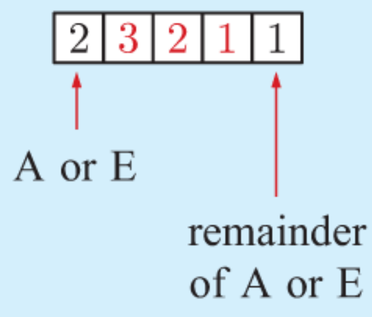
c



A goes into position 2. B goes into position 4. The remaining 3 letters can be placed in 3! ways.

$\therefore$ the number of permutations $= 1 \times 1 \times 3! = 6$.

d



A or E could go in position 1, so there are two options. The other one must go in position 5. The remaining 3 letters can be placed in 3! ways.

$\therefore$ the number of permutations $= 2 \times 1 \times 3! = 12$.

10 Consider the letters P, Q, R, S, and T on 5 alphabet blocks. These blocks are placed in a row. How many permutations are there if:

a there are no restrictions **b** the row must end in RQ

c the row must begin with S and end with P?

11 A 4-digit number is made up using the digits 1, 2, 3, 4, 5, and 6 at most once each. How many different numbers can be formed if:

a there are no other restrictions **b** the number must be divisible by 5

c the number must start with 6 and end with 2?

12 A 3-digit number is made up using the digits 0, 1, 2, 3, 4, 5, 6, and 7 at most once each. The number cannot start with 0. How many different numbers can be formed if:

a there are no other restrictions **b** the number must end in 2, 4, or 6

c the number must end in 0 **d** the number must be even?

13 Sprinters A, B, C, and D are members of a 4×400 m relay team. In how many ways can the sprinters be ordered if:

a there are no restrictions

b B must sprint last

c D must not sprint last

d C must sprint either first or second

e A must be followed directly by B?



14 How many three-digit numbers can be made using the digits 0, 1, 3, 5, and 8 at most once each, if:

a there are no restrictions **b** the numbers must be less than 500

c the numbers must be even and greater than 300?

Example 9 **Self Tutor**

There are 6 different books arranged in a row on a shelf. In how many ways can two of the books, A and B, be together?

Method 1: We could have any of the following locations for A and B

A	B	×	×	×	×
B	A	×	×	×	×
×	A	B	×	×	×
×	B	A	×	×	×
×	×	A	B	×	×
×	×	B	A	×	×
×	×	×	A	B	×
×	×	×	B	A	×
×	×	×	×	A	B
×	×	×	×	B	A

} 10 of these

If we consider any one of these, the remaining 4 books could be placed in 4! different orderings.

$\therefore$ total number of permutations $= 10 \times 4! = 240$.

Method 2: A and B can be put together in 2! ways (AB or BA).

Now consider this pairing as one book (effectively tying a string around them). Together with the other 4 books, they can be ordered in 5! different ways.

$\therefore$ the total number of permutations $= 2! \times 5! = 240$.

15 In how many ways can 5 different books be arranged on a shelf if books X and Y are:

a together **b** separated?

16 10 students sit in a row of 10 chairs. In how many ways can this be done if:

a there are no restrictions **b** students A, B, and C insist on sitting together?

17 Consider the letters of the word MONDAY. How many different permutations of four letters can be chosen if:

a there are no restrictions **b** at least one vowel (A or O) must be used

c the two vowels are not together?

18 Nine cupcakes are each iced with a different whole number from 1 to 9. Five children are allowed to take one cupcake each. In how many different ways can this be done if:

a there are no restrictions

b the first three children all decide that they will choose an even numbered cupcake?

19 Alice has booked ten adjacent front-row seats for a basketball game for herself and nine friends.

a How many different arrangements are possible if there are no restrictions?

b Due to a severe snowstorm, only five of Alice's friends are able to join her for the game. In how many different ways can they be seated in the 10 seats if:

i there are no restrictions

ii they all sit together with no empty seats between

iii any two of Alice's friends sit either side of her?

INVESTIGATION PERMUTATIONS IN A CIRCLE

Click on the icon to obtain this Investigation.



PERMUTATIONS WHERE SOME SYMBOLS ARE REPEATED

Consider the different orderings of the symbols A, A, A, B, and C.

If the 5 symbols were all different, they could be ordered in 5! ways.

However, since some of the symbols are repeated, some of these will be identical.

For example, consider the permutation ABAAC. If we label the 3 As with subscripts, we see there are 6 possible orderings of the As which result in this permutation:

$$\begin{array}{lll} A_1BA_2A_3C & A_2BA_1A_3C & A_3BA_1A_2C \\ A_1BA_3A_2C & A_2BA_3A_1C & A_3BA_2A_1C \end{array}$$

This is true for all of the possible permutations, so the total number of distinguishable permutations is $\frac{5!}{3!} = 20$.

Suppose there are n_1 of the first symbol, n_2 of the second symbol, ..., and n_r of the rth (and final) symbol. The total number of distinguishable permutations is $\dfrac{(n_1 + n_2 + n_3 + ... + n_r)!}{n_1!\, n_2!\, n_3!\, ...\, n_r!}$.

Example 10 **Self Tutor**

Consider the word CALCULUS. How many permutations can be formed:

a using all of the letters **b** which start with A and end with S

c which start with C and end with L?

a There are 2 Cs, 1 A, 2 Ls, 2 Us, and 1 S.

$\therefore$ the total number of distinguishable permutations $= \dfrac{8!}{2!\,1!\,2!\,2!\,1!} = 5040$

b If the word starts with A and ends with S, the number of distinguishable permutations $= \dfrac{6!}{2!\,2!\,2!} = 90$

$$\text{A}\underbrace{_____}_{\text{2 Cs, 2 Ls, 2 Us}}\text{S}$$

c There are 2 ways to choose the initial C and 2 ways to choose the final L, but we ignore these because we cannot tell the Cs apart or the Ls apart anyway.

$\therefore$ the number of distinguishable permutations

$= \dfrac{6!}{1!\,1!\,1!\,2!\,1!} = 360$

$$\text{C}\underbrace{_____}\text{L}$$
1 C, 1 A, 1 L, 2 Us, 1 S

EXERCISE 7D.2

1 List the distinguishable permutations of the symbols A, A, A, B.
Verify the formula for the number of distinguishable permutations is correct.

2 List the distinguishable permutations of the symbols A, A, B, B.
Verify the formula for the number of distinguishable permutations is correct.

3 Find the number of distinguishable permutations formed from the letters of:

a BOOKS **b** LOTTO **c** ASSESS

4 There are 2 copies each of 3 different books on a shelf. In how many distinguishable ways can they be arranged?

5 Two identical Chemistry books and three identical Mathematics books are placed in a pile. In how many distinguishable ways can they be stacked if:

a there are no restrictions **b** the books in each subject are kept together?

6 One red, two white, and three blue marbles are placed in a line. In how many distinguishable ways can this be done if:

a white marbles are at either end **b** blue marbles are at either end

c the red marble is at one end and a white marble is at the other?

7 A Mahjong set contains:

- 4 of each of 9 tiles, in each of 3 suits
- 4 of each of 3 dragons
- 4 of each of 4 winds
- 1 of each of 4 birds, in each of 2 colours.

Explain why the total number of orders in which the tiles can be selected is $\frac{144!}{(4!)^{34}}$.

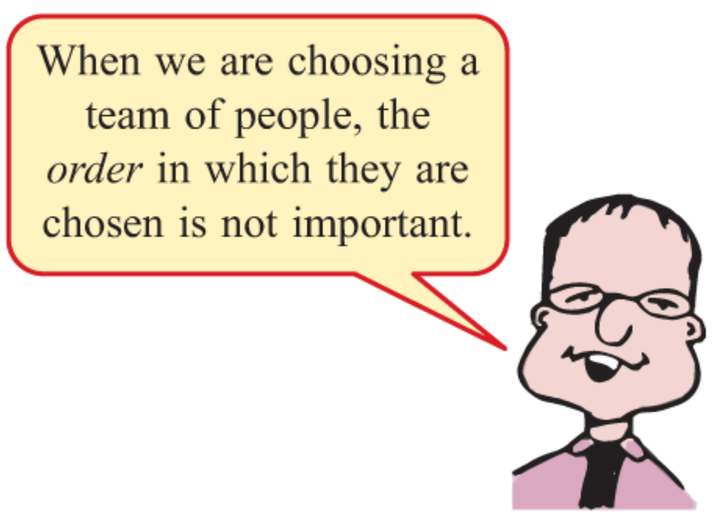
E COMBINATIONS

A **combination** is a selection of objects *without* regard to order or arrangement.

For example, the possible teams of 3 people selected from A, B, C, D, and E are:

ABC ABD ABE ACD ACE ADE
BCD BCE BDE
CDE

We only list ABC and not ACB, BAC, BCA, CAB, and CBA. These are different *permutations* of the players, but they are all the same *combination*.

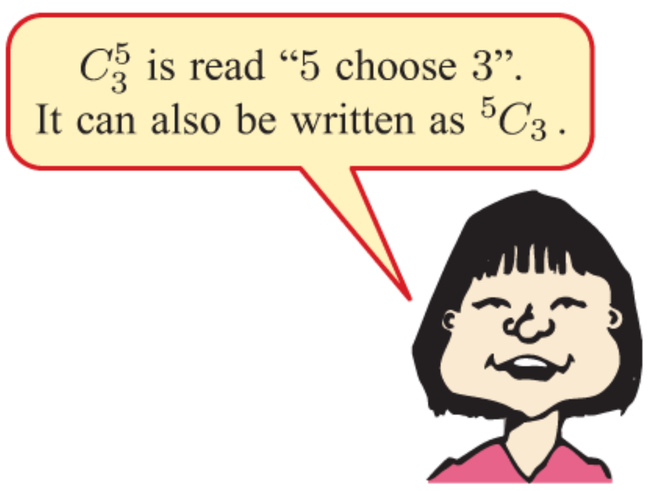


We see that there are 10 combinations on the 5 letters taken 3 at a time.

We write this as $C_3^5 = 10$ or $\binom{5}{3} = 10$.

The number of *permutations* on 5 letters taken 3 at a time is $5 \times 4 \times 3 = 60$, which is 6 times larger than the number of combinations. This is because each combination has $3 \times 2 \times 1$ possible permutations.

So, $C_3^5 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = \frac{5!}{3! \times 2!}$.



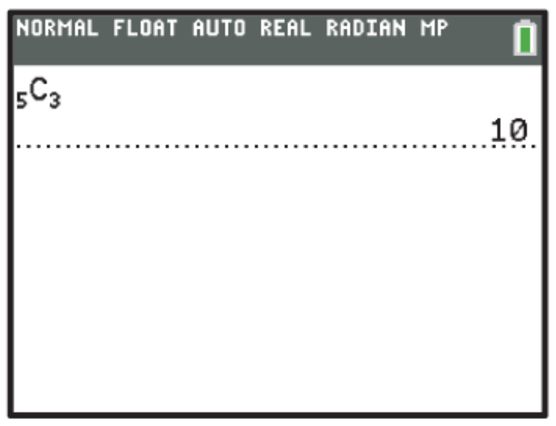
In general: The number of combinations on n distinct symbols taken r at a time is

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

We can use a graphics calculator to find values of $\binom{n}{r}$. Click on the icon for instructions.



GRAPHICS CALCULATOR INSTRUCTIONS



EXERCISE 7E

1 Evaluate:

a $\binom{3}{2}$ **b** $\binom{6}{4}$ **c** $\binom{5}{1}$ **d** $\binom{7}{2}$

Use your calculator to check your answers.

2 **a** List the different teams of 2 that can be chosen from a squad of 5 players named P, Q, R, S, and T.

b Check that $\binom{5}{2}$ gives the total number of teams.

3 **a** List the different teams of 4 that can be chosen from a squad of 6 players named A, B, C, D, E, and F.

b Check that $\binom{6}{4}$ gives the total number of teams.

4 **a** Evaluate:

i $\binom{4}{1}$ **ii** $\binom{7}{1}$ **iii** $\binom{10}{1}$

b Use combinations to explain why $\binom{n}{1} = n$ for any $n \in \mathbb{N}$.

5 **a** Evaluate:

i $\binom{5}{0}$ **ii** $\binom{5}{5}$ **iii** $\binom{9}{0}$ **iv** $\binom{9}{9}$

b Use combinations to explain why, for any $n \in \mathbb{N}$:

i $\binom{n}{0} = 1$ **ii** $\binom{n}{n} = 1$

6 **a** Evaluate:

i $\binom{7}{2}$ and $\binom{7}{5}$ **ii** $\binom{10}{3}$ and $\binom{10}{7}$

b Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to show that $\binom{n}{k} = \binom{n}{n-k}$ for any $n, k \in \mathbb{N}$, $0 \leqslant k \leqslant n$.

c Use combinations to explain why this is true.

7 Find k such that $\binom{9}{k} = 4\binom{7}{k-1}$.

8 Write as a polynomial in n, including an appropriate domain:

a $\binom{n}{n-2}$ **b** $\binom{n}{4}$

Example 11 — Self Tutor

How many different teams of 3 can be selected from a squad of 8 if:

a there are no restrictions **b** the team must include the captain?

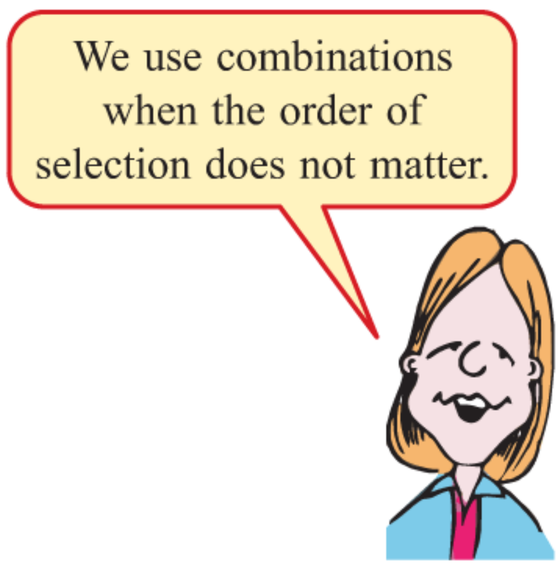
a There are 8 players for selection and we can choose any 3 of them.
This can be done in $\binom{8}{3} = 56$ different ways.

b The captain must be included *and* we need any 2 of the other 7.
This can be done in $\binom{1}{1} \times \binom{7}{2} = 21$ different ways.

9 How many different teams of 7 can be chosen from a training squad of 15?

10 **a** How many different committees of 5 can be selected from a club of 27 members?

b How many of these committees will include the club president?



11 A physics exam contains 7 questions. Students must answer both questions 1 and 2, and any 3 of the remaining questions. How many different selections are possible?

12 **a** How many different teams of 5 can be selected from a squad of 12?

b How many of these teams contain:

i the captain and vice-captain

ii exactly one of the captain or the vice-captain?

13 A team of 9 is selected from a squad of 15. 3 particular players *must* be included, and another must be excluded because of injury. In how many ways can the team be chosen?

Example 12 — Self Tutor

An equestrian team of 4 is to be chosen from 7 male and 6 female riders. How many different teams can be chosen if:

a there are no restrictions **b** there must be two riders of each sex

c at least one rider of each sex is required?

a There are $7+6=13$ riders up for selection, and we need to choose any 4 of them. So there are $\binom{13}{4} = 715$ different teams.

b The males can be chosen in $\binom{7}{2}$ ways and the females in $\binom{6}{2}$ ways.

$\therefore \binom{7}{2} \times \binom{6}{2} = 315$ different teams can be chosen.

c We could have 3 males and 1 female **or** 2 males and 2 females **or** 1 male and 3 females.

$\therefore \binom{7}{3} \times \binom{6}{1} + \binom{7}{2} \times \binom{6}{2} + \binom{7}{1} \times \binom{6}{3} = 665$ different teams can be chosen.

Note: Since the only other alternatives are 4M *and* 0F or 0M *and* 4F, we could have used $\binom{13}{4} - \binom{7}{4} \times \binom{6}{0} - \binom{7}{0} \times \binom{6}{4}$

14 The committee of a table tennis club consists of 3 people chosen from 6 male and 8 female members.

a If there are no restrictions, how many different committees can be chosen?

b How many of the different possible committees consist of:

i 3 males **ii** 2 males and 1 female

iii 1 male and 2 females **iv** 3 females?

15 A committee of 4 singers is chosen from 7 tenors and 9 sopranos.

Find the number of ways of selecting the committee if:

a there are no restrictions

b it must contain 2 tenors and 2 sopranos

c it must contain at least 2 sopranos.



16 Answer the **Opening Problem** on page **176**.

17 A committee of 5 is chosen from 6 doctors, 3 dentists, and 7 nurses.

Determine the number of ways of selecting a committee which contains:

a exactly 2 doctors and 1 dentist **b** exactly 2 doctors

c at least one doctor or dentist.

18 How many diagonals does a 20-sided convex polygon have?

19 Kristen's school offers 6 Group A subjects, 8 Group B subjects, and 5 Group C subjects. Kristen must study 2 Group A, 2 Group B, and 1 Group C subject. Her sixth subject can come from any group. In how many ways can she make her selection?

20 A soccer team consists of 11 players. The team must be sorted into 3 attackers, 3 midfielders, 4 defenders, and 1 goalkeeper for today's match.

a How many possible ways are there to do this?

b Melissa is chosen to play goalkeeper. Linda wants to play attacker, and Robyn does not want to play defender. How many combinations are now possible?

21 12 distinct points A, B, C, D, ..., L lie on a circle. Lines are drawn between each pair of points.

a How many lines are there:

i in total **ii** which pass through B?

b How many triangles are there:

i in total **ii** which include vertex B?

22 How many 4-digit numbers can be constructed in which the digits are in ascending order from left to right? You cannot start a number with 0.

23 **a** Give an example which demonstrates that:

$$\binom{5}{0}\times\binom{6}{4}+\binom{5}{1}\times\binom{6}{3}+\binom{5}{2}\times\binom{6}{2}+\binom{5}{3}\times\binom{6}{1}+\binom{5}{4}\times\binom{6}{0}=\binom{11}{4}.$$

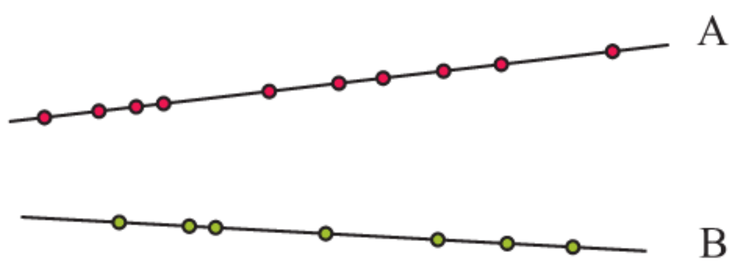
b Copy and complete:

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}=\binom{m}{0}\times\binom{n}{r}+\binom{m}{1}\times\binom{n}{r-1}+\ldots.+\binom{m}{r-1}\times\binom{n}{1}+\binom{m}{r}\times\binom{n}{0}=\ldots.$$

24 In how many ways can 12 people be divided into:

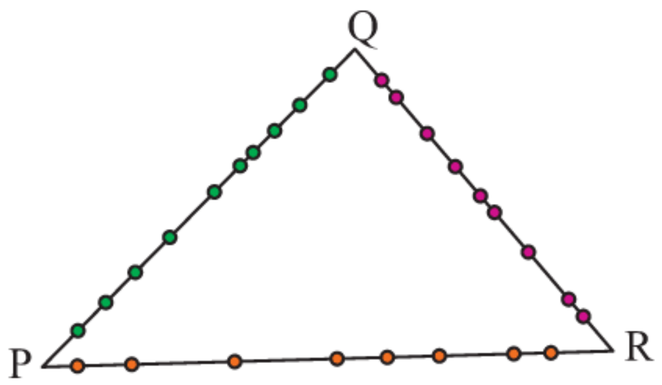
a two equal groups **b** three equal groups?

25 Line A includes 10 points. Line B includes 7 points. The points on line A are all joined to all of the points on line B. Find the maximum number of points of intersection between the new lines constructed.



26 10 points are located on [PQ], 9 on [QR], and 8 on [RP].

All possible lines connecting these 27 points are drawn. Determine the maximum possible number of points of intersection of these lines which lie within triangle PQR.



27 A hand of 13 cards is dealt from a pack of 52 playing cards.

a Find the total number of possible hands.

b Find the number of hands which contain exactly 4 hearts.

c Hence find the probability that a hand containing exactly 4 hearts will be dealt.

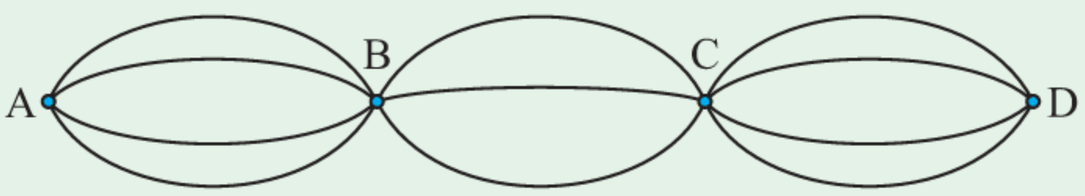
PUZZLE

If we are given 5 integers, there are $\binom{5}{2} = 10$ different ways to find the sum of 2 of the integers.

Is it possible to find a set of 5 integers, such that the 10 sums each have a different last digit?

REVIEW SET 7A

1 How many different paths lead from A to D?



2 Express in factorial form:

a $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ **b** $10 \times 9 \times 8$

3 Simplify: **a** $\dfrac{n!}{(n-2)!}$ **b** $\dfrac{n! + (n+1)!}{n!}$

4 List the permutations on the symbols P, Q, R, and S taken two at a time.

5 How many 4-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, and 7 at most once each if:

a there are no restrictions **b** the number must be even?

6 **a** List the different teams of three people that can be chosen from a squad of six players J, K, L, M, N, and O.

b Show that $\binom{6}{3}$ gives the total number of teams.

7 The letters P, Q, R, S, and T are to be arranged in a row.

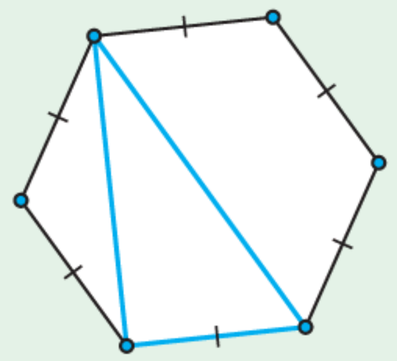
a How many different possible arrangements are there?

b How many of these arrangements:

i begin with P and end with T **ii** begin with S and end with PT

iii have the Q and the S next to each other?

8 **a** How many different triangles can be formed by connecting 3 vertices of a regular hexagon?



b How many of these triangles are scalene?

9 Consider the letters A, A, A, B, B, C, and C taken all at once.

a How many different orderings are possible?

b How many of these orderings have:

i an A at either end **ii** the 3 As together?

10 A committee of 4 is to be chosen from 5 science teachers, 3 language teachers, and 4 mathematics teachers. How many committees can be selected if:

a there are no restrictions

b at least one of each group is represented

c two science teachers X and Y must be chosen, and language teacher Z is unavailable for selection?

11 At an IB Mathematics Teachers' Conference there are 273 delegates present. The organising committee consists of 10 people.

a If each committee member shakes hands with every other committee member, how many handshakes take place?

b If all 273 delegates shake hands with all other delegates, how many handshakes take place now?

c If the organising committee lines up on stage to face the delegates in the audience, in how many different orders can they line up?



12 A hand of 8 cards is dealt from a pack of 52 playing cards.

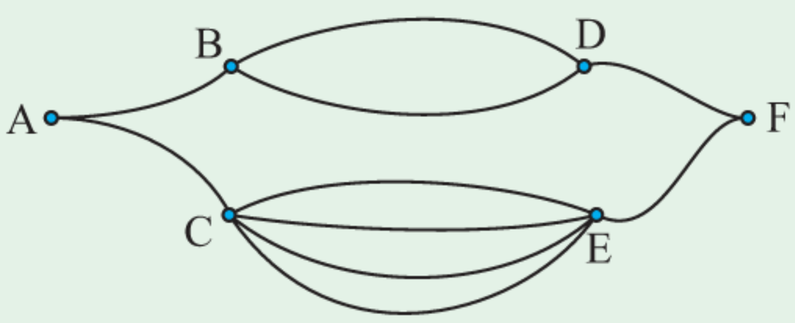
a Find the total number of possible hands.

b Find the number of hands which contain 2 of each of the four suits (clubs, diamonds, hearts, and spades).

c Hence find the probability that a hand containing 2 of each suit will be dealt.

REVIEW SET 7B

1 How many different paths lead from A to F?



2 Simplify:

a $\dfrac{9!}{7!}$ **b** $\dfrac{8!}{3!\,5!}$

3 Simplify:

a $\dfrac{7! + 8!}{9}$ **b** $\dfrac{(n-1)! + (n+1)!}{(n-1)!}$

4 Evaluate:

a $\binom{4}{2}$ **b** $\binom{6}{5}$ **c** $\binom{9}{2}$

5 Eight swimmers are competing in the final of an Olympic event. Assuming there are no ties, in how many different ways can the gold, silver, and bronze medals be awarded?

6 A committee of five is selected from eight engineers, seven scientists, and six mathematicians.

a How many different committees can be selected?

b How many committees consist of:

i all engineers **ii** at least one of each profession?

7 To select 5 people from a group of 9, Miles reasons that he can first select 2 people from the group of 9, then select 3 people from the remaining 7. He therefore concludes that $\binom{9}{5} = \binom{9}{2} \times \binom{7}{3}$.

a Show by direct calculation that Miles is incorrect.

b Explain the error in Miles' reasoning.

8 Eight people enter a room and sit at random in a row of eight chairs. In how many ways can the sisters Cathy, Robyn, and Jane sit together in the row?

9 A team of eight is chosen from 11 men and 7 women. How many different teams are possible if:

a there are no restrictions **b** there must be four of each sex on the team

c there must be at least two women on the team?

10 In an 8-person board game, players are each assigned a character by selecting character cards. There are three "villager" cards, two "sorcerer" cards, two "merchant" cards, and a "queen" card.

a In how many distinguishable ways can the players be assigned characters?

b In how many of these ways:

i is Petra assigned a "villager" and Joel assigned a "merchant"

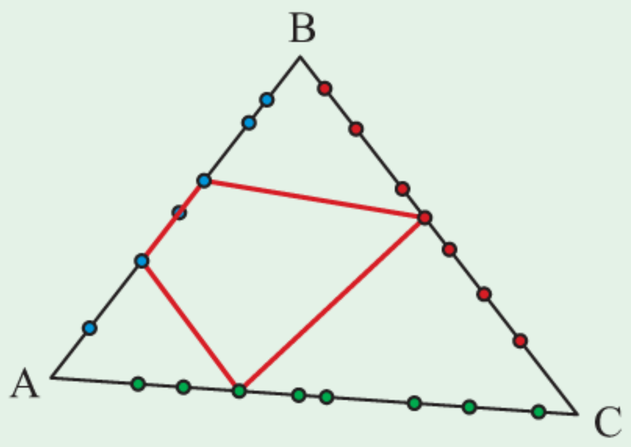
ii are Karen and Nick assigned the same character?

11 In how many ways can 16 people be divided into:

a two equal groups **b** four equal groups?

12 6 points are located on [AB], 7 on [BC], and 8 on [CA].

How many different quadrilaterals can be formed by joining four of these points with line segments?



Chapter 8

The binomial theorem

Contents:

A Binomial expansions

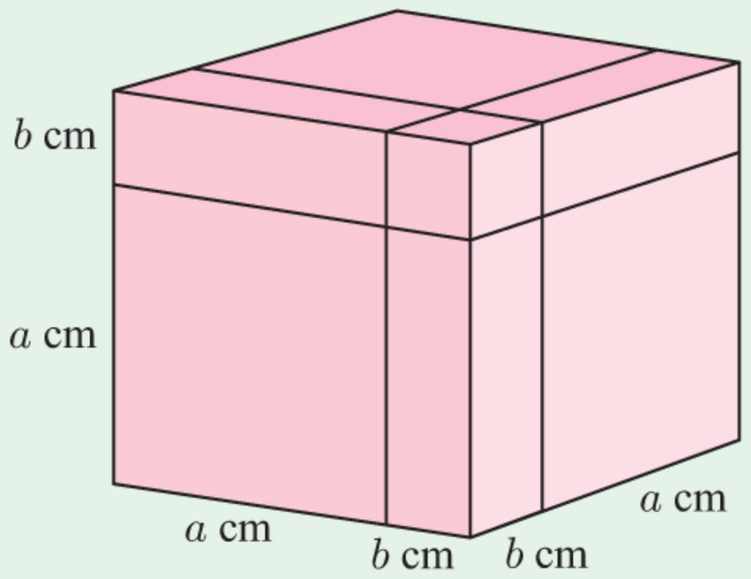
B The binomial theorem for $n \in \mathbb{Z}^+$

C The binomial theorem for $n \in \mathbb{Q}$



OPENING PROBLEM

The cube alongside has sides of length $(a+b)$ cm. Its volume is $(a+b)^3$ cm^3.



The cube has been subdivided into smaller blocks by making 3 cuts parallel to the cube's surfaces as shown.

Things to think about:

a How many blocks have been created?

b How many blocks are:

i a by a by a

ii a by a by b

iii a by b by b

iv b by b by b?

ANIMATION



c By adding the volumes of the blocks, can you write an expression which is equivalent to $(a+b)^3$?

The sum $a+b$ is called a **binomial** as it contains two terms.

Any expression of the form $(a+b)^n$ is called a **power of a binomial**.

We are already familiar with the perfect square expansion $(a+b)^2 = a^2+2ab+b^2$.

In this Chapter we study how $(a+b)^n$ is expanded for other powers.

A BINOMIAL EXPANSIONS

The following algebraic expansion for $(a+b)^3$ should confirm your result from the **Opening Problem**:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) && \{\text{pefect square expansion}\} \\ &= a^3+2a^2b+ab^2 \\ &\quad + \ a^2b+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3 && \{\text{collecting like terms}\}\end{aligned}$$

The **binomial expansion** of $(a+b)^3$ is $a^3+3a^2b+3ab^2+b^3$.

INVESTIGATION 1 THE BINOMIAL EXPANSION

What to do:

1 Expand $(a+b)^4$ using $(a+b)^4 = (a+b)(a+b)^3$.

2 Hence expand $(a+b)^5$ using $(a+b)^5 = (a+b)(a+b)^4$.

3 The cubic expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ contains 4 terms. They are written in order so that the powers of a decrease. We observe that their coefficients are: 1 3 3 1

a With the terms written in this order, what happens to the powers of b?

b Does the pattern in **a** continue for the expansions of $(a+b)^4$ and $(a+b)^5$?

c Use your results to continue this pattern of coefficients up to the case $n = 5$.

$n = 1$ 1 1
$n = 2$ 1 2 1
$n = 3$ 1 3 3 1 ← row 3
⋮

4 The triangle of numbers we are considering is called **Pascal's triangle**.

a How can each row of Pascal's triangle be predicted from the previous one?

b Predict the elements of the 6th row of Pascal's triangle.

c Hence write down the binomial expansion of $(a+b)^6$.

d Check your result algebraically using $(a+b)^6 = (a+b)(a+b)^5$.

From the **Investigation**, you should have observed that in Pascal's triangle:

- the values on the end of each row are always 1
- each of the remaining values is found by adding the two values diagonally above it.

1 1 row 1
1 2 1 row 2
1 3 3 1 row 3
1 4 6 4 1 row 4
1 5 10 10 5 1 row 5

You should have also found that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$= a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4 + 1 = 5$.
- The coefficients of the terms are row 4 of Pascal's triangle.

For the **binomial expansion** of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a *decrease* by 1, while the powers of b *increase* by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n + 1$.
- The coefficients of the terms are row n of Pascal's triangle.

Example 1 — Self Tutor

Use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ to find the binomial expansion of:

a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a = (2x)$ and $b = (3)$.

$$\therefore\ (2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3$$
$$= 8x^3 + 36x^2 + 54x + 27$$

b We substitute $a = (x)$ and $b = (-5)$.

$$\therefore\ (x-5)^3 = (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3$$
$$= x^3 - 15x^2 + 75x - 125$$



Example 2 — Self Tutor

Find the:

a 5th row of Pascal's triangle **b** binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

a
$$1 \quad 1 \longleftarrow \text{the 1st row, for } (a+b)^1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \longleftarrow \text{the 5th row, for } (a+b)^5$$

b Using the coefficients obtained in **a**, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$,

$$\left(x - \frac{2}{x}\right)^5$$
$$= (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$
$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

EXERCISE 8A

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(p+q)^3$ **b** $(x+1)^3$ **c** $(x-3)^3$

d $(2+x)^3$ **e** $(4-x)^3$ **f** $(3x-1)^3$

g $(2x+5)^3$ **h** $(x^2-1)^3$ **i** $(2a-b)^3$

j $(\sqrt{x}-1)^3$ **k** $\left(3x - \frac{1}{3}\right)^3$ **l** $\left(2x + \frac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(1+x)^4$ **b** $(x-2)^4$ **c** $(3-x)^4$

d $(1+2x)^4$ **e** $(2x-3)^4$ **f** $(2x+b)^4$

g $\left(x+\frac{1}{x}\right)^4$ **h** $\left(2x-\frac{1}{x}\right)^4$ **i** $(x+\sqrt{x})^4$

3 **a** Expand and simplify:

i $(a-b)^3$ **ii** $(a-b)^4$

b Compare the expansions in **a** with those of $(a+b)^3$ and $(a+b)^4$. Discuss the signs of the corresponding terms.

4 **a** Write down the 5th row of Pascal's triangle.

b Hence copy and complete: $(a+b)^5 = ...$

c Find the binomial expansion of:

i $(x+2)^5$ **ii** $(x-2y)^5$ **iii** $(1+2x)^5$ **iv** $\left(x-\frac{1}{x}\right)^5$

5 **a** Write down the 6th row of Pascal's triangle.

b Hence copy and complete: $(a+b)^6 = ...$

c Find the binomial expansion of:

i $(x+2)^6$ **ii** $(2x-1)^6$ **iii** $\left(x+\frac{1}{x}\right)^6$

6 Expand and simplify:

a $\left(1+\sqrt{2}\right)^3$ **b** $\left(\sqrt{5}+2\right)^4$ **c** $\left(2-\sqrt{2}\right)^5$

7 **a** Expand $(2+x)^6$. **b** Hence find the value of $(2.01)^6$.

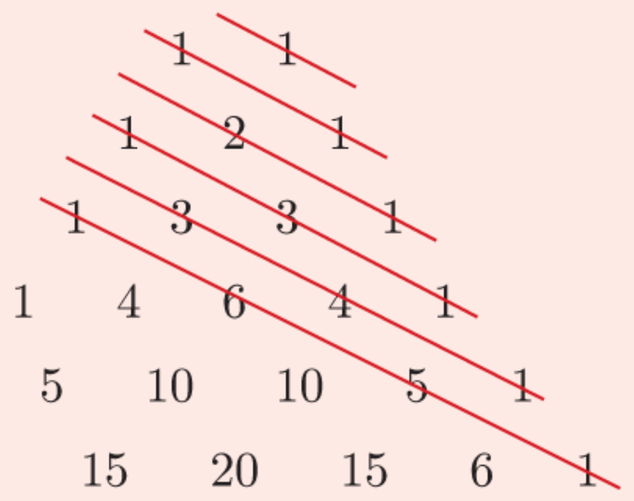
8 Expand and simplify $(2x+3)(x+1)^4$.

9 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$ **b** a^3b^3 in the expansion of $(2a+3b)^6$.

ACTIVITY

Suppose "shallow diagonals" are drawn on Pascal's triangle as shown alongside:



What to do:

1 Find the sum of the numbers in each diagonal.

2 Do you recognise the sequence of numbers formed by your answers to **1**? Can you explain why this occurs?

B THE BINOMIAL THEOREM FOR $n \in \mathbb{Z}^+$

For higher powers of a binomial, generating rows of Pascal's triangle by hand is very tedious. To efficiently expand $(a+b)^n$, we need a general formula for each term of the expansion.

INVESTIGATION 2 THE BINOMIAL COEFFICIENT

Consider the expansion of $(a+b)^n = (a+b)(a+b)(a+b)....(a+b)$.

What to do:

1 Suppose you expanded the brackets completely *without* simplifying "like" terms. How many terms would there be?

2 Each of these terms is generated by selecting one term, either a or b, from each of the n sets of brackets.

a If b is chosen r times, how many times is a chosen?

b In how many ways could b be chosen r times from the n sets of brackets?

c When you collect the "like" terms, how many terms of the form $a^{n-r}b^r$ will there be?

3 **a** Evaluate this triangle of numbers:

$$\begin{array}{ccccccccccccc}
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

b Copy and complete: *The rth number in the nth row of Pascal's triangle is .....*

From the **Investigation**, you should have found that:

- In the expansion of $(a+b)^n$, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.
  $\binom{n}{r}$ is hence called the **binomial coefficient**.
- $\binom{n}{r}$ is the rth number of the nth row of Pascal's triangle, for $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leqslant n$.

The **binomial theorem** states that:

For $n \in \mathbb{Z}^+$,

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n \\ &= \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r\end{aligned}$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ is the **binomial coefficient** for $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leqslant n$.

The binomial theorem allows us to perform a binomial expansion or find a particular term in a binomial expansion, without having to draw Pascal's triangle each time.

The **general term** or $(r+1)$th term in the binomial expansion $(a+b)^n$ is $T_{r+1} = \binom{n}{r} a^{n-r} b^r$.

Example 3 **Self Tutor**

Write down the first three and last two terms of the expansion of $\left(2x + \frac{1}{x}\right)^{12}$.
Do not simplify your answer.

$$\left(2x + \frac{1}{x}\right)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^{n-r} \left(\frac{1}{x}\right)^r$$
$$= (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right)^1 + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^2 + \ldots.$$
$$\ldots. + \binom{12}{11}(2x)^1\left(\frac{1}{x}\right)^{11} + \left(\frac{1}{x}\right)^{12}$$

Example 4 **Self Tutor**

Find the 7th term of $\left(3x - \frac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a = (3x)$, $b = \left(\frac{-4}{x^2}\right)$, and $n = 14$

Given the general term $T_{r+1} = \binom{n}{r} a^{n-r} b^r$, we let $r = 6$

$$\therefore\ T_7 = \binom{14}{6}(3x)^8 \left(\frac{-4}{x^2}\right)^6$$

EXERCISE 8B

1 Write down the first three and last two terms of the following binomial expansions. Do not simplify your answers.

a $(1+2x)^{11}$ **b** $\left(3x + \frac{2}{x}\right)^{15}$ **c** $\left(2x - \frac{3}{x}\right)^{20}$

2 Without simplifying, write down:

a the 6th term of $(2x+5)^{15}$ **b** the 4th term of $\left(x^2 + y\right)^9$

c the 4th term of $(x + y^2)^{10}$ **d** the 10th term of $\left(x - \frac{2}{x}\right)^{17}$

e the 9th term of $\left(2x^2 - \frac{1}{x}\right)^{21}$ **f** the 7th term of $(\sqrt{x}+1)^{16}$.

3 Show that $(a-b)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r a^{n-r} b^r$.

Example 5 **Self Tutor**

In the expansion of $\left(x^2 + \frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6 **b** the constant term.

$a = (x^2)$, $b = \left(\frac{4}{x}\right)$, and $n = 12$

$\therefore$ the general term $T_{r+1} = \binom{12}{r}(x^2)^{12-r}\left(\frac{4}{x}\right)^r$

$= \binom{12}{r}x^{24-2r} \times \frac{4^r}{x^r}$

$= \binom{12}{r}4^r x^{24-3r}$

a If $24 - 3r = 6$

then $3r = 18$

$\therefore\ r = 6$

$\therefore\ T_7 = \binom{12}{6}4^6 x^6$

$\therefore$ the coefficient of x^6 is $\binom{12}{6}4^6 = 3\,784\,704$.

b If $24 - 3r = 0$

then $3r = 24$

$\therefore\ r = 8$

$\therefore\ T_9 = \binom{12}{8}4^8 x^0$

$\therefore$ the constant term is $\binom{12}{8}4^8 = 32\,440\,320$.

4 Consider the expansion of $(x + 2)^8$.

a Write down the general term of the expansion. **b** Find the coefficient of x^5.

5 Consider the expansion of $(x + b)^7$.

a Write down the general term of the expansion.

b Find b given that the coefficient of x^4 is -280.

6 Consider the expansion of $\left(\frac{1}{2}x + \frac{1}{3}\right)^{12}$.

a Write down the general term of the expansion.

b Find the coefficient of:

i x^3 **ii** x^5 **iii** x^9

7 Find the constant term in the expansion of:

a $\left(x + \frac{2}{x^2}\right)^{15}$ **b** $\left(x - \frac{3}{x^2}\right)^9$ **c** $\left(\frac{x}{3} - \frac{2}{\sqrt{x}}\right)^{15}$

8 Find the coefficient of:

a x^{10} in the expansion of $(3 + 2x^2)^{10}$ **b** x^3 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^6$

c x^6y^3 in the expansion of $(2x^2 - 3y)^6$ **d** x^{12} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$.

9 Consider the expression $(x^2y - 2y^2)^6$. Find the term in which x and y are raised to the same power.

10 The third term of $(1 + x)^n$ is $36x^2$. Find n, and hence find the fourth term.

11 Find the value of a given:

a the coefficient of x^6 in the expansion of $(2x+a)^8$ is 448

b the coefficient of $\frac{y^2}{x^6}$ in the expansion of $\left(y-\frac{a}{x}\right)^8$ is 1792

c the fourth term in the expansion of $\left(ax+\sqrt{2}\right)^{10}$ is $30x^7$

d the coefficient of x^{11} in the expansion of $\left(x^2+\frac{1}{ax}\right)^{10}$ is 15.

Example 6 **Self Tutor**

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$
\begin{aligned}
&(x+3)(2x-1)^6 \\
&= (x+3)[(2x)^6 + \tbinom{6}{1}(2x)^5(-1) + \tbinom{6}{2}(2x)^4(-1)^2 + ...] \\
&= (x+3)(2^6x^6 - \tbinom{6}{1}2^5x^5 + \tbinom{6}{2}2^4x^4 - ....)
\end{aligned}
$$

(1) (2)

So, the terms containing x^5 are $\tbinom{6}{2}2^4x^5$ from (1)

and $-3\tbinom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\tbinom{6}{2}2^4 - 3\tbinom{6}{1}2^5 = -336$

12 Find:

a the coefficient of x^4 in the expansion of $(x+4)(x-3)^6$

b the term containing x^6 in the expansion of $(2-x)(3x+1)^9$

c the coefficient of x in the expansion of $(2+x)\left(\frac{1}{x}-2x\right)^4$

d the coefficient of x^3 in the expansion of $(x-2)^2(1-2x)^3$

e the coefficient of x^{-2} in the expansion of $(x+1)^3\left(\frac{1}{x}-2x\right)^6$

f the constant term in the expansion of $\left(x^2+\frac{1}{x}\right)^3\left(x-\frac{2}{x}\right)^4$

13 If $(1+kx)^n = 1-12x+60x^2-...., \ n \in \mathbb{Z}^+$, find the values of k and n.

14 **a** Write down the first 5 rows of Pascal's triangle.

b Find the sum of the numbers in:

i row 1 **ii** row 2 **iii** row 3 **iv** row 4 **v** row 5.

c Copy and complete: "The sum of the numbers in row n of Pascal's triangle is ....."

d Show that $(1+x)^n = \tbinom{n}{0} + \tbinom{n}{1}x + \tbinom{n}{2}x^2 + ... + \tbinom{n}{n-1}x^{n-1} + \tbinom{n}{n}x^n$.

e Hence deduce that:

i $\tbinom{n}{0} + \tbinom{n}{1} + \tbinom{n}{2} + ... + \tbinom{n}{n-1} + \tbinom{n}{n} = 2^n$

ii $\tbinom{n}{0} - \tbinom{n}{1} + \tbinom{n}{2} - \tbinom{n}{3} + ... + (-1)^n\tbinom{n}{n} = 0$

iii $\tbinom{2n+1}{0} + \tbinom{2n+1}{1} + \tbinom{2n+1}{2} + ... + \tbinom{2n+1}{n} = 4^n$

f By considering the binomial expansion of $(1+x)^n$, find $\sum_{r=0}^{n} 2^r\tbinom{n}{r}$.

15 **a** Write down the first four and last two terms of the binomial expansion $(3+x)^n$.

b Hence simplify $3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + ... + 3n + 1$.

16 Expand $(1+2x-x^2)^5$ in ascending powers of x as far as the term containing x^4.

17 Prove **Pascal's rule** $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ for all $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leqslant n$.

Explain what this means in the context of $(a+b)^{n+1} = (a+b)(a+b)^n$.

18 Consider $(x^2-3x+1)^2 = x^4-6x^3+11x^2-6x+1$. The sum of the coefficients in the expansion is $1-6+11-6+1=1$.

Find the sum of the coefficients in the expansion of $(x^3+2x^2+3x-7)^{100}$.

19 By considering $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \binom{n}{3}^2 + ... + \binom{n}{n}^2 = \binom{2n}{n}$.

20 **a** Prove that $r\binom{n}{r} = n\binom{n-1}{r-1}$.

b Hence show that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + ... + n\binom{n}{n} = n2^{n-1}$.

c Suppose the set of numbers $\{P_r\}$ are defined by

$$P_r = \binom{n}{r} p^r (1-p)^{n-r} \text{ for } r = 0, 1, 2, 3, ..., n.$$

Prove that: **i** $\sum_{r=0}^{n} P_r = 1$ **ii** $\sum_{r=1}^{n} rP_r = np$

HISTORICAL NOTE — THE BINOMIAL THEOREM

The binomial theorem is one of the most important results in mathematics.

The process of multiplying out binomial terms dates back to the beginning of algebra. Mathematicians had noticed relationships between the coefficients for many centuries, and Pascal's triangle was certainly widely used long before Pascal.

Sir Isaac Newton discovered the binomial theorem in 1665, but he did not publish his results until much later. Newton was the first person to give a formula for the binomial coefficients. His ground-breaking result included a generalisation of the binomial theorem to the case of $(a+b)^n$ where $n \in \mathbb{Q}$. In doing this, Newton was the first person to confidently use the exponent notation that we recognise today for both negative and fractional powers.



Sir Isaac Newton

C THE BINOMIAL THEOREM FOR $n \in \mathbb{Q}$

We have seen that for $n \in \mathbb{Z}^+$,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ... + b^n$$

Observe how the coefficient of the $(r+1)$th term is $\dfrac{n-(r-1)}{r}$ times the coefficient of the rth term.

Sir Isaac Newton generalised this series expansion to include binomials raised to any rational power. This requires an *infinite* series.

The **binomial expansion** for $n \in \mathbb{Q}$ is $\boldsymbol{(a+b)^n = \sum_{r=0}^{\infty} \binom{n}{r} a^{n-r} b^r}$

where $\binom{n}{r} = \dfrac{n(n-1)(n-2)....(n-r+1)}{r!}$ is the **binomial coefficient** for $n \in \mathbb{Q},\ r \in \mathbb{Z}^+$,

and $\binom{n}{0} = 1$.

In the case where $n \in \mathbb{Z}^+$, the series terminates after $n+1$ terms. This is because with the $(n+2)$th term, $r = n+1$, and so the coefficient is multiplied by $\dfrac{n-(n+1-1)}{n+1} = 0$. Every subsequent term will also have coefficient 0.

THE BINOMIAL EXPANSION OF $(a+bx)^n$ FOR $n \in \mathbb{Q}$

Having observed that an infinite geometric series will only converge under particular conditions, we recognise that this infinite series expansion for $(a+b)^n,\ n \in \mathbb{Q}$ will also only converge under particular conditions.

For example, if we consider the binomial expansion $(a+bx)^n,\ n \in \mathbb{Q}$, the expansion will only converge for values of x on a particular interval called the **interval of convergence**.

To examine this, we first notice that $(a+bx)^n = a^n\left(1+\dfrac{b}{a}x\right)^n$, then apply the binomial expansion.

For $n \in \mathbb{Q}$, $\boldsymbol{(a+bx)^n = a^n \sum_{r=0}^{\infty} \binom{n}{r} \left(\frac{bx}{a}\right)^r}$

where $\binom{n}{r} = \dfrac{n(n-1)(n-2)....(n-r+1)}{r!}$ is the **binomial coefficient** for $n \in \mathbb{Q},\ r \in \mathbb{Z}^+$,

and $\binom{n}{0} = 1$.

The **interval of convergence** for which the expansion converges is $\left|\dfrac{bx}{a}\right| < 1$, which means $-\left|\dfrac{a}{b}\right| < x < \left|\dfrac{a}{b}\right|$.

The proof of this result is beyond the scope of the course. However, we may think of it intuitively as follows:

$$(a+bx)^n = a^n \sum_{r=0}^{\infty} \binom{n}{r} \left(\frac{bx}{a}\right)^r$$

$$= a^n \sum_{k=1}^{\infty} \binom{n}{k-1} \left(\frac{bx}{a}\right)^{k-1} \qquad \{\text{letting } k = r+1\}$$

where the series $\sum_{k=1}^{\infty} \binom{n}{k-1} \left(\frac{bx}{a}\right)^{k-1}$ is similar to the infinite geometric sequence $\sum_{k=1}^{\infty} u_1 r^{k-1}$ with $r = \dfrac{bx}{a}$. The difference is that rather than the constant u_1 in the infinite geometric sequence, we now have the binomial coefficient $\binom{n}{k-1}$ which varies with each term. However, just as the infinite geometric sequence converges for $|r| < 1$, the binomial expansion converges for $\left|\dfrac{bx}{a}\right| < 1$.

Most importantly, for the series to converge, the terms must become smaller in size as k increases. This means that we can approximate the binomial expansion by truncating it at a finite number of terms.

Example 7 **Self Tutor**

Consider the binomial expansion of $\frac{1}{\sqrt{2+x}}$.

a Write down the first 4 terms of the expansion.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\frac{1}{\sqrt{2.1}}$. Check your answer by direct calculation.

a $\frac{1}{\sqrt{2+x}} = (2+x)^{-\frac{1}{2}}$

$= 2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2}}\sum_{r=0}^{\infty}\binom{-\frac{1}{2}}{r}\left(\frac{x}{2}\right)^r$

$= \frac{1}{\sqrt{2}}\left(1+(-\frac{1}{2})\left(\frac{x}{2}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{2}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{x}{2}\right)^3+....\right)$

$= \frac{1}{\sqrt{2}}\left(1-\frac{1}{4}x+\frac{3}{32}x^2-\frac{5}{128}x^3+....\right)$

b The series converges provided $\left|\frac{x}{2}\right| < 1$, which is the interval $-2 < x < 2$.

c Letting $x = 0.1$, $\frac{1}{\sqrt{2.1}} \approx \frac{1}{\sqrt{2}}\left(1-\frac{1}{4}(0.1)+\frac{3}{32}(0.1)^2-\frac{5}{128}(0.1)^3\right)$

$\approx 0.690\,064$

Using technology, $\frac{1}{\sqrt{2.1}} \approx 0.690\,066$.

EXERCISE 8C

1 Evaluate:

a $\binom{\frac{1}{2}}{1}$ **b** $\binom{-\frac{1}{2}}{2}$ **c** $\binom{-2}{2}$ **d** $\binom{-\frac{1}{3}}{0}$ **e** $\binom{-1}{3}$ **f** $\binom{\frac{1}{3}}{3}$

2 Consider the binomial expansion of $\frac{1}{1+x}$.

a Write down the first 4 terms.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\frac{1}{1.1}$. Check your answer by direct calculation.

3 Consider the binomial expansion of $\frac{1}{\sqrt{4-x}}$.

a Write down the first 4 terms.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\frac{1}{\sqrt{3.6}}$. Check your answer by direct calculation.

4 Consider the binomial expansion of $\sqrt[3]{1+3x}$.

a Write down the first 5 terms.

b State the interval of convergence for the complete expansion.

c Estimate $\sqrt[3]{1.3}$ using:

i the first 4 terms **ii** the first 5 terms.

d Compare your estimates with the actual value of $\sqrt[3]{1.3}$.

5 **a** Expand $\dfrac{1}{(1-2x)^2}$ up to the term in x^3.

b State the values of x for which the complete expansion of $\dfrac{1}{(1-2x)^2}$ converges.

c Use the expansion to estimate $\dfrac{1}{(0.96)^2}$.

6 **a** Expand $\sqrt{1+x}$ up to the term in x^3.

b State the values of x for which the complete expansion of $\sqrt{1+x}$ converges.

c To estimate $\sqrt{37}$, explain why you should not simply substitute $x = 36$.

d Estimate $\sqrt{37}$ by first letting $\sqrt{37} = \sqrt{36} \times \sqrt{\frac{37}{36}} = \sqrt{36} \times \sqrt{1+\frac{1}{36}}$.

7 **a** Expand $(1+x)^{\frac{1}{5}}$ up to the term in x^3.

b Hence obtain an approximation for $\sqrt[5]{1.01}$.

c Explain why you should not use your expansion to approximate $\sqrt[5]{2}$.

8 **a** Expand $\dfrac{1}{(1+2x)^2}$ up to the term in x^3.

b Hence find the coefficient of x^2 in the expansion of $\dfrac{(1-3x)^3}{(1+2x)^2}$.

9 **a** Write $\dfrac{x-8}{x^2-x-2}$ in the form $\dfrac{A}{x+1} + \dfrac{B}{x-2}$, where A and B are constants.

b Hence write an expansion for $\dfrac{x-8}{x^2-x-2}$ up to the term in x^3.

c For what values of x does the complete expansion of $\dfrac{x-8}{x^2-x-2}$ converge?

d Use your expansion to estimate the value of $\dfrac{x-8}{x^2-x-2}$ when $x = 0.05$.

10 **a** Expand each expression up to the term in x^3, and state the interval of convergence for the complete expansion:

i $(1+3x)^{\frac{1}{4}}$ **ii** $(2-x)^{-\frac{1}{4}}$

b Hence write an expansion for $\sqrt[4]{\dfrac{1+3x}{2-x}}$ up to the term x^3.

11 Let $f(x) = \dfrac{1}{\sqrt[4]{1+x}}$.

a Write $f(x)$ as a binomial expansion up to the term in x^3.

b State the values of x for which the complete expansion of $f(x)$ converges.

c By considering $f(3x^2)$, write an expansion for $\dfrac{1}{\sqrt[4]{1+3x^2}}$ up to the term in x^6.

REVIEW SET 8A

1 Use the binomial expansion to expand and simplify:

a $(x+3)^3$ **b** $(x-2)^5$

2 Without simplifying, write down:

a the 5th term of $(2x+3)^9$ **b** the 4th term of $(y-3z)^9$

c the 8th term of $\left(3x-\frac{1}{x}\right)^{12}$ **d** the 5th term of $\left(2x-\frac{1}{\sqrt{x}}\right)^{10}$.

3 Expand and simplify:

a $\left(5+\sqrt{3}\right)^3$ **b** $(x+3)(x-1)^4$

4 Use the expansion of $(4+x)^3$ to find the exact value of $(4.02)^3$.

5 **a** Use Pascal's triangle to expand $(a+b)^6$.

b Hence find the binomial expansion of: **i** $(x-3)^6$ **ii** $\left(2+\frac{1}{x}\right)^6$

6 Find the coefficient of:

a x^{-6} in the expansion of $\left(2x-\frac{3}{x^2}\right)^{12}$

b x^5 in the expansion of $(2x+3)(x-2)^6$.

7 Find the coefficient of x^{-1} in the expansion of $(x+2)^4\left(\frac{1}{x}-3x\right)^5$.

8 Find c given that the expansion $(1+cx)(1+x)^4$ includes the term $22x^3$.

9 **a** Write down the first four and last two terms of the binomial expansion $(2+x)^n$ where $n \in \mathbb{Z}^+$.

b Hence simplify $2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} + ... + 2n + 1$.

10 Find the possible values of a if the coefficient of x^3 in $\left(2x+\frac{1}{ax^2}\right)^9$ is 288.

11 Consider the binomial expansion of $\frac{1}{1-x}$.

a Write down the first 4 terms of the expansion.

b For what values of x does the infinite series expansion converge?

c Use the expansion to estimate $\frac{1}{0.95}$. Check your answer by direct calculation.

12 Consider the binomial expansion of $\frac{1}{\sqrt{x+3}}$.

a Write down the first 4 terms of the expansion.

b For what values of x does the infinite series expansion converge?

c Use the expansion to estimate: **i** $\frac{1}{\sqrt{5}}$ **ii** $\frac{1}{\sqrt{3.2}}$

Comment on the accuracy of your estimates.

13 **a** Expand $\sqrt{1-x}$ up to the term in x^3.

b State the values of x for which the infinite series binomial expansion of $\sqrt{1-x}$ converges.

c Hence estimate $\sqrt{15}$.

14 **a** Expand $\dfrac{1}{(1-3x)^2}$ up to the term in x^3.

b Hence find the coefficient of x^2 in the expansion of $\dfrac{(1+4x)^2}{(1-3x)^2}$.

REVIEW SET 8B

1 Use the binomial expansion to find:

a $(x-2y)^3$ **b** $(3x+2)^4$

2 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

3 Find the constant term in the expansion of:

a $\left(2x^2-\dfrac{1}{x}\right)^6$ **b** $\left(x-\dfrac{6}{x^2}\right)^9$ **c** $\left(\sqrt{x}+\dfrac{6}{\sqrt{x}}\right)^5$

4 Expand and simplify:

a $\left(2-\sqrt{2}\right)^6$ **b** $(x+3)(2x+1)^3$

5 Find the coefficient of:

a x^{10} in the expansion of $\left(\dfrac{3}{x^2}-4x\right)^{10}$

b x^5 in the expansion of $(3x-1)(5+2x)^6$.

6 In the expansion of $\left(3x^2+\dfrac{1}{x}\right)^9$, find:

a the coefficient of x^{12} **b** the constant term.

7 Find the coefficient of x^2 in the expansion of $\left(2x+\dfrac{1}{x}\right)^3\left(x^2-\dfrac{2}{x}\right)^4$.

8 The first three terms in the expansion of $(1+kx)^n$, $n \in \mathbb{Z}^+$, are $1-4x+\frac{15}{2}x^2$. Find k and n.

9 Find k in the expansion $(m-2n)^{10} = m^{10} - 20m^9n + km^8n^2 - .... + 1024n^{10}$.

10 Find the possible values of q if the constant terms in the expansions of $\left(x^3+\dfrac{q}{x^3}\right)^8$ and $\left(x^3+\dfrac{q}{x^3}\right)^4$ are equal.

11 Find the value of a given:

a the coefficient of x^5 in the expansion of $(a-2x)^7$ is -42

b the fifth term in the expansion of $\left(\dfrac{1}{x\sqrt{2}}-a\right)^8$ is $\dfrac{1120}{x^4}$.

12 Consider the binomial expansion of $(1+2x)^{-\frac{3}{2}}$.

a Write down the expansion up to the term in x^3.

b State the interval of convergence for the complete binomial expansion.

c Use the expansion up to the term in x^3 to estimate $(1.1)^{-\frac{3}{2}}$.

13 Expand $\dfrac{1+x^2}{\sqrt{1+x}}$ up to the term in x^3. State the values of x for which the expansion is valid.

14 Expand up to the term in x^3, in each case stating the values of x for which the expansion is valid:

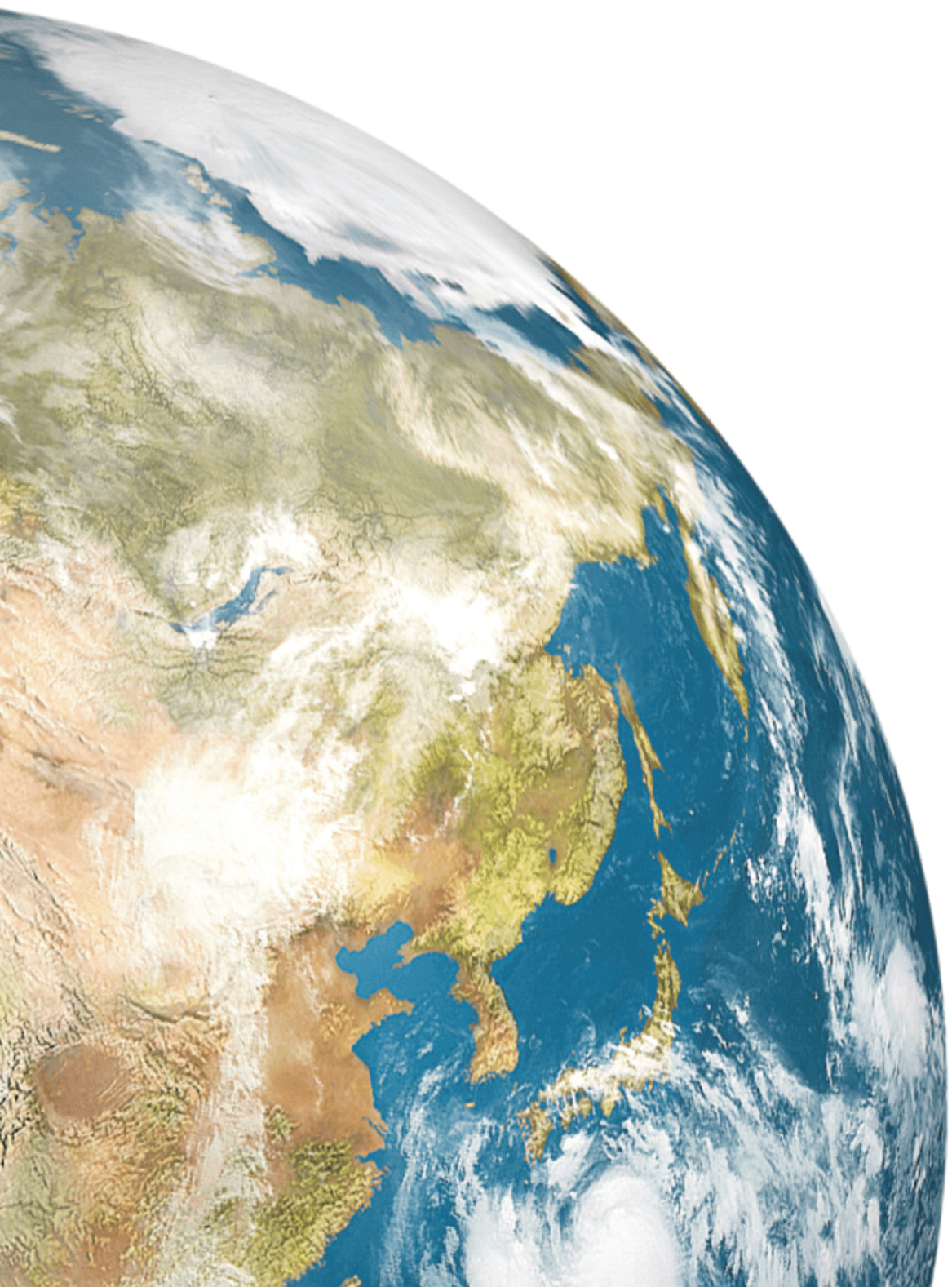
a $\left(1+\dfrac{x}{2}\right)^{\frac{1}{3}}$ **b** $\left(1-\dfrac{x}{3}\right)^{-\frac{1}{3}}$ **c** $\left(\dfrac{6+3x}{6-2x}\right)^{\frac{1}{3}}$

Chapter 9

Reasoning and proof

Contents:

A Logical connectives
B Proof by deduction
C Proof by equivalence
D Definitions
E Proof by exhaustion
F Disproof by counter example
G Proof by contrapositive
H Proof by contradiction: reductio ad absurdum



OPENING PROBLEM

A group of people are involved in a business meeting. Alice looks at Bob and Bob looks at Clare. Alice is married but Clare is not.



Things to think about:

a Do we know whether Bob is married?

b Can we *prove* that a married person looks at an unmarried person?

c What is necessary for a mathematical argument to be convincing and complete?

Greek mathematicians more than 2000 years ago realised that progress in mathematical thinking could be brought about by conscious formulation of the methods of **abstraction** and **proof**.

By considering a few examples, we might notice a certain common quality or pattern from which we could predict a rule or formula for the general case. In mathematics this prediction is known as a **conjecture**. Mathematicians love to find patterns, and try to understand why they occur.

Experiments and further examples might help to convince you that the conjecture is true. However, problems will often contain extra information which can sometimes obscure the essential detail, particularly in applied mathematics. Stripping this away is the process of **abstraction**.

For example, by considering the given table of values we may conjecture:

"If a and b are real numbers, then $a < b$ implies that $a^2 < b^2$."

However, on observing that $-2 < 1$ but $(-2)^2 \not< 1^2$ we have a **counter example**.

a	b	a^2	b^2
1	2	1	4
3	5	9	25
4	5	16	25
5	7	25	49
6	9	36	81

In the light of this we reformulate and refine our conjecture:

"If a and b are *positive* real numbers then $a < b$ implies $a^2 < b^2$."

The difficulty is that this process might continue with reformulations, counter examples, and revised conjectures indefinitely. We can only be certain that the conjecture is true if it is **proven**.

WHAT IS PROOF?

Science relies on experimental evidence. Mathematics relies on logic and reasoning.

A **mathematical proof** is a correct argument which establishes the truth of a mathematical statement.

A mathematical proof:

- starts with assumptions called **hypotheses**
- is a sequence of correct mathematical steps
- ends in a **conclusion**.

A mathematical proof can include logic, calculation, or a combination of the two. We will discuss both of these during the Chapter.

In the **Opening Problem**, we do not know whether or not Bob is married, so we need to consider both cases. We could write the following proof:

Proof:

Bob is either married or he is not married.

- If Bob is married, then when Bob looks at Clare, a married person looks at an unmarried person.
- If Bob is *not* married, then when Alice looks at Bob, a married person looks at an unmarried person.

In both cases, a married person looks at an unmarried person.

These cases exhaust all possibilities, so a married person always looks at an unmarried person.

In this proof, notice that:

- "Alice is married" is a *hypothesis*.
- The *conclusion* is that a married person looks at an unmarried person.
- This *style* of proof is called *proof by exhaustion*, because we have considered all possible cases: Bob is married, and Bob is not married.
- This proof contains logic only. There is no calculation.

THE PURPOSES OF PROOF

Mathematical proof is important:

- **to convince**
  Proofs help you decide if and why a statement is true or false. This is important when the result seems strange.
- **to understand**
  Proofs help you understand how and why all the different assumptions play a part in the result.
- **to communicate**
  Mathematicians use proofs to communicate and debate ideas with each other.
- **to organise**
  Proofs help you organise your thoughts.
- **to discover new mathematics**
  By carefully examining each step of a proof, mathematicians discover new mathematics.

ADVICE ON WRITING PROOFS

- State what you are proving. Make it clear when you have reached your conclusion.
- Include enough detail to make your proof easy to check.
- Your proof should be written in good English, including simple, complete, correct sentences.
- Present your calculations on the page in a manner which makes them easy to follow. Use your layout to clearly separate the parts or different cases considered in your proof.
- Use diagrams when appropriate to give a visual representation of the situation.
- It is often useful to use examples when exploring a problem. However, a single example is usually not sufficient to complete a proof.
- Check to see where you have used each hypothesis. If you have not used a particular hypothesis then either:
  - you did not need it, or
  - your proof should have made use of it, and is incorrect!
- Do not expect to write a complete proof the first time. Expect to use rough working first, and then write a neat version.

A LOGICAL CONNECTIVES

Mathematical language uses **logical connectives** to link mathematical statements together:

Connective	*Symbol*	*Formal name*
and	$\wedge$	conjunction
or	$\vee$	disjunction
not	$\neg$	negation
if .... then	$\Rightarrow$	implication
if, and only if	$\Leftrightarrow$	equivalence

We can illustrate the use of these connectives using *variables* such as A, B, C, ...

For example, $(A \wedge B) \Rightarrow C$ means "A and B implies C". The two arguments $(A \wedge B)$ and C are connected by the connective $\Rightarrow$. The argument $(A \wedge B)$ is itself made up of arguments A and B, connected by the connective $\wedge$. It is therefore helpful to think of statements as mathematical trees, connected by the logical connectives.

NEGATION

The negation of a variable A is its opposite. If A is true then $\neg A$ is false, and vice versa.

For example:

- the negation of "Today is Wednesday" is "Today is not Wednesday"
- if we know $x \in \mathbb{Z}$, the negation of "x is an even integer" is "x is an odd integer"
- if we know $x \in \mathbb{R}$, the negation of "x is an even integer" is "x is not an even integer", since x might be an odd integer, or it might also be a non-integer.

IMPLICATIONS

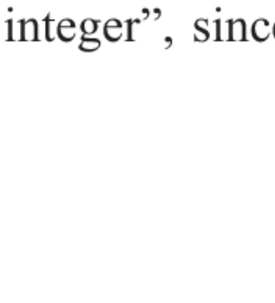
Implications commonly arise when we deduce one thing from another.

For the implication $A \Rightarrow B$, we start with the statement A, and from it we deduce the statement B. The statement A is called the **hypothesis**, and B is called the **conclusion**.

In English there are many words we can use to show an implication.

For example: $A \left\{ \begin{array}{c} \text{implies} \\ \text{so} \\ \text{hence} \\ \text{thus} \\ \text{therefore} \end{array} \right\} B.$

We use $\therefore$ to mean "therefore".



The **converse** of the implication $A \Rightarrow B$ is the statement $B \Rightarrow A$.

It is important to recognise that while the implication $A \Rightarrow B$ may be true, its converse may not be true.

For example, the statement "If $x = 2$ then $x^2 = 4$" is true. However, its converse "If $x^2 = 4$ then $x = 2$" is false, since x may be -2.

EQUIVALENCE

Two statements A and B are **equivalent** if both $A \Rightarrow B$ *and* $B \Rightarrow A$. In this case we can say A is true if and only if B is true.

For example, the statement "If $x = -2$ or 2, then $x^2 = 4$" is true. The statement "If $x^2 = 4$ then $x = -2$ or 2" is also true. Therefore, $x = -2$ or $2 \Leftrightarrow x^2 = 4$.

EXERCISE 9A

1 For each statement, state its negation:

a The cat is black. **b** x is prime. **c** The tree is deciduous.

2 State, with justification, whether each statement is true or false:

a If $x^2 = 9$ then $x = 3$. **b** If $x = 3$ then $x^2 = 9$. **c** $x = 3$ if and only if $x^2 = 9$.

3 State, with justification, whether each statement is true or false:

a If x is positive then $\sqrt{x} \in \mathbb{R}$. **b** If $\sqrt{x} \in \mathbb{R}$ then x is positive.

c x is positive if and only if $\sqrt{x} \in \mathbb{R}$.

4 **a** Write the converse of the statement "If Socrates is a cat then Socrates is an animal".

b Is this converse true or false?

5 Determine whether A and B are equivalent:

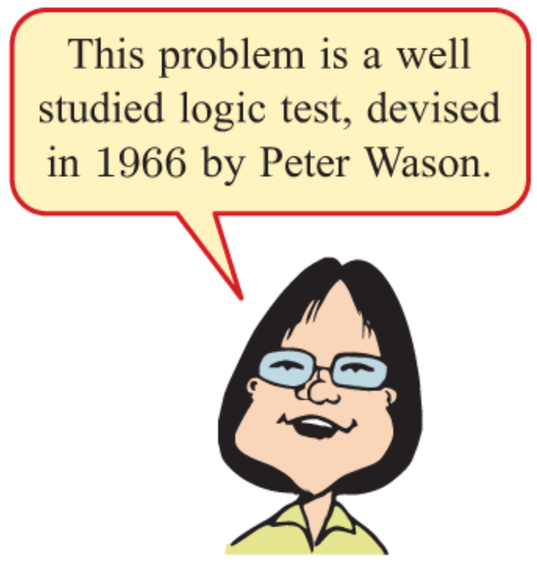
a A: $xyz = 0$, B: $(x = 0) \vee (y = 0) \vee (z = 0)$

b A: x is even, B: x^2 is even

6 There are four cards on a table. Every card has a letter on one side and a number on the other. With the cards placed, you see:



Identify precisely which cards you need to turn over, and which you do not, to establish the truth of the statement: "Every card which has a D on one side has a 7 on the other".



B PROOF BY DEDUCTION

Throughout the course already, we have used algebra to show different results. In cases where we are given some information (the hypothesis) and are asked to prove something, we use a chain of implications. This is called **proof by deduction**.

Proof by deduction is called a **direct proof** because we start with the hypothesis and work forward towards the conclusion.

We have mostly used the therefore symbol "$\therefore$" to indicate our implications, but we could have also used the implication symbol "$\Rightarrow$".

Since these arguments only need to go from the starting information to the conclusion, we also call this type of proof a "left-to-right" proof.

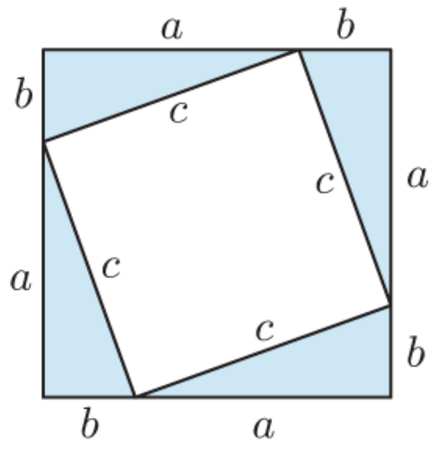
INVESTIGATION PYTHAGORAS' THEOREM

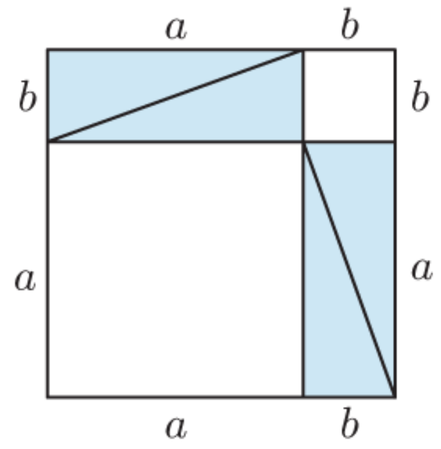
Pythagoras' Theorem is one of the most famous results in mathematics:

If a right angled triangle has sides a, b, and hypotenuse c, then $a^2 + b^2 = c^2$.

Proof by pictures:

The pictures below each show four copies of the original triangle, in different arrangements.





In each figure, the total area is $(a+b)^2$, and the shaded area is $4 \times \frac{1}{2}ab = 2ab$.

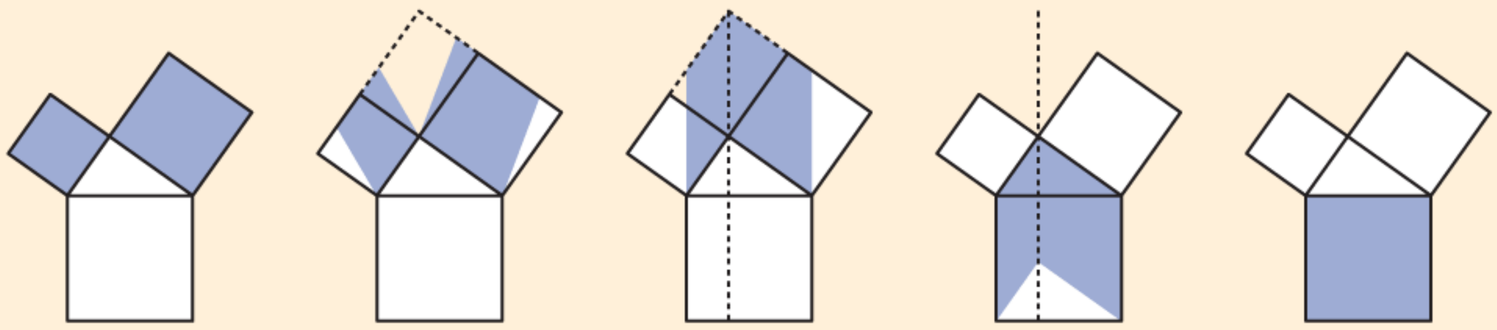
$\therefore$ the unshaded areas are equal

$\therefore$ $c^2 = a^2 + b^2$

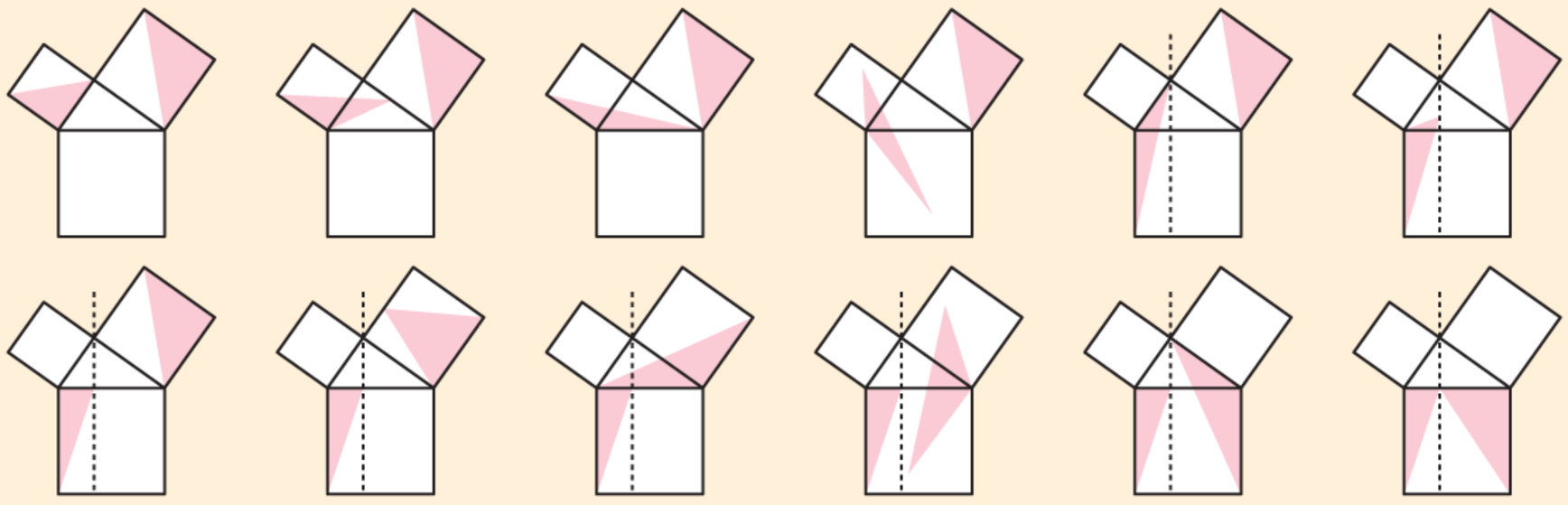
In 1968, E. Loomis collected around 365 different proofs in his book *The Pythagorean Proposition* (National Council of Teachers of Mathematics, Washington). It might seem strange to have so many different proofs of the same thing, given that just one correct proof establishes the theorem. However, one purpose of proof is to help us understand the result, and different proofs can help develop this understanding.

What to do:

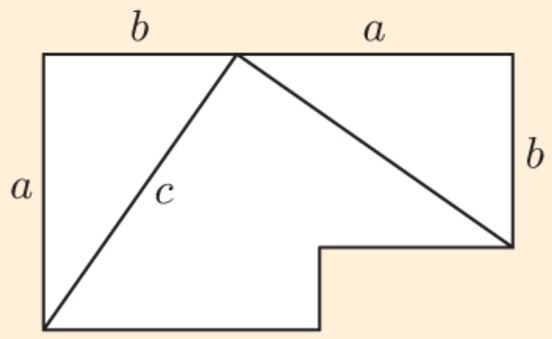
1 The sequence of pictures below shows shears and translations of parallelograms. Explain how the sequence proves Pythagoras' Theorem.

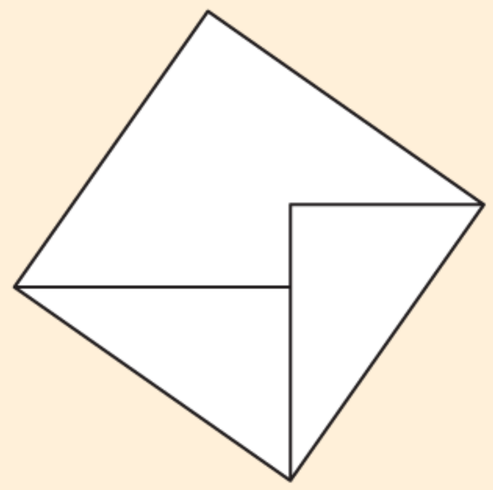


2 The sequence of pictures below shows shears and rotations of triangles. Explain how the sequence proves Pythagoras' Theorem.



3 The figure below shows a jigsaw made from three pieces, in two arrangements. Explain how these prove Pythagoras' Theorem.





In a numerical or algebraic argument, we perform operations at each deductive step. For a proof by deduction to be valid, only a correct **implication** is required, rather than equivalence.

Example 1 — Self Tutor

a Prove by deduction: "If $x < -5$ then $x^2 > 25$."

b Explain why the converse is *not* true.

a If $x < -5$ then $|x| > 5$ {geometric definition of absolute value}

$\therefore |x|^2 > 5^2$ {squaring both sides}

$\therefore x^2 > 25$ $\{|x|^2 = x^2\}$

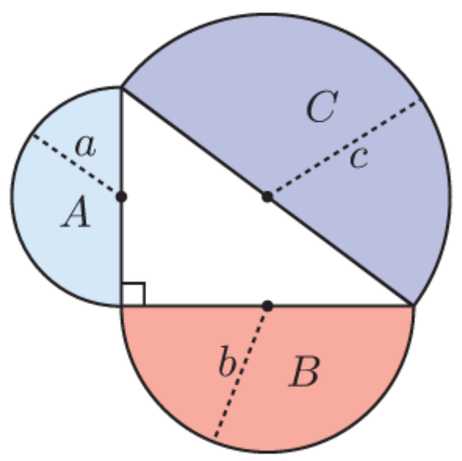
b The converse would be: "If $x^2 > 25$ then $x < -5$."
This is not necessarily true because $6^2 > 25$, but $6 > -5$.

EXERCISE 9B

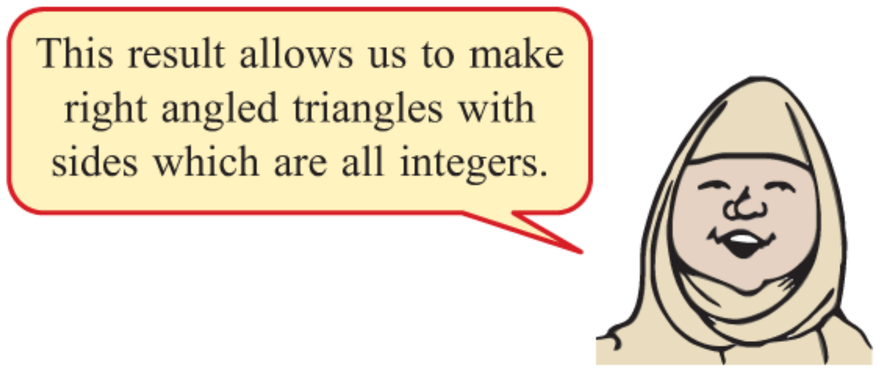
1 **a** Prove by deduction: "If $x = -2$ then $x^2 - x - 6 = 0$."

b Explain why the converse is *not* true.

2 In the diagram alongside, three semi-circles have been constructed using the sides of a right angled triangle as diameters. The areas of the semi-circles are A, B, and C as shown. Prove that $A + B = C$.



3 Suppose $a, b \in \mathbb{N}$. Let $x = a^2 - b^2$, $y = 2ab$, and $z = a^2 + b^2$. Prove that $x^2 + y^2 = z^2$.



4 Prove that the sum of three consecutive integers is divisible by 3.

5 The product of three consecutive integers is increased by the middle integer. Prove that the result is a perfect cube.

Hint: Let the middle number be x.

6 Prove that $\dfrac{a^2+b^2}{2} \geqslant ab$ for all $a, b \in \mathbb{R}$.

7 Prove by deduction that $\sin 2\theta \tan\theta = 2\sin^2\theta$.

8 Consider a 3-digit number "abc", $a \neq c$. Written backwards, it is "cba". Let S be the result when the smaller of the two numbers is subtracted from the larger. When S is written backwards and the result is added to S, prove that the sum is always 1089.
For example: 276 backwards is 672, so $S = 396$ and $396 + 693 = 1089$.

9 The following "proofs" by deduction are incorrect. Identify the incorrect step(s) in each case:

a
$$4x^2 = 3x$$
$$\therefore\ 4x = 3$$
$$\therefore\ x = \tfrac{3}{4}$$

b
$$(x+3)(2-x) = 4$$
$$\therefore\ x+3 = 4 \text{ or } 2-x = 4$$
$$\therefore\ x = 1 \text{ or } x = -2$$

HISTORICAL NOTE

Charles Lutwidge Dodgson (1832 - 1898) was a lecturer of mathematics at Christ Church college, Oxford. A curious and controversial figure, he is best known by his pen name, **Lewis Carroll**, under which he wrote a number of books including *Alice's Adventures in Wonderland* (1865) and *Through the Looking-Glass and What Alice Found There* (1871).



Charles Lutwidge Dodgson

In his latter years, Carroll wrote several more mathematical books, including *The Game of Logic* (1886) and *Symbolic Logic* (1896). These books include numerous "nonsense" statements written to challenge the reader as to what, if anything, could be deduced from them.

For example, consider these statements from *The Game of Logic*:

- "No bald person needs a hair-brush; No lizards have hair."
  We can deduce that since lizards do not have hair, they do not need a hair-brush.
- "Some oysters are silent; No silent creatures are amusing."
  We can deduce that silent oysters are not amusing, but we cannot say anything about oysters which are not silent.
- "No riddles interest me that can be solved; All these riddles are insoluble."
  We can deduce nothing from these statements, because no comment is made about whether insoluble riddles interest him.

Think about these for yourself:

- Toothache is never pleasant; Warmth is never unpleasant.
- All uneducated men are shallow; All these students are educated.
- No lobsters are unreasonable; No reasonable creatures expect impossibilities.
- No misers are generous; Some old men are not generous.
- Sugar is sweet; Some sweet things are liked by children.
- "I saw it in a newspaper; All newspapers tell lies."
- All wasps are unfriendly; No puppies are unfriendly.

C PROOF BY EQUIVALENCE

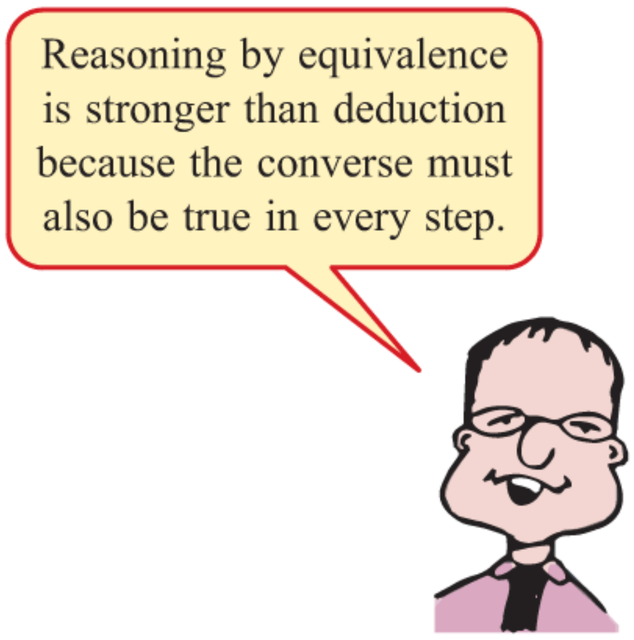
In proof by deduction, we were only concerned with moving in one direction.

This means that if we want to show that two statements A and B are *equivalent*, then we would need to prove both $A \Rightarrow B$ and $B \Rightarrow A$ separately.

However, if we can construct an argument so that each deductive step is an equivalence, then we will prove $A \Rightarrow B$ and $B \Rightarrow A$ at the same time.

For example, to deduce the solutions of $(x+1)^3 = x^3+1$ we can write the following series of equivalent statements:

$$
\begin{aligned}
& (x+1)^3 = x^3+1 \\
\Leftrightarrow \quad & x^3+3x^2+3x+1 = x^3+1 \\
\Leftrightarrow \quad & 3x^2+3x = 0 \\
\Leftrightarrow \quad & 3x(x+1) = 0 \\
\Leftrightarrow \quad & x = 0 \quad \vee \quad x = -1
\end{aligned}
$$



This chain of reasoning relies on the facts that:

- two equations are equivalent if they have precisely the same solutions
- two functions $p(x)$ and $q(x)$ are equivalent if $p(x) = q(x)$ for all $x \in \mathbb{R}$.

To maintain mathematical equivalence, we can:

- add or subtract the same term from both sides of an equation
- multiply or divide both sides of an equation by a non-zero term
- substitute for an equivalent sub-term, for example by replacing a term by its factored or expanded form.

There are some other operations which maintain equivalence, for example some of the rules of logarithms with positive terms.

However, squaring or taking square roots of both sides of an equation does not maintain equivalence!

Given $a^2 = b^2$, taking the square root of both sides gives $a = b$. However, in doing this we have ignored the possibility that $a = -b$.

Instead, we use the difference between two squares result as follows:

$$
\begin{aligned}
& a^2 = b^2 \\
\Leftrightarrow \quad & a^2 - b^2 = 0 \\
\Leftrightarrow \quad & (a-b)(a+b) = 0 && \{\text{difference between two squares}\} \\
\Leftrightarrow \quad & a-b = 0 \quad \vee \quad a+b = 0 && \{\text{null factor law}\} \\
\Leftrightarrow \quad & a = b \quad \vee \quad a = -b \\
\Leftrightarrow \quad & a = \pm b
\end{aligned}
$$

Using this procedure, equivalence is maintained. With experience, mathematicians *compress* this working into a single step.

Example 2 **Self Tutor**

Find the smallest positive integer a for which $x^2 + (a-2)x + a = 0$ has real solutions.

$$\begin{aligned}
& x^2 + (a-2)x + a = 0 \\
\Leftrightarrow \quad & \left(x + \frac{a-2}{2}\right)^2 - \left(\frac{a-2}{2}\right)^2 + a = 0 && \{\text{completing the square}\} \\
\Leftrightarrow \quad & \left(x + \frac{a-2}{2}\right)^2 = \frac{(a-2)^2}{4} - a
\end{aligned}$$

The equation has real solutions if and only if

$$\begin{aligned}
& \frac{(a-2)^2}{4} - a \geqslant 0 \\
\Leftrightarrow \quad & (a-2)^2 - 4a \geqslant 0 \\
\Leftrightarrow \quad & a^2 - 8a + 4 \geqslant 0 \\
\Leftrightarrow \quad & (a-4)^2 - 16 + 4 \geqslant 0 \\
\Leftrightarrow \quad & (a-4)^2 \geqslant 12 \\
\Leftrightarrow \quad & a - 4 \geqslant \sqrt{12} \quad \vee \quad a - 4 \leqslant -\sqrt{12} \\
\Leftrightarrow \quad & a \geqslant 4 + \sqrt{12} \approx 7.46 \quad \vee \quad a \leqslant 4 - \sqrt{12} \approx 0.536
\end{aligned}$$

$\therefore$ the smallest positive integer a for which the equation has real solutions is $a = 8$.

EXERCISE 9C

1 Prove that:

a $(a+b)^2 - (a-b)^2 = 4ab$ **b** $(a+b)^2 - 4(a-b)^2 = (3b-a)(3a-b)$

2 Find the smallest positive integer a for which $x^2 + (a-3)x + 2a = 0$ has real solutions.

3 Prove that $(x-y)^5 + (x-y)^3 = 0$ if and only if $x = y$.

4 **a** Expand and collect like terms: $(n^2 - 2n + 2)(n^2 + 2n + 2)$

b Hence find all integers n such that $n^4 + 4$ is prime.

5 **a** Prove that $(k+1)^2 - (k-1)^2 = 4k$.

b Hence find two square numbers that have difference:

i 40 **ii** 100.

6 The following "proofs" end in nonsensical results. Identify the incorrect step(s) in each case:

a
$$\begin{aligned}
& a = b \\
\Leftrightarrow \quad & a^2 = ab \\
\Leftrightarrow \quad & a^2 - b^2 = ab - b^2 \\
\Leftrightarrow \quad & (a-b)(a+b) = b(a-b) \\
\Leftrightarrow \quad & a + b = b \\
\Leftrightarrow \quad & 2a = a \\
\Leftrightarrow \quad & 2 = 1
\end{aligned}$$

b
$$\begin{aligned}
& \frac{x+10}{x-6} - 5 = \frac{4x-40}{13-x} \\
\Leftrightarrow \quad & \frac{x+10-5(x-6)}{x-6} = \frac{4x-40}{13-x} \\
\Leftrightarrow \quad & \frac{4x-40}{6-x} = \frac{4x-40}{13-x} \\
\Leftrightarrow \quad & 6 - x = 13 - x \\
\Leftrightarrow \quad & 6 = 13
\end{aligned}$$

7 The following "proofs" give correct results but their methods are incorrect. Identify the incorrect step(s) in each case:

a
$$\begin{aligned} & 6x - 12 = 3(x-2) \\ \Leftrightarrow\ & 6x - 12 + 3(x-2) = 0 \\ \Leftrightarrow\ & 12x - 24 = 0 \\ \Leftrightarrow\ & x = 2 \end{aligned}$$

b
$$\begin{aligned} & x^2 - 6x + 9 = 0 \\ \Leftrightarrow\ & x^2 - 6x = -9 \\ \Leftrightarrow\ & x(x-6) = 3(-3) \\ \Leftrightarrow\ & x = 3 \text{ or } x - 6 = -3 \\ \Leftrightarrow\ & x = 3 \end{aligned}$$

8 The sum of fractions $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$ has the form $\dfrac{1}{m} + \dfrac{1}{3m} = \dfrac{4}{3m}$ where $m = 4$, or alternatively the form $\dfrac{1}{n+1} + \dfrac{1}{n^2+n} = \dfrac{1}{n}$ where $n = 3$.

a Prove by equivalence:

i $\dfrac{1}{m} + \dfrac{1}{3m} = \dfrac{4}{3m}$

ii $\dfrac{1}{n+1} + \dfrac{1}{n^2+n} = \dfrac{1}{n}$ for $n \neq 0, -1$.

b Is it accurate to say that $\dfrac{1}{n+1} + \dfrac{1}{n^2+n}$ is equivalent to $\dfrac{1}{n}$? Explain your answer.

9 Use the Factor Theorem to prove that $x + y + z$ is a factor of $x^3 + y^3 + z^3 - 3xyz$.

D DEFINITIONS

THEORY OF KNOWLEDGE

In mathematics we clearly *define* terms so there is no misunderstanding of their exact meaning.

For example:

- A **rational number** is a number which can be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.
- An integer n is **even** if $n = 2k$ for some integer k.
- An integer n is **odd** if $n = 2k + 1$ for some integer k.
- An integer n is **divisible** by an integer m if $n = km$ for some integer k.

We can understand the need for specific definitions by considering integers and rational numbers:

- 2 is an integer, and is also a rational number since $2 = \frac{4}{2}$.
- $\frac{4}{2}$ is a rational number, and is also an integer since $\frac{4}{2} = 2$.
- $\frac{4}{3}$ is a rational number, but is *not* an integer.

1 Why is it important that mathematicians use the same definitions?

2 Words such as *similar*, *or*, *function*, *domain*, *range*, *period*, and *wave* are common words in English, but also have different or more specific mathematical definitions.
For each of these words, discuss the difference between their mathematical definition and their common use in English.

3 What is the difference between *equal*, *equivalent*, and *the same*? Why is it important to distinguish between these terms?

4 Are there any words which we use only in mathematics? What does this tell us about the nature of mathematics and the world around us?

Example 3 — Self Tutor

Prove that $0.\overline{13} \in \mathbb{Q}$.

Let $x = 0.\overline{13} = 0.1313131313....$

$\Rightarrow \quad 100x = 13.13131313....$

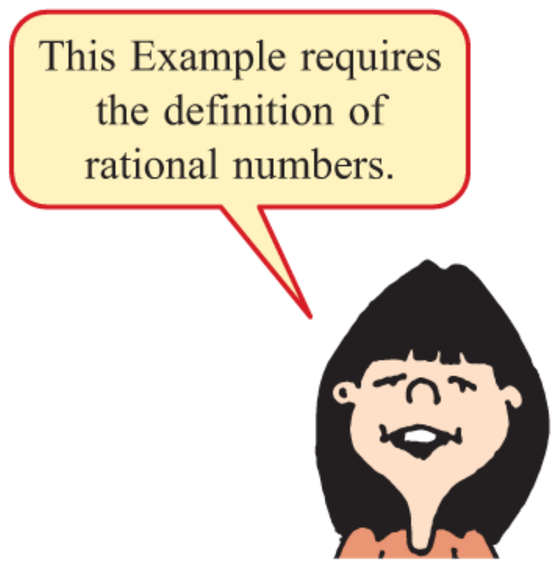
$\Rightarrow \quad 100x = 13 + x$

$\Rightarrow \quad 99x = 13$

$\Rightarrow \quad x = \frac{13}{99}$

$\Rightarrow \quad x \in \mathbb{Q}$

$\Rightarrow \quad 0.\overline{13} \in \mathbb{Q}$



Example 4 — Self Tutor

Prove that the sum of any two rational numbers is also a rational number.

Proof:

Let x and y be two rational numbers.

By definition, there exists $p, q \in \mathbb{Z}, \ q \neq 0$ so that $x = \frac{p}{q}$.

By definition, there exists $r, s \in \mathbb{Z}, \ s \neq 0$ so that $y = \frac{r}{s}$.

So, $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$

Since p, q, r, s are all integers, $ps + rq$ is an integer which we call P.

Since q, s are non-zero integers, qs is a non-zero integer which we call Q.

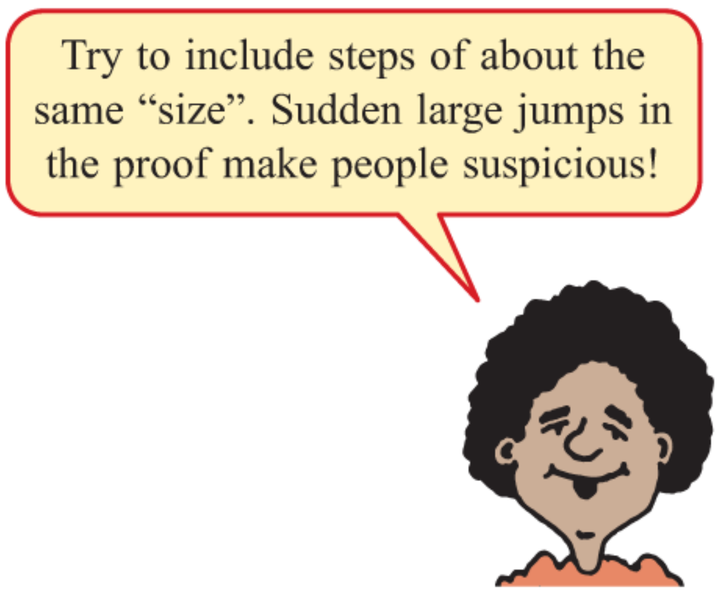
$\therefore \quad x + y = \frac{ps + rq}{qs} = \frac{P}{Q}$ where $P, Q \in \mathbb{Z}, \ Q \neq 0$

$\therefore$ by definition, $x + y$ is a rational number.

We have written this proof in great detail. However, the whole point of a proof is that you should be able to check each step easily.

Mathematicians sometimes choose to leave out steps to make the argument shorter and easier to read. If the reader trusts the writer this is fine. However, it should always be possible to fill in the missing steps.

Learning which steps to include is part of learning how to write a mathematical proof.



EXERCISE 9D

1 Prove that $0.\overline{9} \in \mathbb{Z}$.

2 Prove that:

a $0.\overline{4} \in \mathbb{Q}$ b $0.\overline{23} \in \mathbb{Q}$ c $0.0\overline{79} \in \mathbb{Q}$

3 Prove that the infinite geometric series $(4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + ... \in \mathbb{Z}$.

4 Prove that the difference between any two rational numbers is also a rational number.

5 Prove that the product of any two rational numbers is also a rational number.

6 Prove that the product of two odd integers is odd.

7 Prove that if p and q are odd integers then $p^2 - q^2$ is divisible by 8.

8 Prove that if p and q are consecutive odd integers, $p > q$, then $p^3 - q^3 - 2$ is divisible by 24.

9 If a, b, c are integers and $ax^2 + bx + c = 0$ has rational root $\frac{r}{s}$ in lowest terms, prove that s is a factor of a, and r is a factor of c.

DISCUSSION

Mathematical definitions are very difficult to write. They are often the result of careful thought by many experienced mathematicians. Once commonly accepted, they rarely change, because if they did it would cause great confusion.

Definitions collect similar examples together. When attempting to understand something, it is always worth building up a collection of examples which do and do not satisfy the definition.

Discuss how you could define a *chair*.

THEORY OF KNOWLEDGE

An **axiom** is a statement that is taken to always be true. Axioms form the foundation for proofs of all results and theorems in mathematics.

For example, these are the axioms for addition and multiplication on the real numbers:

Axiom 1: Associativity: $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$

Axiom 2: Commutativity: $a + b = b + a$ and $a \times b = b \times a$

Axiom 3: Distributivity: $a \times (b + c) = a \times b + a \times c$

Axiom 4: Additive identity: $a + 0 = a$ and Multiplicative identity: $a \times 1 = a$

Axiom 5: Additive inverse: $a + (-a) = 0$ and Multiplicative inverse: $a \times a^{-1} = 1$.

Using only these axioms, we can prove that $0 \times x = x \times 0 = 0$ for all $x \in \mathbb{R}$.

Proof:

$$0 = 0 + 0$$

$\therefore\ 0 \times x = (0 + 0) \times x$ {multiplying both sides by x}

$\therefore\ 0 \times x = 0 \times x + 0 \times x$ {Axiom 3}

$\therefore\ 0 \times x + (-(0 \times x)) = (0 \times x + 0 \times x) + (-(0 \times x))$ {adding the additive inverse of $0 \times x$}

$\therefore\ 0 \times x + (-(0 \times x)) = (0 \times x) + [0 \times x + (-(0 \times x))]$ {Axiom 1}

$\therefore\ 0 = 0 \times x + 0$

$\therefore\ 0 = 0 \times x$ {Axiom 4}

$\therefore\ 0 \times x = 0 = x \times 0$ {Axiom 2}

1 Is an axiom a definition, an assumption, both, or neither?

2 Is it necessarily practical to *only* use axioms to prove results?

Axioms are important because they cannot be proven. In some ways, they are similar to the fundamental Laws of Physics and other sciences.

3 Some fundamental laws of physics, such as Newton's Laws, have been verified via strong experimental evidence and therefore are generally accepted. Why does this not work for mathematical axioms?

In 1939, Austrian mathematician **Kurt Gödel** proved that there does not exist any set of consistent axioms that is able to prove everything. In other words, there will be results that cannot be proved. This result is known as **Gödel's incompleteness theorem**.



Kurt Gödel

4 Does this mean that we must accept unprovable results as axioms?

5 Are there an infinite number of axioms?

In 1637, French mathematician **Pierre de Fermat** famously claimed that the equation $a^n + b^n = c^n$ where $n > 2$, $n \in \mathbb{N}$ has no integer solutions for a, b, and c. He scrawled this in the margin of a copy of *Arithmetica*, claiming that his proof was too large for the margin to contain.

This statement became known as **Fermat's Last Theorem**. It remained unproved for 350 years until British mathematician **Andrew Wiles** proved it in 1994. Whether or not Fermat's supposed proof was valid is debatable, as Wiles required modern mathematical methods from after Fermat's time.

6 Are we *discovering* mathematics or *inventing* it?

7 Can two people "discover" the same thing?

E PROOF BY EXHAUSTION

A proof can sometimes be split into a finite number of cases, each of which is then proven separately. It is important to justify why the list of cases covers *all* possible cases. Since the list *exhausts* all possibilities, we call this style of proof a **proof by exhaustion**.

For example, every natural number $n \in \mathbb{Z}$ is either even or odd.

- If n is even it is divisible by two, and so $n = 2k$ for some $k \in \mathbb{Z}$.
- If n is odd, it has remainder 1 when divided by 2. In this case $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Example 5 **Self Tutor**

Given that $n^5 - n = (n-1)n(n+1)(n^2+1)$, prove that $n^5 - n$ is divisible by 5 for all $n \in \mathbb{Z}$.

Proof by exhaustion:

If we divide n by 5 then the remainder will be 0, 1, 2, 3, or 4. Hence every integer can be written in one of the forms $5k$, $5k+1$, $5k+2$, $5k+3$, or $5k+4$, for some $k \in \mathbb{Z}$.

Let $N = n^5 - n = (n-1)n(n+1)(n^2+1)$.

If $n = 5k$ then the factor n is divisible by 5, and so N is divisible by 5.

If $n = 5k + 1$ then the factor $(n - 1) = 5k$ is divisible by 5, so N is divisible by 5.

If $n = 5k + 2$ then the factor $(n^2 + 1) = (5k + 2)^2 + 1$
$= 25k^2 + 20k + 5$
$= 5(5k^2 + 4k + 1)$
which is divisible by 5, so N is divisible by 5.

If $n = 5k + 3$ then the factor $(n^2 + 1) = (5k + 3)^2 + 1$
$= 25k^2 + 30k + 10$
$= 5(5k^2 + 6k + 2)$
which is divisible by 5, so N is divisible by 5.

If $n = 5k + 4$ then the factor $(n + 1) = 5(k + 1)$ which is divisible by 5, so N is divisible by 5.

In all cases, N is divisible by 5, so $n^5 - n$ is divisible by 5 for all $n \in \mathbb{Z}$.

EXERCISE 9E

1 Prove that if n is an integer such that n^2 is even, then n is even.

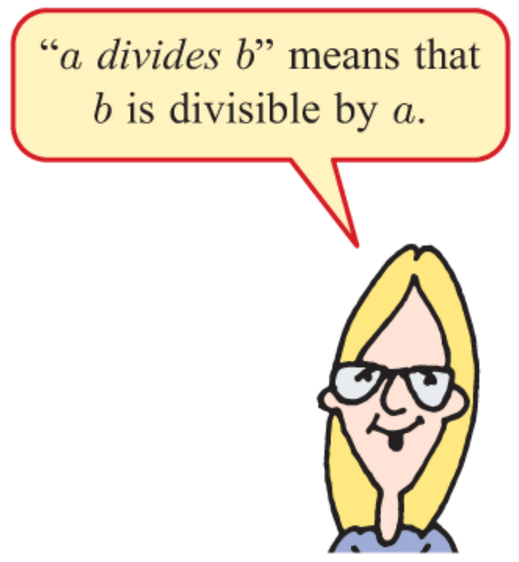
2 Prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{Z}$.

3 Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.

4 Prove that $n^2 - 1$ is divisible by 8 for all odd integers n.

5 Prove that 7 never divides $n^2 + 4$ for all $n \in \mathbb{Z}$.

6 Let n be a positive integer and define $x = (n + 1)! + 2$.
Prove that none of x, $x + 1$, $x + 2$, ..., $x + n - 1$ are prime.



7 You are presented with 3 boxes, A, B, and C, one of which contains a prize. On the lid of each box there is a statement, and you are told that only one of the statements is true. They say:

A	B	C
The prize is in this box.	The prize is not in this box.	The prize is in box **A**.

Which box contains the prize?

8 13 points are equally spaced around a circle. Prove that if any 5 are chosen, then the set will include 3 points which form an isosceles triangle.

F DISPROOF BY COUNTER EXAMPLE

Experimental evidence is very convincing, and scientific research relies on experimental evidence. However, in mathematics we do not consider experimental evidence to be sufficient to constitute a *proof*. The reason for this is that one single **counter example** which shows a general statement to be false, is sufficient to *disprove* the statement.

Example 6 **Self Tutor**

The values of $n^2 - n + 41$ for some values of $n \in \mathbb{N}$ are shown in the table alongside. We observe that all of these values are primes, so we conjecture that:

$n^2 - n + 41$ is prime for all $n \in \mathbb{N}$.

Find an example which proves this conjecture is false.

n	$n^2 - n + 41$
1	41
2	43
3	47
4	53
5	61
6	71
7	83
8	97
9	113
10	131

n	$n^2 - n + 41$
11	151
12	173
13	197
14	223
⋮	⋮
30	911
⋮	⋮
99	9743
⋮	⋮

For $n = 41$, we have $n^2 - n + 41 = 41^2 - 41 + 41 = 41^2$ which is not prime.

$\therefore$ the conjecture is false.

EXERCISE 9F

1 Find a counter example which disproves:

a $(a + b)^2 = a^2 + b^2$

b If p is prime then $2p + 1$ is prime.

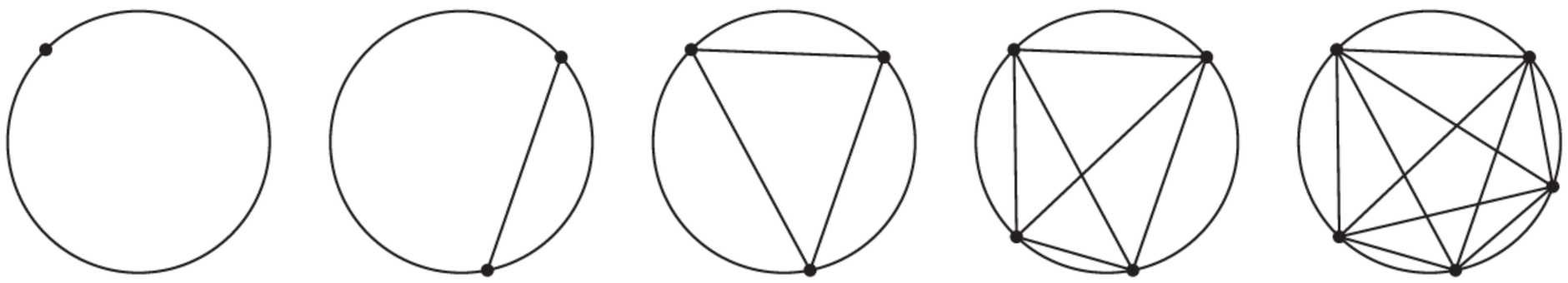
c For each k, at least one of $6k - 1$ or $6k + 1$ is prime.

2 Find a counter example which disproves:

a If $p_1, p_2, ..., p_n$ are distinct prime numbers then $p = (p_1 \times p_2 \times .... \times p_n) + 1$ is also prime.

b If $p_1, p_2, ..., p_n$ is the list of the first n prime numbers then $p = (p_1 \times p_2 \times .... \times p_n) + 1$ is also prime.

3 Suppose n distinct dots are placed around the outside of a circle, and every pair of dots is joined with a straight line segment. We can then count the number of regions in which the circle has been divided. The first five cases are illustrated below. We observe that the circle is divided into 1, 2, 4, 8, and 16 regions, and therefore conjecture that for the case of n dots, the circle will be divided into 2^{n-1} regions.



By considering the case $n = 6$, prove the conjecture is false.

4 Consider the sequence 31, 331, 3331, ... from which we conjecture that each number in the sequence is prime. Find a counter example to disprove this conjecture.

5 Prime numbers of the form $2^n - 1$, $n \in \mathbb{Z}^+$ are called **Mersenne primes** after **Marin Mersenne** (1588 - 1647). As of December 2018 the largest known Mersenne prime was $2^{82\,589\,933} - 1$ which has 24 862 048 digits.

a Calculate $2^n - 1$ for $n = 1, 2, ..., 10$. For what values of n is $2^n - 1$ prime?

b Consider the conjecture: If n is prime then $2^n - 1$ is prime.
Find a counter example to disprove this conjecture.

c Consider the conjecture: If n is composite then $2^n - 1$ is composite. Prove this conjecture is true by expanding the brackets and calculating $(2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + ... + 2^{(a-1)b})$.

G PROOF BY CONTRAPOSITIVE

The **contrapositive** of $A \Rightarrow B$ is the statement $\neg B \Rightarrow \neg A$.

For example, the contrapositive of "if x is rational then $2^x \neq 3$" is "if $2^x = 3$ then x is not rational".

The implication $A \Rightarrow B$ and its contrapositive are said to be *logically equivalent* because they tell us exactly the same information.

We can explain this using a **truth table** which considers every possibility for whether A and B are true.

As an example, suppose A is "Sam is in the library" and B is "Sam is reading".

We will mark $A \Rightarrow B$ and $\neg B \Rightarrow \neg A$ as true if they are consistent with the truth values of A and B, and false if they are not.

A	B	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$
T	T	Sam is in the library and he is reading. This is consistent with $A \Rightarrow B$. T	Since Sam is reading, $\neg B$ is false. The hypothesis is false, so the implication cannot be inconsistent. T
T	F	Sam is in the library but he is *not* reading. This is inconsistent with $A \Rightarrow B$. F	Sam is not reading but he *is* in the library. This is inconsistent with $\neg B \Rightarrow \neg A$. F
F	T	Sam is not in the library. The hypothesis is false, so the implication cannot be inconsistent. T	Since Sam is reading, $\neg B$ is false. The hypothesis is false, so the implication cannot be inconsistent. T
F	F	Sam is not in the library. The hypothesis is false, so the implication cannot be inconsistent. T	Sam is not reading and he is not in the library. This is consistent with $\neg B \Rightarrow \neg A$. T

An implication only says something if the hypothesis is true. If the hypothesis is false, it cannot be inconsistent.



MATHEMATICAL LOGIC

The implication $A \Rightarrow B$ and its contrapositive $\neg B \Rightarrow \neg A$ are logically equivalent because they always have the same truth value.

You can learn more about truth tables and mathematical logic by clicking on the icon.

The important conclusion for us here is that because the implication and its contrapositive are logically equivalent, proving one is equivalent to proving the other.

If the implication $A \Rightarrow B$ is proven by proving its contrapositive $\neg B \Rightarrow \neg A$, we call this a **proof by contrapositive**.

Proof by contrapositive is worth considering because proving a contrapositive may be significantly easier than proving the original implication directly.

Example 7 — Self Tutor

Let n be an integer. Prove that if $n^2 - 4n + 5$ is even, then n is odd.

Proof by contrapositive:

The contrapositive is: If n is even then $n^2 - 4n + 5$ is odd.

If n is even then there exists an integer k for which $n = 2k$.

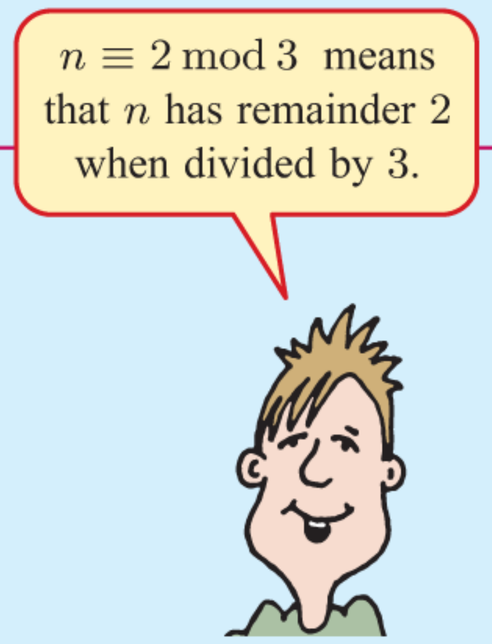
Hence $$\begin{aligned} n^2 - 4n + 5 &= (2k)^2 - 4(2k) + 5 \\ &= 4k^2 - 8k + 5 \\ &= 2(2k^2 - 4k + 2) + 1 \quad \text{which is odd.} \end{aligned}$$

This proves the contrapositive and hence the original statement.

In a proof by contrapositive, you can still use all the normal proof techniques. In the following Example we use proof by exhaustion to prove the contrapositive statement.

Example 8 — Self Tutor

Let n be a positive integer. Prove that if $n \equiv 2 \bmod 3$ then n is not a perfect square.



Proof by contrapositive:

The contrapositive is: If n is a perfect square then $n \equiv 0$ or $1 \bmod 3$.
{if the remainder when divided by 3 is not 2, then it must be 0 or 1}

Suppose n is a perfect square, so $n = k^2$ for some integer k.
There are three cases to consider:

If $k \equiv 0 \bmod 3$ then $k = 3q$ for some integer q.

$$\therefore \quad n = k^2 = (3q)^2 = 9q^2 = 3(3q^2)$$
$$\therefore \quad n \equiv 0 \bmod 3$$

If $k \equiv 1 \bmod 3$ then $k = 3q + 1$ for some integer q.

$$\begin{aligned} \therefore \quad n = k^2 &= (3q+1)^2 \\ &= 9q^2 + 6q + 1 \\ &= 3(3q^2 + 2q) + 1 \end{aligned}$$
$$\therefore \quad n \equiv 1 \bmod 3$$

If $k \equiv 2 \bmod 3$ then $k = 3q + 2$ for some integer q.

$$\begin{aligned} \therefore \quad n = k^2 &= (3q+2)^2 \\ &= 9q^2 + 12q + 4 \\ &= 3(3q^2 + 4q + 1) + 1 \end{aligned}$$
$$\therefore \quad n \equiv 1 \bmod 3$$

$\therefore$ if n is a perfect square then $n \equiv 0$ or $1 \bmod 3$.

This proves the original statement: If $n \equiv 2 \bmod 3$ then n is not a perfect square.

EXERCISE 9G

1 Write the contrapositive statement of each of the following:

a If it is raining then you get wet.

b If penguins can fly then it is Thursday.

c If you drink alcohol then you must be aged 18 or over.

d If k is divisible by 5 then k is not prime.

e If $a^2 + b^2 = c^2$ then a or b is even.

2 Let m and n be integers. Prove, by constructing the contrapositive, that:

a if n^2 is even then n is even

b if mn is even than at least one of n or m is even

c if $m + n$ is even then m and n have the same *parity* (are either both even or both odd).

3 Let a and b be real numbers. Prove that if the product ab is irrational then either a or b must be irrational.

4 Suppose n is a positive integer. Prove that if $n \equiv 2$ or $3 \bmod 4$ then n is not a perfect square.

H PROOF BY CONTRADICTION: REDUCTIO AD ABSURDUM

When people talk about proof by contradiction, they are normally referring to *reductio ad absurdum* which is Latin for "reduction to absurdity". In this form of argument we show that attempting to disprove a statement inevitably leads to a ridiculous, absurd, or impractical conclusion. For example, if we end up with $1 = 0$, or that an odd number is also an even number, we regard this as absurd. If the result is absurd or impossible, and all the reasoning is correct, then the only conclusion is that the hypothesis is false.

HISTORICAL NOTE

Godfrey Harold Hardy (1877 - 1947) was an English mathematician famous for his contributions to number theory and mathematical analysis.

A prominent lecturer of Trinity College, Cambridge, Hardy collaborated with **John Littlewood** and the Indian mathematician **Srinivasa Ramanujan**.

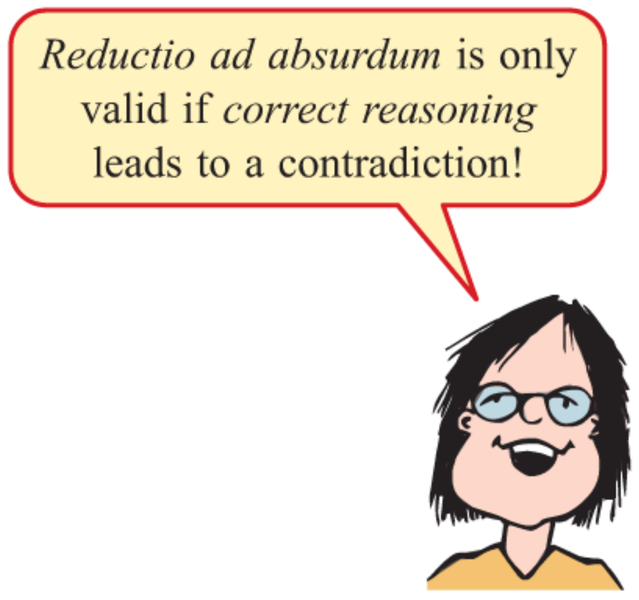
He wrote of proof by contradiction:

> *It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.*



Godfrey Harold Hardy

To use *reductio ad absurdum*, we start with the *negation* (opposite) of the conclusion, and correctly prove that this leads to a contradiction with the hypothesis. We hence deduce that the assumption was false, so the conclusion must be true.



For a hypothesis A and conclusion B, we show that

$$(A \wedge \neg B) \Rightarrow (C \wedge \neg C)$$

where C is some statement which is contradicted by $\neg C$.

Since $C \wedge \neg C$ is always false, $A \wedge \neg B$ is false.

So, if A is true, then B must also be true. This is the implication $A \Rightarrow B$.

Proof by contradiction is an **indirect proof** because we do not start with the hypothesis, but rather the opposite of the conclusion.

Example 9 Self Tutor

Prove that $\log_2 3$ is irrational.

Proof by contradiction:

Suppose $\log_2 3$ is rational

$\therefore \log_2 3 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$

$\therefore 3 = 2^{\frac{p}{q}}$

$\therefore 3^q = 2^p$

The left hand side is always odd and the right hand side is always even. This is a contradiction.

$\therefore$ our original supposition is false, and $\log_2 3$ is irrational.

Example 10 Self Tutor

The number $\sqrt{2}$ is the length of a diagonal of a square with side length 1. Prove that $\sqrt{2}$ is irrational.

Proof by contradiction:

Suppose $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ for some (positive) integers p and q, $q \neq 0$.

We assume this fraction has been written in lowest terms, so p and q have no common factors.

Squaring both sides, $2 = \frac{p^2}{q^2}$

$\therefore p^2 = 2q^2$ .... (1) $\therefore p^2$ is even, and so p must be even.

Thus $p = 2k$ for some $k \in \mathbb{Z}^+$.

Substituting into (1), $4k^2 = 2q^2$

$\therefore q^2 = 2k^2$ $\therefore q^2$ is even, and so q must be even.

This is a contradiction, as p and q have no common factors.

Thus our original supposition is false, and $\sqrt{2}$ is irrational.

Example 11 **Self Tutor**

Prove that there are infinitely many prime numbers.

Proof by contradiction:

Suppose there are only a finite number n of prime numbers, which we label $p_1, p_2, ..., p_n$.

Every positive integer greater than 1 is either a member of this list, or is divisible by a member of this list.

Let $N = p_1 \times p_2 \times .... \times p_n + 1$.

Notice that:

- $N > p_k$ for all $k = 1, ..., n$
  $\therefore$ N is not a member of the list.
- If we divide N by any p_k, $k = 1, ..., n$, then the remainder is 1.
  $\therefore$ N is not divisible by any p_k.

This is a contradiction, so our supposition is false, and there are infinitely many prime numbers.

Notice that this proof does *not* prove that $N = p_1 \times p_2 \times .... \times p_n + 1$ is actually a prime number. It only proves that N is not in the list, and not divisible by members of the list. In fact, N is not always prime. For example, consider $N = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\,031 = 59 \times 509$.

EXERCISE 9H

1 Prove that the following numbers are irrational:

a $\log_2 5$ **b** $\log_3 5$ **c** $\sqrt{3}$ **d** $\sqrt[3]{2}$

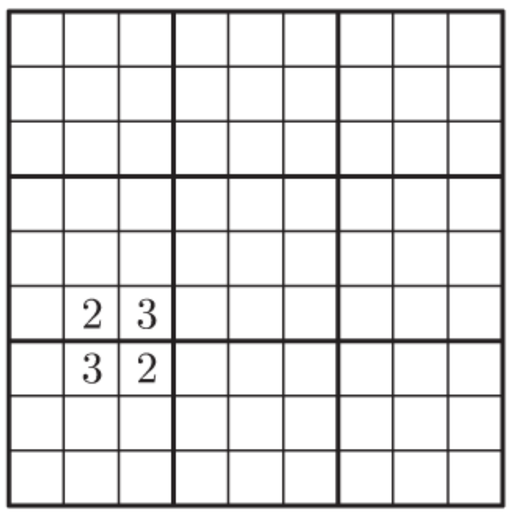
2 Attempt to prove that $\sqrt{4}$ is irrational using the method in **Example 10**. Explain why the argument fails.

3 Prove that if $a < b$ then $a < \dfrac{a+b}{2}$ using:

a a direct proof **b** proof by contradiction.

4 Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$ then a or b is even.

5 A Sudoku is a puzzle played on a 9×9 grid. The goal is to fill the grid with digits so that each column, each row, and each of the nine 3×3 subgrids contains each of the digits from 1 to 9.
The person posing the puzzle will provide a starting grid with some numbers. A puzzle is *well-posed* if it has a unique solution.
Suppose the numbers shown are *not* part of the starting grid. Can these numbers be part of a solution to a well-posed Sudoku? Explain your answer.



6 In the game Noughts and Crosses, two players O (noughts) and X (crosses) take turns marking their symbol in the spaces in a 3×3 grid. The first player who places three of their marks in a horizontal, vertical, or diagonal row wins the game.
Can the second player devise a winning strategy S which guarantees they will win?

7 **Dirichlet's theorem on arithmetic progressions** states that for any two positive integers a and d which have no common factors, there are infinitely many prime numbers of the form $a+nd$ where $n \in \mathbb{Z}^+$.

Prove Dirichlet's theorem on arithmetic progressions for the case where $a = 3$ and $d = 4$.

DISCUSSION

1 Can a proof by contradiction be considered a proof by deduction or proof by equivalence?

2 Is an indirect proof "cheating"?

3 Why is it so hard to prove that π is irrational?

DISCUSSION

A proof is a useful way to present a solution. However, sometimes it is not clear from the proof how someone came up with the idea. The mathematician **Carl Gauss** supposedly said "the architect does not leave scaffolding behind".

For example, consider this proof that an irrational number raised to an irrational power can be rational.

Proof:

Let $I = \sqrt{2}^{\sqrt{2}}$. I is either rational or irrational.

If I is rational then since $\sqrt{2}$ is irrational, I is itself an example of an irrational number to an irrational power being rational.

If I is irrational, consider $I^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^{2} = 2.$

In this case $I^{\sqrt{2}}$ is an example of an irrational number to an irrational power being rational.

By exhaustion, there must be an irrational number which raised to an irrational power, is rational.

In this proof, we do not actually establish whether I is rational. We do not need to.

List the different forms of proof you have learned. Discuss what features of a problem point you towards using particular forms of proof.

REVIEW SET 9A

1 The roots of $f(x) = x^2 + px + q$ are a and b. Prove that $q = ab$ and $p = -(a+b)$.

2 Prove that:

a $2.\overline{9} \in \mathbb{Z}$ **b** $0.\overline{38} \in \mathbb{Q}$

3 State the negation of each statement:

a The boy has blue eyes. **b** x is larger than 4.

4 State, with justification, whether each statement is true or false.

a If a function f is periodic with period p, then $f(x+p) = f(x)$ for all x.

b If $f(x+p) = f(x)$ for all x, then f is periodic with period p.

c f is periodic with period p if and only if $f(x+p) = f(x)$ for all x.

5 Prove that $9a^2 + b^2 \geqslant 6ab$ for all $a, b \in \mathbb{R}$.

6 **a** Prove by equivalence that $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.

b **i** Prove by deduction that $\dfrac{1-\sin^2\theta}{\cos\theta} = \cos\theta$.

ii Explain why the argument in **i** cannot be strengthened to equivalence for all θ.

7 Prove that the quotient of two non-zero rational numbers is also a rational number.

8 **a** Evaluate the following, and state whether the result is prime or composite:

i $1^3 + 2^3$ **ii** $2^3 + 3^3$ **iii** $3^3 + 4^3$ **iv** $4^3 + 5^3$

b Prove that $k^3 + (k+1)^3 = (2k+1)(k^2+k+1)$.

c Hence prove that the sum of two consecutive positive cubes is always composite.

9 Prove that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{Z}$.

10 **a** Prove that the fifth powers of the numbers $k = 1, 2, 3, ..., 9$ all have last digit k.

b Hence prove that n always has the same last digit as its 5th power n^5, for all $n \in \mathbb{Z}$.
Hint: Consider writing an integer in the form $10m + k$.

11 Find a counter example which disproves: If p is prime then $p! + 1$ is prime.

12 Let a, b, and n be integers. Prove that if ab is not divisible by n then a is not divisible by n and b is not divisible by n.

13 Prove that there are infinitely many prime numbers.

14 Prove that $\log_3 2$ is irrational.

15 Prove by contradiction that there exist no integers n and m for which:

a $14n + 7m = 1$ **b** $4n^2 - m^2 = 1$

REVIEW SET 9B

1 Let $p(x) = x^2 + 2bx + c$ and define $q(x) = p(x-b)$. Prove that $q(x) = q(-x)$ for all x.

2 Prove that $\dfrac{a+b}{2} \geqslant \sqrt{ab}$ for all $a, b \in \mathbb{R}^+$.

3 Consider the statement: "If x is acute then $\sin x$ is positive".

a Is the statement true or false? **b** Write the converse of the statement.

c Is the converse true or false?

4 Determine whether A and B are equivalent:

a A: x is not prime, B: x is composite

b A: x and y are both odd integers, B: xy is odd

5 Prove that if k is an odd integer then $k^3 + k^2 - k - 1$ is divisible by 8.

6 Prove that, in an equilateral triangle, the sum of the distances from any point in the triangle to the three sides is a constant.

7 Prove that the difference between the two digit numbers "ab" and "ba" is always divisible by 9.

8 Identify the incorrect step in the following "proof":

$$\begin{aligned}
& -6 = -6 \\
\Leftrightarrow \quad & 9 - 15 = 4 - 10 \\
\Leftrightarrow \quad & 3^2 - 3 \times 5 = 2^2 - 2 \times 5 \\
\Leftrightarrow \quad & 3^2 - 2 \times 3 \times \tfrac{5}{2} + \left(\tfrac{5}{2}\right)^2 = 2^2 - 2 \times 2 \times \tfrac{5}{2} + \left(\tfrac{5}{2}\right)^2 \\
\Leftrightarrow \quad & \left(3 - \tfrac{5}{2}\right)^2 = \left(2 - \tfrac{5}{2}\right)^2 \\
\Leftrightarrow \quad & 3 - \tfrac{5}{2} = 2 - \tfrac{5}{2} \\
\Leftrightarrow \quad & 3 = 2
\end{aligned}$$

9 Prove that the product of 3 consecutive integers is divisible by 6.

10 Find a counter example which disproves: If 10 divides n and 15 divides n then 150 divides n.

11 Prove that for any $a, b \in \mathbb{R}$:

- the greater of a and b is $\max(a, b) = \dfrac{a + b + |a - b|}{2}$
- the lesser of a and b is $\min(a, b) = \dfrac{a + b - |a - b|}{2}$.

12 Let n be an integer. Prove, by constructing the contrapositive, that if n^3 is even then n is even.

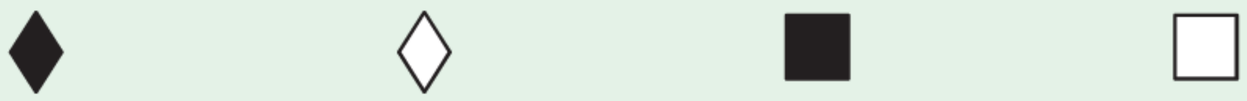
13 Prove that $\sqrt[3]{4}$ is irrational.

14 **a** Let n be an integer. Prove by contrapositive that if n^2 is divisible by 6 then n is divisible by 6.

b Prove by contradiction that $\sqrt{6}$ is irrational.

c Prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational.

15 The following problem is called the Wason THOG task, named after **Peter Wason**. In front of you are four designs: a black diamond, a white diamond, a black square, and a white square.



Meanwhile, I have written down one of the colours (black or white) and one of the shapes (diamond or square). Read the following rule carefully:

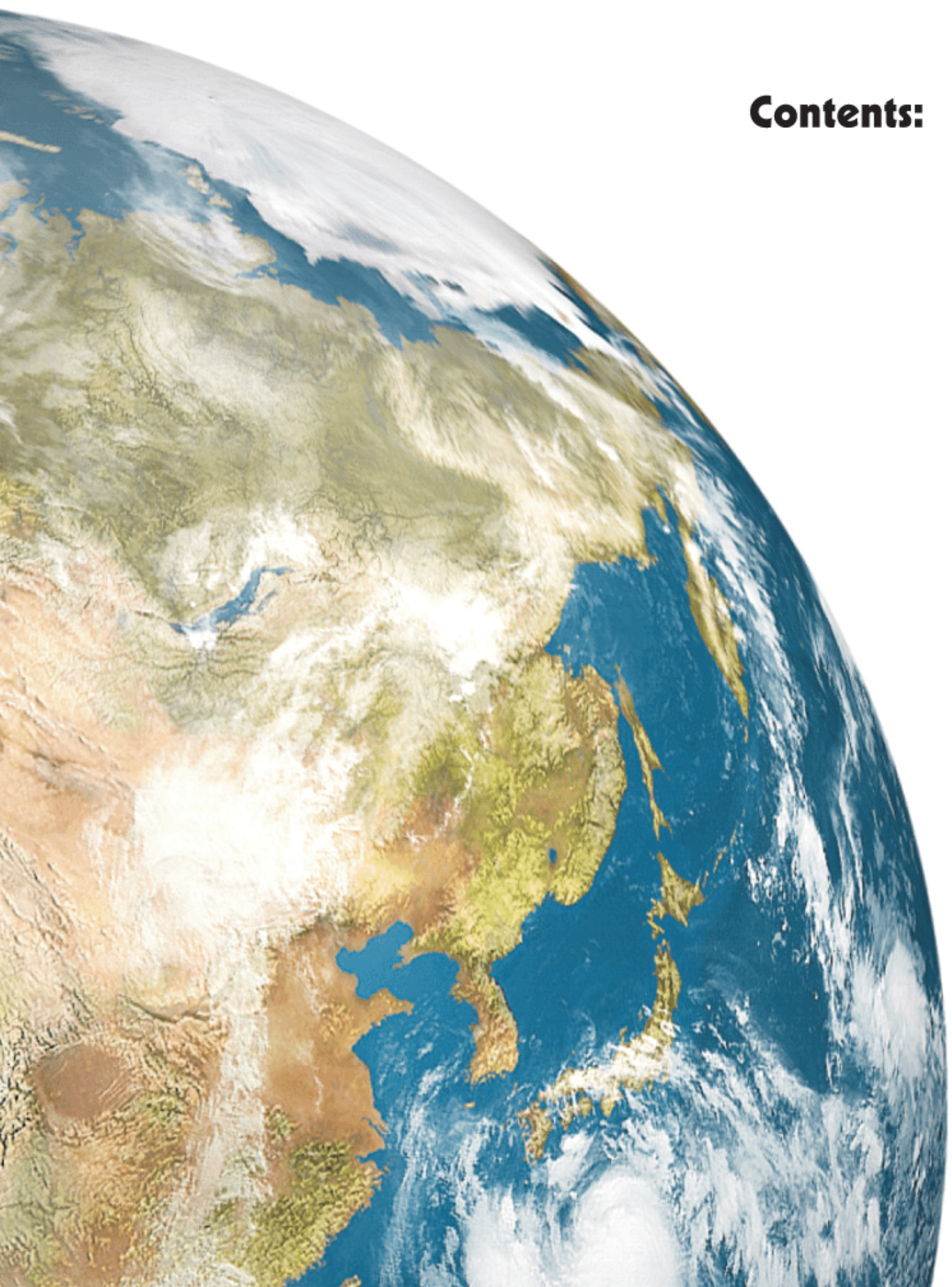
> If, and only if, any of the designs includes either the colour I have written down, or the shape I have written down, but not both, then it is called a THOG.

I will tell you that the black diamond is a THOG. Classify each of the following designs into one of the categories: definitely a THOG, insufficient information to decide, or definitely not a THOG. Justify your answers.

a **b** **c** **d**

Chapter 10

Proof by mathematical induction



Contents:

A The process of induction
B The principle of mathematical induction

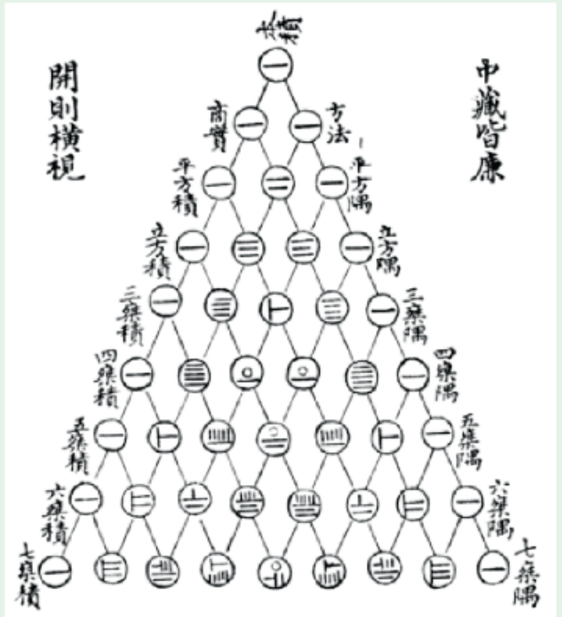
OPENING PROBLEM

Pascal's triangle is named after the French mathematician **Blaise Pascal** who recorded it in 1653 in his treatise *Traité du triangle arithmétique*.

However, this triangle was studied by mathematicians long before Pascal. Indian and Persian mathematicians used it in the 10th century, as did the Chinese mathematician **Yang Hui** in the 13th century.



Yang Hui



Yang Hui's Triangle

The first few rows of Pascal's triangle are shown alongside.

1 ← row 0
1 1 ← row 1
1 2 1 ← row 2
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
⋮

Things to think about:

a Find the sum of the numbers in:

i row 0 **ii** row 1 **iii** row 2
iv row 3 **v** row 4.

b What do you suspect is the sum of the numbers in row n?

c Suppose your suggestion in **b** is true for the nth row of Pascal's triangle.

i Can you show that your suggestion must then be true for the $(n+1)$th row also?

ii How can we formally prove that your suggestion in **b** is correct?

In this Chapter we consider another method of proof called **mathematical induction**. It is a common method for proving that an infinite collection of statements are all true.

HISTORICAL NOTE

The concept of mathematical induction has been used, probably independently, by numerous mathematicians in history, such as **Euler** in his proof of the infinity of primes.

The formalisation of mathematical induction or "complete induction" as a method of proof, began with the Sicilian mathematician **Francesco Maurolico** (1494 - 1575) and was extended by **Pascal**.

A THE PROCESS OF INDUCTION

The process of formulating a general result from close examination of the simplest cases is called **mathematical induction**.

For example:
the first positive even number is $2 = 2 \times 1$
the second positive even number is $4 = 2 \times 2$
the third positive even number is $6 = 2 \times 3$
the fourth positive even number is $8 = 2 \times 4$

and from these results we *induce* that "the nth positive even number is $2 \times n$ or $2n$".

The statement that "the nth positive even number is $2n$" is a summary of the observations of the simple cases $n = 1, 2, 3, 4$ and is a statement which we *believe* is true. We call such a statement a **conjecture** or **proposition**.

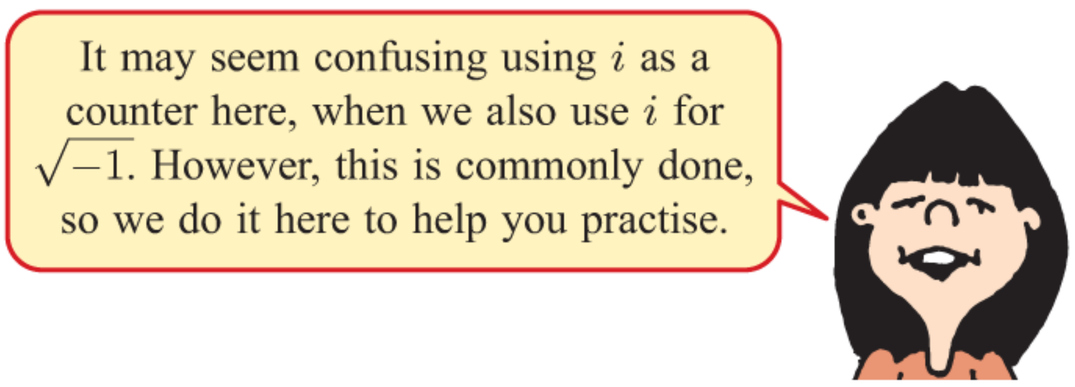
When we write a conjecture, it is important to state the values of the variables for which we believe the conjecture is true.

For example: "the nth positive odd number is $2n - 1$ where $n \in \mathbb{Z}^+$".

If we are making a conjecture about the sum or product of the first n terms of a sequence, we either use sum or product notation.

For example, for the arithmetic sequence $u_n = 3n - 1$, we write:

- $u_1 + u_2 + u_3 + u_4 = \sum_{i=1}^{4} u_i = \sum_{i=1}^{4} (3i - 1)$
- $u_1 \times u_2 \times u_3 \times u_4 = \prod_{i=1}^{4} u_i = \prod_{i=1}^{4} (3i - 1)$



Example 1 **Self Tutor**

By examining the cases $n = 1, 2, 3, 4$, make a conjecture about the sum of

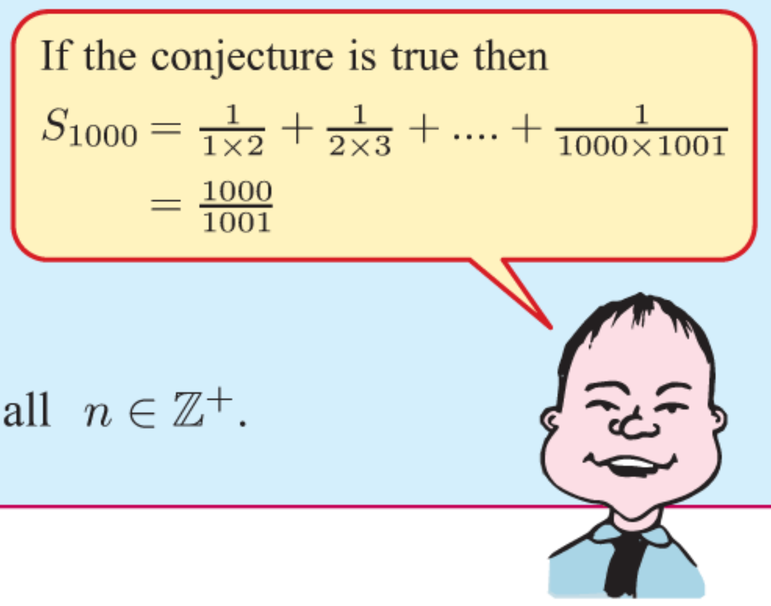
$S_n = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + ... + \frac{1}{n(n+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)}$.

$S_1 = \frac{1}{1\times2} = \frac{1}{2}$

$S_2 = \frac{1}{1\times2} + \frac{1}{2\times3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$S_3 = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

$S_4 = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$



From these results we conjecture that $S_n = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$.

EXERCISE 10A

1 Copy and complete the conjecture:

a The nth term of the sequence 3, 7, 11, 15, 19, ... is ____ for $n \in$ ____.

b The nth term of the sequence 25, 22, 19, 16, ... is ____ for $n \in$ ____.

c The nth term of the sequence 8, 12, 18, 27, ... is ____ for $n \in$ ____.

2 Examine the following using substitutions like $n = 1, 2, 3, 4, ...$ Hence complete the proposition or conjecture.

a $3^n > 1 + 2n$ for ____.

b $11^n - 1$ is divisible by ____ for ____.

c $7^n + 2$ is divisible by ____ for ____.

d $8^n - 3^n$ is divisible by ____ for ____.

Example 2 **Self Tutor**

By examining the cases $n = 1, 2, 3, 4,$ make a conjecture about the sum

$\sum_{i=1}^{n} i \times 2^{i-1} = 1 + 4 + 12 + 32 + ... + n \times 2^{n-1}$.

$\sum_{i=1}^{1} i \times 2^{i-1} = \quad 1 \times 2^0 \quad = \; 1 = 0 \times 2^1 + 1$

$\sum_{i=1}^{2} i \times 2^{i-1} = \; 1 + 2 \times 2^1 = \; 1 + 4 \; = \; 5 = 1 \times 2^2 + 1$

$\sum_{i=1}^{3} i \times 2^{i-1} = \; 5 + 3 \times 2^2 = \; 5 + 12 = 17 = 2 \times 2^3 + 1$

$\sum_{i=1}^{4} i \times 2^{i-1} = 17 + 4 \times 2^3 = 17 + 32 = 49 = 3 \times 2^4 + 1$

We conjecture that $\sum_{i=1}^{n} i \times 2^{i-1} = (n-1) \times 2^n + 1$ for all $n \in \mathbb{Z}^+$.

3 Copy and complete each conjecture, then write it using sum or product notation:

a $2 + 4 + 6 + 8 + 10 + ... + 2n =$ ____ for ____.

b $1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! =$ ____ for ____.

c $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n}{(n+1)!} =$ ____ for ____.

d $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + ...$ to n terms $=$ ____ for ____.

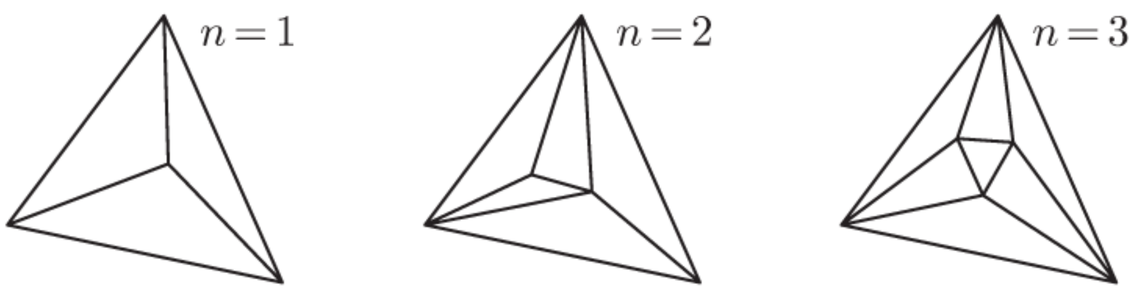
e $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) .... \left(1 - \frac{1}{n+1}\right) =$ ____ for ____.

f $2 + 2^2 + 2^3 + ... + 2^n =$ ____ for ____.

g $\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right) .... \left(1 - \frac{2}{n+2}\right) =$ ____ for ____.

4 n points are placed inside a triangle.

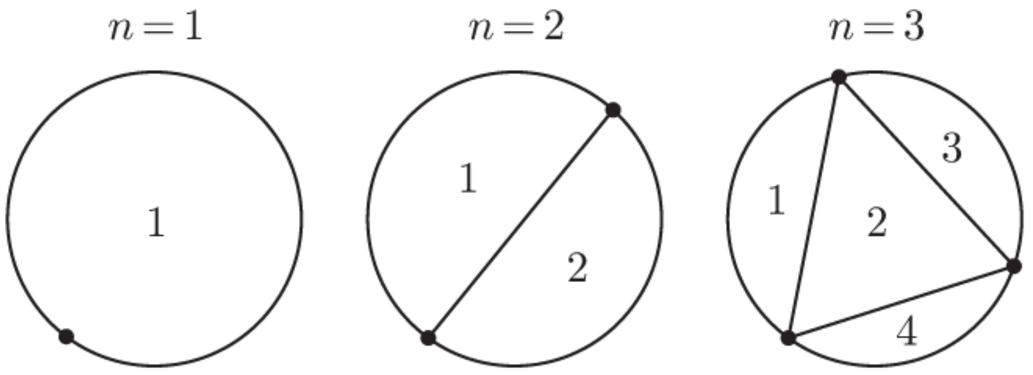
Non-intersecting line segments are drawn connecting the 3 vertices of the triangle and the points within it, to partition the given triangle into smaller triangles.



a Draw the cases $n = 4$ and $n = 5$.

b Make a proposition concerning the maximum number of triangles obtained in the general case.

5 n points are placed around a circle so that when line segments are drawn between every pair of the points, no three line segments intersect at the same point inside the circle. We consider the number of regions formed within the circle.



a Draw the cases $n = 4$ and $n = 5$.

b Use the cases $n = 1, 2, 3, 4, 5$ to form a conjecture about the number of regions formed in the general case.

c Draw the case $n = 6$. Do you still believe your conjecture?

B THE PRINCIPLE OF MATHEMATICAL INDUCTION

PROPOSITION NOTATION

We use P_n to represent a proposition which is defined for every integer n where $n \geqslant a$, $a \in \mathbb{Z}$.

For example, in the case of **Example 1**, our proposition P_n is:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+....+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \text{for all } n \in \mathbb{Z}^+.$$

We see that:

- P_1 is: $\dfrac{1}{1\times 2}=\dfrac{1}{2}$
- P_2 is: $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}=\dfrac{2}{3}$
- P_k is: $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+....+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

The **principle of mathematical induction** constitutes a formal proof that a particular proposition is true.

> Suppose P_n is a proposition which is defined for every integer $n \geqslant a$, $a \in \mathbb{Z}$.
>
> If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geqslant a$.

Suppose for a given proposition that P_1 is true. If we can show that P_{k+1} is true whenever P_k is true, then the truth of P_1 implies that P_2 is true, which implies that P_3 is true, which implies that P_4 is true, and so on.

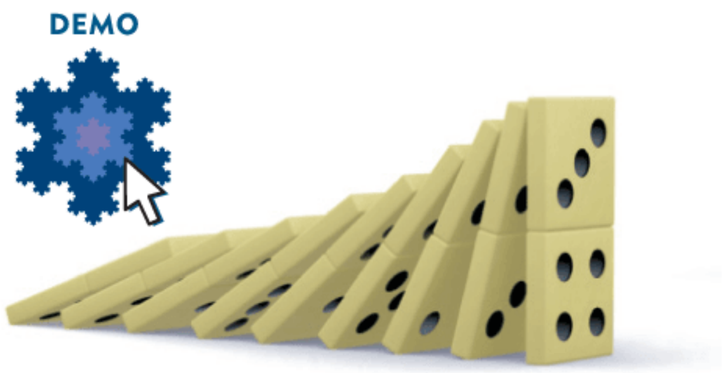
One can liken the principle of mathematical induction to the **domino effect**. We imagine an infinite set of dominoes all lined up.

Provided that:

- the first domino topples to the right, and
- the $(k+1)$th domino will topple if the kth domino topples,

then given sufficient time, every domino will topple.

This is why mathematical induction can be used to show that an infinite collection of statements P_n are all true.



Example 3 — Self Tutor

Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}$, $n \geqslant 0$, by using:

a the binomial expansion of $(1+3)^n$ **b** the principle of mathematical induction.

a If $n = 0$, $4^n + 2 = 4^0 + 2 = 3$ which is divisible by 3.

$$\begin{aligned}4^n + 2 &= (1+3)^n + 2\\ &= 1^n + \binom{n}{1}3 + \binom{n}{2}3^2 + \binom{n}{3}3^3 + ... + \binom{n}{n-1}3^{n-1} + \binom{n}{n}3^n + 2\\ &= 3 + \binom{n}{1}3 + \binom{n}{2}3^2 + \binom{n}{3}3^3 + ... + \binom{n}{n-1}3^{n-1} + \binom{n}{n}3^n\\ &= 3\left(1 + \binom{n}{1} + \binom{n}{2}3 + \binom{n}{3}3^2 + ... + \binom{n}{n-1}3^{n-2} + \binom{n}{n}3^{n-1}\right)\end{aligned}$$

where the contents of the brackets is an integer.

$\therefore$ $4^n + 2$ is divisible by 3.

b P_n is: $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}$, $n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $4^0 + 2 = 3 = 1 \times 3$ $\therefore$ P_0 is true.

(2) If P_k is true, then $4^k + 2 = 3A$ where A is an integer, and $A \geqslant 1$.

Now $4^{k+1} + 2 = 4^1 4^k + 2$

$= 4(3A - 2) + 2$ {using P_k}

$= 12A - 8 + 2$

$= 12A - 6$

$= 3(4A - 2)$ where $4A - 2$ is an integer as $A \in \mathbb{Z}$.

Thus $4^{k+1} + 2$ is divisible by 3 if $4^k + 2$ is divisible by 3.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,

P_n is true for all $n \in \mathbb{Z}$, $n \geqslant 0$. {principle of mathematical induction}

EXERCISE 10B

1 Prove that $3^n + 1$ is divisible by 2 for all integers $n \geqslant 0$, by using:

a the binomial expansion of $(1+2)^n$ **b** the principle of mathematical induction.

2 Prove that $6^n - 1$ is divisible by 5 for all integers $n \geqslant 0$, by using:

a the binomial expansion of $(5+1)^n$ **b** the principle of mathematical induction.

3 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $n^3 + 2n$ is divisible by 3 **b** $7^n - 1$ is divisible by 3

c $8^n - 3^n$ is divisible by 5 **d** $7^n - 4^n - 3^n$ is divisible by 12

e $7^n + 2 \times 4^n$ is divisible by 3 **f** $n^3 + 3n^2 + 2n$ is divisible by 6.

Example 4 — Self Tutor

Prove that $2^{n+2} + 3^{3n}$ is divisible by 5 for $n \in \mathbb{Z}$, $n \geqslant 0$.

P_n is: $2^{n+2} + 3^{3n}$ is divisible by 5 for $n \in \mathbb{Z}$, $n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $2^{0+2} + 3^0 = 5$ $\therefore$ P_0 is true.

(2) If P_k is true, then $2^{k+2} + 3^{3k} = 5A$ where A is an integer.

Now $2^{(k+1)+2} + 3^{3(k+1)}$

$= 2^{k+2} \times 2^1 + 3^{3k} \times 3^3$

$= 2 \times 2^{k+2} + 27 \times 3^{3k}$

$= 2\left[5A - 3^{3k}\right] + 27 \times 3^{3k}$ {using P_k}

$= 5 \times 2A - 2 \times 3^{3k} + 27 \times 3^{3k}$

$= 5 \times 2A + 25 \times 3^{3k}$

$= 5(2A + 5 \times 3^{3k})$ where $2A + 5 \times 3^{3k}$ is an integer as A is an integer and $k \geqslant 0$ is an integer.

Thus $2^{(k+1)+2} + 3^{3(k+1)}$ is divisible by 5 whenever $2^{k+2} + 3^{3k}$ is divisible by 5.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}$, $n \geqslant 0$. {principle of mathematical induction}

4 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $3^{2n+4} - 2^{2n}$ is divisible by 5 **b** $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

5 Prove that $\dfrac{2^n - (-1)^n}{3}$ is an odd number for all $n \in \mathbb{Z}^+$.

6 Attempt to prove the following *incorrect* statements using induction:

a $n^2 + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

b $3^n + 4$ is divisible by 7 for all integers $n \geqslant 1$.

c $2^{n+1} - 3^n - 1$ is divisible by 5 for all integers $n \geqslant 0$.

Example 5 **Self Tutor**

Prove that $x^n - y^n$ has a factor $(x - y)$ for all $n \in \mathbb{Z}^+$.

P_n is: $x^n - y^n$ has a factor $(x - y)$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $x^1 - y^1 = x - y$ which has $(x - y)$ as a factor
$\therefore$ P_1 is true.

(2) If P_k is true, then $x^k - y^k$ has a factor $(x - y)$.

Now $x^{k+1} - y^{k+1}$

$= x \times x^k - y \times y^k$

$= x(x^k - y^k) + x \times y^k - y \times y^k$

$= \underbrace{x(x^k - y^k)}_{\text{using } P_k \text{ this term has factor } (x-y)} + \underbrace{y^k(x - y)}_{\text{this term has factor } (x-y)}$

Thus $x^{k+1} - y^{k+1}$ has a factor $(x - y)$ whenever $x^k - y^k$ has a factor $(x - y)$.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

7 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $x^{2n+1} + a^{2n+1}$ has a factor $(x+a)$ **b** $x^{2n} - 1$ has a factor $(x+1)$.

Example 6 Self Tutor

a Prove that $1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$.

b Find $1^2 + 2^2 + 3^2 + ... + 100^2$.

a P_n is: $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^2 = 1$ and RHS $= \dfrac{1 \times 2 \times 3}{6} = 1$

$\therefore$ P_1 is true.

(2) If P_k is true for $k \in \mathbb{Z}^+$, then

$$\sum_{i=1}^{k} i^2 = 1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Thus $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \{\text{using } P_k\}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \times \frac{6}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{[k+1]([k+1]+1)(2[k+1]+1)}{6}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

$\sum_{i=1}^{n}$ reads "the sum from i equals 1 to n, of"



Always look for common factors.



b Using P_{100}, $1^2 + 2^2 + 3^2 + ... + 100^2 = \dfrac{100 \times 101 \times 201}{6} = 338\,350$

8 Use the principle of mathematical induction to prove that the following propositions are true for all positive integers n:

a $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$ **b** $\sum_{i=1}^{n} i(i+1) = \dfrac{n(n+1)(n+2)}{3}$ **c** $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

9 Prove that the sum of the first n odd numbers is n^2 using:

a the sum of an arithmetic series formula

b the principle of mathematical induction.

10 Prove that for all $n \in \mathbb{Z}^+$:

a $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$

b $\sum_{i=1}^{n} i \times 2^{i-1} = (n-1) \times 2^n + 1$

c $\sum_{i=1}^{n} i \times 3^i = \frac{3}{4}[(2n-1)3^n + 1]$

d $\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$

Example 7 **Self Tutor**

Prove that $\sum_{i=1}^{n} \frac{1}{(3i-1)(3i+2)} = \frac{n}{6n+4}$ for all $n \in \mathbb{Z}^+$.

P_n is: $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + ... + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \frac{1}{2 \times 5} = \frac{1}{10}$ and RHS $= \frac{1}{6 \times 1 + 4} = \frac{1}{10}$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sum_{i=1}^{k} \frac{1}{(3i-1)(3i+2)} = \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + ... + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$$

Now $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + ... + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$

$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$ {using P_k}

$= \frac{k}{2(3k+2)} \times \left(\frac{3k+5}{3k+5}\right) + \frac{1}{(3k+2)(3k+5)} \times \left(\frac{2}{2}\right)$

$= \frac{3k^2 + 5k + 2}{2(3k+2)(3k+5)}$

$= \frac{\cancel{(3k+2)}(k+1)}{2\cancel{(3k+2)}(3k+5)}$

$= \frac{k+1}{6k+10}$

$= \frac{(k+1)}{6(k+1)+4}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

11 **a** Prove that $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$.

b Hence find $\frac{1}{10 \times 11} + \frac{1}{11 \times 12} + \frac{1}{12 \times 13} + ... + \frac{1}{20 \times 21}$.

12 **a** Prove that $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + ... + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all $n \in \mathbb{Z}^+$.

b Hence find $\frac{1}{4 \times 5 \times 6} + \frac{1}{5 \times 6 \times 7} + ... + \frac{1}{8 \times 9 \times 10}$.

13 **a** Prove that $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$, where $n!$ is the product of the first n positive integers.

b Prove that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all $n \in \mathbb{Z}^+$.

c Hence find the sum $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{9}{10!}$ in rational form.

14 Prove that $n + 2(n-1) + 3(n-2) + ... + (n-2)3 + (n-1)2 + n = \frac{n(n+1)(n+2)}{6}$ for all integers $n \geqslant 1$.

Hint: $1 \times 6 + 2 \times 5 + 3 \times 4 + 4 \times 3 + 5 \times 2 + 6 \times 1$
$= 1 \times 5 + 2 \times 4 + 3 \times 3 + 4 \times 2 + 5 \times 1 + (1 + 2 + 3 + 4 + 5 + 6)$

15 The nth term of the sequence $3, \frac{4}{3}, \frac{11}{18}, ...$ is given by $u_n = a\left(\frac{1}{3}\right)^n + b\left(\frac{1}{2}\right)^n$.

a Find the values of a and b.

b Use the principle of mathematical induction to prove that the sum of the first n terms of this sequence is, $\sum_{i=1}^{n} u_i = \frac{3}{2}\left(1 - \frac{1}{3^n}\right) + 4\left(1 - \frac{1}{2^n}\right)$ for all $n \in \mathbb{Z}^+$.

c Hence find the sum of the infinite series.

Example 8 Self Tutor

A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$.

Prove that $u_n = 2^n - 1$ for all $n \in \mathbb{Z}^+$.

P_n is: if $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$, then $u_n = 2^n - 1$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 2^1 - 1 = 1$

$\therefore$ P_1 is true.

(2) If P_k is true, then $u_k = 2^k - 1$

Now $u_{k+1} = 2u_k + 1$
$= 2(2^k - 1) + 1$ {using P_k}
$= 2^{k+1} - 2 + 1$
$= 2^{k+1} - 1$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

16 Use the principle of mathematical induction to prove these propositions:

a If a sequence $\{u_n\}$ is defined by $u_1 = 5$ and $u_{n+1} = u_n + 8n + 5$ for all $n \in \mathbb{Z}^+$, then $u_n = 4n^2 + n$.

b If the first term of a sequence is 1, and subsequent terms are defined by the recursion formula $u_{n+1} = 2 + 3u_n$, then $u_n = 2(3^{n-1}) - 1$.

c If a sequence is defined by $u_1 = 2$ and $u_{n+1} = \dfrac{u_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{2^{2-n}}{n!}$.

d If a sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + (-1)^n(n+1)^2$ for all $n \in \mathbb{Z}^+$, then $u_n = \dfrac{(-1)^{n-1}n(n+1)}{2}$.

e If a sequence $\{u_n\}$ satisfies $u_{n+1} = \dfrac{u_n}{u_n + 1}$, then $u_n = \dfrac{u_0}{n\,u_0 + 1}$ for all $n \in \mathbb{Z}^+$.

17 Use the principle of mathematical induction to prove that:

a the sum of the first n terms in an arithmetic series with first term u_1 and common difference d is $\dfrac{n}{2}(2u_1 + (n-1)d)$ for all $n \in \mathbb{Z}^+$

b the sum of the first n terms in a geometric series with first term u_1 and common ratio r is $\dfrac{u_1(1-r^n)}{1-r}$, $r \neq 1$ for all $n \in \mathbb{Z}^+$.

18 A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + (2n+1)$ for all $n \in \mathbb{Z}^+$.

a By finding u_n for $n = 2$, 3, and 4, conjecture a formula for u_n in terms of n only.

b Prove that your conjecture is true using the principle of mathematical induction.

c Find the value of u_{20}.

19 A sequence is defined by $u_1 = \frac{1}{3}$ and $u_{n+1} = u_n + \dfrac{1}{(2n+1)(2n+3)}$ for all $n \in \mathbb{Z}^+$.

a By finding u_n for $n = 2$, 3, and 4, conjecture a formula for u_n in terms of n only.

b Prove that your conjecture is true using the principle of mathematical induction.

c Find the value of u_{50}.

20 $(2+\sqrt{3})^n = A_n + B_n\sqrt{3}$ for all $n \in \mathbb{Z}^+$, where A_n and B_n are integers.

a Find A_n and B_n for $n = 1$, 2, 3, and 4.

b Without using induction, show that $A_{n+1} = 2A_n + 3B_n$ and $B_{n+1} = A_n + 2B_n$.

c Calculate $(A_n)^2 - 3(B_n)^2$ for $n = 1$, 2, 3, and 4 and hence conjecture a result.

d Prove that your conjecture is true.

21 Sometimes, in order to prove the next inductive step, we require the previous *two* cases. Another form of the principle of mathematical induction is therefore:

> Suppose P_n is a proposition defined for all $n \in \mathbb{Z}^+$.
> If P_1 and P_2 are true, and also P_{k+2} is true whenever P_k and P_{k+1} are true, then P_n is true for all $n \in \mathbb{Z}^+$.

Use this form to prove that:

a If a sequence u_n is defined by $u_1 = 11$, $u_2 = 37$, and $u_{n+2} = 5u_{n+1} - 6u_n$ for all $n \in \mathbb{Z}^+$, then $u_n = 5(3^n) - 2^{n+1}$.

b If $u_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$ where $n \in \mathbb{Z}^+$, then u_n is a multiple of 2^n.
Hint: First find a and b such that $u_{n+2} = au_{n+1} + bu_n$.

Example 9 **Self Tutor**

Use the principle of mathematical induction to prove that:

$$2\cos 2x + 2\cos 4x + 2\cos 6x + ... + 2\cos 2nx = \frac{\sin[(2n+1)x]}{\sin x} - 1 \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

P_n is: $2\cos 2x + 2\cos 4x + 2\cos 6x + ... + 2\cos 2nx = \dfrac{\sin[(2n+1)x]}{\sin x} - 1$ for all $n \in \mathbb{Z}^+$.

(1) If $n = 1$, LHS $= 2\cos 2x$ and RHS $= \dfrac{\sin 3x}{\sin x} - 1$

$$= \frac{\sin(2x+x)}{\sin x} - 1$$

$$= \frac{\sin 2x\cos x + \cos 2x \sin x}{\sin x} - 1$$

$$= \frac{(2\cancel{\sin x}\cos x)\cos x + \cos 2x \cancel{\sin x}}{\cancel{\sin x}} - 1$$

$$= 2\cos^2 x + \cos 2x - 1$$

$$= (2\cos^2 x - 1) + \cos 2x$$

$$= 2\cos 2x \qquad \therefore \ P_1 \text{ is true.}$$

(2) If P_k is true, then $2\cos 2x + 2\cos 4x + 2\cos 6x + ... + 2\cos 2kx = \dfrac{\sin[(2k+1)x]}{\sin x} - 1$

$\therefore \quad 2\cos 2x + 2\cos 4x + 2\cos 6x + ... + 2\cos 2kx + 2\cos[2(k+1)x]$

$$= \left(\frac{\sin[(2k+1)x]}{\sin x} - 1\right) + 2\cos[2(k+1)x] \times \left(\frac{\sin x}{\sin x}\right)$$

$$= \frac{\sin[(2k+1)x] + 2\sin x\cos[2(k+1)x]}{\sin x} - 1$$

$$= \frac{\sin[(2k+1)x] + \sin[x + 2(k+1)x] + \sin[x - 2(k+1)x]}{\sin x} - 1$$

$$\{2\sin A\cos B = \sin(A+B) + \sin(A-B)\}$$

$$= \frac{\sin[(2k+1)x] + \sin[(2(k+1)+1)x] + \sin[-(2k+1)x]}{\sin x} - 1$$

$$= \frac{\cancel{\sin[(2k+1)x]} + \sin[(2(k+1)+1)x] - \cancel{\sin[(2k+1)x]}}{\sin x} - 1$$

$$= \frac{\sin[(2(k+1)+1)x]}{\sin x} - 1$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

22 **a** Simplify $1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + ... + \sin^{n-1} x$.

b Find the sum of the infinite series $1 + \sin x + \sin^2 x + \sin^3 x + ...$

c Find the value of x between 0 and 2π such that the series in **b** has sum $\frac{2}{3}$.

23 We know that $2\sin x\cos x = \sin 2x$.

a Show that:

i $2\sin x(\cos x + \cos 3x) = \sin 4x$ **ii** $2\sin x(\cos x + \cos 3x + \cos 5x) = \sin 6x$

b Predict what the following would simplify to:

i $2\sin x(\cos x + \cos 3x + \cos 5x + \cos 7x)$

ii $\cos x + \cos 3x + \cos 5x + ... + \cos 19x$ (10 terms)

iii $\cos x + \cos 3x + \cos 5x + ... + \cos[(2n-1)x]$ (n terms)

c Use the principle of mathematical induction to prove that:

$$\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta} \quad \text{for all } n \in \mathbb{Z}^+.$$

24 **a** Use the principle of mathematical induction to prove that:

$$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta} \quad \text{for all positive integers } n.$$

b Hence find the value of $\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} + \sin\frac{5\pi}{7} + \sin\pi + \sin\frac{9\pi}{7} + \sin\frac{11\pi}{7} + \sin\frac{13\pi}{7}$.

25 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + ... + \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x$ provided $x \neq m\pi$

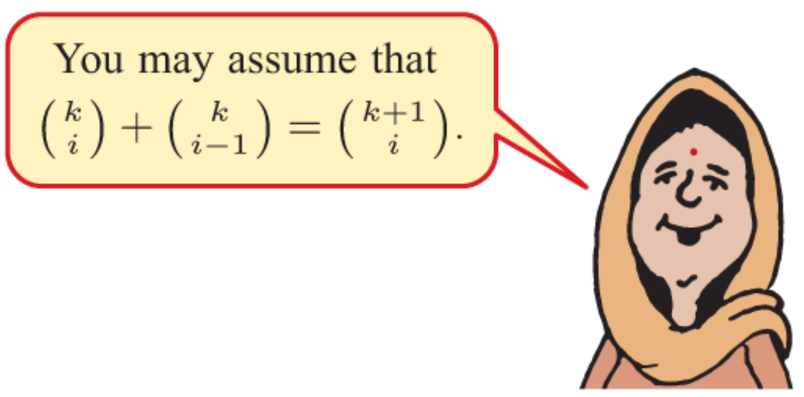
b $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \cos^2 4\theta + ... + \cos^2(n\theta) = \frac{1}{2}\left[n + \frac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$.

26 Use the principle of mathematical induction to prove:

a the binomial expansion

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i \quad \text{for all } n \in \mathbb{Z}^+.$$

b Fermat's Small Theorem "$n^p - n$ is divisible by p for any prime p and $n \in \mathbb{Z}^+$".



Example 10 Self Tutor

Prove that if $x \neq 1$, then

$$\prod_{i=1}^{n}(1+x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4)....(1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x} \quad \text{for all } n \in \mathbb{Z}^+.$$

P_n is: For $x \neq 1$, $\prod_{i=1}^{n}(1+x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4)....(1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}$, $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1+x$

$$\text{RHS} = \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x} = 1+x \quad \text{as } x \neq 1.$$

$\therefore$ P_1 is true.

(2) If P_k is true, then $(1+x)(1+x^2)(1+x^4)....(1+x^{2^{k-1}}) = \frac{1-x^{2^k}}{1-x}$

Now $(1+x)(1+x^2)(1+x^4)\ldots.(1+x^{2^{k-1}})(1+x^{2^k})$

$= \left(\dfrac{1-x^{2^k}}{1-x}\right) \times (1+x^{2^k})$ {using P_k}

$= \dfrac{1-(x^{2^k})^2}{1-x}$ $\{(a-b)(a+b) = a^2-b^2\}$

$= \dfrac{1-x^{2^k \times 2^1}}{1-x}$

$= \dfrac{1-x^{2^{k+1}}}{1-x}$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

27 Use the principle of mathematical induction to prove that:

a $\prod_{i=1}^{n}\left(1-\dfrac{1}{i+1}\right) = \left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\left(1-\dfrac{1}{4}\right)\ldots.\left(1-\dfrac{1}{n+1}\right) = \dfrac{1}{n+1}, \quad n \in \mathbb{Z}^+$

b $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\ldots.\left(1-\dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$ for all integers $n \geqslant 2$.

28 Prove that the product of n odd integers is odd for $n \geqslant 2$.

29 From $\sin 2x = 2\sin x \cos x$ we observe that $\sin x \cos x = \dfrac{\sin 2x}{2} = \dfrac{\sin(2^1 x)}{2^1}$.

a Prove that:

i $\sin x \cos x \cos 2x = \dfrac{\sin(2^2 x)}{2^2}$ **ii** $\sin x \cos x \cos 2x \cos 4x = \dfrac{\sin(2^3 x)}{2^3}$

b Assuming the pattern in **a** continues, simplify:

i $\sin x \cos x \cos 2x \cos 4x \cos 8x$ **ii** $\sin x \cos x \cos 2x \ldots. \cos 32x$

c Generalise the results from **a** and **b**. Prove your generalisation using the principle of mathematical induction.

30 Use the principle of mathematical induction to prove that:

a $(z_1 + z_2 + \ldots. + z_n)^* = {z_1}^* + {z_2}^* + \ldots. + {z_n}^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, \ldots., z_n$

b $(z_1 z_2 \ldots. z_n)^* = {z_1}^* {z_2}^* \ldots. {z_n}^*$ for all $n \in \mathbb{Z}^+$ and complex $z_1, z_2, \ldots., z_n$

c $(z^n)^* = (z^*)^n$ for all $n \in \mathbb{Z}^+$ and complex z.

Example 11 Self Tutor

Prove that $2^{-n} < \dfrac{1}{n}$ for all $n \in \mathbb{Z}^+$.

P_n is: $2^{-n} < \dfrac{1}{n}$ for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 2^{-1} = \frac{1}{2}$ and RHS $= \frac{1}{1} = 1$

$\therefore$ P_1 is true.

(2) If P_k is true, then $2^{-k} < \dfrac{1}{k}$ for some $k \geqslant 1$.

Now $2^{-(k+1)} = 2^{-k} \times \frac{1}{2}$

$< \frac{1}{k} \times \frac{1}{2}$ {using P_k}

$< \frac{1}{k+1}$ {$k \geqslant 1$, so $2k \geqslant k+1$}

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

31 Use the principle of mathematical induction to prove that:

a $3^n \geqslant 1 + 2n$ for all integers $n \geqslant 0$

b $n! \geqslant 2^n$ for all $n \in \mathbb{Z}$, $n \geqslant 4$

c $8^n \geqslant n^3$ for all $n \in \mathbb{Z}^+$

d $\sum_{i=1}^{n} \frac{1}{i} \leqslant \frac{n}{2} + 1$ for all $n \in \mathbb{Z}^+$

e $2^n > 3n^2 + 3n + 1$ for all $n \in \mathbb{Z}^+$, $n \geqslant 8$

f $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - .... + (-1)^{n-1} \frac{1}{n}$ is always positive for all $n \in \mathbb{Z}^+$.

32 Use the principle of mathematical induction to prove that $(1-h)^n \leqslant \frac{1}{1+nh}$ for $0 \leqslant h \leqslant 1$ and all $n \in \mathbb{Z}^+$.

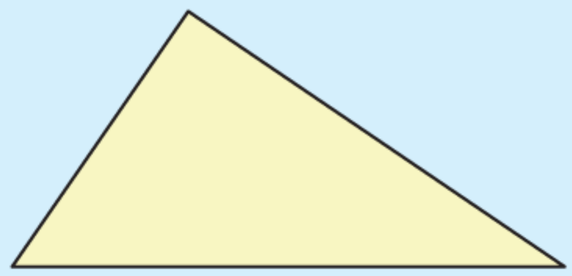
Example 12 **Self Tutor**

Prove that a convex n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals for all $n \geqslant 3$.

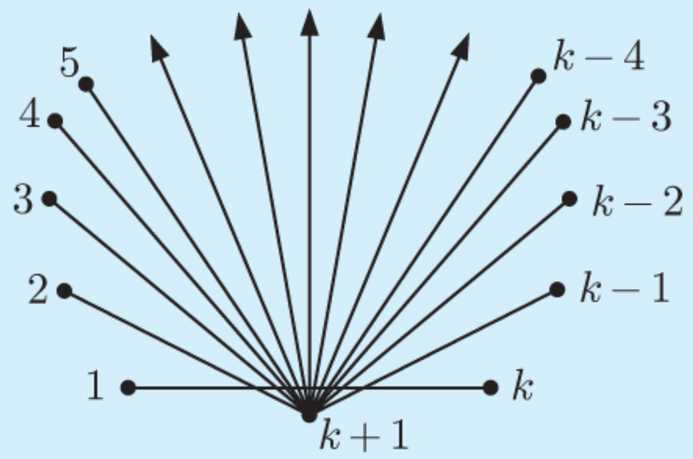
P_n is: A convex n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals for all $n \geqslant 3$.

Proof: (By the principle of mathematical induction)

(1) If $n = 3$ we have a triangle with 0 diagonals,
and $\frac{1}{2} \times 3 \times 0 = 0$
$\therefore$ P_3 is true.



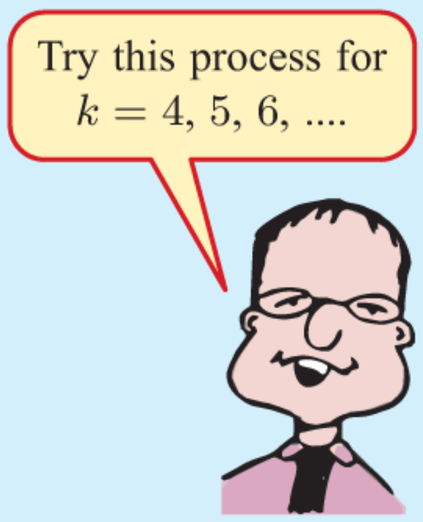
(2) If P_k is true, a convex k-sided polygon has $\frac{1}{2}k(k-3)$ diagonals.
We label the vertices
1, 2, 3, 4, 5, ..., $k-1$, k
and $k+1$ as an additional vertex.



The number of diagonals when $n = k+1$ is
the number of diagonals when $n = k$, $+ \underbrace{k-2}_{\text{the number of diagonals from vertex } k+1 \text{ to the vertices } 2, 3, 4, 5, ..., k-1} \overbrace{+1}^{\text{the line from vertex 1 to vertex } k \text{ was previously a side and is now a diagonal}}$

$\therefore$ the number of diagonals when $n = k+1$ is

$$\begin{aligned}
&\tfrac{1}{2}k(k-3) + k - 1 \\
&= \tfrac{1}{2}k(k-3) + \tfrac{2}{2}(k-1) \\
&= \tfrac{1}{2}[k^2 - 3k + 2k - 2] \\
&= \tfrac{1}{2}[k^2 - k - 2] \\
&= \tfrac{1}{2}(k+1)(k-2) \\
&= \tfrac{1}{2}[k+1]([k+1]-3)
\end{aligned}$$



Since P_3 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \geqslant 3$. {principle of mathematical induction}

33 Use the principle of mathematical induction to prove that:

a If n straight lines are drawn on a plane such that each line intersects every other line and no three lines have a common point of intersection, then the plane is divided into $\frac{n(n+1)}{2} + 1$ regions.

b If n points are placed inside a triangle, and non-intersecting lines are drawn connecting the 3 vertices of the triangle and the points within it to partition the triangle into smaller triangles, then the maximum number of triangles resulting is $2n + 1$.

INVESTIGATION 1 SEQUENCES, SERIES, AND INDUCTION

This Investigation involves the principle of mathematical induction as well as concepts from sequences, series, and counting.

The sequence of numbers $\{u_n\}$ is defined by $u_n = n \times n!$.

What to do:

1 Let $S_n = u_1 + u_2 + u_3 + ... + u_n$. Investigate S_n for several different values of n.

2 Hence conjecture an expression for S_n.

3 Prove your conjecture to be true using the principle of mathematical induction.

4 Show that u_n can be written as $(n+1)! - n!$ and hence devise an alternative direct proof of your conjecture for S_n.

5 Let $C_n = u_n + u_{n+1}$. Write an expression for C_n in factorial notation, and simplify it.

6 Let $T_n = C_1 + C_2 + C_3 + ... + C_n$. Find T_n for $n = 1, 2, 3, 4$, and 5.

7 Conjecture an expression for T_n.

8 Prove your conjecture for T_n by any method.

INVESTIGATION 2 EQUIVALENT STATEMENTS

Already in the course we have seen *equivalent statements* which are connected by "if and only if".

For example, if the statements A and B are equivalent we write $A \Leftrightarrow B$.

In order to prove that two statements are equivalent, we need to prove that *both*:

- if A is true then B is also true, written $A \Rightarrow B$
- if B is true then A is also true, written $B \Rightarrow A$

For example, consider the equivalent statements for $n \in \mathbb{Z}^+$:

$$(n+2)^2 - n^2 \text{ is a multiple of } 8 \quad \Leftrightarrow \quad n \text{ is odd}$$

Proof:

$A \Rightarrow B$: If $(n+2)^2 - n^2$ is a multiple of 8 then

$$\begin{aligned} n^2 + 4n + 4 - n^2 &= 8k \quad \text{for some integer } k \\ \therefore \quad 4n + 4 &= 8k \\ \therefore \quad 4(n+1) &= 8k \\ \therefore \quad n + 1 &= 2k \\ \therefore \quad n &= 2k - 1 \\ \therefore \quad n &\text{ is odd.} \end{aligned}$$

$B \Rightarrow A$: If n is odd then $n = 2k - 1$ for some integer k

$$\begin{aligned} \therefore \quad n + 1 &= 2k \\ \therefore \quad 4(n+1) &= 8k \\ \therefore \quad 4n + 4 &= 8k \\ \therefore \quad n^2 + 4n + 4 - n^2 &= 8k \\ \therefore \quad (n+2)^2 - n^2 &\text{ is a multiple of } 8 \end{aligned}$$

What to do:

1 Prove using the method of mathematical induction that $(n+2)^2 - n^2$ is a multiple of 8 for all odd integers n.

2 Prove that for all $x \in \mathbb{Z}^+$, x is not divisible by 3 if and only if $x^2 - 1$ is divisible by 3.

3 Can you prove using the method of mathematical induction that $x^2 - 1$ is divisible by 3 for all positive integers x that are *not* divisible by 3?

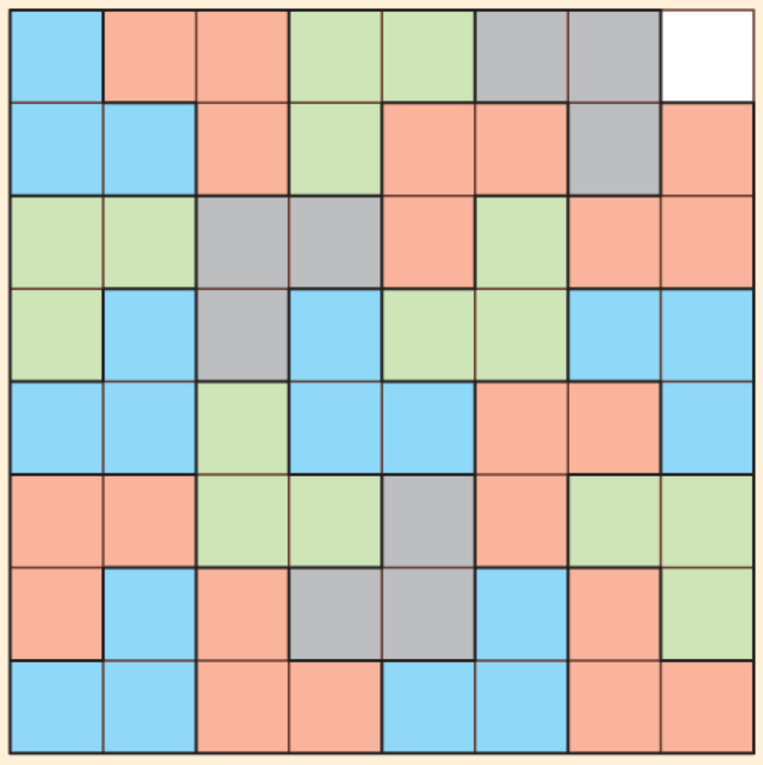
We need to show that $A \Rightarrow B$ *and* that $B \Rightarrow A$.



INVESTIGATION 3 TRIOMINOES

Meg is trying to cover an 8×8 square chessboard with triominoes of shape  .

She finds that she is always left with a missing square in the corner.



What to do:

1 Explain why it is impossible to exactly cover an 8×8 grid of squares with these triominoes.

2 Use mathematical induction to show that any $2^n \times 2^n$ grid of squares can be covered by the triominoes, except for one square missing in the corner.

3 Use mathematical induction to show that any $2^n \times 2^n$ grid of squares can be covered by the triominoes, except for any one particular square missing.

ACTIVITY FERMAT'S METHOD OF INFINITE DESCENT

The French lawyer and councillor **Pierre de Fermat** (1601 - 1665) is most famous for his contributions as an amateur mathematician. He developed a technique called *infinite descent* which he used primarily to prove negative assertions. For example, he used it to prove that:

- the equation $x^2 + y^4 = z^4$ has no integer solutions (other than the trivial $x = y = z = 0$)
- $\sqrt{2}$ is irrational.

Infinite descent was a historical precursor to mathematical induction. However, it is in fact a special form of proof by contraction.

To understand the method of infinite descent, consider a statement that P_n is true for all $n \in \mathbb{Z}^+$ like we would normally see in mathematical induction. In this case, we:

(1) Start with the assumption that P_n is *false* for at least one value of n. There must therefore exist a *smallest* value of n for which P_n is false.

(2) Seek to prove that if a value of n exists for which P_n is false, there must also be a *smaller* value n for which it is false. If we can prove this, then there cannot be a *smallest* value of n. This would present a contradiction, implying that P_n can never be false at all.

For example, here is a proof using infinite descent that $\sqrt{2}$ is irrational:

Proof:

Assume $\sqrt{2}$ is rational, so $\sqrt{2} = \dfrac{p_1}{q_1}$ for some $p_1, q_1 \in \mathbb{Z}^+$.

We know that $1 < \sqrt{2} < 2$, so $p_1 > q_1$ and $2q_1 > p_1$ .... $(*)$

Now $$\sqrt{2} = \sqrt{2} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$
$$= \frac{2-\sqrt{2}}{\sqrt{2}-1}$$
$$= \frac{2 - \frac{p_1}{q_1}}{\frac{p_1}{q_1} - 1}$$
$$= \frac{2q_1 - p_1}{p_1 - q_1}$$

Letting $p_2 = 2q_1 - p_1$ and $q_2 = p_1 - q_1$,

$$\sqrt{2} = \frac{p_2}{q_2} \quad \text{where } p_2, q_2 \in \mathbb{Z}^+, \ p_2 > q_2 \qquad \{\text{since } \sqrt{2} > 1\}$$
$$\text{and } p_2 < p_1 \text{ and } q_2 < q_1 \qquad \{\text{using } (*)\}$$

This process can be repeated an infinite number of times, so there is an infinite descent through the positive integers p, q satisfying $\sqrt{2} = \dfrac{p}{q}$.

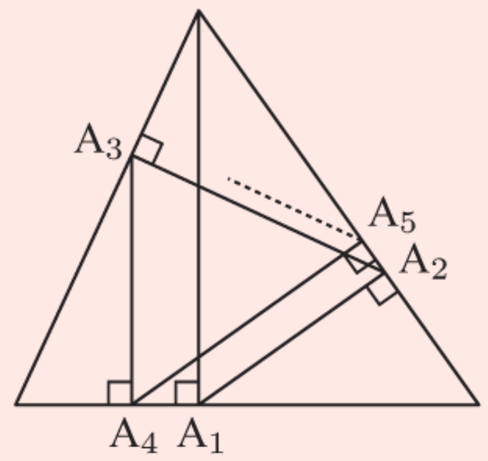
$\therefore$ no *smallest* p, q exist satisfying $\sqrt{2} = \dfrac{p}{q}$

$\therefore$ $\sqrt{2}$ cannot be rational.

What to do:

1 Use the method of infinite descent to prove that $\sqrt{3}$ is irrational.

2 Starting from a vertex of an acute triangle, the perpendicular is drawn to meet the opposite side at A_1. From A_1, a perpendicular is drawn to meet another side at A_2. This process is repeated so the sides are each visited in turn.
Prove that points A_1, A_2, A_3, ... are all distinct.



3 Let $a_1, a_2, ..., a_{2n+1}$ be a set of integers such that if any one of them is removed, the remaining $2n$ integers can be divided into two sets of n integers with equal sums.
Prove that $a_1 = a_2 = ... = a_{2n+1}$.

4 As a class, find all positive primes p for which there exist positive integers x, y, and n such that $p^n = x^3 + y^3$.
Hint: You need to *prove* you have found them all!

REVIEW SET 10A

1 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $7^n + 2$ is divisible by 3

b $\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$

c $1 + r + r^2 + r^3 + r^4 + ... + r^{n-1} = \frac{1-r^n}{1-r}$ provided $r \neq 1$

d $5^{2n} - 1$ is divisible by 24

e $5^n \geqslant 1 + 4n$

f $3^{3n+1} + 9 \times 2^{n+3}$ is divisible by 25.

2 Prove that if $u_1 = 1$ and $u_{n+1} = 3u_n + 2^n$, then $u_n = 3^n - 2^n$ for all $n \in \mathbb{Z}^+$.

3 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $7^n - 1$ is divisible by 6

b $\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2-1)$

c $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

d $\sqrt[n]{n!} \leqslant \frac{n+1}{2}$

e $3^{2n-1} + 2^{n+1}$ is divisible by 7

f $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

4 Prove that $3^n - 1 - 2n$ is divisible by 4 for all non-negative integers n.

5 Use the principle of mathematical induction to prove that:

$\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2(n\theta) = \frac{1}{2}\left[n - \frac{\cos[(n+1)\theta]\sin n\theta}{\sin\theta}\right]$ for all $n \in \mathbb{Z}^+$.

6 Use the principle of mathematical induction to prove that $5^{2n} + 17 \times 7^n$ is divisible by 18 for all non-negative integers n.

7 Consider the sequence defined by $u_0 = 1$, $u_1 = 3$, and $u_{n+2} = 4u_{n+1} - 3u_n$ for all $n \in \mathbb{Z}$, $n \geqslant 0$.

a Find the values of u_2, u_3, and u_4. Hence conjecture a formula for u_n involving n only.

b Prove your conjecture is true using mathematical induction.

8 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

$$\prod_{i=1}^{n} \cos(2^{n-1}x) = \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times .... \times \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \times \sin x}$$

REVIEW SET 10B

1 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$:

a $\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n+1)(2n-1)}{3}$

b $3^{2n+2} - 8n - 9$ is divisible by 64

c $\sum_{i=1}^{n} (2i+1)2^{i-1} = 1 + (2n-1) \times 2^n$

d $\sum_{i=1}^{n} i(i+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$

e $5^n + 3^n \geqslant 2^{2n+1}$

f $5^{2n+1} + 13 \times 3^{n+1}$ is divisible by 22.

2 Prove that $5^n + 3$ is divisible by 4 for all integers $n \geqslant 0$.

3 Prove that if $u_1 = 9$ and $u_{n+1} = 2u_n + 3(5^n)$, then $u_n = 2^{n+1} + 5^n$ for all $n \in \mathbb{Z}^+$.

4 Use the principle of mathematical induction to prove that:

a $9^n - 3^n$ is divisible by 6 for all $n \in \mathbb{Z}^+$

b $\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + ... + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$ for all $n \in \mathbb{Z}^+$

c $6n^3 + 5n^2 + n$ is even for all integers $n \geqslant 0$

d $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1} \times n^2 = \frac{1}{2}(-1)^{n+1} n(n+1)$ for all integers $n \geqslant 1$

e $8^{n+2} - 7n - 15$ is divisible by 49 for all $n \in \mathbb{Z}^+$

f $\frac{3}{2^2 \times 1^2} + \frac{5}{3^2 \times 2^2} + ... + \frac{2n-1}{n^2(n-1)^2} = 1 - \frac{1}{n^2}$ for all integers $n \geqslant 2$

g $\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right) .... \left(1+\frac{1}{n}\right) = n+1$ for all $n \in \mathbb{Z}^+$.

5 Use the principle of mathematical induction to prove that for all $n \in \mathbb{Z}^+$,

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + ... + \frac{1}{\sin(2^n x)} = \cot x - \cot(2^n x)$$ whenever these terms are defined.

6 Prove that if $u_1 = 5$ and $u_{n+1} = 2u_n - 3(-1)^n$, then $u_n = 3(2^n) + (-1)^n$ for all $n \in \mathbb{Z}^+$.

7 **a** Show that $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin\alpha \sin\beta$.

b Hence show that $\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

c Hence show that $\sin[(k+1)\theta] \sin\frac{\theta}{2} + \sin\frac{k\theta}{2} \sin\frac{(k+1)\theta}{2} = \sin\frac{(k+1)\theta}{2} \sin\frac{(k+2)\theta}{2}$.

d Use the principle of mathematical induction to prove that:

$$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin(n\theta) = \frac{\sin\left[\frac{1}{2}(n+1)\theta\right] \sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$ for all $n \in \mathbb{Z}^+$.

8 Use the principle of mathematical induction to prove the *pigeonhole principle*:

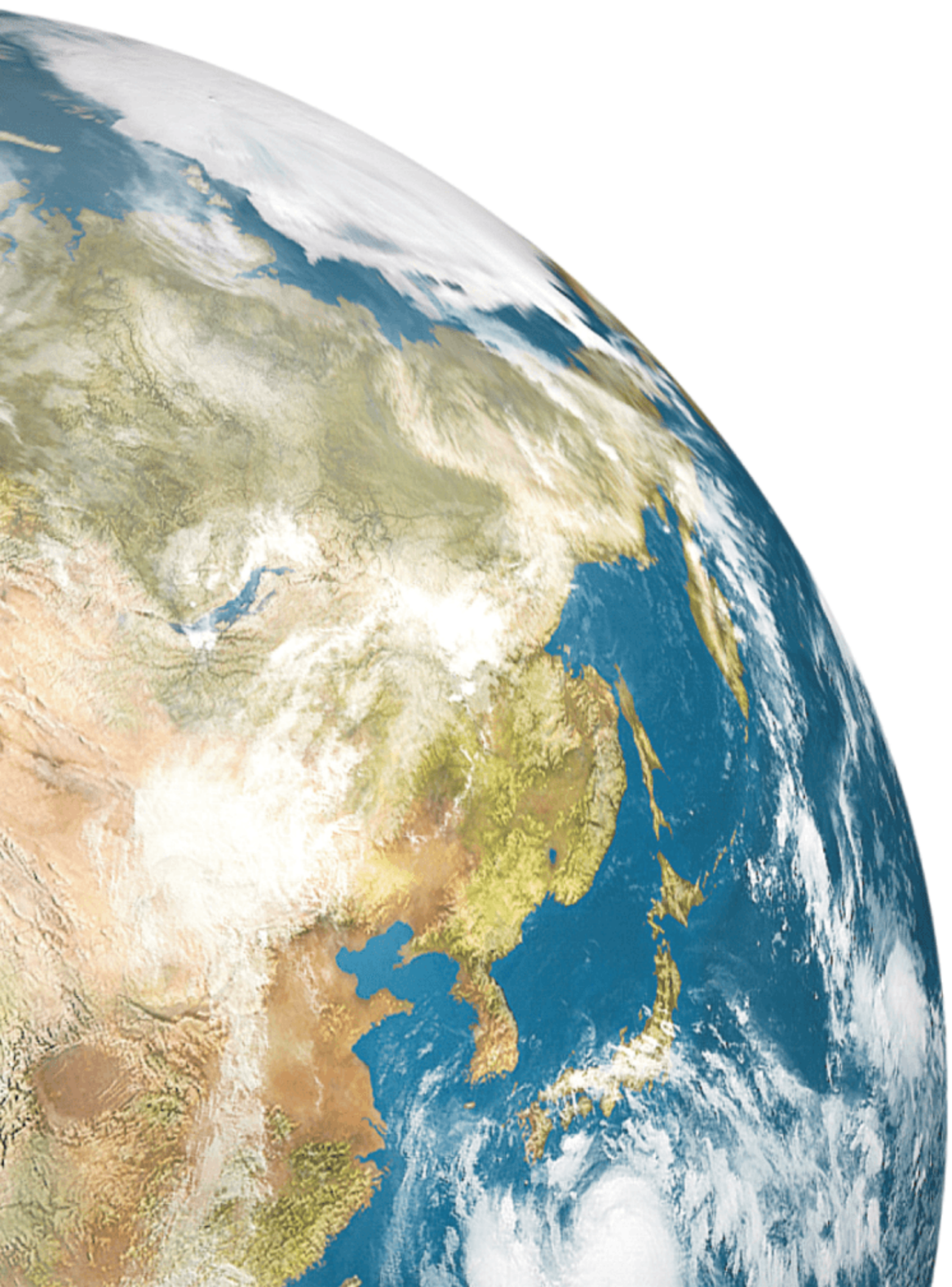
"If n items are distributed amongst m pigeonholes with $n, m \in \mathbb{Z}^+$ and $n > m$, then at least one pigeonhole will contain at least $\frac{n}{m}$ items."

Chapter 11

Linear algebra

Contents:

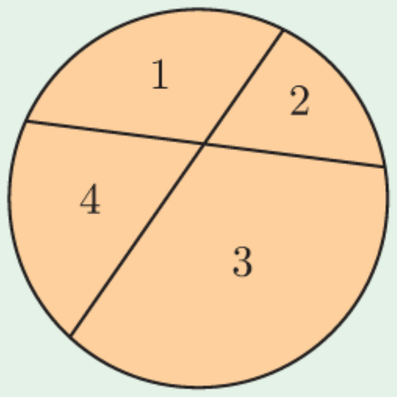
A Systems of linear equations
B Row operations
C Solving 2×2 systems of linear equations
D Solving 3×3 systems of linear equations



OPENING PROBLEM

When a circular pizza is cut using n straight cuts, the maximum number of pieces that can be made is given by $P(n) = an^2 + bn + c$ where a, b, c are constants.

For example, with two cuts we can make a maximum of 4 pieces, so $P(2) = 4a + 2b + c = 4$.



Things to think about:

a Can you find the maximum number of pieces that can be made using:

i 3 cuts **ii** 4 cuts?

b Can you write a *system* of three linear equations involving a, b, and c?

c Can you find a, b, and c using technology?

d For what values of n is $P(n)$ valid?

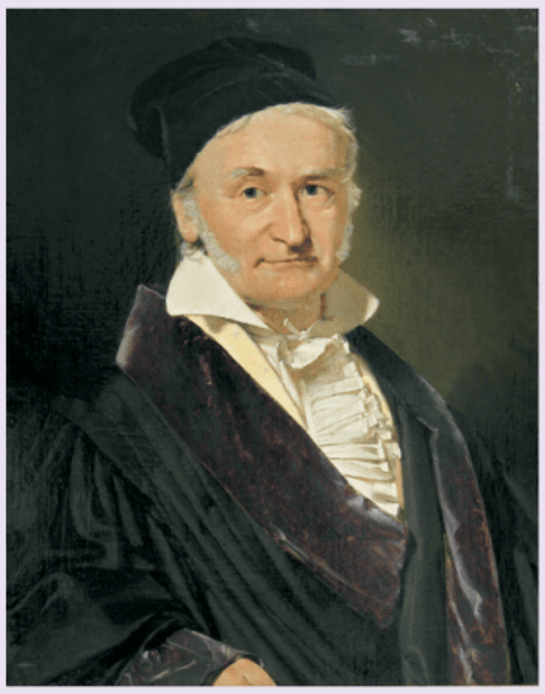
e Is there a logical procedure for solving systems of linear equations using algebra?

We have previously seen several methods for solving systems of two linear equations in two unknowns. We now consider more general systems and how they can be represented using **augmented matrices**. This study is just a small glimpse of the branch of mathematics called **linear algebra**.

Linear algebra is fundamental to all modern technology, being the basis for all computer science. Its applications extend to engineering, physics, natural sciences, analytic geometry, and economics.

HISTORICAL NOTE

Carl Friedrich Gauss (1777 - 1855) was born in Brunswick, now a town in Germany. His parents were very poor. His mother could not read or write, so she did not record the date of Gauss' birth. However, Gauss himself worked out his birthday from his mother remembering that he was born on a Wednesday, 8 days before the Feast of the Ascension, a Christian celebration which occurs 40 days after Easter. At the same time, Gauss found a method for finding the date of Easter in both past and future years.



Carl Gauss

Gauss was a child prodigy. At the age of ten, Gauss' teacher set his class the task of adding the first hundred integers. Expecting the students to take a long time, the teacher was surprised when Gauss derived a formula to solve the problem, and presented his correct solution in mere minutes. Gauss started university at age 15, and attended for three years. His tuition was paid for by the Duke of Brunswick, who had heard of Gauss' mathematical talents. Gauss' greatest work, *Disquisitiones Arithmeticae*, was completed when he was 21 years old. When it was published three years later, it was dedicated to the Duke.

Gauss made many mathematical discoveries during his doctoral studies. These include the Fundamental Theorem of Algebra (which he proved in four different ways) and the matrix method for solving systems of linear equations which we now call Gaussian elimination.

In the late 18th century, the dwarf planet Ceres was tracked by a number of Italian astronomers until it was "lost" behind the sun. Gauss was able to predict a position for Ceres in December 1801, which turned out to be accurate within a half-degree when it was rediscovered by German astronomers.

A SYSTEMS OF LINEAR EQUATIONS

A **linear equation** in the variables $x_1, x_2, ..., x_n$ is an equation which can be written in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ where $a_1, a_2, ..., a_n, b$ are constants.

For example, $2x = 3$, $4y = 7$, and $3x - y = 4$ are all linear equations.

A **solution** to the linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is a set of values $x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ which together satisfy the equation.

For example, consider the linear equation $x_1 + 2x_2 - 3x_3 = 8$.

- $x_1 = 3$, $x_2 = 4$, $x_3 = 1$ is a solution to the equation since $3 + 2(4) - 3(1) = 8$.
- $x_1 = -1$, $x_2 = 3$, $x_3 = -1$ is a solution to the equation since $-1 + 2(3) - 3(-1) = 8$.
- There are actually an infinite number of solutions to the equation. They have the form $x_1 = 8 - 2s + 3t$, $x_2 = s$, $x_3 = t$ where $s, t \in \mathbb{R}$.

The set of all solutions to a linear equation is called the **solution set**.

The solution set for a linear equation can often be written in many different forms.

For example, the solution set for $x_1 + 2x_2 - 3x_3 = 8$ can also be written in **parametric form** as $x_1 = s$, $x_2 = t$, $x_3 = \dfrac{s + 2t - 8}{3}$ where $s, t \in \mathbb{R}$ are called **parameters**.

Example 1 — Self Tutor

Find a solution set for:

a $x - 3y = 1$

b $2x_1 - x_2 + 4x_3 = 11$

a Let $y = t$

$\therefore\ x - 3t = 1$

$\therefore\ x = 1 + 3t$

$\therefore$ a solution set is $x = 1 + 3t$, $y = t$ where $t \in \mathbb{R}$.

b Let $x_1 = s$ and $x_3 = t$

$\therefore\ 2s - x_2 + 4t = 11$

$\therefore\ 2s + 4t - 11 = x_2$

$\therefore$ a solution set is $x_1 = s$, $x_2 = 2s + 4t - 11$, $x_3 = t$ where $s, t \in \mathbb{R}$.

An $m \times n$ **system of linear equations** consists of m linear equations in n unknowns.

For the variables $x_1, x_2, ..., x_n$, the system can be written in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

where $\{a_{ij}\}$, $\{b_i\}$ are constants.

A **solution** to the system of linear equations is any set of values $x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ which satisfy *all* m equations simultaneously.

For example, consider the system $\begin{cases} 2x_1 - x_2 + 4x_3 = 11 & .... (1) \\ x_1 + x_2 - 3x_3 = 2 & .... (2) \end{cases}$

- $x_1 = 3,\ x_2 = -1,\ x_3 = 1$ satisfies (1) since $2(3) - (-1) + 4(1) = 11$
  but does not satisfy (2) since $(3) + (-1) - 3(1) = -1 \neq 2$
  $\therefore\ x_1 = 3,\ x_2 = -1,\ x_3 = 1$ is not a solution to the system.

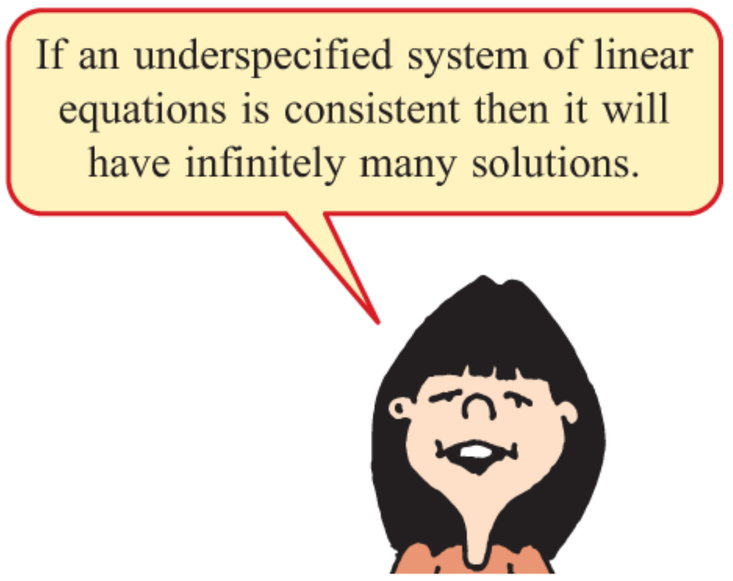
- $x_1 = \frac{11}{3},\ x_2 = \frac{13}{3},\ x_3 = 2$ satisfies (1) since $2(\frac{11}{3}) - (\frac{13}{3}) + 4(2) = 11$
  and satisfies (2) since $(\frac{11}{3}) + (\frac{13}{3}) - 3(2) = 2$
  $\therefore\ x_1 = \frac{11}{3},\ x_2 = \frac{13}{3},\ x_3 = 2$ is a solution to the system.

An $m \times n$ system of linear equations is:
- **inconsistent** if it has no solution
- **consistent** if it has at least one solution.

For example, the system $\begin{cases} x + 2y = 3 \\ x + 2y = 4 \end{cases}$ is inconsistent because there are no solutions which satisfy both equations.

An $m \times n$ system of linear equations is:
- **underspecified** if it has more unknowns than equations $(n > m)$
- **overspecified** if it has more equations than unknowns $(m > n)$.

For example, $\begin{cases} x + y + 2z = 2 \\ 2x + y - z = 4 \end{cases}$ is an underspecified system of linear equations, since there are 2 equations in 3 unknowns.

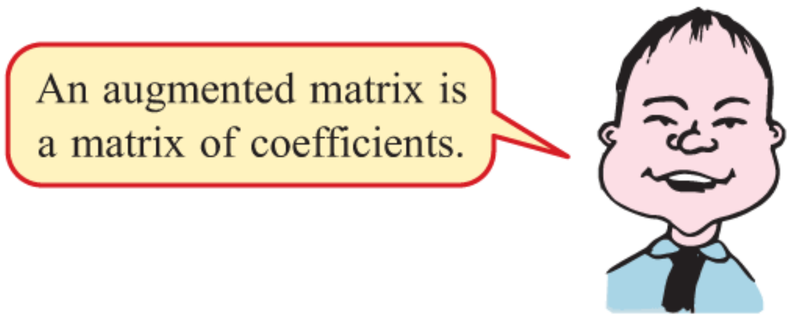


AUGMENTED MATRICES

A system of $m \times n$ linear equations can be written as a rectangular array of m by $(n+1)$ numbers by leaving out the $+$ and $=$ signs, and the variables. We call this an **augmented matrix**.

The general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$



can be written as the augmented matrix $\left(\begin{array}{cccc|c} a_{11} & a_{12} & .... & a_{1n} & b_1 \\ a_{21} & a_{22} & .... & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} & b_m \end{array}\right)$.

Example 2 **Self Tutor**

Write down the augmented matrix for the system of linear equations $\begin{cases} 2x - y + z = -1 \\ x - 3z = 4 \\ -x + 2y - z = 0 \end{cases}$.

The system has augmented matrix $\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 0 & -3 & 4 \\ -1 & 2 & -1 & 0 \end{array}\right)$.

EXERCISE 11A

1 Explain why each of the following is not a linear equation:

a $2x + y + xy = 3$ **b** $x_1 - x_2 - 2x_3^2 = 0$ **c** $x = 7 - \sqrt{y}$

2 Find a solution set for:

a $8x - y = 3$ **b** $x_1 - 2x_2 + x_3 = 10$ **c** $x + y - 2z = -2$

3 Decide whether each system is consistent, giving reasons for your answers:

a $\begin{cases} x + y = 2 \\ x + y = 3 \end{cases}$ **b** $\begin{cases} 2x - y = 1 \\ 2x + y = 1 \end{cases}$ **c** $\begin{cases} x - y + z = 3 \\ -x + y + z = 3 \end{cases}$

4 Write down the augmented matrix for the system of linear equations, and state if the system is underspecified or overspecified:

a $\begin{cases} x + 2y = 0 \\ 2x - y = -1 \\ -x + y = 2 \end{cases}$ **b** $\begin{cases} x + 2y - z = 1 \\ -2x + y + 2z = -4 \end{cases}$ **c** $\begin{cases} x_1 + x_2 - x_3 = 4 \\ x_1 - x_2 + x_3 = 8 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases}$

5 Write, in terms of x, y, and z, the system of linear equations with augmented matrix:

a $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 1 & 3 & -1 & 5 \end{array}\right)$ **b** $\left(\begin{array}{ccc|c} 3 & 1 & -2 & -3 \\ 0 & -1 & 2 & 8 \\ 2 & 0 & -1 & 2 \end{array}\right)$ **c** $\left(\begin{array}{ccc|c} 4 & -1 & 2 & -3 \\ -1 & 3 & 3 & 2 \\ 2 & 1 & -2 & 5 \end{array}\right)$

B ROW OPERATIONS

From our previous study of simultaneous equations, we know we can do the following things without changing the solutions:

- interchange the equations

  For example, $\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases}$ has the same solution as $\begin{cases} x - 3y = 17 \\ 2x + y = -1 \end{cases}$

- replace an equation by any non-zero multiple of itself

  For example, $\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases}$ has the same solution as $\begin{cases} 6x + 3y = -3 \\ x - 3y = 17 \end{cases}$ {multiplying by 3}

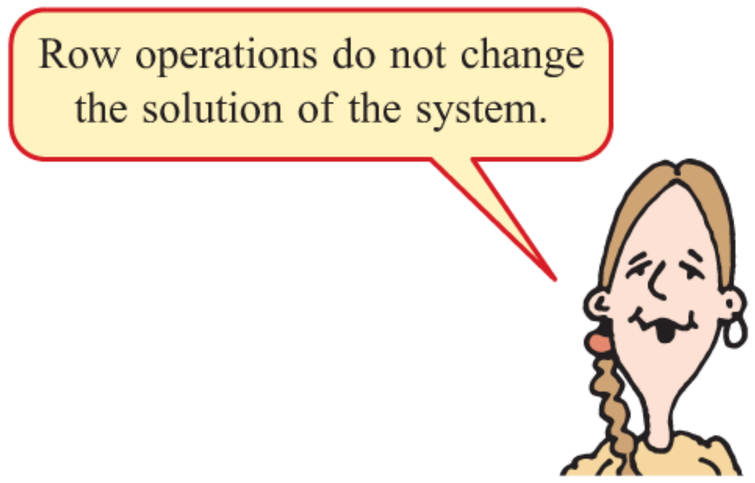
- replace an equation by a multiple of itself plus a multiple of another equation

For example, given $\begin{cases} 2x + y = -1 & .... (1) \\ x - 3y = 17 & .... (2) \end{cases}$, $2 \times (2) - (1)$ will be: $\begin{array}{r} 2x - 6y = 34 \\ -(2x + \ y = -1) \\ \hline -7y = 35 \end{array}$

$\therefore \begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases}$ has the same solution as $\begin{cases} 2x + y = -1 \\ -7y = 35 \end{cases}$.

These principles can also be applied to augmented matrices as **row operations**. We can hence:

- interchange rows
- replace any row by a non-zero multiple of itself
- replace any row by itself plus a multiple of another row.



For example, the system $\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases}$ has augmented matrix $\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right)$.

- If we interchanged rows 1 and 2, we would write:

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 17 \\ 2 & 1 & -1 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$\sim$ means "which has the same solution as"

$R_1 \leftrightarrow R_2$ indicates rows 1 and 2 have been interchanged

- If we multiplied row 1 by -3, we would write:

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} -6 & -3 & 3 \\ 1 & -3 & 17 \end{array}\right) \quad -3R_1 \to R_1$$

$-3R_1 \to R_1$ indicates row 1 has been replaced by "$-3 \times$ row 1"

- If we replaced row 2 by "twice row 2 minus row 1", we would write:

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & -7 & 35 \end{array}\right) \quad 2R_2 - R_1 \to R_2$$

$$\left\{\begin{array}{ccc} 2 & -6 & 34 \\ -2 & -1 & 1 \\ \hline 0 & -7 & 35 \end{array}\right\}$$

$2R_2 - R_1 \to R_2$ indicates row 2 has been replaced by "twice row 2 minus row 1"

EXERCISE 11B

1 Consider the system of linear equations $\begin{cases} x - 3y = 3 \\ 2x + y = -2 \end{cases}$.

a Write the system as an augmented matrix.

b Rewrite the augmented matrix with:

i the first row multiplied by 2

ii the first row replaced by "twice the first row, minus the second row"

iii the second row replaced by "the first row plus three times the second row".

2 Consider the system of linear equations $\begin{cases} x+y=4 \\ 3x+3y=a \end{cases}$.

a Write the system as an augmented matrix.

b Multiply the first row by 3.

c For what value(s) of a is the system consistent?

3 Consider the system of linear equations $\begin{cases} 2x-y=5 \\ -x+ay=3 \end{cases}$.

a Write the system as an augmented matrix.

b Multiply the second row by -2.

c For what value(s) of a is the system inconsistent?

4 Consider the system $\begin{cases} x+y+\ \ z=p \\ x\ \ \ \ \ \ +2z=q \\ 2x+y+3z=r \end{cases}$.

a Write the system as an augmented matrix.

b Replace the second row with "the sum of the first two rows".

c Hence find the relationship between p, q, and r such that the system is consistent.

C SOLVING 2×2 SYSTEMS OF LINEAR EQUATIONS

In the process of **row reduction**, we use elementary row operations to eliminate variables from selected rows of an augmented matrix. This allows us to systematically solve the corresponding system of linear equations.

To solve a 2×2 system of linear equations by row reduction, we aim to obtain a 1 in the top left corner and a 0 in the bottom left corner of the augmented matrix. This is equivalent to eliminating x from the corresponding equation.

When we perform this process we may obtain one of the forms below, with corresponding numbers of solutions:

$\left(\begin{array}{cc\|c} 1 & \square & \square \\ 0 & 1 & \square \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & \square & \square \\ 0 & 0 & \square \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & \square & \square \\ 0 & 0 & 0 \end{array}\right)$
1 solution	no solutions	infinitely many solutions

DISCUSSION

Discuss why each of the forms above corresponds to its particular number of solutions.

Example 3 — Self Tutor

Use elementary row operations to solve: $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$

In augmented matrix form, the system is

$$\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 5 & 4 & 17 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & -7 & 14 \end{array}\right) \quad 2R_2 - 5R_1 \to R_2$$

$$\left\{\begin{array}{ccc} 10 & 8 & 34 \\ -10 & -15 & -20 \\ \hline 0 & -7 & 14 \end{array}\right\}$$

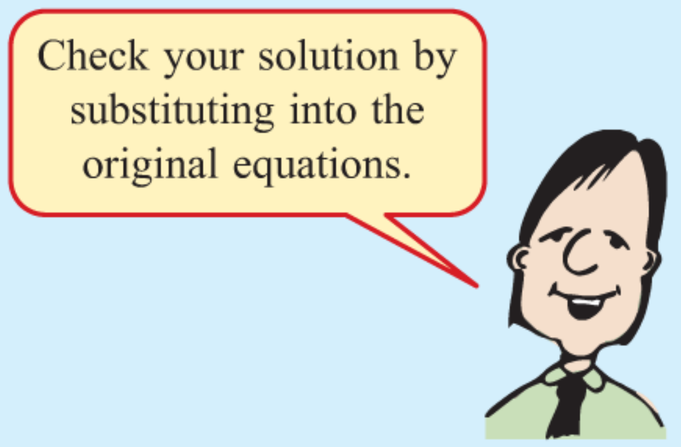
$$\sim \left(\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 0 & 1 & -2 \end{array}\right) \quad \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -\frac{1}{7}R_2 \to R_2 \end{array}$$

Using row 2, $y = -2$

Substituting into row 1, $x + \frac{3}{2}(-2) = 2$

$\therefore \quad x = 5$

$\therefore$ the solution is $x = 5, \ y = -2$.



EXERCISE 11C

1 Consider the system of linear equations $\begin{cases} x - 3y = 2 \\ 2x + y = -3 \end{cases}$.

a Write the system as an augmented matrix.

b Replace the second row with "the second row minus twice the first row".

c Hence solve the system.

2 By inspection, decide whether the pair of lines are intersecting, parallel, or coincident, and state the number of solutions to the system.

a $\begin{cases} x + 2y = 1 \\ 3x + 6y = 3 \end{cases}$ **b** $\begin{cases} 2x - y = -1 \\ x + 4y = 13 \end{cases}$

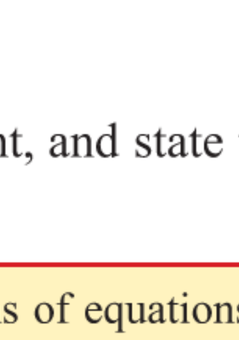
c $\begin{cases} x - 5y = 8 \\ 2x = 10y + 14 \end{cases}$ **d** $\begin{cases} x + y = 4 \\ x + y = a, \ \ a \in \mathbb{R} \end{cases}$

3 Use row operations to solve:

a $\begin{cases} x - 3y = -8 \\ 4x + 5y = 19 \end{cases}$ **b** $\begin{cases} x + 7y = -17 \\ 2x - y = 11 \end{cases}$

c $\begin{cases} 2x + 3y = -8 \\ x + 4y = -9 \end{cases}$ **d** $\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$

2×2 systems of equations can easily be solved by substitution. However, row operations work efficiently for any number of variables when substitution becomes extremely complicated.



4 Consider the system $\begin{cases} x + 3y = 4 \\ 2x + 6y = 8 \end{cases}$.

a Explain why there are infinitely many solutions.

b Try to solve the system using row operations. Explain what happens.

c Let $y = t, \ \ t \in \mathbb{R}$. Solve the system in terms of t.

d Let $x = s$, $s \in \mathbb{R}$. Solve the system in terms of s.

e Explain why your solutions in **c** and **d** are equivalent.

5 Consider the system $\begin{cases} x - 5y = 8 \\ 2x - 10y = a \end{cases}$ where $a \in \mathbb{R}$.

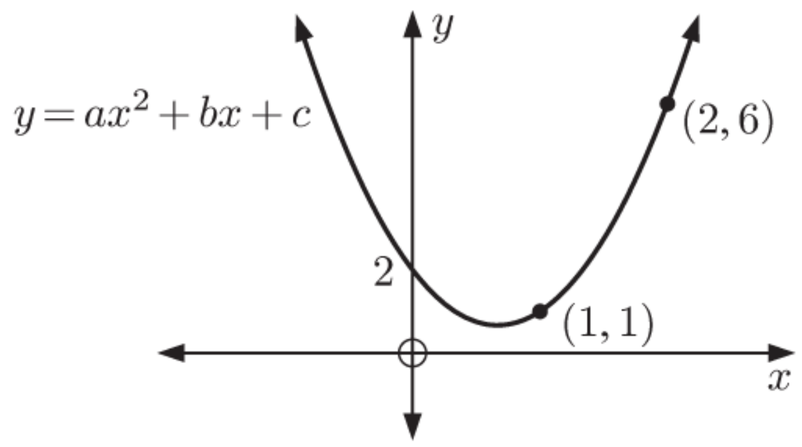
a Write the system as an augmented matrix, and perform a row operation to make the bottom left corner element 0.

b Explain what the second row means for the cases where $a \neq 16$. How many solutions does the system have in this case?

c Find all solutions for the case where $a = 16$.

6 Discuss the solutions to $\begin{cases} x + 3y = 4 \\ 2x + ay = b \end{cases}$ for all $a, b \in \mathbb{R}$.

7 Consider the quadratic function $y = ax^2 + bx + c$ alongside.



a State the value of c.

b Write two equations in terms of a and b.

c Use row operations to find a and b, and hence state the equation of the quadratic function.

8 Consider the general system of linear equations $\begin{cases} ax + by = m \\ cx + dy = n \end{cases}$.

a Write the system as an augmented matrix, and perform a row operation to make the bottom left corner element 0.

b Find the relationship between a, b, c, and d such that the system has a unique solution. Explain your answer.

c Suppose the system has infinitely many solutions. State the value of $cm - an$, giving a reason for your answer.

D SOLVING 3×3 SYSTEMS OF LINEAR EQUATIONS

General 3×3 systems of linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$ can be written as the augmented matrix $\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$.

We can use row operations to reduce the matrix to the form $\left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & \square & \square & \square \\ 0 & 0 & \square & \square \end{array}\right)$ in which there is a 1 in the top left corner, and a triangle of zeros in the bottom left corner. We call this **row echelon form**.

As with 2×2 systems of linear equations, 3×3 systems may have a unique solution, no solution, or infinitely many solutions.

- If we can reduce the matrix to the form $\left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 1 & \square & \square \\ 0 & 0 & 1 & \square \end{array}\right)$, the system has a **unique solution**.

Example 4 **Self Tutor**

Solve using row operations: $\begin{cases} x + 3y - z = 15 \\ 2x + y + z = 7 \\ x - y - 2z = 0 \end{cases}$

The system has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & -5 & 3 & -23 \\ 0 & -4 & -1 & -15 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & 1 & 1 & 7 \\ -2 & -6 & 2 & -30 \\ \hline 0 & -5 & 3 & -23 \end{array}\right\} \qquad \left\{\begin{array}{cccc} 1 & -1 & -2 & 0 \\ -1 & -3 & 1 & -15 \\ \hline 0 & -4 & -1 & -15 \end{array}\right\}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & -5 & 3 & -23 \\ 0 & 0 & -17 & 17 \end{array}\right) \quad 5R_3 - 4R_2 \to R_3$$

$$\left\{\begin{array}{cccc} 0 & -20 & -5 & -75 \\ 0 & 20 & -12 & 92 \\ \hline 0 & 0 & -17 & 17 \end{array}\right\}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & 1 & -\frac{3}{5} & \frac{23}{5} \\ 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{array}{l} \frac{1}{5}R_2 \to R_2 \\ \frac{1}{17}R_3 \to R_3 \end{array}$$

The first two row operations generate the zeros in the first column. The third row operation generates the zero in the second column.



Using row 3, $z = -1$

Substituting into row 2, $y - \frac{3}{5}(-1) = \frac{23}{5}$

$\therefore \quad y = 4$

Substituting into row 1, $x + 3(4) - (-1) = 15$

$\therefore \quad x = 2$

$\therefore$ the unique solution is $x = 2, \ y = 4, \ z = -1$.

- When aiming to reduce the matrix to the form required for a unique solution, we may obtain one of the forms below.

$$\left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 1 & \square & \square \\ 0 & 0 & 0 & \square \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 0 & 1 & \square \\ 0 & 0 & 0 & \square \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 \end{array}\right)$$

not zero (for each)

In each of these forms, we have an equation in which the LHS is zero, and the RHS is not zero. In this case, there is **no solution**, and the system is **inconsistent**.

Example 5 **Self Tutor**

Solve using row operations: $\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 8 \\ 3x - 4y + z = 18 \end{cases}$

The system has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 8 \\ 3 & -4 & 1 & 18 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & 2 \\ 0 & -10 & -2 & 9 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 5 \end{array}\right) \quad \begin{array}{l} -\frac{1}{5}R_2 \to R_2 \\ R_3 - 2R_2 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & -1 & 1 & 8 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -5 & -1 & 2 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 3 & -4 & 1 & 18 \\ -3 & -6 & -3 & -9 \\ \hline 0 & -10 & -2 & 9 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 0 & -10 & -2 & 9 \\ 0 & 10 & 2 & -4 \\ \hline 0 & 0 & 0 & 5 \end{array}\right\}$$

Row 3 means that $0x + 0y + 0z = 5$, which is absurd.

$\therefore$ there is no solution, and the system is inconsistent.

- Alternatively, we may obtain one of the forms below.

$$\left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 1 & \square & \square \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 0 & 1 & \square \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & \square & \square & \square \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case, the system has **infinitely many solutions**. We express the solutions in terms of **parameters** such as s and t.

Example 6 **Self Tutor**

Solve using row operations: $\begin{cases} 2x - y + z = 5 \\ x + y - z = 2 \\ 3x - 3y + 3z = 8 \end{cases}$

The system has augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 1 & -1 & 2 \\ 3 & -3 & 3 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 3 & -3 & -1 \\ 0 & -3 & 3 & 1 \end{array}\right) \quad \begin{array}{l} 2R_2 - R_1 \to R_2 \\ 2R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} \frac{1}{3}R_2 \to R_2 \\ R_2 + R_3 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & 2 & -2 & 4 \\ -2 & 1 & -1 & -5 \\ \hline 0 & 3 & -3 & -1 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 6 & -6 & 6 & 16 \\ -6 & 3 & -3 & -15 \\ \hline 0 & -3 & 3 & 1 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 0 & 3 & -3 & -1 \\ 0 & -3 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right\}$$

Row 3 indicates there are infinitely many solutions.

If we let $z = t$, then using row 2, $y - t = -\frac{1}{3}$

$$\therefore \quad y = t - \tfrac{1}{3}$$

Substituting into row 1, $2x - (t - \frac{1}{3}) + t = 5$

$$\therefore \quad 2x = \tfrac{14}{3}$$

$$\therefore \quad x = \tfrac{7}{3}$$

$\therefore$ the solutions have the form $x = \frac{7}{3}$, $y = t - \frac{1}{3}$, $z = t$, $t \in \mathbb{R}$.

EXERCISE 11D

1 Solve each system of linear equations:

a $\begin{cases} x + 4y + 11z = 7 \\ x + 6y + 17z = 9 \\ x + 4y + 8z = 4 \end{cases}$ **b** $\begin{cases} 2x - y + 3z = 17 \\ 2x - 2y - 5z = 4 \\ 3x + 2y + 2z = 10 \end{cases}$ **c** $\begin{cases} 2x + 3y + 4z = 1 \\ 5x + 6y + 7z = 2 \\ 8x + 9y + 10z = 4 \end{cases}$

d $\begin{cases} x - 2y + 5z = 1 \\ 2x - y + 8z = 2 \\ -3x \quad - 11z = -3 \end{cases}$ **e** $\begin{cases} x + 2y - z = 4 \\ 3x + 2y + z = 7 \\ 5x + 2y + 3z = 11 \end{cases}$ **f** $\begin{cases} 2x + 4y + z = 1 \\ 3x - 5y - 3z = 19 \\ 5x + 13y + 7z = 1 \end{cases}$

2 Find the equation of the quadratic function which passes through the points:

a $(1, -2)$, $(2, 4)$, and $(3, 12)$ **b** $(-1, 3)$, $(2, 9)$, and $(4, -7)$.

3 Try to find the equation of the quadratic which passes through the points $(-1, 7)$, $(2, 1)$, and $(3, -1)$. Explain your answer.

4 Consider the **Opening Problem** on page **254**.

a Find a quadratic function $P(n)$ for the maximum number of pieces that can be made using n cuts. Include the domain of $P(n)$ in your answer.

b Hence find the maximum number of pieces that can be made using 12 cuts.

Example 7 **Self Tutor**

Consider the system $\begin{cases} x - 2y - z = -1 \\ 2x + y + 3z = 13 \\ x + 8y + 9z = a \end{cases}$ where $a \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b For what values of a does the system have no solutions?

c Under what conditions does the system have infinitely many solutions? Find the solutions in this case.

a The system has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 2 & 1 & 3 & 13 \\ 1 & 8 & 9 & a \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 15 \\ 0 & 10 & 10 & a+1 \end{array}\right) \qquad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & a-29 \end{array}\right) \qquad \begin{array}{l} \frac{1}{5}R_2 \to R_2 \\ R_3 - 2R_2 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & 1 & 3 & 13 \\ -2 & 4 & 2 & 2 \\ \hline 0 & 5 & 5 & 15 \end{array}\right\} \qquad \left\{\begin{array}{cccc} 1 & 8 & 9 & a \\ -1 & 2 & 1 & 1 \\ \hline 0 & 10 & 10 & a+1 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 0 & 10 & 10 & a+1 \\ 0 & -10 & -10 & -30 \\ \hline 0 & 0 & 0 & a-29 \end{array}\right\}$$

b Using row 3, the system has no solutions if $a \neq 29$.

c The system has infinitely many solutions if the last row is all zeros. This occurs when $a = 29$.
In this case we let $z = t$.
Using row 2, $y + t = 3$
$\therefore\ y = 3 - t$

Substituting into row 1, $x - 2(3 - t) - t = -1$
$\therefore\ x - 6 + t = -1$
$\therefore\ x = 5 - t$

$\therefore$ the solutions have the form $x = 5 - t,\ y = 3 - t,\ z = t,\ t \in \mathbb{R}$.

5 Consider the system $\begin{cases} x + 2y + z = 3 \\ 2x - y + 4z = 1 \\ x + 7y - z = k \end{cases}$ where $k \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b For what values of k does the system have no solutions?

c Under what condition does the system have infinitely many solutions? Find the solutions in this case.

d Explain why the system never has a unique solution.

6 Consider the system $\begin{cases} x + 2y - 2z = 5 \\ x - y + 3z = -1 \\ x - 7y + kz = -k \end{cases}$ where $k \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b Show that for one value of k, the system has infinitely many solutions. Find the solutions in this case.

c Show that there is a unique solution for all other values of k. Find this solution in terms of k.

7 Consider the system $\begin{cases} x + 3y + 3z = a - 1 \\ 2x - y + z = 7 \\ 3x - 5y + az = 16 \end{cases}$ where $a \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b Show that for one value of a, the system has infinitely many solutions. Find the solutions in this case.

c Show that there is a unique solution for all other values of a. Find the solution in terms of a.

8 Consider the system $\begin{cases} x + 4y + mz = -m \\ (m+1)x + 4y + z = 1 \\ 4x + 4y + z = 1 \end{cases}$ where $m \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b Show that for one value of m, the system has no solutions.

c Show that for a different value of m, the system has infinitely many solutions. Find the solutions in this case.

d Show that there is a unique solution for all other values of m. Find the solution in terms of m.

ACTIVITY

A graphics calculator can also be used to solve systems of linear equations.



GRAPHICS CALCULATOR INSTRUCTIONS

What to do:

1 Use technology to solve the following systems. Make sure you understand how your calculator displays cases of no solutions and infinitely many solutions.

a $\begin{cases} 2x+3y=1 \\ x-2y=11 \end{cases}$ **b** $\begin{cases} 3x-y+2z=18 \\ y+4z=18 \\ 2x+3y-2z=-12 \end{cases}$ **c** $\begin{cases} x-2y-4z=3 \\ 5x+3y+z=4 \\ -2x+y-3z=15 \end{cases}$

d $\begin{cases} x-2y+5z=-1 \\ 3x+y-4z=5 \\ 5x-3y+6z=-2 \end{cases}$ **e** $\begin{cases} 3x+y+2z=-7 \\ -2x+8y-7z=0 \\ 2x-3z=10 \end{cases}$ **f** $\begin{cases} -4x+y+6z=1 \\ x+5y+2z=-2 \\ 6x+9y-2z=-5 \end{cases}$

2 Which of the questions in the previous Exercise could you *not* answer using a graphics calculator?

3 Research **Gaussian elimination** for solving systems of linear equations.
What is the difference between row echelon form and reduced row echelon form? Why is reduced row echelon form useful?

REVIEW SET 11A

1 Find a solution set for:

a $5x+y=-2$ **b** $x-4y+3z=1$

2 Write down the augmented matrix for the system of linear equations, and state if the system is underspecified or overspecified.

a $\begin{cases} 3x-2y=4 \\ x+5y=-8 \end{cases}$ **b** $\begin{cases} 2x+y+2z=-1 \\ 3x-2y=6 \end{cases}$ **c** $\begin{cases} x-4y=0 \\ 5x-y=2 \\ -3x+y=-7 \end{cases}$

3 Consider the system of linear equations $\begin{cases} 4x-6y=-1 \\ ax+2y=3 \end{cases}$.

a Write the system as an augmented matrix.

b Multiply the second row by -3.

c For what value(s) of a is the system inconsistent?

4 Use row operations to solve:

a $\begin{cases} x+2y=3 \\ 3x+5y=11 \end{cases}$ **b** $\begin{cases} 5x+y=-1 \\ x-2y=-9 \end{cases}$

5 Consider the system $\begin{cases} 2x-5y=3 \\ 6x+ky=9 \end{cases}$.

a For what value(s) of k does the system have a unique solution? Find this unique solution.

b Discuss the solutions for the remaining value(s) of k.

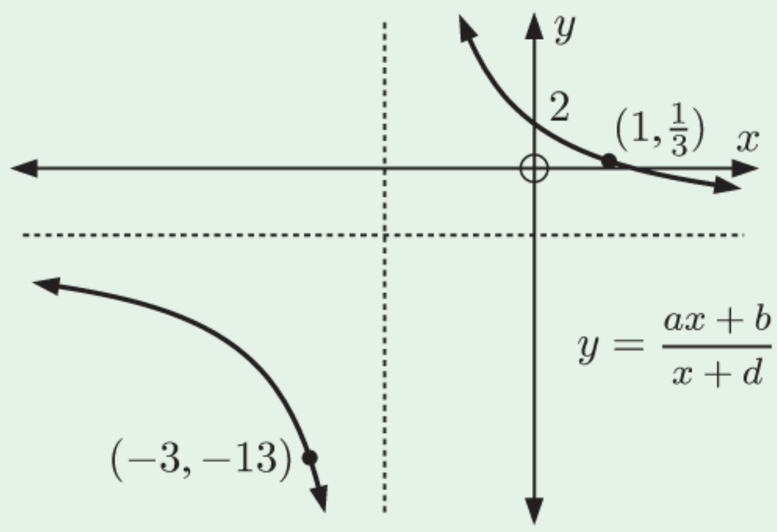
6 Solve each system of equations:

a $\begin{cases} x - 2y + 5z = 1 \\ 2x - 4y + 8z = 2 \\ -3x + 6y + 7z = -3 \end{cases}$ **b** $\begin{cases} x + 4y + 11z = 7 \\ x + 6y + 17z = 9 \\ x + 4y + 8z = 4 \end{cases}$ **c** $\begin{cases} 2x + 3y + 4z = 1 \\ 5x + 6y + 7z = 2 \\ 8x + 9y + 10z = 3 \end{cases}$

7 Consider the rational function $y = \dfrac{ax + b}{x + d}$ alongside.

a Use the given points to write three linear equations involving a, b, and d.

b Find the equation of the rational function.



8 Consider the system $\begin{cases} x - 3y + 2z = -7 \\ 2x + y - z = 5 \\ 5x - 8y + kz = 12 \end{cases}$ where $k \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b Show that for one value of k, the system has no solutions.

c **i** Show that for all other values of k, there is a unique solution. Write this solution in terms of k.

ii Hence find the unique solution for $k = 3$.

REVIEW SET 11B

1 Decide whether each system is consistent, giving reasons for your answers:

a $\begin{cases} 3x + y = 2 \\ 3x - y = 2 \end{cases}$ **b** $\begin{cases} x + 4y - z = 1 \\ -x - 4y + z = 1 \end{cases}$

2 Write, in terms of x, y, and z, the system of linear equations with augmented matrix:

a $\left(\begin{array}{ccc|c} 5 & -1 & 4 & 3 \\ 1 & 0 & -2 & -1 \end{array}\right)$ **b** $\left(\begin{array}{ccc|c} 2 & 0 & -3 & -2 \\ -1 & 6 & 0 & 3 \\ 3 & 3 & -4 & -7 \end{array}\right)$

3 Consider the system $\begin{cases} x - y + 3z = a \\ 5x + y - z = b \\ 2x + 4y - 10z = c \end{cases}$.

a Write the system as an augmented matrix.

b Replace the second row with "the second row, minus three times the first row".

c Hence find the relationship between a, b, and c such that the system is consistent.

4 Consider the system $\begin{cases} 4x + y = -1 \\ -8x - 2y = a \end{cases}$ where $a \in \mathbb{R}$.

a For what value(s) of a does the system have infinitely many solutions? Find the solutions in this case.

b Discuss the solutions for the remaining value(s) of a.

5 Discuss the solutions to $\begin{cases} x - 2y = 5 \\ 4x + ay = b \end{cases}$ for all $a, b \in \mathbb{R}$.

6 Solve each system of linear equations:

a $\begin{cases} 2x - y + 3z = 17 \\ 2x - 2y - 5z = 4 \\ 3x + 2y + 2z = 10 \end{cases}$ **b** $\begin{cases} x - 3y + z = -2 \\ 3x + y - 2z = -5 \\ 2x + 4y - 3z = 4 \end{cases}$

7 The two points $(-2, 4)$ and $(1, 3)$ lie on a circle $x^2 + y^2 + ax + by + c = 0$.

a Find two equations in a, b, and c, and solve the system of equations.

b Explain why infinitely many solutions are obtained in **a**.

c Suppose the point $(4, q)$ also lies on the circle.

i Find the value of q for which the system of equations in a, b, and c has no solutions. Explain this result geometrically.

ii Find the equation of the circle when $q = 12$.

8 Consider the system $\begin{cases} x + 4y - z = k \\ -2x + z = 3 \\ 5x + 4y + (k-6)z = -3 \end{cases}$ where $k \in \mathbb{R}$.

a Use row operations to reduce the system to row echelon form.

b Show that for one value of k, the system has infinitely many solutions. Find the solutions in this case.

c Show that there is a unique solution for all other values of k. Write this solution in terms of k.

Chapter 12

Vectors

Contents:

A Vectors and scalars
B Geometric operations with vectors
C Vectors in the plane
D The magnitude of a vector
E Operations with plane vectors
F Vectors in space
G Operations with vectors in space
H Vector algebra
I The vector between two points
J Parallelism
K The scalar product of two vectors
L The angle between two vectors
M Proof using vector geometry
N The vector product of two vectors



OPENING PROBLEM

An aeroplane in calm conditions is flying at 800 km h^{-1} due east. A cold wind suddenly blows from the south-west at 35 km h^{-1}, pushing the aeroplane slightly off course.

Things to think about:

a How can we illustrate the plane's movement and the wind using a scale diagram?

b What operation do we need to perform to find the effect of the wind on the aeroplane?

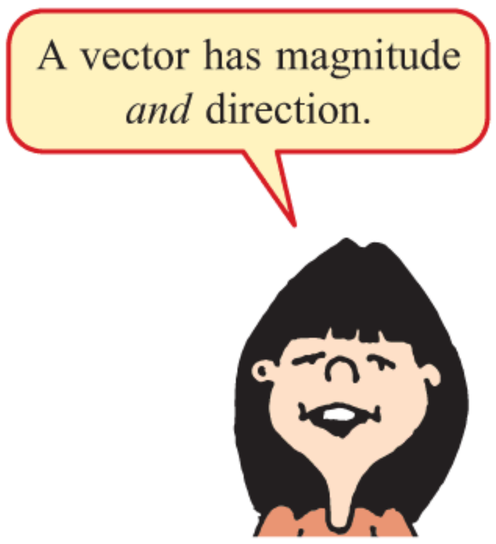
c Can you use a scale diagram to determine the resulting speed and direction of the aeroplane?

A VECTORS AND SCALARS

In the **Opening Problem**, the effect of the wind on the aeroplane is determined by both its speed *and* its direction. The effect would be different if the wind was blowing against the aeroplane rather than from behind it.

Quantities which have only magnitude are called **scalars**.

Quantities which have both magnitude and direction are called **vectors**.



The *speed* of the plane is a scalar. It describes how fast the plane is moving.

The *velocity* of the plane is a vector. It includes both its speed and also its direction.

Other examples of vector quantities are:

- acceleration
- displacement
- force
- momentum

For example, farmer Giles needs to remove a fence post. He starts by pushing on the post sideways to loosen the ground. Giles has a choice of how hard to push the post and in which direction. The force he applies is therefore a vector.



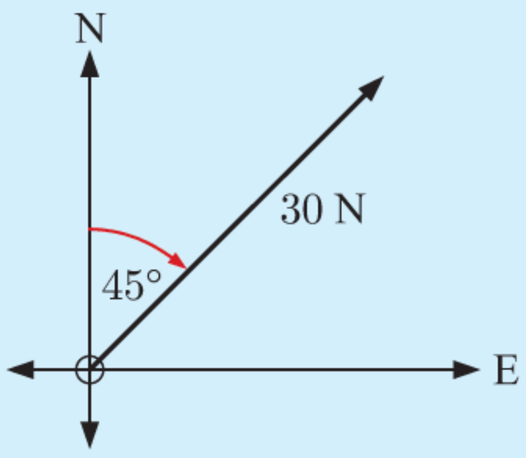
DIRECTED LINE SEGMENT REPRESENTATION

We can represent a vector quantity using a **directed line segment** or **arrow**.

The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows its direction.

Example 1 **Self Tutor**

Draw a scale diagram to represent a force of 30 newtons in a north-easterly direction.



Scale:
1 cm ≡ 10 N

EXERCISE 12A.1

1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

a 30 newtons in a south-easterly direction

b 25 m s^{-1} in a northerly direction

c an excavator digging a tunnel at a rate of 30 cm min^{-1} at an angle of 30° to the ground

d an aeroplane taking off at an angle of 10° to the runway with a speed of 50 m s^{-1}.

2 If $\xrightarrow{\quad 25\text{ mm} \quad}$ represents a velocity of 50 m s^{-1} due east, draw a directed line segment representing a velocity of:

a 100 m s^{-1} due west **b** 75 m s^{-1} north-east.

3 Draw a scale diagram to represent:

a a force of 30 newtons in a north-westerly direction

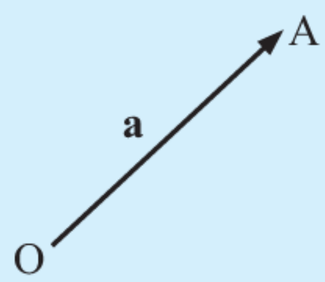
b a velocity of 36 m s^{-1} vertically downwards

c a displacement of 4 units at an angle of 15° to the positive x-axis

d the velocity of an aeroplane taking off at an angle of 8° to the runway at a speed of 150 km h^{-1}.

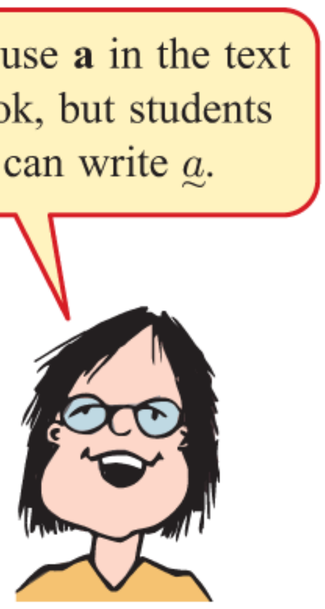
VECTOR NOTATION

- The vector from the origin O to point A is called the **position vector** of point A. It is written as $\overrightarrow{\text{OA}}$, **a** or $\underset{\sim}{a}$.
  The **magnitude** or **length** of vector $\overrightarrow{\text{OA}}$ is written as $|\overrightarrow{\text{OA}}|$ or OA or $|\mathbf{a}|$ or $|\underset{\sim}{a}|$.



- $\overrightarrow{\text{AB}}$ is the vector which **originates** at A and **terminates** at B, $\overrightarrow{\text{AB}}$ is called the **displacement vector** of B relative to A.



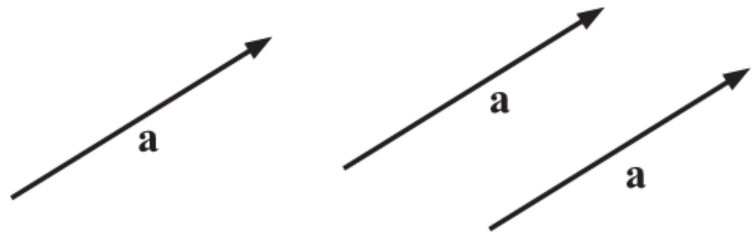


GEOMETRIC VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude *and* direction.

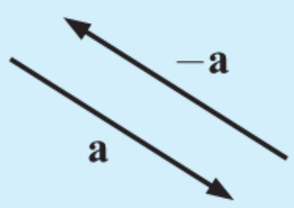
Equal vectors are **parallel** and **in the same direction**, and are **equal in length**. The arrows that represent them are translations of one another.

We can draw a vector with given magnitude and direction from *any* point, so we consider vectors to be **free**.



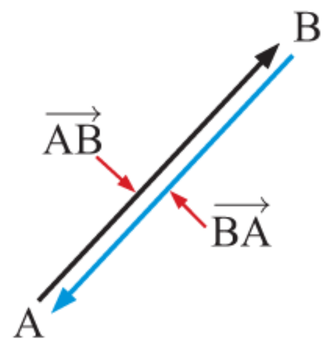
GEOMETRIC NEGATIVE VECTORS

The **negative** of a vector **a** is the vector parallel to **a** and with the same length, but in the opposite direction. It is denoted $-\mathbf{a}$.

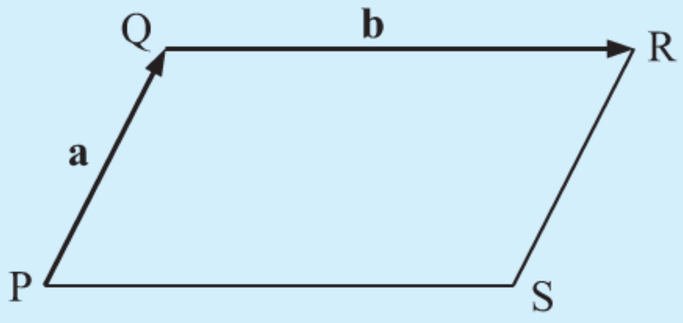


For example, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

$\overrightarrow{BA}$ is the **negative** of $\overrightarrow{AB}$, and we write $\overrightarrow{BA} = -\overrightarrow{AB}$.



Example 2 **Self Tutor**



PQRS is a parallelogram in which $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$.

Find vector expressions for:

a $\overrightarrow{QP}$ **b** $\overrightarrow{RQ}$ **c** $\overrightarrow{SR}$ **d** $\overrightarrow{SP}$

a $\overrightarrow{QP} = -\mathbf{a}$ {the negative vector of $\overrightarrow{PQ}$}

b $\overrightarrow{RQ} = -\mathbf{b}$ {the negative vector of $\overrightarrow{QR}$}

c $\overrightarrow{SR} = \mathbf{a}$ {parallel to and the same length as $\overrightarrow{PQ}$}

d $\overrightarrow{SP} = -\mathbf{b}$ {parallel to and the same length as $\overrightarrow{QR}$, but opposite in direction}

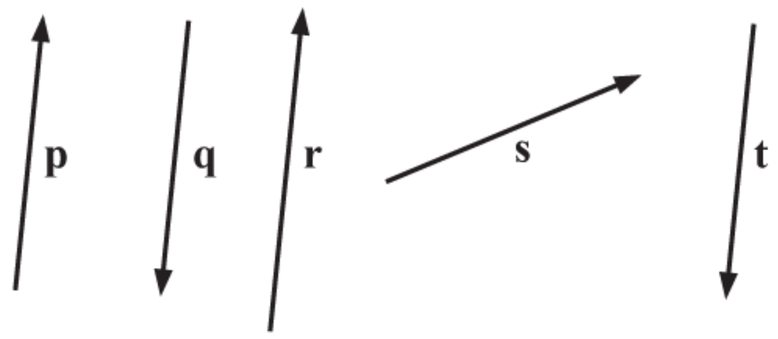
EXERCISE 12A.2

1 State the vectors which are:

a equal in magnitude **b** parallel

c in the same direction **d** equal

e negatives of one another.



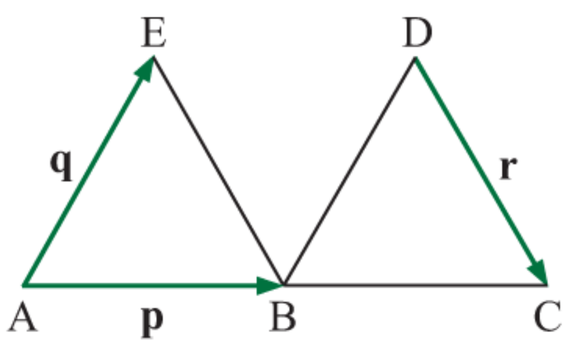
2 The figure shown consists of two equilateral triangles. A, B, and C are collinear.

$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AE} = \mathbf{q}$, and $\overrightarrow{DC} = \mathbf{r}$.

Which of the following statements are true?

a $\overrightarrow{EB} = \mathbf{r}$ **b** $|\mathbf{p}| = |\mathbf{q}|$ **c** $\overrightarrow{BC} = \mathbf{r}$

d $\overrightarrow{DB} = \mathbf{q}$ **e** $\overrightarrow{ED} = \mathbf{p}$ **f** $\mathbf{p} = \mathbf{q}$



3 ABCDEF is a regular hexagon.

a Write down the vector which:

i originates at B and terminates at C

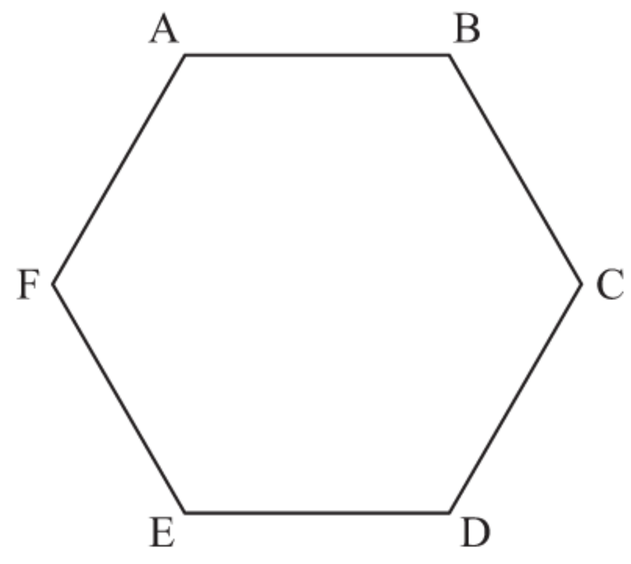
ii is equal to $\overrightarrow{AB}$.

b Write down *all* vectors which:

i are the negative of $\overrightarrow{EF}$

ii have the same length as $\overrightarrow{ED}$.

c Write down a vector which is parallel to $\overrightarrow{AB}$ and twice its length.



DISCUSSION

- Does a vector of length 0 have a direction?
- Is it possible to represent such a vector geometrically?

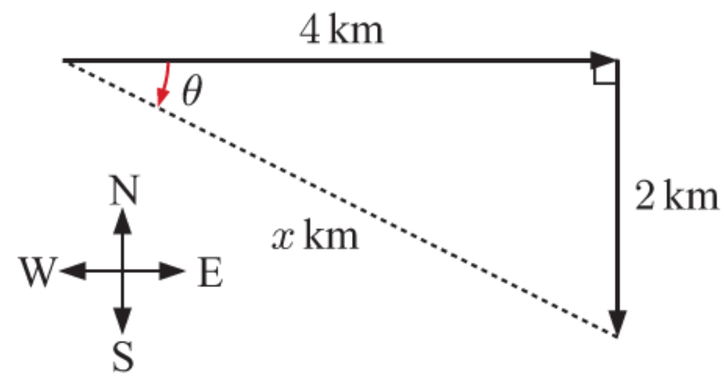
B GEOMETRIC OPERATIONS WITH VECTORS

In previous years we have often used trigonometry and Pythagoras' theorem to solve problems involving distances and directions.

A typical problem could be:

A runner runs east for 4 km and then south for 2 km.

How far is she from her starting point and in what direction?



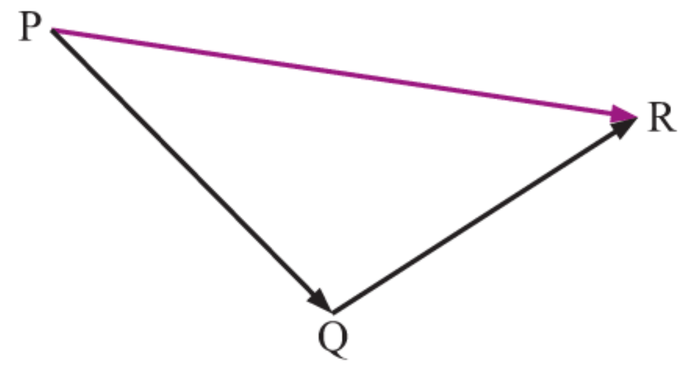
Problems like these can be well represented by vectors. In this case, the vectors are displacements. There is a displacement of 4 km to the east followed by a displacement of 2 km to the south. We want to know the *sum* of these displacements, including distance *and* direction.

GEOMETRIC VECTOR ADDITION

Suppose we have three towns P, Q, and R.

A trip from P to Q followed by a trip from Q to R has the same origin and destination as a trip from P to R.

This can be expressed in vector form as the sum
$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.



From examples like this, we can define vector addition geometrically as follows:

To construct $\mathbf{a} + \mathbf{b}$:

Step 1: Draw $\mathbf{a}$.

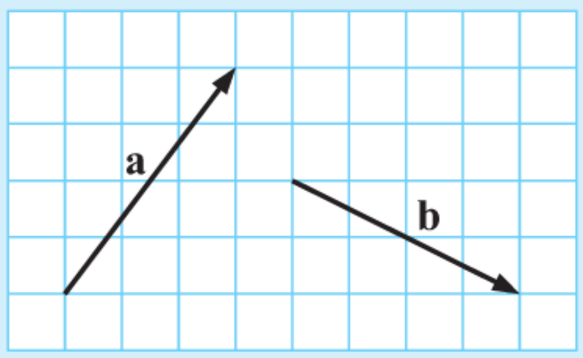
Step 2: At the arrowhead end of $\mathbf{a}$, draw $\mathbf{b}$.

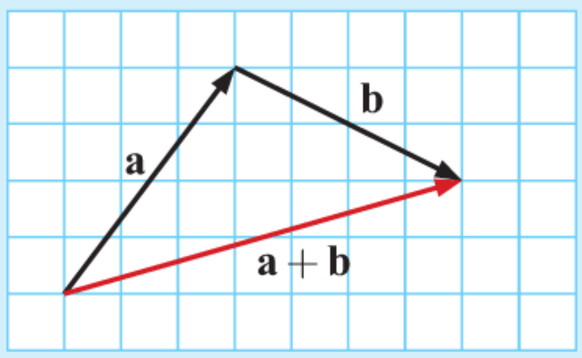
Step 3: Join the beginning of $\mathbf{a}$ to the arrowhead end of $\mathbf{b}$. This is vector $\mathbf{a} + \mathbf{b}$.



Example 3 **Self Tutor**

Given vectors $\mathbf{a}$ and $\mathbf{b}$ shown, construct $\mathbf{a} + \mathbf{b}$.





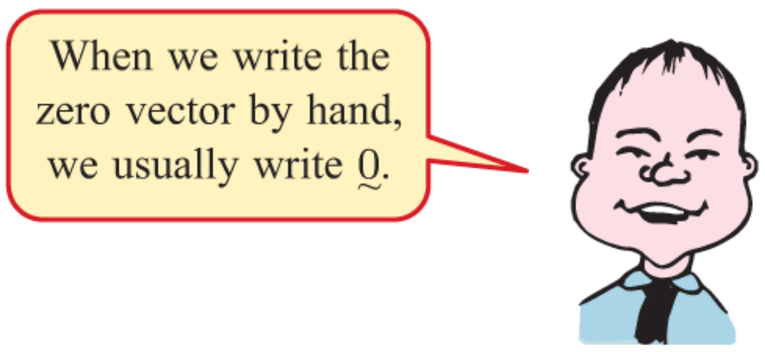
THE ZERO VECTOR

Having defined vector addition, we are now able to state that:

The **zero vector 0** is a vector of length 0.

For any vector **a**: $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.



Example 4 — Self Tutor

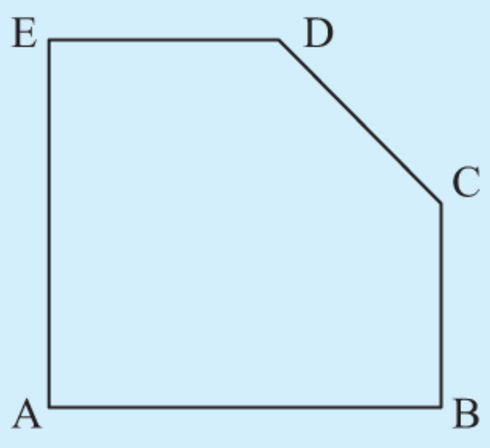
Find a single vector which is equal to:

a $\overrightarrow{BC} + \overrightarrow{CA}$

b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC}$

c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$



a $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

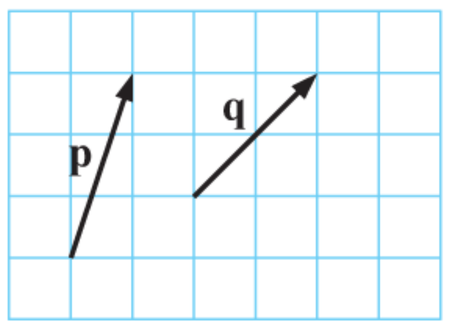
b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{BC}$

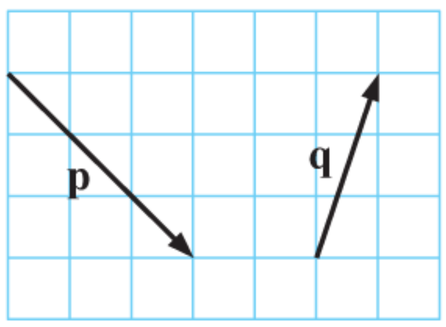
c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \mathbf{0}$

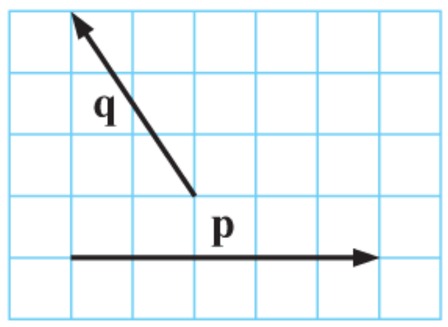
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

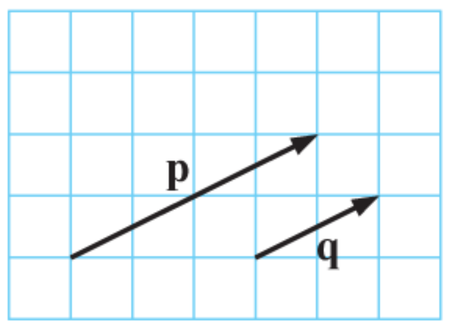
EXERCISE 12B.1

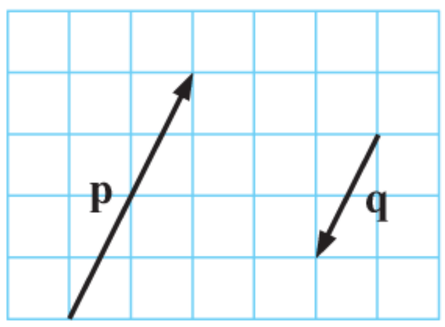
1 Use the given vectors $\mathbf{p}$ and $\mathbf{q}$ to construct $\mathbf{p} + \mathbf{q}$:

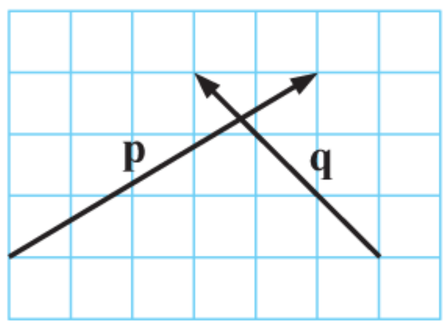
a 

b 

c 

d 

e 

f 

2 Find a single vector which is equal to:

a $\overrightarrow{AB} + \overrightarrow{BC}$

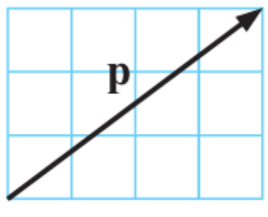
b $\overrightarrow{BC} + \overrightarrow{CD}$

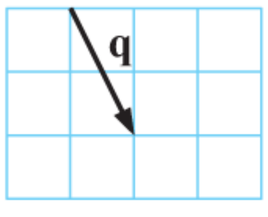
c $\overrightarrow{AB} + \overrightarrow{BA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

e $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

f $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$

3 a Given 

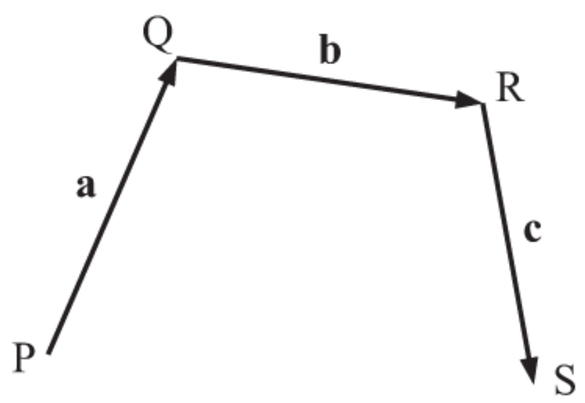
and 

use vector diagrams to find:

i $\mathbf{p} + \mathbf{q}$ ii $\mathbf{q} + \mathbf{p}$.

b For any two vectors $\mathbf{p}$ and $\mathbf{q}$, is $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$? Illustrate your answer.

4 Consider:



One way of finding $\overrightarrow{PS}$ is:

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PR} + \overrightarrow{RS} \\ &= (\mathbf{a} + \mathbf{b}) + \mathbf{c}.\end{aligned}$$

Use the diagram to show that $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

5 Answer the **Opening Problem** on page **270**.

6 Susan and Cathy are pushing a heavy trolley containing groceries. Susan pushes the trolley with force 40 N in the direction 017°. Cathy pushes the trolley with force 35 N in the direction 335°.

a Use a scale diagram to estimate the resultant force from the two girls pushing.

b Use trigonometry to find the resultant force more accurately.

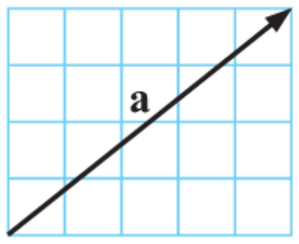


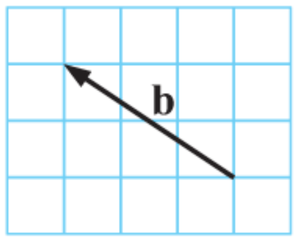
GEOMETRIC VECTOR SUBTRACTION

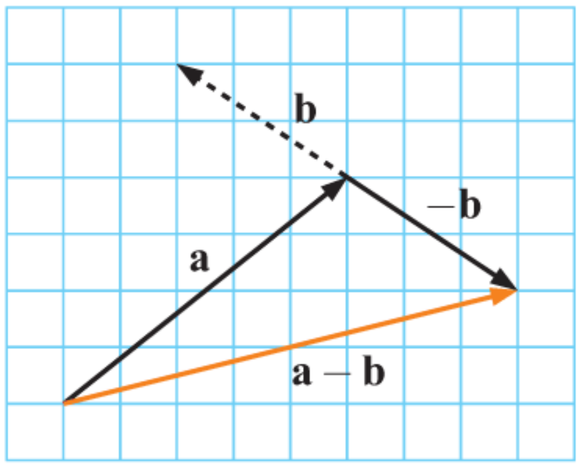
To subtract one vector from another, we simply **add its negative**.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

For example,

given 

and 

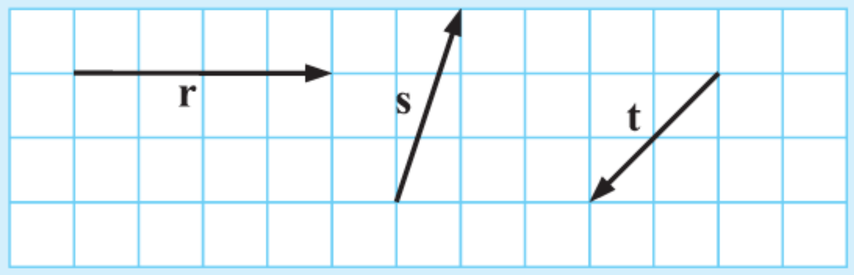
then 

Example 5 **Self Tutor**

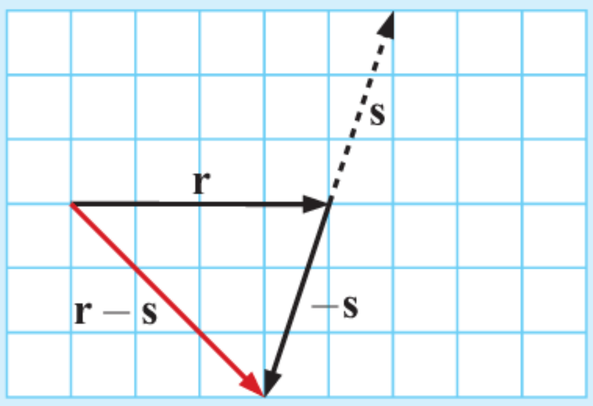
For the vectors illustrated, show how to construct:

a $\mathbf{r} - \mathbf{s}$

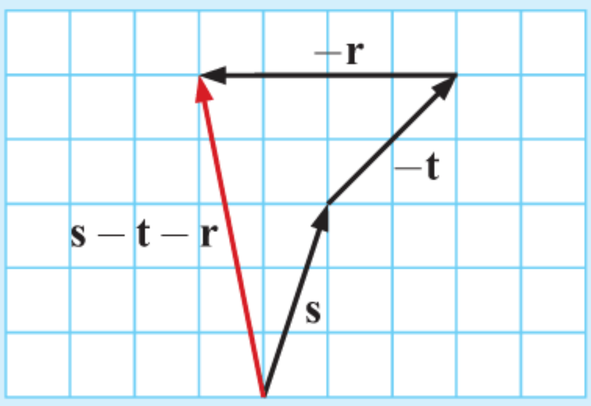
b $\mathbf{s} - \mathbf{t} - \mathbf{r}$



a



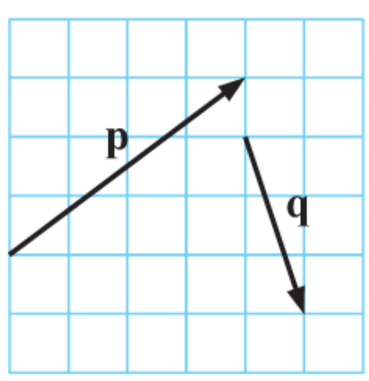
b



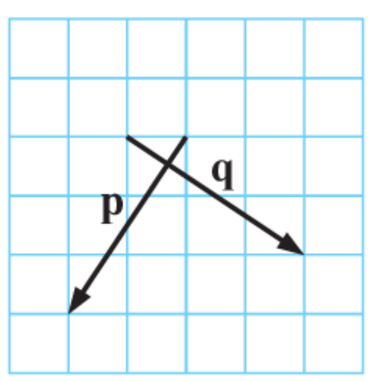
EXERCISE 12B.2

1 For the following vectors $\mathbf{p}$ and $\mathbf{q}$, show how to construct $\mathbf{p} - \mathbf{q}$:

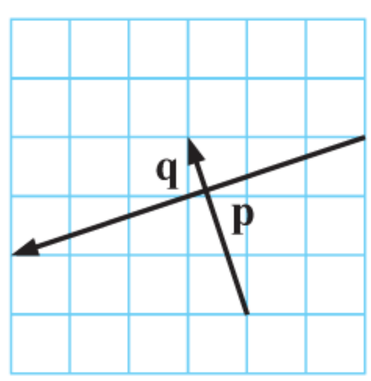
a



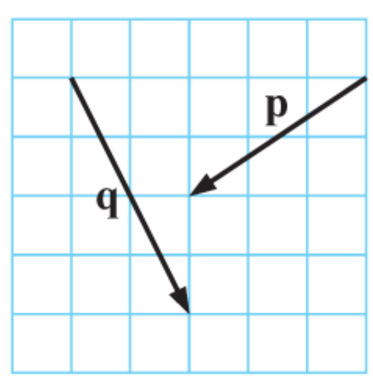
b



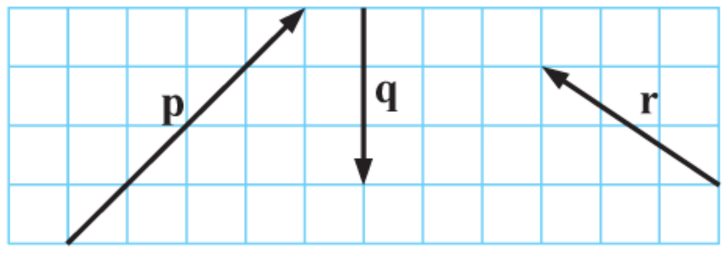
c



d



2 For the vectors illustrated, show how to construct:



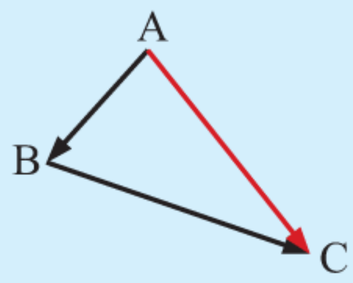
a $\mathbf{p} + \mathbf{q} - \mathbf{r}$ **b** $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **c** $\mathbf{r} - \mathbf{q} - \mathbf{p}$

Example 6 — Self Tutor

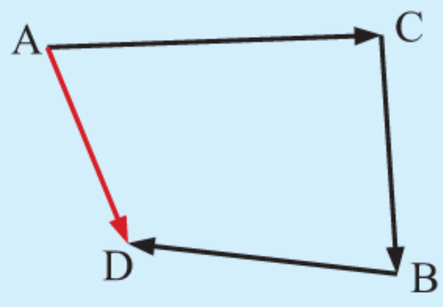
For points A, B, C, and D, simplify the following vector expressions:

a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

a
$$\begin{aligned} &\overrightarrow{AB} - \overrightarrow{CB} \\ &= \overrightarrow{AB} + \overrightarrow{BC} \quad \{\text{as } \overrightarrow{BC} = -\overrightarrow{CB}\} \\ &= \overrightarrow{AC} \end{aligned}$$



b
$$\begin{aligned} &\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB} \\ &= \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} \\ &= \overrightarrow{AD} \end{aligned}$$



3 For points A, B, C, and D, simplify the following vector expressions:

a $\overrightarrow{AC} + \overrightarrow{CB}$ **b** $\overrightarrow{AD} - \overrightarrow{BD}$ **c** $\overrightarrow{AC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **e** $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$ **f** $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

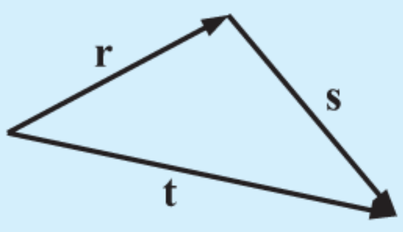
VECTOR EQUATIONS

Whenever we have vectors which form a closed polygon, we can write a **vector equation**. The vector equation can usually be written in several ways, but they are all equivalent.

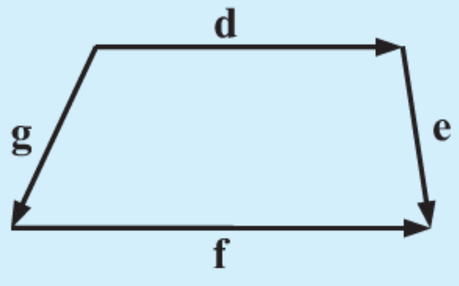
Example 7 — Self Tutor

Construct vector equations for:

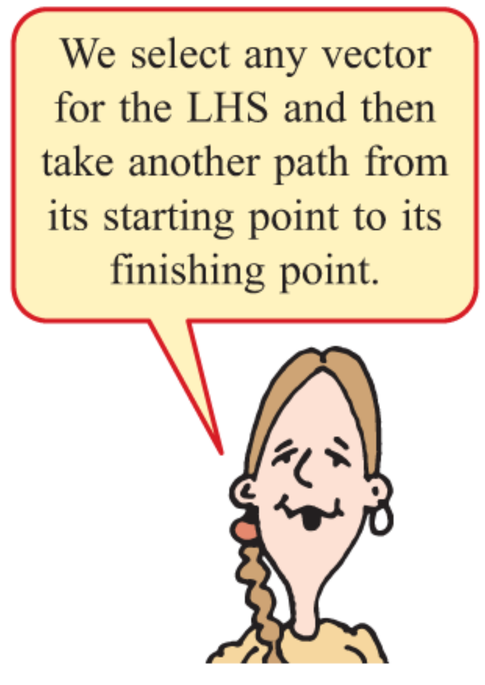
a



b



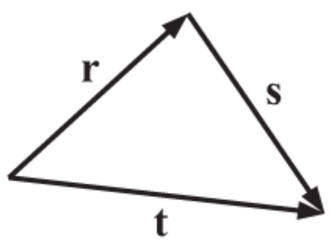
a $\mathbf{t} = \mathbf{r} + \mathbf{s}$ **b** $\mathbf{f} = -\mathbf{g} + \mathbf{d} + \mathbf{e}$



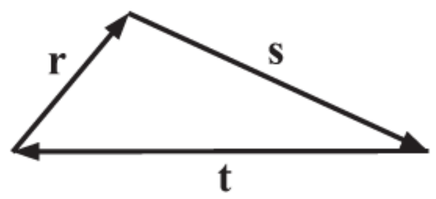
EXERCISE 12B.3

1 Construct vector equations for:

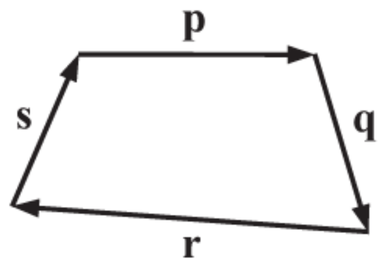
a



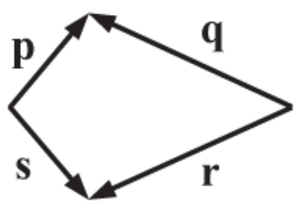
b



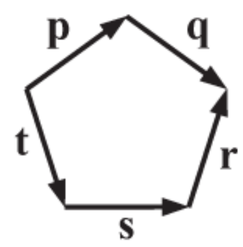
c



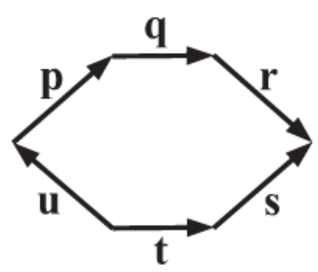
d



e



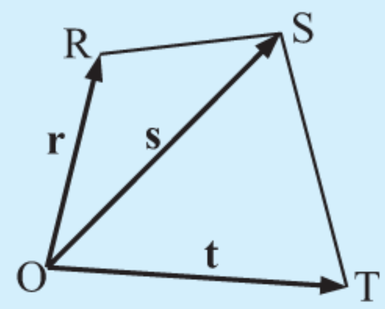
f



Example 8 — **Self Tutor**

Find, in terms of **r**, **s**, and **t**:

a $\overrightarrow{RS}$ **b** $\overrightarrow{SR}$ **c** $\overrightarrow{ST}$



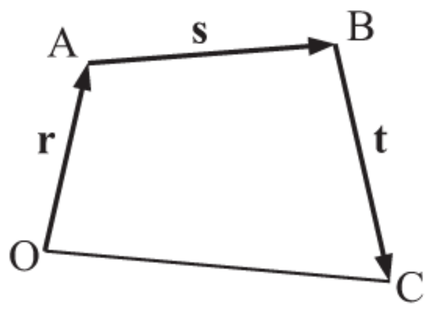
a $\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS}$
$= -\mathbf{r} + \mathbf{s}$
$= \mathbf{s} - \mathbf{r}$

b $\overrightarrow{SR} = \overrightarrow{SO} + \overrightarrow{OR}$
$= -\mathbf{s} + \mathbf{r}$
$= \mathbf{r} - \mathbf{s}$

c $\overrightarrow{ST} = \overrightarrow{SO} + \overrightarrow{OT}$
$= -\mathbf{s} + \mathbf{t}$
$= \mathbf{t} - \mathbf{s}$

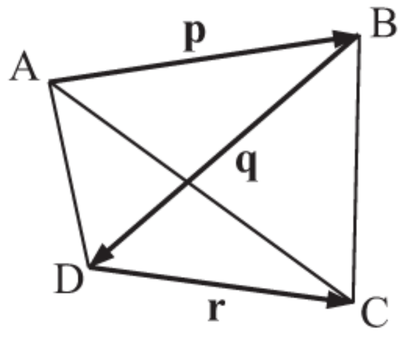
2 **a** Find, in terms of **r**, **s**, and **t**:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$ **iii** $\overrightarrow{OC}$



b Find, in terms of **p**, **q**, and **r**:

i $\overrightarrow{AD}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{AC}$



GEOMETRIC SCALAR MULTIPLICATION

A **scalar** is a non-vector quantity. It has magnitude but no direction.

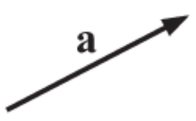
A scalar multiple of a vector is always **parallel** to the vector.



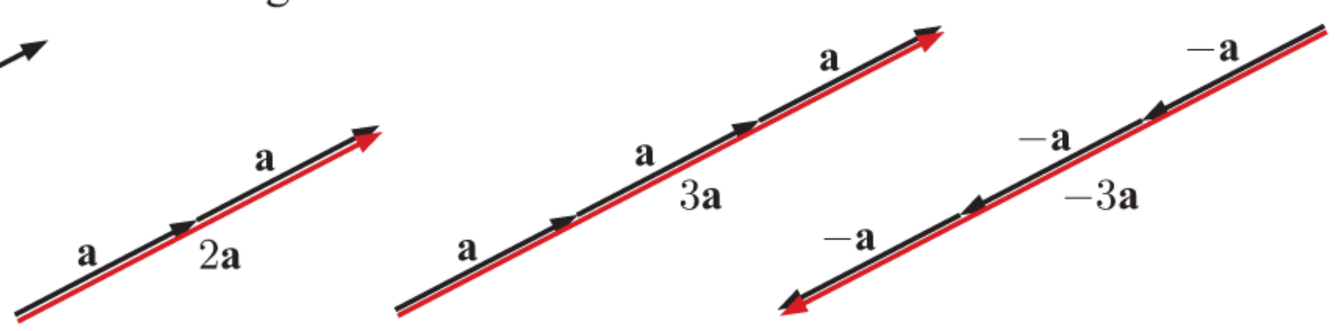
We can multiply vectors by scalars such as 2 and -3, or in fact any $k \in \mathbb{R}$.

If **a** is a vector, we define:

- $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ has the same direction as **a** but is twice the length
- $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$ has the same direction as **a** but is three times the length
- $-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$ has the opposite direction to **a** and is three times the length.

If **a** is 

then we have:



If $\mathbf{a}$ is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

- If $k > 0$, $k\mathbf{a}$ and $\mathbf{a}$ have the same direction.
- If $k < 0$, $k\mathbf{a}$ and $\mathbf{a}$ have opposite directions.
- If $k = 0$, $k\mathbf{a} = \mathbf{0}$, the zero vector.

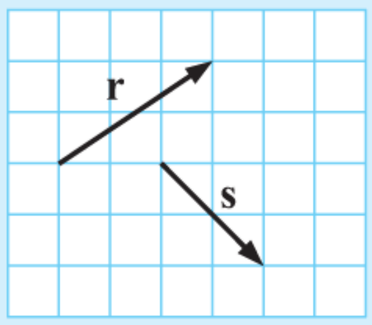
VECTOR SCALAR MULTIPLICATION



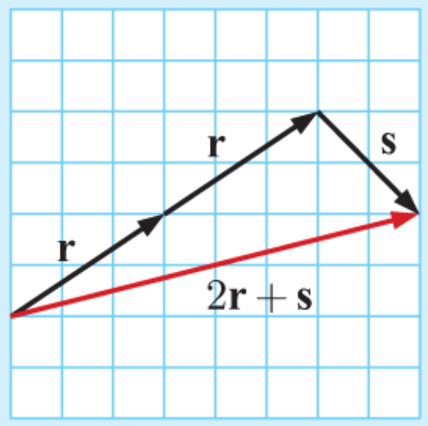
Example 9 — Self Tutor

Given the vectors below, construct:

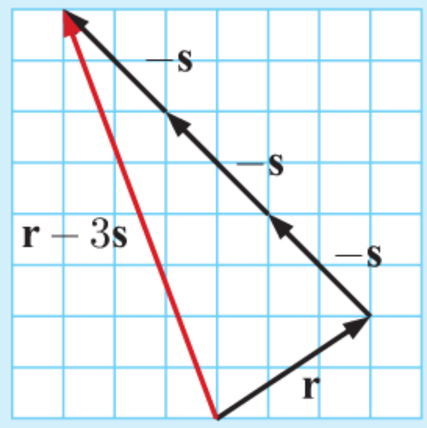
a $2\mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} - 3\mathbf{s}$



a



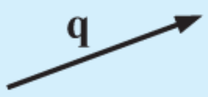
b



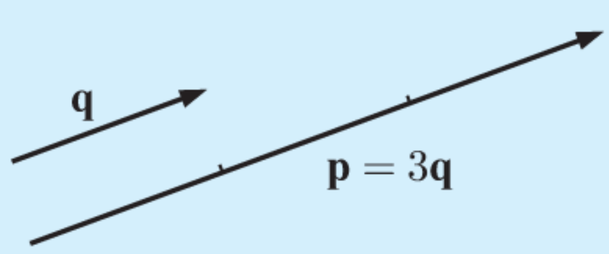
Example 10 — Self Tutor

Sketch vectors $\mathbf{p}$ and $\mathbf{q}$ if: **a** $\mathbf{p} = 3\mathbf{q}$ **b** $\mathbf{p} = -\frac{1}{2}\mathbf{q}$

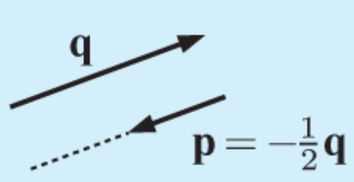
Suppose $\mathbf{q}$ is:



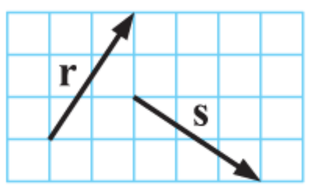
a



b



EXERCISE 12B.4

1 Given vectors 

, construct geometrically:

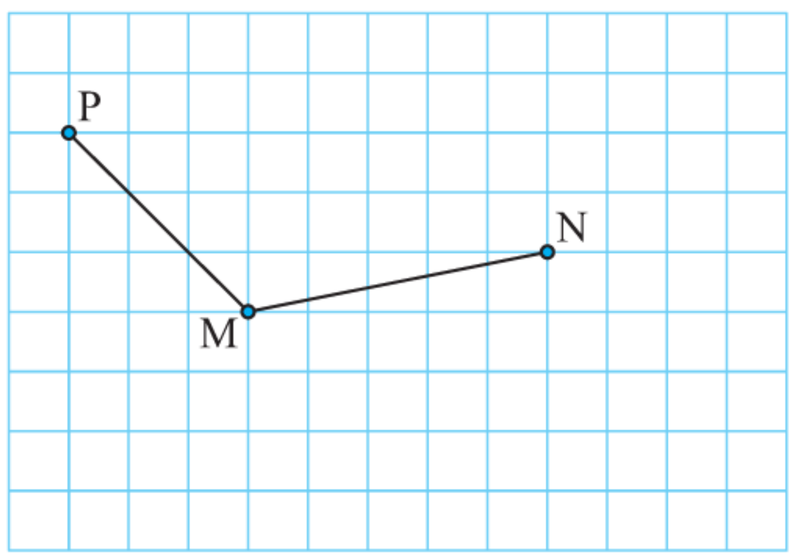
a $-\mathbf{r}$ **b** $2\mathbf{s}$ **c** $\frac{1}{2}\mathbf{r}$ **d** $-\frac{3}{2}\mathbf{s}$

e $2\mathbf{r} - \mathbf{s}$ **f** $2\mathbf{r} + 3\mathbf{s}$ **g** $\frac{1}{2}\mathbf{r} + 2\mathbf{s}$ **h** $\frac{1}{2}(\mathbf{r} + 3\mathbf{s})$

2 Sketch vectors $\mathbf{p}$ and $\mathbf{q}$ if:

a $\mathbf{p} = \mathbf{q}$ **b** $\mathbf{p} = -\mathbf{q}$ **c** $\mathbf{p} = 2\mathbf{q}$ **d** $\mathbf{p} = \frac{1}{3}\mathbf{q}$ **e** $\mathbf{p} = -3\mathbf{q}$

3 **a** Copy this diagram and on it mark the points:



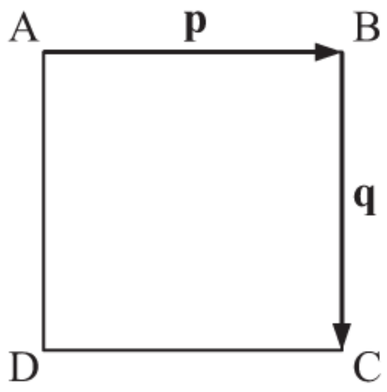
i X such that $\overrightarrow{MX} = \overrightarrow{MN} + \overrightarrow{MP}$

ii Y such that $\overrightarrow{MY} = \overrightarrow{MN} - \overrightarrow{MP}$

iii Z such that $\overrightarrow{PZ} = 2\overrightarrow{PM}$.

b What type of figure is MNYZ?

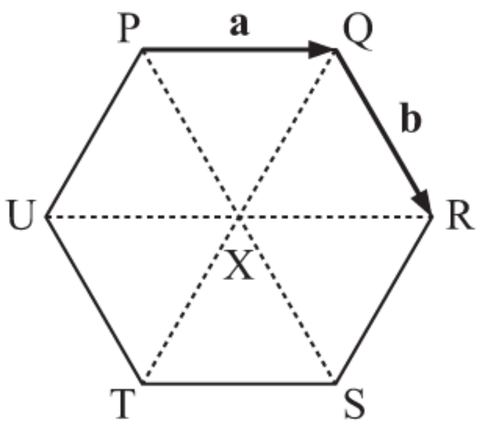
4



ABCD is a square. Its diagonals [AC] and [BD] intersect at M.
If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, find in terms of $\mathbf{p}$ and $\mathbf{q}$:

a $\overrightarrow{CD}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{AM}$ **d** $\overrightarrow{BM}$

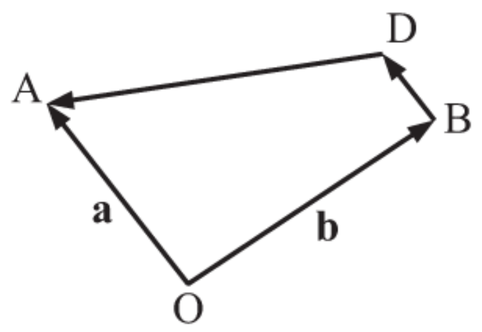
5



PQRSTU is a regular hexagon.
If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$, find in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{PX}$ **b** $\overrightarrow{PS}$ **c** $\overrightarrow{QX}$ **d** $\overrightarrow{RS}$

6



In the given figure [BD] is parallel to [OA] and half its length.
Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, vector expressions for:

a $\overrightarrow{BD}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$
d $\overrightarrow{OD}$ **e** $\overrightarrow{AD}$ **f** $\overrightarrow{DA}$

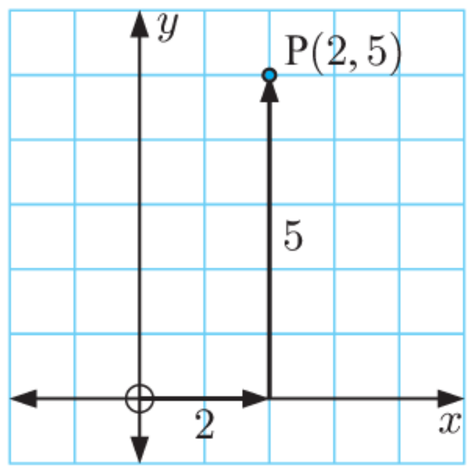
DISCUSSION

- What difficulties are there in using scale diagrams to solve vector problems?
- Can you extend your vector diagrams to higher dimensions?
- Can you draw the zero vector $\mathbf{0}$?

C VECTORS IN THE PLANE

When we plot points in the Cartesian plane, we move first in the x-direction and then in the y-direction.

For example, to plot the point $P(2, 5)$, we start at the origin, move 2 units in the x-direction, and then 5 units in the y-direction.

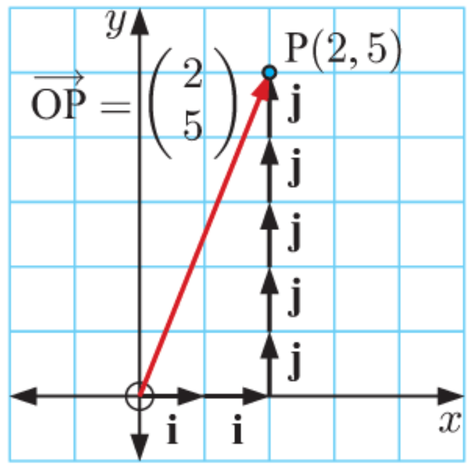


We have previously seen in transformation geometry that translating a point a units in the x-direction and b units in the y-direction can be achieved using the *translation vector* $\begin{pmatrix} a \\ b \end{pmatrix}$.

So, the vector from O to P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be a translation 1 unit in the positive x-direction

and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be a translation 1 unit in the positive y-direction.



Moving from O to P is equivalent to two lots of $\mathbf{i}$ plus 5 lots of $\mathbf{j}$.

$$\overrightarrow{OP} = 2\mathbf{i} + 5\mathbf{j}$$

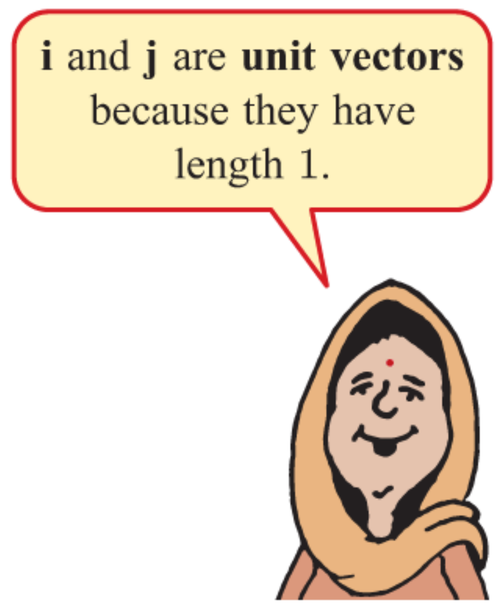
$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the 2-dimensional plane, the point $P(x, y)$ has **position vector**

$$\overrightarrow{OP} = \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{component form}} = \underbrace{x\mathbf{i} + y\mathbf{j}}_{\text{unit vector form}}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the **base unit vector** in the x-direction

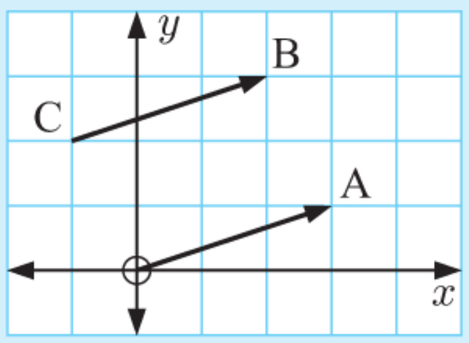
and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the **base unit vector** in the y-direction.



VECTOR EQUALITY

Two vectors are **equal** if their components are equal.

Example 11 **Self Tutor**



a Write vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ in component form and in unit vector form.

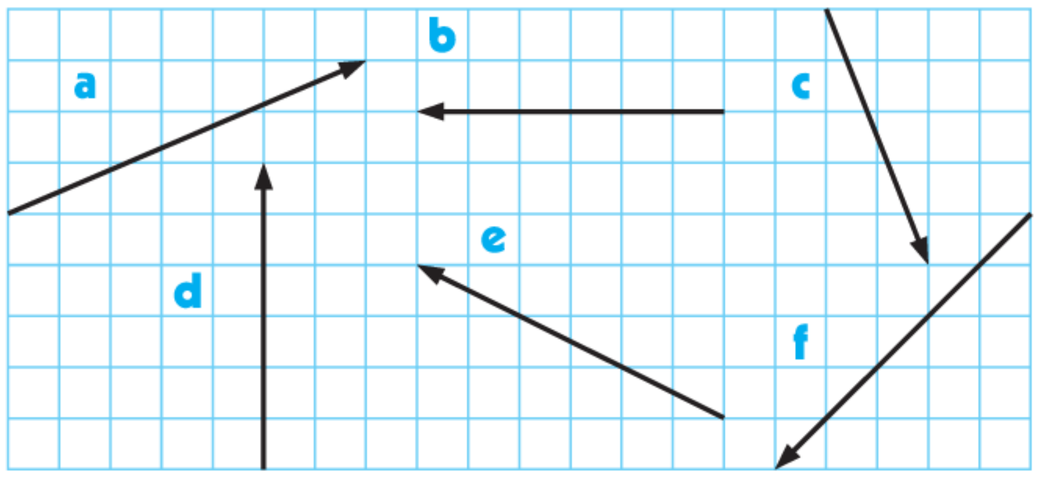
b Comment on your answers in **a**.

a $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$ $\qquad$ $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$

b The vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ are equal.

EXERCISE 12C

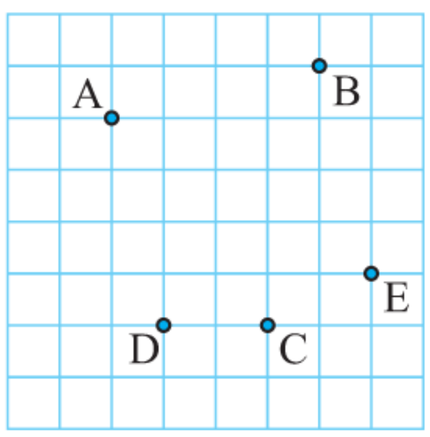
1 Write the illustrated vectors in component form and in unit vector form:



2 Write each vector in unit vector form, and illustrate it using an arrow diagram:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\qquad$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\qquad$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ $\qquad$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

3



a Write in component form and in unit vector form:

i $\overrightarrow{BA}$ $\qquad$ **ii** $\overrightarrow{BC}$ $\qquad$ **iii** $\overrightarrow{DC}$

iv $\overrightarrow{AC}$ $\qquad$ **v** $\overrightarrow{CA}$ $\qquad$ **vi** $\overrightarrow{DB}$

vii $\overrightarrow{AE}$ $\qquad$ **viii** $\overrightarrow{CE}$ $\qquad$ **ix** $\overrightarrow{ED}$

b Use the given points to describe as many pairs of equal vectors as you can.

4 Write in component form, and illustrate using a directed line segment:

a $\mathbf{i} + 2\mathbf{j}$ **b** $-\mathbf{i} + 3\mathbf{j}$ **c** $-5\mathbf{j}$ **d** $4\mathbf{i} - 2\mathbf{j}$

5 Write the zero vector $\mathbf{0}$ in component form.

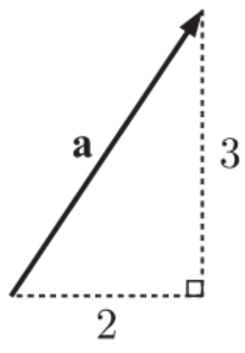
6 Find the unknowns if:

a $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ **b** $\begin{pmatrix} a+1 \\ 2b-8 \end{pmatrix} = \begin{pmatrix} 9-a \\ a \end{pmatrix}$ **c** $\begin{pmatrix} 2x+3y \\ x-2 \end{pmatrix} = \begin{pmatrix} 11 \\ 2y \end{pmatrix}$

D THE MAGNITUDE OF A VECTOR

Consider vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\mathbf{i} + 3\mathbf{j}$.

The **magnitude** or **length** of $\mathbf{a}$ is represented by $|\mathbf{a}|$.



By Pythagoras, $|\mathbf{a}|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$\therefore \quad |\mathbf{a}| = \sqrt{13}$ units $\quad \{\text{since } |\mathbf{a}| > 0\}$

The **magnitude** or **length** of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2}$.

Example 12 Self Tutor

If $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j}$ find: **a** $|\mathbf{p}|$ **b** $|\mathbf{q}|$

a $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\therefore \quad |\mathbf{p}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{34}$ units

b As $2\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$,

$|\mathbf{q}| = \sqrt{2^2 + (-5)^2}$

$= \sqrt{29}$ units

UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

For example, we have seen that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the base unit vectors in the positive x and y-directions respectively.

Example 13 Self Tutor

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector, $\sqrt{(-\frac{1}{3})^2 + k^2} = 1$

$\therefore \quad \sqrt{\frac{1}{9} + k^2} = 1$

$\therefore \quad \frac{1}{9} + k^2 = 1$

$\therefore \quad k^2 = \frac{8}{9}$

$\therefore \quad k = \pm\frac{\sqrt{8}}{3}$

EXERCISE 12D

1 Find the magnitude of:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2 Find the length of:

a $\mathbf{i} + \mathbf{j}$ **b** $5\mathbf{i} - 12\mathbf{j}$ **c** $-\mathbf{i} + 4\mathbf{j}$ **d** $3\mathbf{i}$ **e** $k\mathbf{j}$

3 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

4 Find k such that the given vector is a unit vector:

a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} k \\ k \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

5 Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$ units, find the possible values of p.

E OPERATIONS WITH PLANE VECTORS

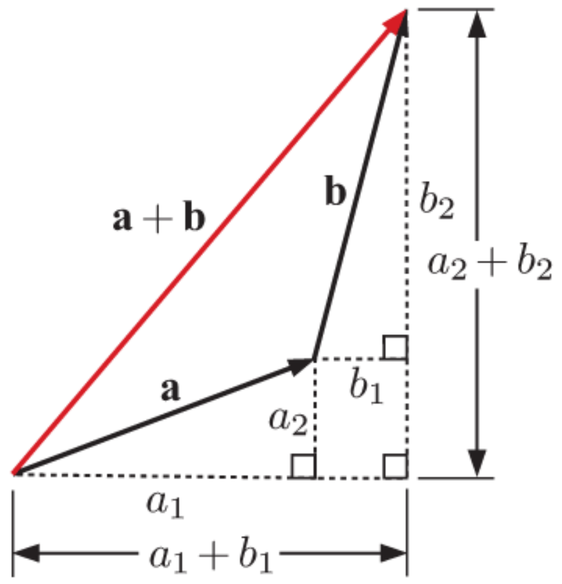
ALGEBRAIC VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Notice that:

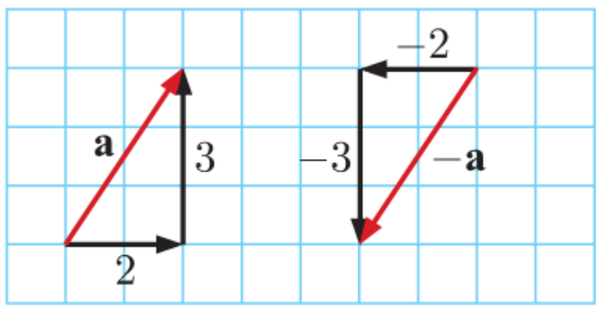
- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.



ALGEBRAIC NEGATIVE VECTORS

In the diagram we see the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and its negative $-\mathbf{a} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

ALGEBRAIC VECTOR SUBTRACTION

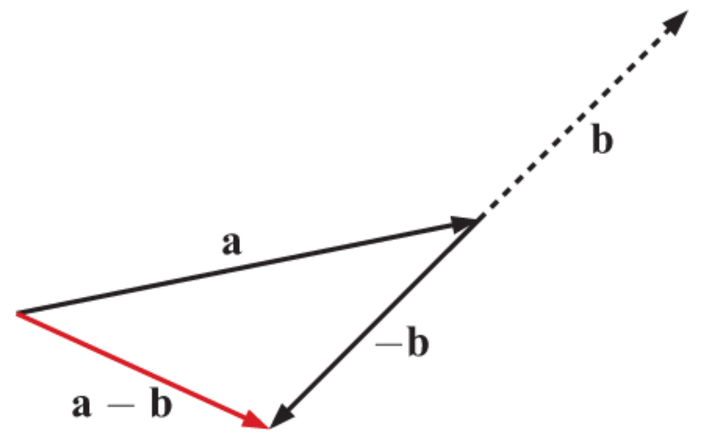
To subtract one vector from another, we simply **add its negative**.

Suppose $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$\therefore \quad \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 14 — Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$, find:

a $\mathbf{p} + \mathbf{r}$ **b** $\mathbf{q} - \mathbf{p}$ **c** $\mathbf{p} - \mathbf{q} - \mathbf{r}$

a $\mathbf{p} + \mathbf{r}$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + (-2) \\ -2 + (-5) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -7 \end{pmatrix}$$

b $\mathbf{q} - \mathbf{p}$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3 \\ 4 - (-2) \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

c $\mathbf{p} - \mathbf{q} - \mathbf{r}$

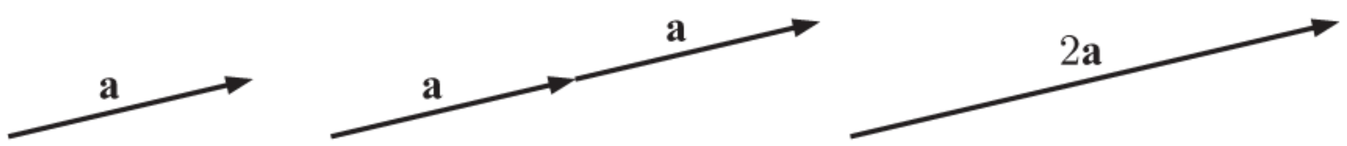
$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 1 - (-2) \\ -2 - 4 - (-5) \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

ALGEBRAIC SCALAR MULTIPLICATION

We have already seen a geometric approach for integer scalar multiplication:



Consider $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. $\quad 2\mathbf{a} = \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \times 1 \\ 2 \times 3 \end{pmatrix}$

$$3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \times 1 \\ 3 \times 3 \end{pmatrix}$$

If k is any scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 15 Self Tutor

For $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ find:

a $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$
$= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$

b $\mathbf{p} + 2\mathbf{q}$
$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix}$
$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$
$= \frac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix}$
$= \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$

EXERCISE 12E

1 If $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$

e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find:

a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$

d $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$ **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

a Use vector addition to show that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.

b Use vector subtraction to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

4 For $\mathbf{p} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$

e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

5 Consider $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a Find geometrically:

i $\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q}$ **ii** $\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q}$ **iii** $\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q}$

b Comment on your results in **a**.

6 Find k such that:

a $\begin{pmatrix} 2 \\ -5 \end{pmatrix} + k\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -11 \end{pmatrix}$ **b** $\frac{1}{3}\begin{pmatrix} -1 \\ k \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 7 \\ k+1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$

Example 16 **Self Tutor**

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$.

$$\begin{aligned}\mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j}) \\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j} \\ &= 5\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\therefore \ |\mathbf{p} - 2\mathbf{q}| = \sqrt{5^2 + (-1)^2} = \sqrt{26} \text{ units}$$

7 If $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{s} = -\mathbf{i} + 4\mathbf{j}$, find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|\mathbf{r} - \mathbf{s}|$

e $|\mathbf{s} - 2\mathbf{r}|$ **f** $|2\mathbf{r}|$ **g** $|\mathbf{r} + 2\mathbf{s}|$ **h** $|2\mathbf{r} - \mathbf{s}|$

8 **a** If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find:

i $|\mathbf{p}|$ **ii** $|2\mathbf{p}|$ **iii** $|-2\mathbf{p}|$ **iv** $|3\mathbf{p}|$ **v** $|-3\mathbf{p}|$

b By letting $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, prove that $|k\mathbf{v}| = |k|\,|\mathbf{v}|$.

9 Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Show by equating components, that if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$.

ACTIVITY **SKI RACER**

Click on the icon to practise your skills with vectors.



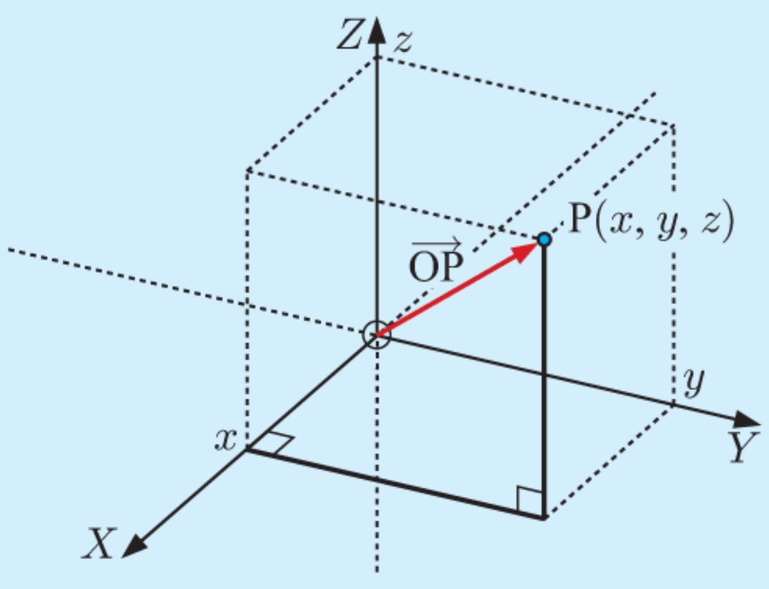
F VECTORS IN SPACE

In **Chapter 10** of the **Core Topics HL** book, we saw that we can specify a point in 3-dimensional space using 3 mutually perpendicular axes called the X, Y, and Z axes.

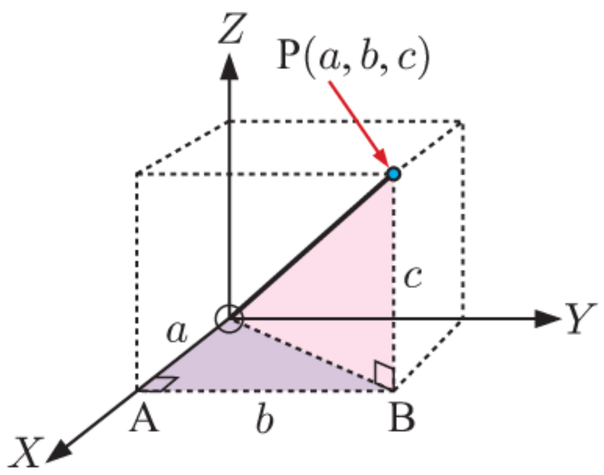
In 3-dimensional space, the **position vector** of $\mathrm{P}(x, y, z)$ is $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

are the **base unit vectors** in the X, Y, and Z directions respectively.



THE MAGNITUDE OF A VECTOR



Triangle OAB is right angled at A

$\therefore \quad \text{OB}^2 = a^2 + b^2 \quad \ldots. \ (1) \quad \{\text{Pythagoras}\}$

Triangle OBP is right angled at B

$\therefore \quad \text{OP}^2 = \text{OB}^2 + c^2 \quad \{\text{Pythagoras}\}$

$\therefore \quad \text{OP}^2 = a^2 + b^2 + c^2 \quad \{\text{using (1)}\}$

$\therefore \quad \text{OP} = \sqrt{a^2 + b^2 + c^2}$

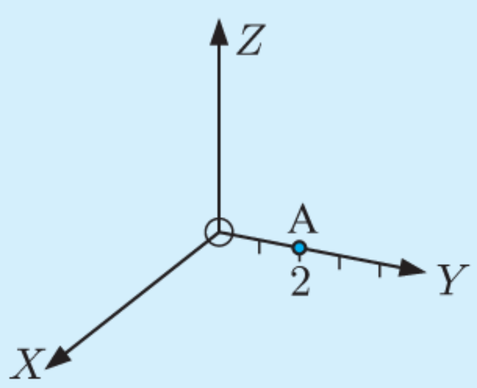
The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Example 17 **Self Tutor**

Illustrate each point and find its distance from the origin O:

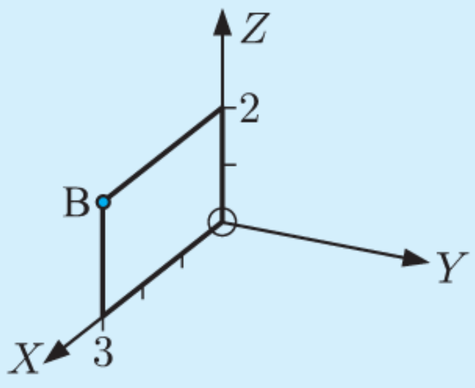
a A(0, 2, 0) **b** B(3, 0, 2) **c** C(−1, 2, 3)

a



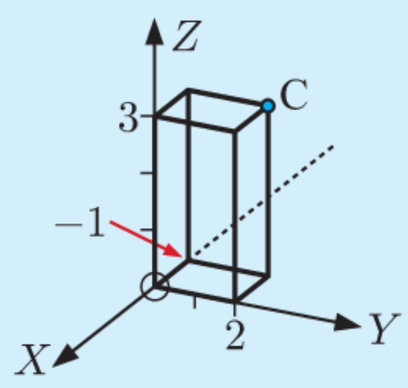
$|\overrightarrow{\text{OA}}|$
$= \sqrt{0^2 + 2^2 + 0^2}$
$= 2$ units

b



$|\overrightarrow{\text{OB}}|$
$= \sqrt{3^2 + 0^2 + 2^2}$
$= \sqrt{13}$ units

c



$|\overrightarrow{\text{OC}}|$
$= \sqrt{(-1)^2 + 2^2 + 3^2}$
$= \sqrt{14}$ units

EXERCISE 12F

1 Consider the point T(3, −1, 4).

a Draw a diagram to locate the position of T in space. **b** Find $\overrightarrow{\text{OT}}$.

c Find the distance from O to T.

2 For each point P, write $\overrightarrow{\text{OP}}$ in component form and unit vector form:

a P(2, 0, 4) **b** P(−3, 1, 2) **c** P(0, −2, 0)

3 Find the length of:

a $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

4 Illustrate P and find its distance from the origin O:

a P(0, 0, −3) **b** P(0, −1, 2) **c** P(3, 1, 4) **d** P(−1, −2, 3)

5 Write each vector in component form, and find its length:

a $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ **b** $\mathbf{b} = -3\mathbf{i} + 4\mathbf{k}$ **c** $\mathbf{c} = 2\mathbf{j} - 3\mathbf{k}$

6 Find a, b, and c if:

a $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ **b** $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$ **c** $\begin{pmatrix} 3a+4 \\ b-c \\ 4-5b \end{pmatrix} = \begin{pmatrix} 12-a \\ a-4 \\ 5a-22 \end{pmatrix}$

7 Which of the following are unit vectors?

a $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ **d** $\begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix}$



8 Find k given the unit vector:

a $\begin{pmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{pmatrix}$ **b** $\begin{pmatrix} k \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ **c** $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ **d** $\begin{pmatrix} \frac{1}{k} \\ \frac{2}{k} \\ \frac{3}{k} \end{pmatrix}$

9 Find the possible values of m, given that:

a $\begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix}$ has length $\sqrt{14}$ units **b** $\begin{pmatrix} 4 \\ -3 \\ m \end{pmatrix}$ has length 6 units.

10 Suppose $\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = 5$ units and $\left|\begin{pmatrix} b \\ c \end{pmatrix}\right| = 3$ units. Find the range of possible values of $\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right|$.

G OPERATIONS WITH VECTORS IN SPACE

For 3-dimensional vectors, it is difficult to illustrate operations. Instead, we perform operations using component form.

The rules for algebra with vectors readily extend from 2 dimensions to 3 dimensions:

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

- $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$
- $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$
- $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$
- $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ for any scalar k.

Example 18 Self Tutor

If $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - \mathbf{b}$ **c** $|\mathbf{a}|\mathbf{b}$

a
$$\mathbf{a}+\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1+3 \\ 3+0 \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

b
$$2\mathbf{a}-\mathbf{b} = 2\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-3 \\ 6-0 \\ 4-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 5 \end{pmatrix}$$

c
$$|\mathbf{a}|\mathbf{b} = \sqrt{(-1)^2+3^2+2^2}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \sqrt{14}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3\sqrt{14} \\ 0 \\ -\sqrt{14} \end{pmatrix}$$

EXERCISE 12G

1 Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{b} + 2\mathbf{c}$

d $\mathbf{c} - \frac{1}{2}\mathbf{a}$ **e** $\mathbf{a} - \mathbf{b} - \mathbf{c}$ **f** $2\mathbf{b} - \mathbf{c} + \mathbf{a}$

2 Let $\mathbf{a} = \mathbf{i} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find:

a $|\mathbf{a}|$ **b** $|\mathbf{b}|$ **c** $2|\mathbf{a}|$ **d** $|2\mathbf{a}|$

e $-3|\mathbf{b}|$ **f** $|-3\mathbf{b}|$ **g** $|\mathbf{a}+\mathbf{b}|$ **h** $|\mathbf{a}-\mathbf{b}|$

3 Let $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$. Find:

a $2\mathbf{a} - \mathbf{c}$ **b** $3\mathbf{b} + \frac{1}{2}\mathbf{c}$ **c** $-\mathbf{a} + 3\mathbf{b}$ **d** $\mathbf{a} - 5\mathbf{b} + 4\mathbf{c}$

e $|\mathbf{b}+\mathbf{c}|$ **f** $|\mathbf{a}-\mathbf{c}|$ **g** $|\mathbf{a}|\mathbf{b}$ **h** $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$

4 Find scalars a, b, and c such that:

a $2\begin{pmatrix} 1 \\ 0 \\ 3a \end{pmatrix} = \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix}$ **b** $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

c $a\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ c \\ 2 \end{pmatrix}$ **d** $2\begin{pmatrix} a \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix}$

H VECTOR ALGEBRA

INVESTIGATION 1 PROPERTIES OF VECTORS IN SPACE

There are several properties which are valid for the addition of real numbers. For example, we know that $a + b = b + a$. In this Investigation we identify similar properties of vectors.

What to do:

1 Use general vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ to prove that:

a $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ **b** $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

c $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ **d** $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

2 Prove that for scalar k and vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

a $|k\mathbf{a}| = |k|\,|\mathbf{a}|$ **b** $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

From the **Investigation** you should have found that for vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and scalar $k \in \mathbb{R}$:

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ {commutative property}
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {associative property}
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ {additive identity}
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ {additive inverse}
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ {distributive property}
- $|k\mathbf{a}| = |k|\,|\mathbf{a}|$ where $k\mathbf{a}$ is parallel to $\mathbf{a}$

($|k\mathbf{a}|$: length of $k\mathbf{a}$; $|k|$: absolute value of k; $|\mathbf{a}|$: length of $\mathbf{a}$)

The rules for solving vector equations are similar to those for solving real number equations. However, there is no such thing as dividing a vector by a scalar. Instead, we multiply by the reciprocal scalar.

For example, if $2\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{2}\mathbf{a}$ and *not* $\frac{\mathbf{a}}{2}$. $\frac{\mathbf{a}}{2}$ has no meaning in vector algebra.

Two useful rules are:

- If $\mathbf{x} + \mathbf{a} = \mathbf{b}$ then $\mathbf{x} = \mathbf{b} - \mathbf{a}$.
- If $k\mathbf{x} = \mathbf{a}$, $k \neq 0$, then $\mathbf{x} = \frac{1}{k}\mathbf{a}$.

Proof:

- If $\mathbf{x} + \mathbf{a} = \mathbf{b}$
  then $\mathbf{x} + \mathbf{a} + (-\mathbf{a}) = \mathbf{b} + (-\mathbf{a})$
  $\therefore \mathbf{x} + \mathbf{0} = \mathbf{b} - \mathbf{a}$
  $\therefore \mathbf{x} = \mathbf{b} - \mathbf{a}$
- If $k\mathbf{x} = \mathbf{a}$
  then $\frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{a}$
  $\therefore 1\mathbf{x} = \frac{1}{k}\mathbf{a}$
  $\therefore \mathbf{x} = \frac{1}{k}\mathbf{a}$

Example 19 — Self Tutor

Solve for $\mathbf{x}$:

a $3\mathbf{x} - \mathbf{r} = \mathbf{s}$ **b** $\mathbf{c} - 2\mathbf{x} = \mathbf{d}$

a
$3\mathbf{x} - \mathbf{r} = \mathbf{s}$
$\therefore\ 3\mathbf{x} = \mathbf{s} + \mathbf{r}$
$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{s} + \mathbf{r})$

b
$\mathbf{c} - 2\mathbf{x} = \mathbf{d}$
$\therefore\ \mathbf{c} - \mathbf{d} = 2\mathbf{x}$
$\therefore\ \frac{1}{2}(\mathbf{c} - \mathbf{d}) = \mathbf{x}$

EXERCISE 12H

1 Solve the following vector equations for $\mathbf{x}$:

a $2\mathbf{x} = \mathbf{q}$ **b** $\frac{1}{2}\mathbf{x} = \mathbf{n}$ **c** $-3\mathbf{x} = \mathbf{p}$

d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$ **e** $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$ **f** $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

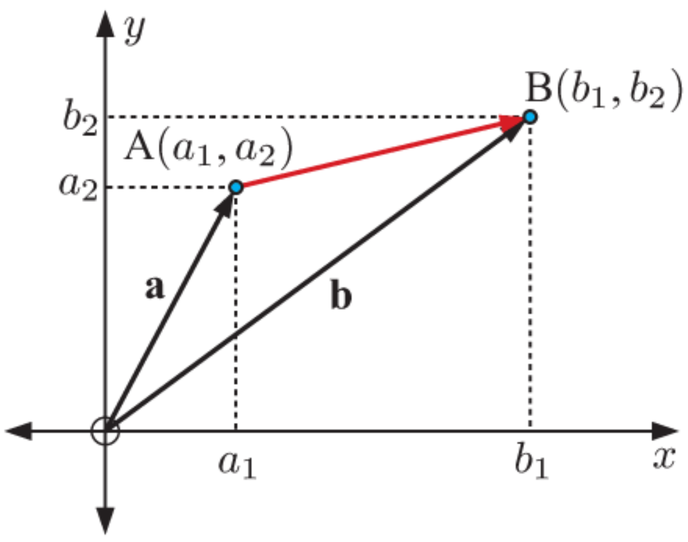
2 Suppose $\mathbf{p} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$. Find $\mathbf{x}$ such that:

a $2\mathbf{x} = \mathbf{p}$ **b** $\frac{1}{3}\mathbf{x} = \mathbf{p}$ **c** $4\mathbf{x} + \mathbf{p} = \mathbf{0}$

3 Suppose $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find $\mathbf{x}$ such that:

a $2\mathbf{a} + \mathbf{x} = \mathbf{b}$ **b** $3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$ **c** $2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$

I THE VECTOR BETWEEN TWO POINTS

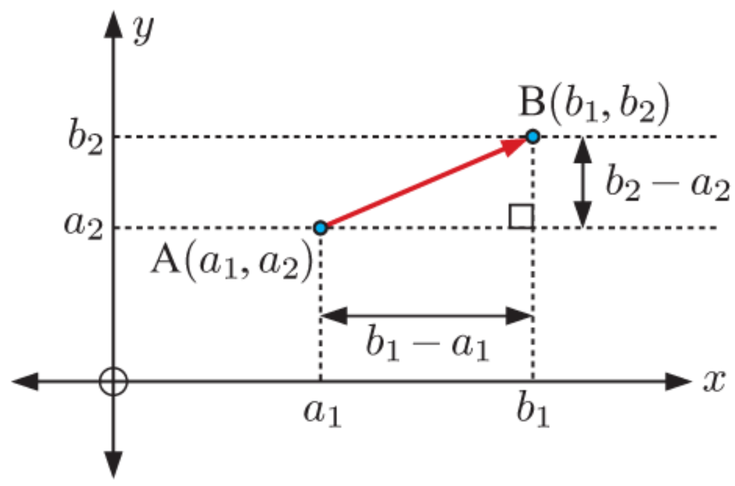


In the diagram, point A has position vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and point B has position vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\therefore\ \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$= \mathbf{b} - \mathbf{a}$$
$$= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

The **displacement vector of B relative to A** is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

Notice that $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ is the *translation vector* required to move point A to point B.



Example 20 — Self Tutor

Given points A$(-1, 2)$, B$(3, 4)$, and C$(4, -5)$, find the displacement vector of:

a B from O **b** B from A **c** A from C.

a The displacement vector of B relative to O is $\overrightarrow{OB} = \begin{pmatrix} 3-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b The displacement vector of B relative to A is $\overrightarrow{AB} = \begin{pmatrix} 3--1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c The displacement vector of A relative to C is $\overrightarrow{CA} = \begin{pmatrix} -1-4 \\ 2--5 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

We can define a similar result for vectors in space:

If A(a_1, a_2, a_3) and B(b_1, b_2, b_3) are two points in space with position vectors **a** and **b** respectively, then the **displacement vector of B relative to A** is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \begin{matrix} \leftarrow X\text{-step} \\ \leftarrow Y\text{-step} \\ \leftarrow Z\text{-step} \end{matrix}$$

Example 21 — Self Tutor

Given P$(-3, 1, 2)$ and Q$(1, -1, 3)$, find: **a** $\overrightarrow{OP}$ **b** $\overrightarrow{PQ}$ **c** $|\overrightarrow{PQ}|$

a $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} 1-(-3) \\ -1-1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

c $|\overrightarrow{PQ}| = \sqrt{4^2 + (-2)^2 + 1^2}$
$= \sqrt{21}$ units

EXERCISE 12I

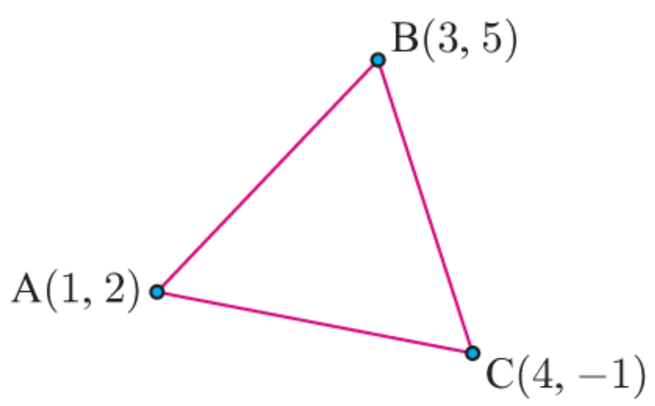
1 Find $\overrightarrow{AB}$ given:

a A$(2, 3)$ and B$(4, 7)$ **b** A$(3, -1)$ and B$(1, 4)$ **c** A$(-2, 7)$ and B$(1, 4)$.

2 Given $A(2, 3)$ and $B(6, 2)$, find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$ **c** $|\overrightarrow{AB}|$

3



a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Explain why $\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$.

c Hence find $\overrightarrow{BC}$.

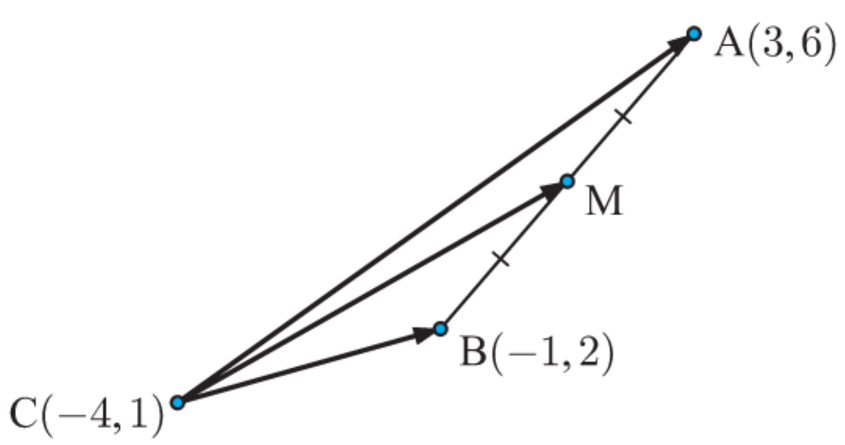
d Check your answer to **c** by direct evaluation.

4 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$.

b Given $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\overrightarrow{CB}$.

c Given $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

5



a Find the coordinates of M.

b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$, and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

6 Find $\overrightarrow{PQ}$ given:

a $P(1, 0, 2)$ and $Q(3, 5, 4)$ **b** $P(5, 2, 3)$ and $Q(6, 2, -1)$

c $P(-2, 3, 0)$ and $Q(1, 4, -3)$ **d** $P(4, -1, 5)$ and $Q(-1, -5, 3)$.

7 If $\overrightarrow{OA} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, find $\overrightarrow{AB}$ and hence the distance from A to B.

8 Given $A(-3, 1, 2)$ and $B(1, 0, -1)$, find:

a $\overrightarrow{AB}$ and $\overrightarrow{BA}$ **b** the lengths $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$.

9 Given $M(4, -2, -1)$ and $N(-1, 2, 0)$, find:

a the displacement vector of M relative to N **b** the displacement vector of N relative to M

c the distance between M and N.

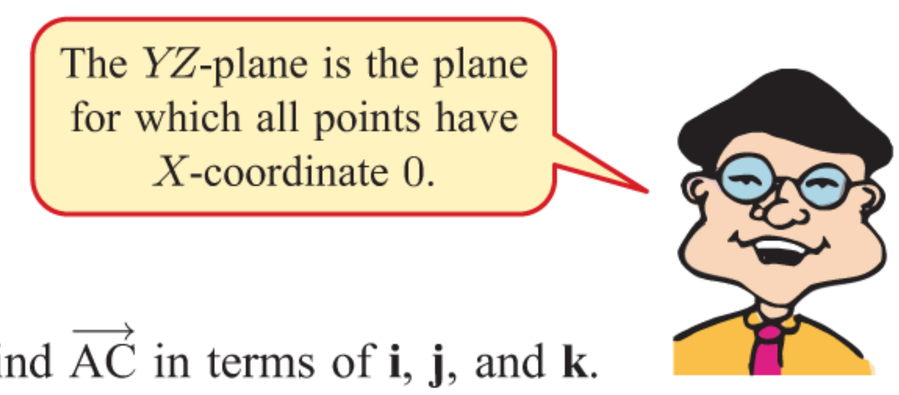
10 For $A(-1, 3, -2)$ and $B(3, -2, 1)$ find:

a $\overrightarrow{AB}$ in terms of **i**, **j**, and **k** **b** the magnitude of $\overrightarrow{AB}$.

11 Find the shortest distance from $Q(3, 1, -2)$ to:

a the Y-axis **b** the origin

c the YZ-plane.



12 If $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\overrightarrow{BC} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, find $\overrightarrow{AC}$ in terms of **i**, **j**, and **k**.

13 If $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, and $\overrightarrow{BD} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, find:

a $\overrightarrow{AD}$ **b** $\overrightarrow{CB}$ **c** $\overrightarrow{CD}$

14 Consider the points $A(-1, 3)$ and $B(3, k)$.

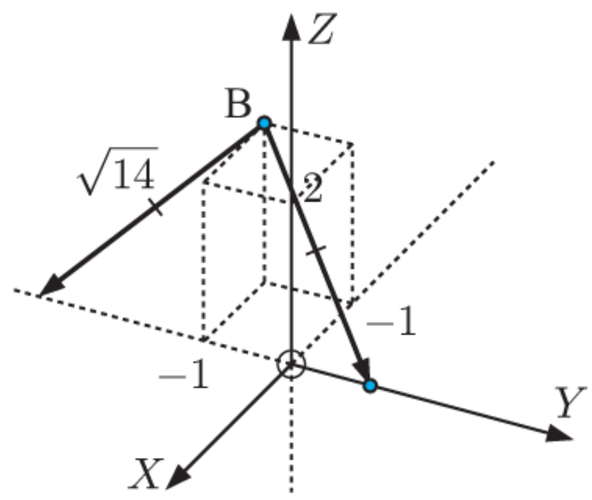
a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Given that A and B are 5 units apart, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

15 **a** State the coordinates of any general point A on the Y-axis.

b Hence find the coordinates of two points on the Y-axis which are $\sqrt{14}$ units from $B(-1, -1, 2)$.



Example 22 **Self Tutor**

[AB] is the diameter of a circle with centre $C(-1, 2)$. If B is $(3, 1)$, find:

a $\overrightarrow{BC}$ **b** the coordinates of A.

a $\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b If A has coordinates (a, b), then

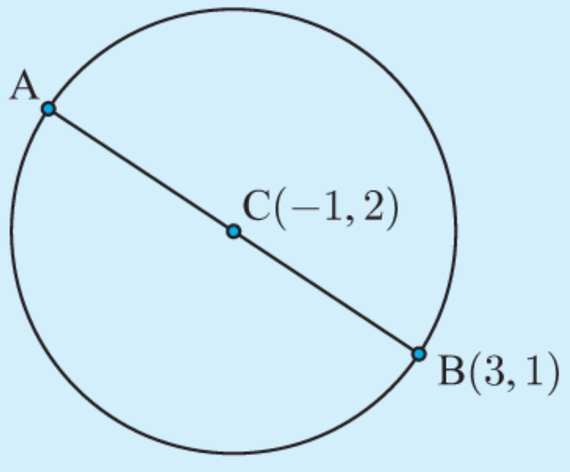
$\overrightarrow{CA} = \begin{pmatrix} a-(-1) \\ b-2 \end{pmatrix} = \begin{pmatrix} a+1 \\ b-2 \end{pmatrix}$

But $\overrightarrow{CA} = \overrightarrow{BC}$, so $\begin{pmatrix} a+1 \\ b-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore \quad a+1 = -4 \quad \text{and} \quad b-2 = 1$

$\therefore \quad a = -5 \quad \text{and} \quad b = 3$

$\therefore$ A is $(-5, 3)$.



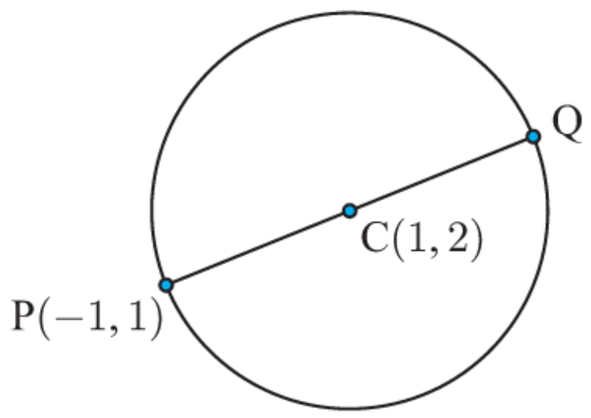
16 Consider the point $A(1, 4)$. Find the coordinates of:

a B given $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** C given $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

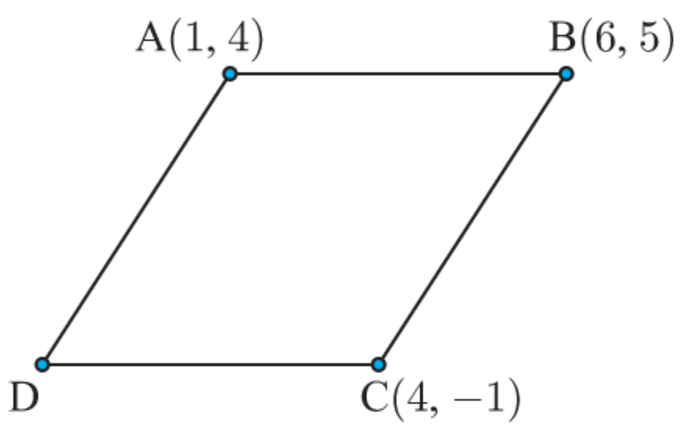
17 [PQ] is the diameter of a circle with centre C.

a Find $\overrightarrow{PC}$.

b Hence find the coordinates of Q.



18



ABCD is a parallelogram.

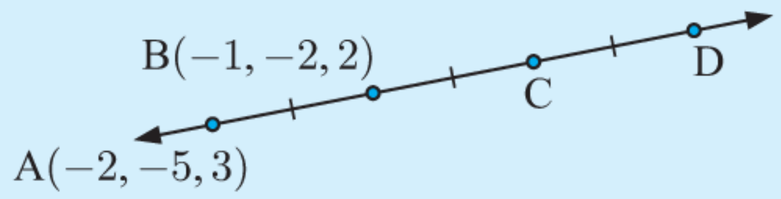
a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

Example 23 **Self Tutor**

Find the coordinates of C and D:



$$\overrightarrow{AB} = \begin{pmatrix} -1-(-2) \\ -2-(-5) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= \overrightarrow{OA} + 2\overrightarrow{AB}$$
$$= \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

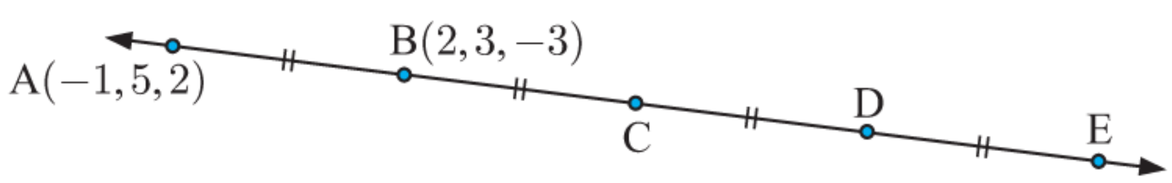
$\therefore$ C is $(0, 1, 1)$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$= \overrightarrow{OA} + 3\overrightarrow{AB}$$
$$= \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$\therefore$ D is $(1, 4, 0)$

19 Consider the points $A(2, 1, -2)$, $B(0, 3, -4)$, $C(1, -2, 1)$, and $D(-2, -3, 2)$.
Show that $\overrightarrow{BD} = 2\overrightarrow{AC}$.

20 Find the coordinates of C, D, and E.



21 A sphere has centre $C(-1, 2, 4)$ and diameter [AB] where A is $(-2, 1, 3)$.
Find the coordinates of B and the radius of the sphere.

22 Consider $A(-1, 2, 5)$, $B(2, 0, 3)$, and $C(-3, 1, 0)$.

a Find the displacement vector $\overrightarrow{AB}$ and its length AB.

b Find the displacement vector $\overrightarrow{AC}$ and its length AC.

c Find the displacement vector $\overrightarrow{CB}$ and its length CB.

d Hence classify triangle ABC.

23 Show that $P(0, 4, 4)$, $Q(2, 6, 5)$, and $R(1, 4, 3)$ are vertices of an isosceles triangle.

24 Use side lengths to classify triangle ABC given the coordinates:

a $A(0, 0, 3)$, $B(2, 8, 1)$, and $C(-9, 6, 18)$

b $A(1, 0, -3)$, $B(2, 2, 0)$, and $C(4, 6, 6)$.

25 The vertices of triangle ABC are $A(5, 6, -2)$, $B(6, 12, 9)$, and $C(2, 4, 2)$.

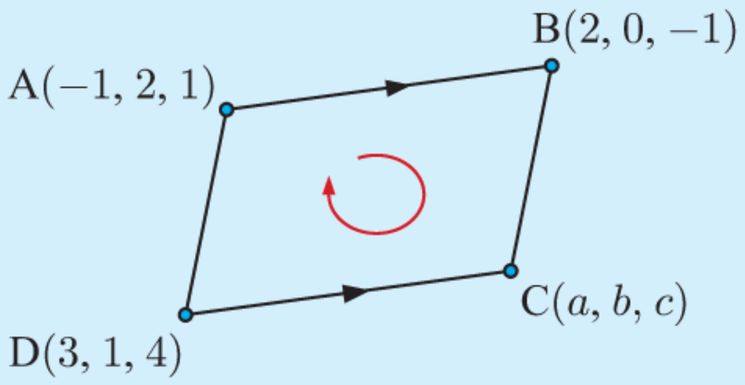
a Show that the triangle is right angled. **b** Hence find the area of the triangle.

Example 24 **Self Tutor**

ABCD is a parallelogram. A is $(-1, 2, 1)$, B is $(2, 0, -1)$, and D is $(3, 1, 4)$.
Find the coordinates of C.

Let C be (a, b, c).

[AB] is parallel to [DC], and they have the same length, so $\overrightarrow{DC} = \overrightarrow{AB}$



$$\therefore \begin{pmatrix} a-3 \\ b-1 \\ c-4 \end{pmatrix} = \begin{pmatrix} 2-(-1) \\ 0-2 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$\therefore\ a-3=3, \quad b-1=-2, \quad$ and $\quad c-4=-2$

$\therefore\ a=6, \quad b=-1, \quad$ and $\quad c=2$

So, C is $(6, -1, 2)$.

Check: The midpoint of [DB] is $\left(\frac{3+2}{2}, \frac{1+0}{2}, \frac{4+-1}{2}\right)$ which is $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

The midpoint of [AC] is $\left(\frac{-1+6}{2}, \frac{2+-1}{2}, \frac{1+2}{2}\right)$ which is $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

The midpoints are the same, so the diagonals of the parallelogram bisect. ✓

26 $A(-1, 3, 4)$, $B(2, 5, -1)$, $C(-1, 2, -2)$, and $D(r, s, t)$ are four points in space. Find r, s, and t if:

a $\overrightarrow{AC} = \overrightarrow{BD}$ **b** $\overrightarrow{AB} = \overrightarrow{DC}$

27 A quadrilateral has vertices $A(1, 2, 3)$, $B(3, -3, 2)$, $C(7, -4, 5)$, and $D(5, 1, 6)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

b What can be deduced about the quadrilateral ABCD?

28 PQRS is a parallelogram. P is $(-1, 2, 3)$, Q is $(1, -2, 5)$, and R is $(0, 4, -1)$.

a Use vectors to find the coordinates of S.

b Use the midpoints of the diagonals to check your answer.

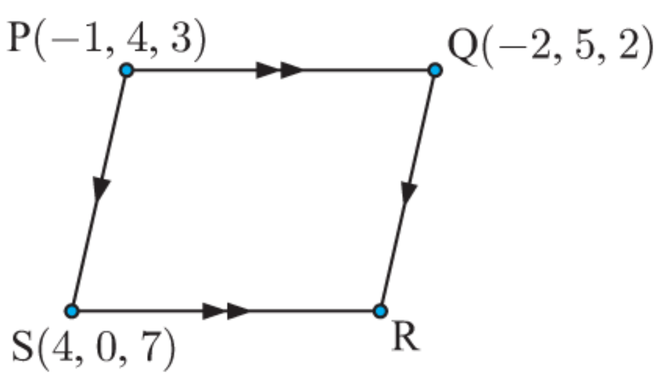
29 Use vectors to determine whether ABCD is a parallelogram:

a $A(5, 0, 3)$, $B(-1, 2, 4)$, $C(4, -3, 6)$, and $D(10, -5, 5)$

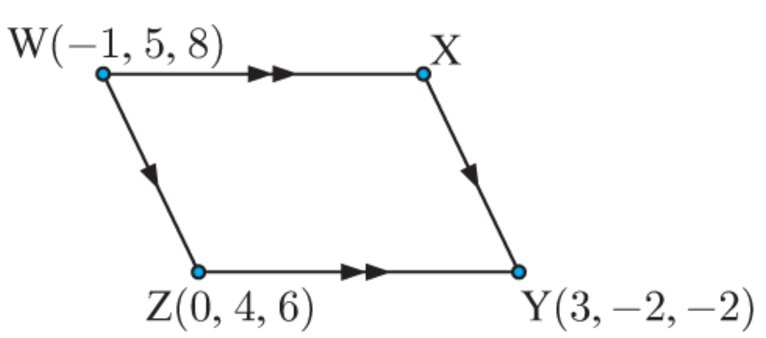
b $A(2, -3, 2)$, $B(1, 4, -1)$, $C(-2, 6, -2)$, and $D(-1, -1, 2)$.

30 Use vector methods to find the remaining vertex of:

a



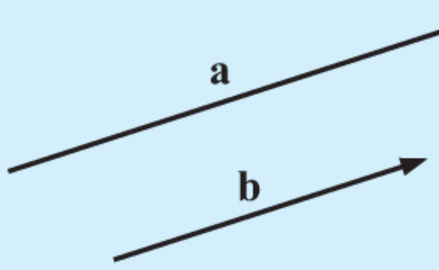
b



J PARALLELISM

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{v}$ and non-zero scalar k, the vector $k\mathbf{v}$ is parallel to $\mathbf{v}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k||\mathbf{b}|$.

$|k|$ is the absolute value of k, whereas $|\mathbf{a}|$ is the length of vector $\mathbf{a}$.



Example 25 **Self Tutor**

Find r and s given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} = k\mathbf{b}$ for some scalar k.

$$\therefore \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix} = k \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$$

$\therefore \ 2 = ks, \ -1 = 2k,$ and $r = -3k$

Consequently, $k = -\frac{1}{2}$

$\therefore \ 2 = -\frac{1}{2}s$ and $r = -3(-\frac{1}{2})$

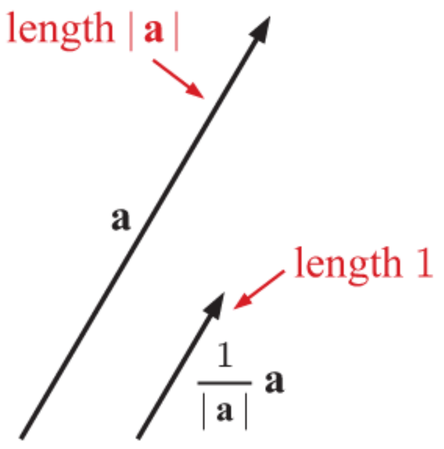
$\therefore \ s = -4$ and $r = \frac{3}{2}$

UNIT VECTORS

Given a non-zero vector $\mathbf{a}$, its magnitude $|\mathbf{a}|$ is a scalar quantity.

If we multiply $\mathbf{a}$ by the scalar $\frac{1}{|\mathbf{a}|}$, we obtain the parallel vector $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

The length of this vector is $\left|\frac{1}{|\mathbf{a}|}\right| |\mathbf{a}| = \frac{|\mathbf{a}|}{|\mathbf{a}|} = 1$, so $\frac{1}{|\mathbf{a}|}\mathbf{a}$ is a unit vector in the direction of $\mathbf{a}$. We write this unit vector as $\widehat{\mathbf{a}}$.



- A unit vector in the direction of $\mathbf{a}$ is $\widehat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = k\widehat{\mathbf{a}} = \frac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k which is *parallel to* $\mathbf{a}$ could be $\mathbf{b} = \pm k\widehat{\mathbf{a}} = \pm\frac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 26 — Self Tutor

Find a vector **b** of length 7 units in the opposite direction to $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

The unit vector in the direction of **a** is

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{\sqrt{4+1+1}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

$$\therefore \quad \mathbf{b} = -\frac{7}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

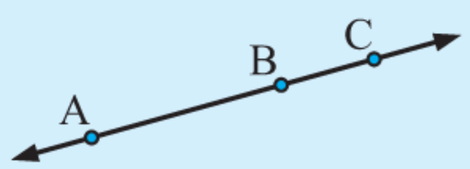
We multiply the unit vector by -7. The negative reverses the direction. The 7 gives the required length.



COLLINEAR POINTS

Three or more points are said to be **collinear** if they lie on the same straight line.

A, B, and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.



Example 27 — Self Tutor

Prove that A$(-1, 2, 3)$, B$(4, 0, -1)$, and C$(14, -4, -9)$ are collinear.

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 10 \\ -4 \\ -8 \end{pmatrix} = 2\begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$$

$\therefore \quad \overrightarrow{BC} = 2\overrightarrow{AB}$

$\therefore$ [BC] is parallel to [AB].

Since B is common to both, A, B, and C are collinear.

EXERCISE 12J

1 Determine whether each pair of vectors is parallel:

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 15 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ -12 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

d $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$ **e** $\begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$ **f** $\begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$

2 Find a such that:

a $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ a \end{pmatrix}$ are parallel **b** $\begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are parallel.

3 Find r and s given that $\mathbf{a}$ and $\mathbf{b}$ are parallel:

a $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ r \\ s \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ 2 \\ s \end{pmatrix}$

4 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

5 The position vectors of P, Q, R, and S are $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ respectively.

Show that [PR] and [QS] are parallel, and state the ratio PR : QS.

6 If $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, write down the vector:

a in the same direction as $\mathbf{a}$ and with twice its length

b in the opposite direction to $\mathbf{a}$ and with half its length.

7 Find the 2-dimensional unit vector in the direction:

a $\mathbf{i} + 2\mathbf{j}$ **b** $2\mathbf{i} - 3\mathbf{j}$ **c** $-2\mathbf{i} + 2\mathbf{j}$

8 Find the 3-dimensional unit vector in the direction:

a $2\mathbf{i} + 3\mathbf{k}$ **b** $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ **c** $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

9 Find a vector $\mathbf{v}$ which has:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and with length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and with length 2 units

c the same direction as $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and with length 6 units

d the opposite direction to $-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and with length 5 units.

10 **a** Find the vectors of length 1 unit which are parallel to $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

b Find vectors of length 2 units which are parallel to $\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

11 A is $(3, 2)$ and point B is 4 units from A in the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a Find $\overrightarrow{AB}$. **b** Find $\overrightarrow{OB}$ using $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

c Hence deduce the coordinates of B.

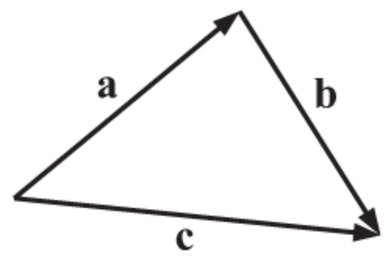
12 Find the coordinates of point X which is:

a 5 units from $(1, -5)$ in the direction $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b 6 units from $(1, 3, -2)$ in the direction $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$

c 4 units from $(2, -1, 4)$ in the direction $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

13 The **triangle inequality** states that:
"In any triangle, the sum of any two sides must always be greater than the third side."
Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$ using a geometrical argument.



Hint: Consider:
- **a** not parallel to **b** and use the triangle inequality
- **a** and **b** parallel
- any other cases.

14 Show that the following sets of points are collinear:

a $A(-1, 1)$, $B(4, 6)$, $C(1, 3)$

b $P(3, 3)$, $Q(6, -3)$, $R(1, 7)$

c $A(-2, 1, 4)$, $B(4, 3, 0)$, $C(19, 8, -10)$

d $P(2, 1, 1)$, $Q(5, -5, -2)$, $R(-1, 7, 4)$

15 Find a and b if:

a $A(2, -3, 4)$, $B(11, -9, 7)$, and $C(-13, a, b)$ are collinear

b $K(1, -1, 0)$, $L(4, -3, 7)$, and $M(a, 2, b)$ are collinear.

16 Let $P(x, y, z)$ be a point on a sphere with radius r units and centre $C(X, Y, Z)$.

a Write down the vector $\overrightarrow{CP}$.

b Use the distance formula for CP to write a Cartesian equation for the sphere.

INVESTIGATION 2 LINEAR COMBINATIONS

Any vector $\begin{pmatrix} r \\ s \end{pmatrix}$ in the plane can be written in the form $r\mathbf{i} + s\mathbf{j}$. We say that $\begin{pmatrix} r \\ s \end{pmatrix}$ is a **linear combination** of **i** and **j**.



Click on the icon to explore linear combinations further.

K THE SCALAR PRODUCT OF TWO VECTORS

For ordinary numbers a and b we can write the product of a and b as ab or $a \times b$. There is only one interpretation for this product, so we can use power notation $a^2 = a \times a$, $a^3 = a \times a \times a$, and so on as shorthand.

However, there are *two* different types of product involving two vectors. These are:

- The **scalar product** of 2 vectors, which results in a **scalar** answer and has the notation $\mathbf{a} \bullet \mathbf{b}$ (read "**a** dot **b**").
- The **vector product** of 2 vectors, which results in a **vector** answer and has the notation $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**").

Consequently, for vector $\mathbf{a}$, we do not use the terms $\mathbf{a}^2$ or $(\mathbf{a})^2$, as it is not clear which of the vector products it refers to.

For 2-dimensional vectors in a plane, we only need to talk about scalar product. The vector cross product will be important for 3-dimensional vectors in space.

SCALAR PRODUCT

The **scalar product** of two vectors is also known as the **dot product** or **inner product**.

- $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2$
- $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

Be careful not to confuse the **scalar product**, which is the product of two vectors to give a scalar answer, with **scalar multiplication**, which is the product of a scalar and a vector to give a parallel vector. They are quite different.

Example 28 Self Tutor

Find: **a** $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

a $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
$= 2 \times 1 + 3 \times 4$
$= 14$

b $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$
$= -1 \times 3 + 2 \times 1 + 4 \times 2$
$= 7$

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

1 $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

2 $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$

3 $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

4 $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

EXERCISE 12K

1 Find:

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ **d** $\begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix}$

2 For $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find:

a $\mathbf{q} \bullet \mathbf{p}$ **b** $\mathbf{q} \bullet \mathbf{r}$ **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$ **d** $3\mathbf{r} \bullet \mathbf{q}$

e $2\mathbf{p} \bullet 2\mathbf{p}$ **f** $\mathbf{i} \bullet \mathbf{p}$ **g** $\mathbf{q} \bullet \mathbf{j}$ **h** $\mathbf{i} \bullet \mathbf{i}$

3 For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** $\mathbf{b} \bullet \mathbf{a}$ **c** $|\mathbf{a}|^2$

d $\mathbf{a} \bullet \mathbf{a}$ **e** $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$ **f** $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

4 Use the general vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to prove that:

a $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ **b** $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$

5 Find:

a $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$ **b** $\mathbf{i} \bullet \mathbf{i}$ **c** $(3\mathbf{i} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j})$

d $(2\mathbf{i} + \mathbf{k}) \bullet (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ **e** $\mathbf{i} \bullet \mathbf{j}$ **f** $(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \bullet (3\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$

6 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$.

a Find:

i $\mathbf{a} \bullet \mathbf{c}$ **ii** $\mathbf{a} \bullet \mathbf{d}$ **iii** $\mathbf{b} \bullet \mathbf{c}$ **iv** $\mathbf{b} \bullet \mathbf{d}$

b Hence show that $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$.

7 Use $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ to prove that $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$.

Hence prove that $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$.

8 Use the properties of the scalar product to show that:

a $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a} \bullet \mathbf{b} + |\mathbf{b}|^2$ **b** $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$

c $|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4\,\mathbf{a} \bullet \mathbf{b}$

9 Explain why $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is meaningless.

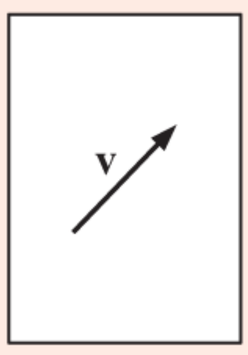
10 If $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 3$, find $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$.

11 Show that if vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$.

L THE ANGLE BETWEEN TWO VECTORS

DISCUSSION

1

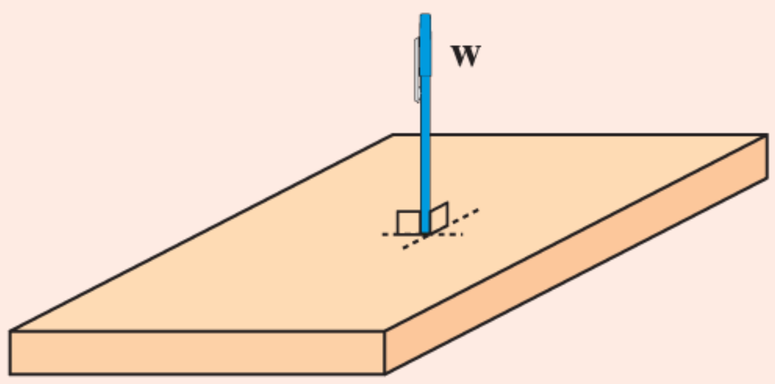


Elaine has drawn a vector $\mathbf{v}$ on a *plane* which is a sheet of paper. It is therefore a 2-dimensional vector.

a How many vectors can she draw which are *perpendicular* to $\mathbf{v}$?

b Are all of these vectors parallel?

2 Edward is thinking about vectors in *space*. These are 3-dimensional vectors. He is holding his pen vertically on his desk to represent a vector **w**.



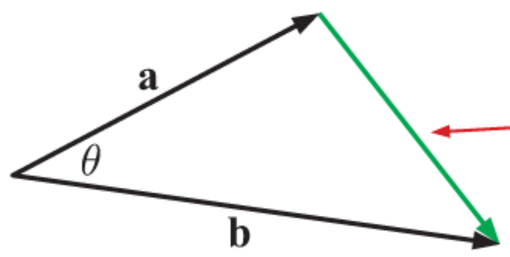
a How many vectors are there which are *perpendicular* to **w**?

b If Edward was to draw a vector (in pencil) on his desk, would it be perpendicular to **w**?

c Are all of the vectors which are perpendicular to **w**, parallel to one another?

THE ANGLE BETWEEN TWO VECTORS

Suppose θ is the angle between the vectors **a** and **b**.



This vector is $-\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$.
Its length is $|\mathbf{b} - \mathbf{a}|$.

Using the cosine rule, $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

Letting $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\therefore \quad (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\therefore \quad 2a_1b_1 + 2a_2b_2 + 2a_3b_3 = 2|\mathbf{a}||\mathbf{b}|\cos\theta \qquad \{\text{simplifying}\}$$

$$\therefore \quad a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\therefore \quad \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

The angle θ between two vectors **a** and **b** can be found using $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Example 29 — Self Tutor

If $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, find the angle between **p** and **q**.

$$\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$
$$= 2(-1) + 3(0) + (-1)2$$
$$= -2 + 0 - 2$$
$$= -4$$

Now $\cos\theta = \dfrac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

$$= \frac{-4}{\sqrt{4+9+1}\sqrt{1+0+4}}$$

$$= \frac{-4}{\sqrt{70}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right) \approx 119^\circ$$

GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

Using the formula $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we notice that:

► If $\mathbf{a}$ is perpendicular to $\mathbf{b}$ then $\theta = 90°$.

$$\therefore \quad \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$= |\mathbf{a}||\mathbf{b}|\cos 90°$$
$$= 0$$

► If $\mathbf{a}$ is parallel to $\mathbf{b}$ then $\theta = 0°$ or $180°$.

$$\therefore \quad \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$= |\mathbf{a}||\mathbf{b}|\cos 0° \quad \text{or} \quad |\mathbf{a}||\mathbf{b}|\cos 180°$$
$$= \pm|\mathbf{a}||\mathbf{b}|$$
$$\therefore \quad |\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$$

We are hence able to state the following **geometric properties of the scalar product**:

For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$:

1 $\mathbf{a} \bullet \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{a}$ and $\mathbf{b}$ are **perpendicular**.

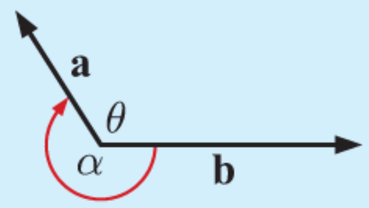
2 $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \quad \Leftrightarrow \quad \mathbf{a}$ and $\mathbf{b}$ are **parallel vectors**.

3 If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

If θ is acute, $\cos\theta > 0$ and so $\mathbf{a} \bullet \mathbf{b} > 0$

If θ is obtuse, $\cos\theta < 0$ and so $\mathbf{a} \bullet \mathbf{b} < 0$.

The angle between two vectors is always taken as the angle θ such that $0° \leqslant \theta \leqslant 180°$, rather than reflex angle α.



EXERCISE 12L

1 If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** the angle between $\mathbf{a}$ and $\mathbf{b}$.

2 Find the angle between $\mathbf{r}$ and $\mathbf{s}$, given:

a $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

c $\mathbf{r} = \mathbf{i} - \mathbf{j}$ and $\mathbf{s} = 2\mathbf{i} - \mathbf{j}$ **d** $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{i} + \mathbf{j}$

3 Show that $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are perpendicular.

4 Show that $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ are mutually perpendicular.

Vectors are *mutually perpendicular* if each one is perpendicular to all the others.



5 Find the angle between:

a $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

c $2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 2\mathbf{k}$

d $\mathbf{j}$ and $-\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

6 Find t given that the following pairs of vectors are perpendicular:

a $\mathbf{p} = \begin{pmatrix} 3 \\ t \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

c $\mathbf{a} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2-3t \\ t \end{pmatrix}$

d $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix}$

e $\mathbf{p} = \begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix}$

f $\mathbf{a} = 2\mathbf{i} + t\mathbf{j} + (t-2)\mathbf{k}$, $\mathbf{b} = t\mathbf{i} + 3\mathbf{j} + t\mathbf{k}$

7 Find r, s, and t such that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular.

Example 30 — Self Tutor

Find a vector of length 3 units which is perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 1 \end{pmatrix} = -4 + 4 = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ are perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Now $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ has length $\sqrt{16+1} = \sqrt{17}$ units

$\therefore$ $k = \frac{1}{\sqrt{17}}$ for a unit vector.

$\therefore$ a vector of length 3 units is $\frac{3}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, which is $\begin{pmatrix} \frac{12}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \end{pmatrix}$.

8 Find a vector of length:

a 10 units which is perpendicular to $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

b $3\sqrt{2}$ units which is perpendicular to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\sqrt{20}$ units which is perpendicular to $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

9 Find *two* vectors of length 5 units which are perpendicular to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Example 31 **Self Tutor**

$\mathbf{a}$ and $\mathbf{b}$ have lengths 5 units and $\sqrt{7}$ units respectively, and the angle between them is $110°$. Find $\mathbf{a} \bullet \mathbf{b}$.

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\ &= 5 \times \sqrt{7} \times \cos 110° \\ &\approx -4.525\end{aligned}$$

10 $\mathbf{a}$ and $\mathbf{b}$ have lengths $\sqrt{10}$ units and 3 units respectively and the angle between them is $71°$. Find $\mathbf{a} \bullet \mathbf{b}$.

11 Find $\mathbf{p} \bullet \mathbf{q}$ for:

a $|\mathbf{p}| = 2$, $|\mathbf{q}| = 5$, and $\theta = 60°$ **b** $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$, and $\theta = \frac{2\pi}{3}$

12 Prove that two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \bullet \mathbf{b} = 0$.

Hint: You need to show that $\mathbf{a}$ and $\mathbf{b}$ being perpendicular implies $\mathbf{a} \bullet \mathbf{b} = 0$ *and* that $\mathbf{a} \bullet \mathbf{b} = 0$ implies $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

13 Prove that if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \Leftrightarrow \mathbf{a}$ and $\mathbf{b}$ are parallel.

14 Suppose $\mathbf{a} \bullet \mathbf{b} = -12$ and $\mathbf{b}$ is a unit vector.

a Explain why $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular.

b Given that $\mathbf{a}$ and $\mathbf{b}$ are parallel, find $|\mathbf{a}|$.

15 Suppose $|\mathbf{c}| = |\mathbf{d}| = \sqrt{5}$. What can be deduced about $\mathbf{c}$ and $\mathbf{d}$ if:

a $\mathbf{c} \bullet \mathbf{d} = 5$ **b** $\mathbf{c} \bullet \mathbf{d} = -5$?

16 Find three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in space such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ but $\mathbf{b} \neq \mathbf{c}$.

17 Find any two vectors which are not parallel, but which are *both* perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

18 Consider triangle ABC in which A is $(5, 1, 2)$, B is $(6, -1, 0)$, and C is $(3, 2, 0)$. Using scalar product only, show that the triangle is right angled.

19 $A(2, 4, 2)$, $B(-1, 2, 3)$, $C(-3, 3, 6)$, and $D(0, 5, 5)$ are vertices of a quadrilateral.

a Prove that ABCD is a parallelogram.

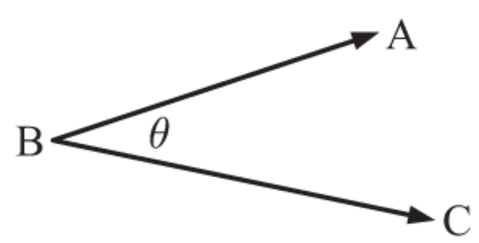
b Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$. What can be said about ABCD?

c Find $\overrightarrow{AC} \bullet \overrightarrow{BD}$ and discuss your answer.

20 Find the angle ABC of triangle ABC for $A(3, 0, 1)$, $B(-3, 1, 2)$, and $C(-2, 1, -1)$.

Hint: To find the angle at B, use $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

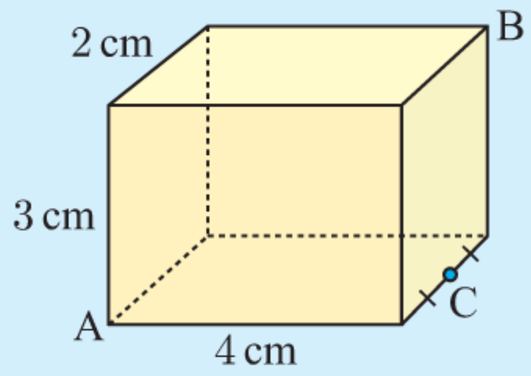
What angle is found if $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used?



Example 32

Self Tutor

Use vector methods to determine the measure of $A\hat{B}C$.



Placing the coordinate axes as illustrated, A is $(2, 0, 0)$, B is $(0, 4, 3)$, and C is $(1, 4, 0)$.

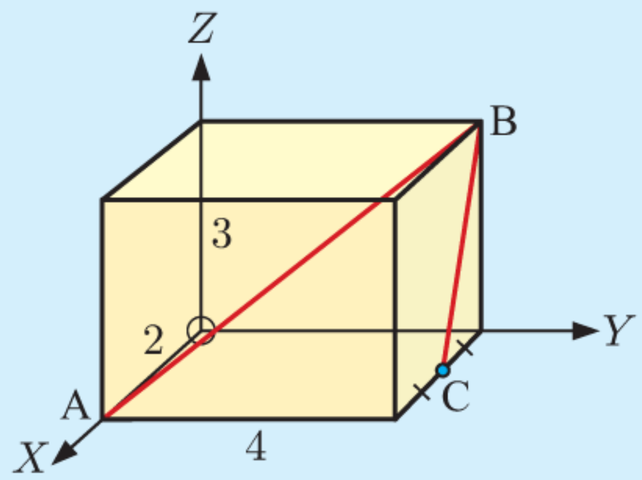
$$\therefore \quad \overrightarrow{BA} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{Now} \quad \cos(A\hat{B}C) = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}| \ |\overrightarrow{BC}|}$$

$$= \frac{2(1) + (-4)(0) + (-3)(-3)}{\sqrt{4+16+9}\sqrt{1+0+9}}$$

$$= \frac{11}{\sqrt{290}}$$

$$\therefore \quad A\hat{B}C = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.8°$$

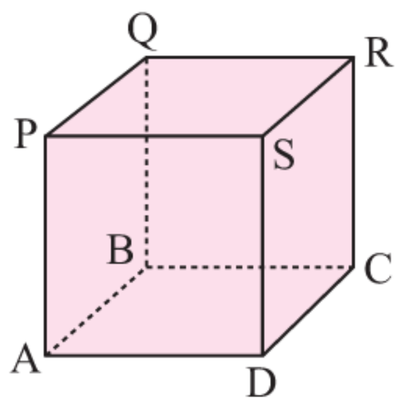


The vectors used must both be away from B (or towards B). If this is not done you will be finding the exterior angle at B.



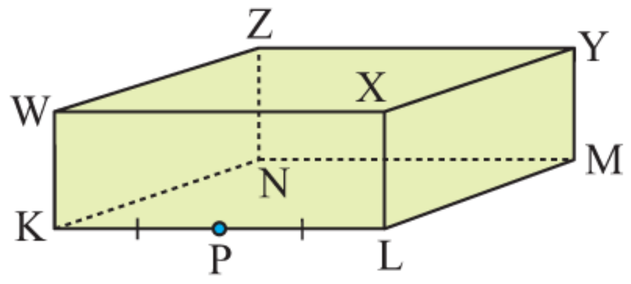
21 The cube alongside has edges of length 2 cm. Find, using vector methods, the measure of:

a $A\hat{B}S$ **b** $R\hat{B}P$ **c** $P\hat{B}S$



22 [KL], [LM], and [LX] are 8, 5, and 3 units long respectively. P is the midpoint of [KL]. Find, using vector methods, the measure of:

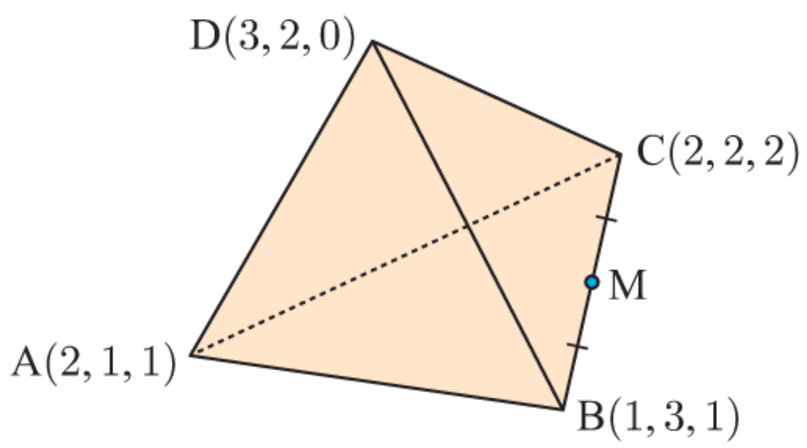
a $Y\hat{N}X$ **b** $Y\hat{N}P$



23 Consider the tetrahedron ABCD.

a Find the coordinates of M.

b Find the measure of $D\hat{M}A$.

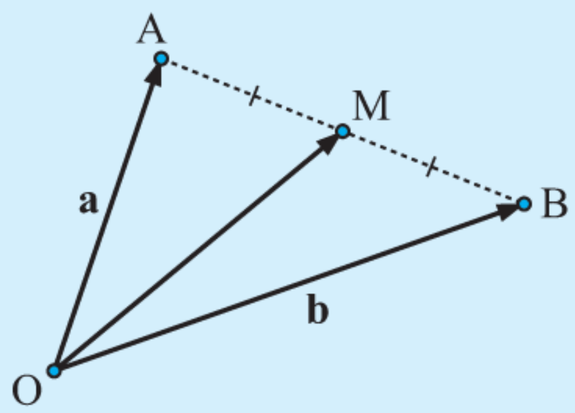


M PROOF USING VECTOR GEOMETRY

We can use vectors to prove important geometrical facts. The use of vectors in such proofs provides us with an alternative to using deductive or coordinate arguments, and can often simplify the proofs.

Example 33 **Self Tutor**

If M is the midpoint of [AB], prove that $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.



$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

USEFUL TOOLS IN VECTOR PROOF

- If $\mathbf{a} = k\mathbf{b}$ where k is a scalar then $\mathbf{a}$ and $\mathbf{b}$ are parallel, and $|\mathbf{a}| = |k||\mathbf{b}|$.
- If M is the midpoint of [AB] then $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.
- To prove $\mathbf{a}$ is perpendicular to $\mathbf{b}$, show that $\mathbf{a} \bullet \mathbf{b} = 0$.
- Properties of scalar product:

1 $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ for any two vectors $\mathbf{a}$ and $\mathbf{b}$.

2 $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$ for any vector $\mathbf{a}$.

3 $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

4 $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$.

Example 34 **Self Tutor**

Prove that the line joining the apex to the midpoint of the base of an isosceles triangle is perpendicular to the base.

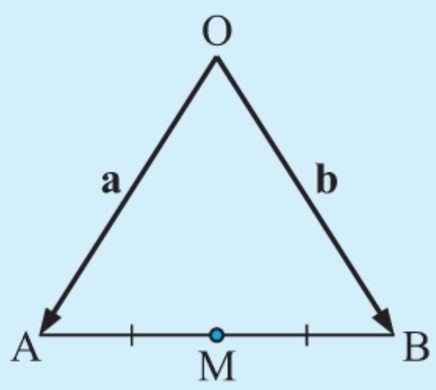
Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

$\therefore \overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

and $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB}$

$= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$= \frac{1}{2}(\mathbf{a} + \mathbf{b})$



$\therefore \overrightarrow{OM} \bullet \overrightarrow{AB} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{b} - \mathbf{a})$

$= \frac{1}{2}(\mathbf{a} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a})$

$= \frac{1}{2}(\mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a})$

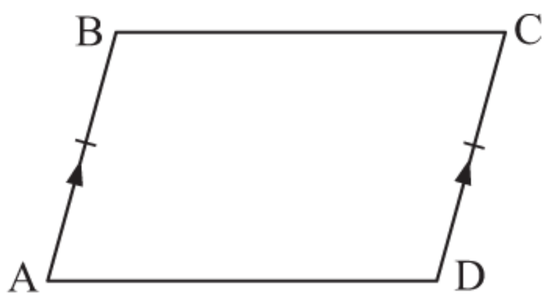
$= 0$

$\therefore \overrightarrow{OM} \perp \overrightarrow{AB}$

EXERCISE 12M

1 Prove that: *If a pair of opposite sides of a quadrilateral are parallel and equal in length, then the quadrilateral is a parallelogram.*

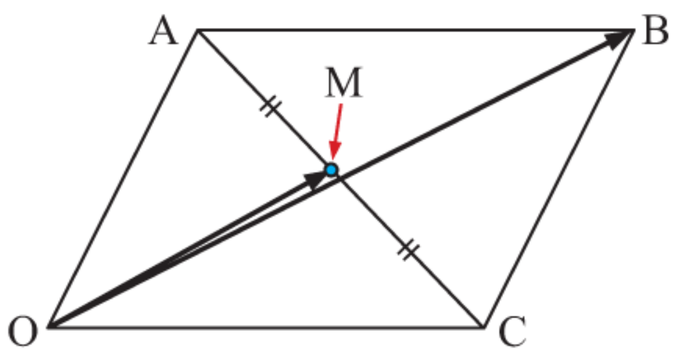
Hint:



Let $\overrightarrow{AB} = \mathbf{r}$ and $\overrightarrow{AD} = \mathbf{s}$.
Find $\overrightarrow{DC}$ and $\overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC}$ in terms of $\mathbf{r}$ and $\mathbf{s}$.

2 Prove that: *The diagonals of a parallelogram bisect each other.*

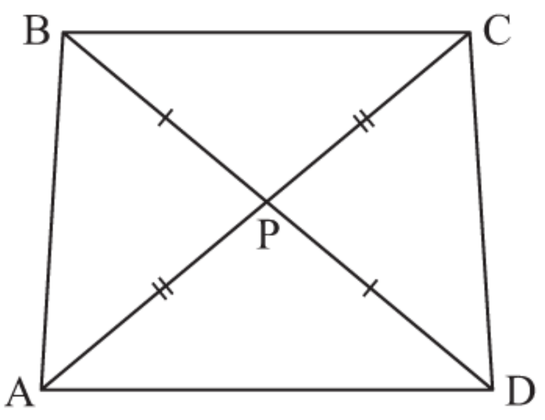
Hint:



M is the midpoint of [AC].
Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.
Find $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

3 Prove that: *The line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.*

4 ABCD is a quadrilateral in which $BP = PD$ and $AP = PC$.
Let $\overrightarrow{AP} = \mathbf{r}$ and $\overrightarrow{PB} = \mathbf{s}$.



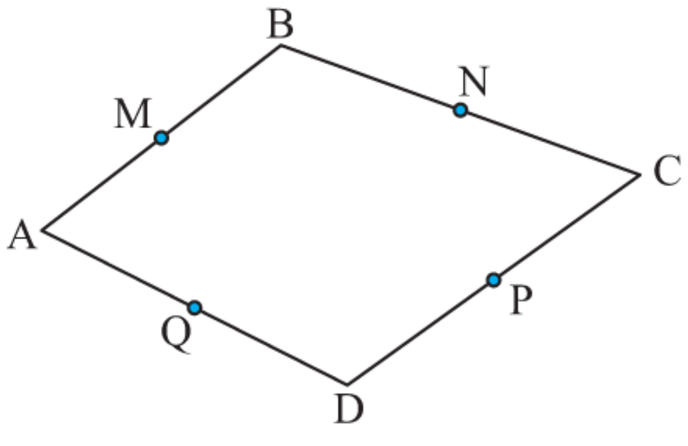
a Find, in terms of $\mathbf{r}$ and $\mathbf{s}$:

i $\overrightarrow{PC}$ and $\overrightarrow{DP}$ **ii** $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

b Discuss the significance of your answers to **a ii**.

c Copy and complete: *If the diagonals of a quadrilateral bisect each other .....*

5 Consider a quadrilateral ABCD where the midpoints of the sides are M, N, P, and Q as shown.
Prove that MNPQ is a parallelogram.



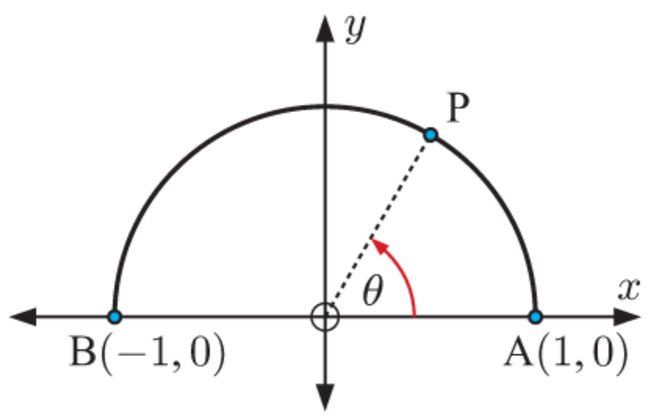
6 In the given figure:

a State the coordinates of P.

b Find $\overrightarrow{BP}$ and $\overrightarrow{AP}$.

c Find $\overrightarrow{AP} \bullet \overrightarrow{BP}$ using **b**.

d What property of a semi-circle has been deduced in **c**?



7 $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of two distinct points A and B, neither of which is the origin.
Show that if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ then $\mathbf{a}$ is perpendicular to $\mathbf{b}$, using:

a a vector algebraic method **b** a geometric argument.

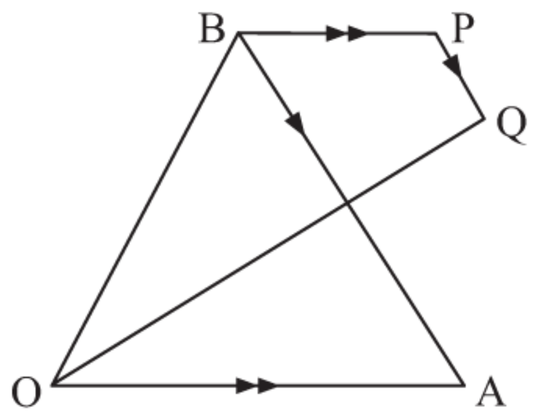
8 Prove that the diagonals of a rhombus bisect the angles of the rhombus using:

a a geometric argument **b** vector methods.

9 Prove that: *The diagonals of a parallelogram meet at right angles if and only if it is a rhombus.*
Hint: For an "if and only if" proof, you need to prove it in both directions.

10 Triangle OAB is equilateral. [BP] is parallel to [OA] and a half of its length. [PQ] is parallel to [BA] and a quarter of its length.

Let $\overrightarrow{OA} = \mathbf{r}$ and $\overrightarrow{OB} = \mathbf{s}$.



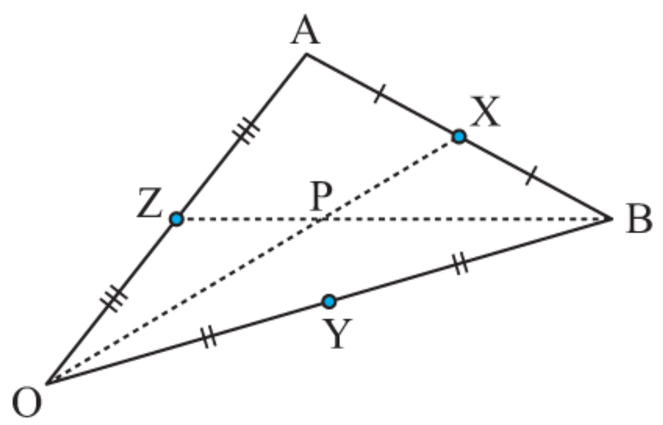
a Find vector expressions for $\overrightarrow{AB}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{r}$ and $\mathbf{s}$.

b Prove that [AB] is perpendicular to [OQ].

11 Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. Prove that if $k_1\mathbf{a} + k_2\mathbf{b} = l_1\mathbf{a} + l_2\mathbf{b}$, then $k_1 = l_1$ and $k_2 = l_2$.

12 Prove that: *The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.*

13 Consider a triangle OAB where the midpoints of the sides are X, Y, and Z as shown. [OX], [AY], and [BZ] are called the *medians* of the triangle.



Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a Write $\overrightarrow{OX}$ and $\overrightarrow{BZ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Suppose [OX] and [BZ] intersect at P. Write $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

c Write $\overrightarrow{YP}$ and $\overrightarrow{PA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

d Show that the medians of a triangle meet at a common point.

N THE VECTOR PRODUCT OF TWO VECTORS

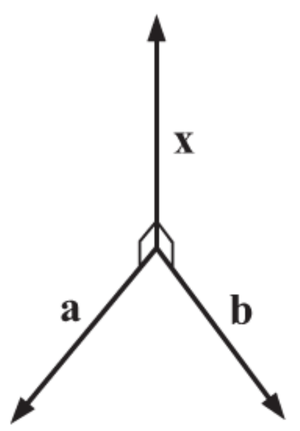
We have seen how the scalar product of two vectors results in a **scalar**.

The second form of product between two vectors is the **vector product** or **vector cross product**, and this results in a **vector**.

The vector cross product is only defined for 3-dimensional vectors.

The vector product arises when we attempt to find a vector which is *perpendicular to two other known vectors*.

Suppose $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is perpendicular to both $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.



$$\therefore \begin{cases} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \end{cases} \quad \{\text{as dot products are zero}\}$$

$$\therefore \begin{cases} a_1x + a_2y = -a_3z & \ldots. (1) \\ b_1x + b_2y = -b_3z & \ldots. (2) \end{cases}$$

We will now try to solve these two equations simultaneously to obtain expressions for x and y in terms of z.

To eliminate x, we multiply (1) by $-b_1$ and (2) by a_1.

$$\begin{aligned} -a_1b_1x - a_2b_1y &= a_3b_1z && \{-b_1 \times (1)\} \\ a_1b_1x + a_1b_2y &= -a_1b_3z && \{a_1 \times (2)\} \end{aligned}$$

Adding these gives $(a_1b_2 - a_2b_1)y = (a_3b_1 - a_1b_3)z$

$\therefore \quad y = (a_3b_1 - a_1b_3)t$ and $z = (a_1b_2 - a_2b_1)t$ for any non-zero t.

Substituting into (1), $a_1x + a_2(a_3b_1 - a_1b_3)t = -a_3(a_1b_2 - a_2b_1)t$

$$\therefore \quad a_1x = (-a_1a_3b_2 + \cancel{a_2a_3b_1} - \cancel{a_2a_3b_1} + a_1a_2b_3)t$$

$$\therefore \quad a_1x = a_1(a_2b_3 - a_3b_2)t$$

$$\therefore \quad x = (a_2b_3 - a_3b_2)t$$

Letting $t = 1$, we obtain $\mathbf{x} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$, and we call this vector the **vector cross product** of **a** and **b**.

The **vector cross product** of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

A formula like this will be given in your exam formula booklet. However, the formula may be confusing to use if, for example, you were finding $\mathbf{b} \times \mathbf{a}$. For this reason we recommend an alternative method for calculating vector cross product. We will discover it in the following **Investigation**.

INVESTIGATION 3 THE VECTOR CROSS PRODUCT FORMULA

In matrix algebra, we can define a **determinant** for any square matrix. In particular:

- The determinant of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

  It is the product of the terms on the main (red) diagonal minus the product of the other terms.

- The determinant of the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Notice that:

- The sign of the middle term is negative.
- The *size* of the ith term is a_{1i} times the 2×2 determinant seen by covering the top row and the ith column.

VIDEO

What to do:

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be vectors.

Show that $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$

is equal to the vector cross product $\mathbf{a} \times \mathbf{b}$.

From the **Investigation**, you should have found that:

The vector cross product of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$



Example 35 — Self Tutor

For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, find:

a $\mathbf{b} \times \mathbf{c}$ **b** $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$

a $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & 4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \mathbf{k}$

$= \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

or $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$

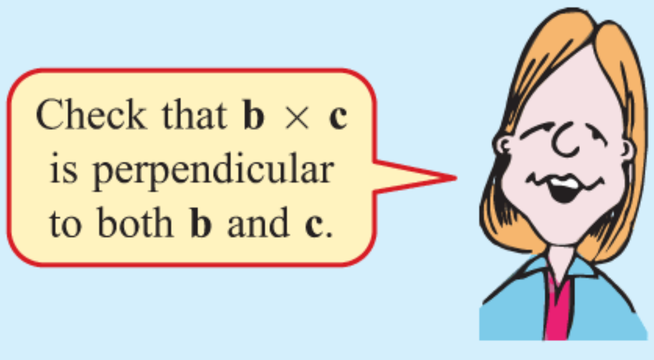
$= \begin{pmatrix} (2)(4) - (3)(0) \\ (3)(2) - (1)(4) \\ (1)(0) - (2)(2) \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

b $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

$= 16 + 2 + 4$

$= 22$



ALGEBRAIC PROPERTIES OF THE VECTOR CROSS PRODUCT

The vector cross product has the following algebraic properties for non-zero 3-dimensional vectors **a**, **b**, **c**, and **d**:

1 $\mathbf{a} \times \mathbf{b}$ is a vector which is perpendicular to both **a** and **b**.

2 $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all **a**.

3 $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all **a** and **b**.
This means that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length but opposite direction.

4 $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product**.

5 $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

6 $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})$.

You will prove or verify these results in the next Exercise.

EXERCISE 12N.1

1 Calculate:

a $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix}$

d $(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} - \mathbf{k})$ **e** $(2\mathbf{i} - \mathbf{k}) \times (\mathbf{j} + 3\mathbf{k})$ **f** $(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

2 Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$.

a Find $\mathbf{a} \times \mathbf{b}$. **b** Hence determine $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$.

c Explain your results.

3 **i**, **j**, and **k** are the base unit vectors parallel to the coordinate axes.

a Find $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$, and $\mathbf{k} \times \mathbf{k}$. Comment on your results.

b Find $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$. Comment on your results.

c Predict what the following vector cross products will be, then check your prediction:

i $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{j}$ **ii** $\mathbf{i} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$.

4 Use $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to prove that:

a $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all 3-dimensional vectors **a**

b $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ for all 3-dimensional vectors **a** and **b**.

5 Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$. Find:

a $\mathbf{b} \times \mathbf{c}$ **b** $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$

6 Suppose $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} - \mathbf{k}$. Find:

a $\mathbf{a} \times \mathbf{b}$ **b** $\mathbf{a} \times \mathbf{c}$

c $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ **d** $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$

7 **a** Use $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ to prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

b *Hence* prove that $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})$.

Hint: Since $\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x})$, the order of the vectors must be maintained.

8 Use the properties of vector cross product to simplify:

a $\mathbf{a} \times (\mathbf{a} + \mathbf{b})$ **b** $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b})$

c $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$ **d** $2\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b})$

9 **a** Use general vectors to show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$.

b Hence show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Example 36 **Self Tutor**

Find *all* vectors perpendicular to both $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{k}$$

$$= -6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

$$= -2(3\mathbf{i} - \mathbf{j} + \mathbf{k})$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(-3) - (-1)(0) \\ (-1)(1) - (1)(-3) \\ (1)(0) - (2)(1) \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 2 \\ -2 \end{pmatrix}$$

$$= -2\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

The vectors have the form $k\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ where k is any non-zero real number.

10 Find *all* vectors perpendicular to both:

a $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$

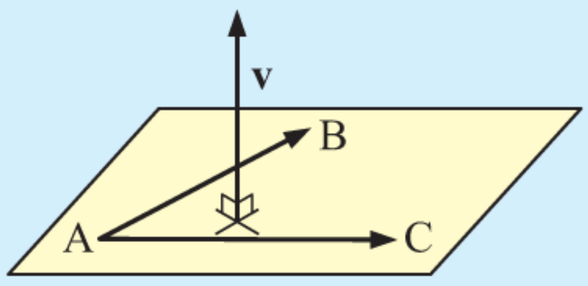
c $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} - \mathbf{k}$ **d** $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

11 Find all vectors perpendicular to both $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Hence find *two* vectors of length 5 units which are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Example 37 **Self Tutor**

Find a vector which is perpendicular to the plane passing through the points $A(1, -1, 2)$, $B(3, 1, 0)$, and $C(-1, 2, -3)$.



$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$

The vector $\mathbf{v}$ must be perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\therefore \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -2 \\ -2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 3 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ -2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix}\mathbf{k}$$
$$= -4\mathbf{i} + 14\mathbf{j} + 10\mathbf{k}$$
$$= -2(2\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$$

Any non-zero multiple of $\begin{pmatrix} 2 \\ -7 \\ -5 \end{pmatrix}$ will be perpendicular to the plane.

12 Find a vector which is perpendicular to the plane passing through the points:

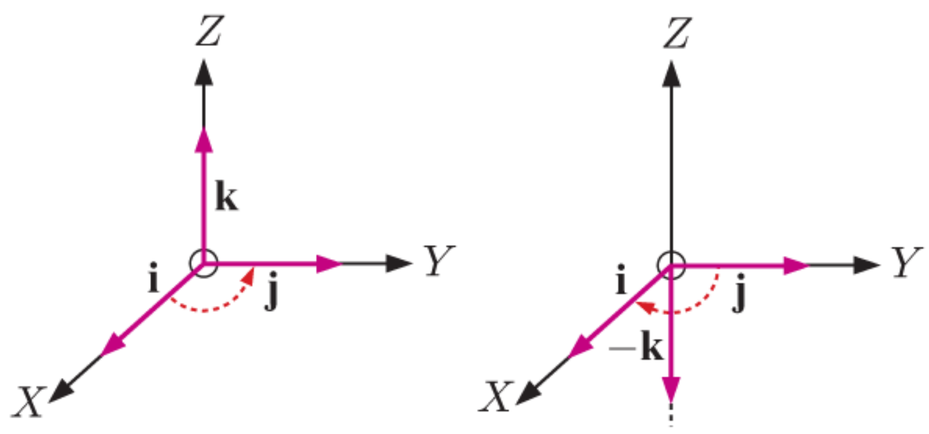
a $A(1, 3, 2)$, $B(0, 2, -5)$, $C(3, 1, -4)$ **b** $P(2, 0, -1)$, $Q(0, 1, 3)$, $R(1, -1, 1)$

THE DIRECTION OF $\mathbf{a} \times \mathbf{b}$

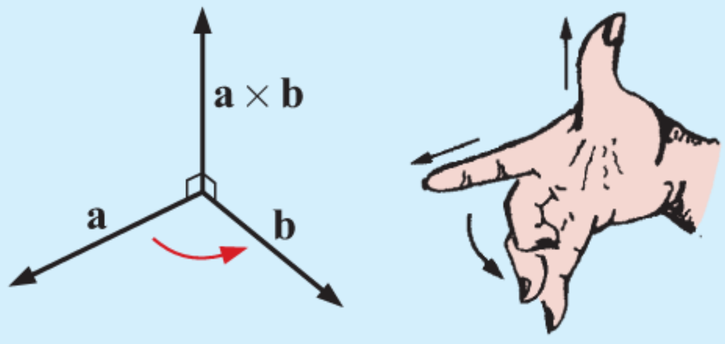
We have already observed that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$, so $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are in opposite directions.

However, what is the direction of each?

In the last Exercise, we saw that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.



In general, the **direction** of $\mathbf{a} \times \mathbf{b}$ is determined by the **right hand rule**:



If the fingers on your right hand turn from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

THE LENGTH OF $\mathbf{a} \times \mathbf{b}$

Using $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$, $|\mathbf{a} \times \mathbf{b}| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$.

However, another very useful form of the length of $\mathbf{a} \times \mathbf{b}$ exists:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \quad \text{where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}.$$

Proof:

$$\begin{aligned}
|\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \bullet \mathbf{b})^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2
\end{aligned}$$

which on expanding and then factorising gives

$$\begin{aligned}
&= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= |\mathbf{a} \times \mathbf{b}|^2
\end{aligned}$$

$$\therefore \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \qquad \{\text{as} \quad \sin\theta > 0\}$$

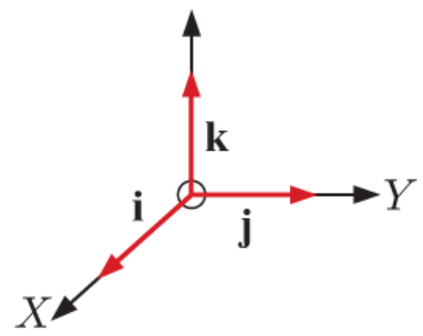
The immediate consequences of this result are:

- If $\mathbf{u}$ is a **unit vector** in the direction of $\mathbf{a} \times \mathbf{b}$ then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \mathbf{u}$. In some texts this is the **geometric definition** of $\mathbf{a} \times \mathbf{b}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}$ is parallel to $\mathbf{b}$.

EXERCISE 12N.2

1 **a** Find $\mathbf{i} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$ using the original definition of $\mathbf{a} \times \mathbf{b}$.

Check that the **right hand rule** correctly gives the direction of $\mathbf{i} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$.



b Verify that $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \mathbf{u}$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$, is true for $\mathbf{i} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$.

2 Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{k}$.

a Find $\mathbf{a} \bullet \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

b Find $\cos\theta$ using $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

c Find $\sin\theta$ using $\sin^2\theta + \cos^2\theta = 1$.

d Find $\sin\theta$ using $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

3 Prove the property:
"If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}$ is parallel to $\mathbf{b}$."

4 Suppose $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

a Show that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

b Find $\mathbf{p} \times \mathbf{q}$.

c Verify that $|\mathbf{p} \times \mathbf{q}| = |\mathbf{p}||\mathbf{q}|$.

5 A, B, and C are 3 distinct points with non-zero position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

a If $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, what can be deduced about $\overrightarrow{OC}$ and $\overrightarrow{AB}$?

b If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, what relationship exists between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{c}$?

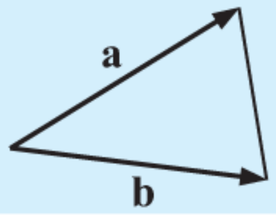
c If $\mathbf{c} \neq \mathbf{0}$ and $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$, prove that $\mathbf{a} + \mathbf{b} = k\mathbf{c}$ for some scalar k.

GEOMETRIC INTERPRETATION OF $|\mathbf{a} \times \mathbf{b}|$

We can use the vector cross product to find the area of a triangle or parallelogram defined by two vectors.

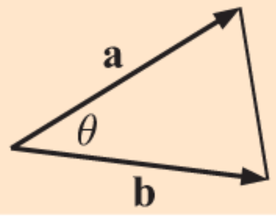
TRIANGLES

If a triangle has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units^2.



Proof:

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times \text{product of two sides} \times \text{sine of included angle} \\ &= \tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|\end{aligned}$$



Example 38 — Self Tutor

Find the area of triangle ABC given A$(-1, 2, 3)$, B$(2, 1, 4)$, and C$(0, 5, -1)$.

The defining vectors from A are $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{k}$

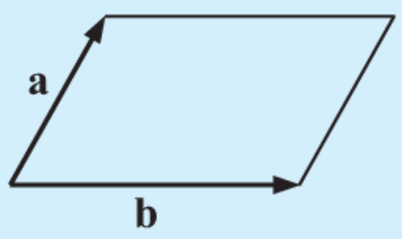
$= \mathbf{i} + 13\mathbf{j} + 10\mathbf{k}$

$\therefore$ area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{1 + 169 + 100}$

$= \frac{1}{2}\sqrt{270}$ units^2

PARALLELOGRAMS

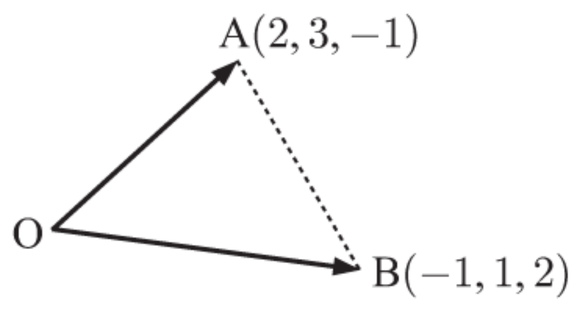
If a parallelogram has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $|\mathbf{a} \times \mathbf{b}|$ units^2.



The proof follows directly from that of a triangle, as the parallelogram consists of two congruent triangles with defining vectors $\mathbf{a}$ and $\mathbf{b}$.

EXERCISE 12N.3

1 Consider the triangle OAB shown. Find:



a $\overrightarrow{OA}$ and $\overrightarrow{OB}$

b $\overrightarrow{OA} \times \overrightarrow{OB}$

c the area of $\triangle OAB$.

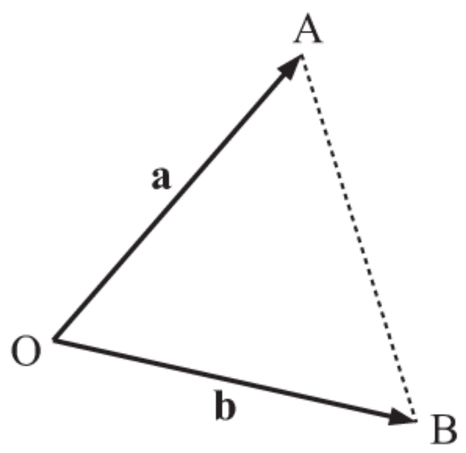
2 Calculate the area of triangle ABC given:

a A$(2, 1, 1)$, B$(4, 3, 0)$, C$(1, 3, -2)$

b A$(0, 0, 0)$, B$(-1, 2, 3)$, C$(1, 2, 6)$

c A$(1, 3, 2)$, B$(2, -1, 0)$, C$(1, 10, 6)$

3



In the triangle OAB alongside, the defining vectors from O are **a** and **b**.

a Write down the area of $\triangle OAB$ in terms of **a** and **b**.

b The defining vectors from B are $\overrightarrow{BO}$ and $\overrightarrow{BA}$.

i Write these vectors in terms of **a** and **b**.

ii Hence write down the area of $\triangle OAB$ using the defining vectors from B.

c Use the properties of the vector product to show that the areas in **a** and **b ii** are equal.

4 $A(-1, 1, 2)$, $B(2, 0, 1)$, and $C(x, 2, -1)$ are three points in space. The area of triangle ABC is $\sqrt{88}$ units^2. Find x.

5 $A(2, 4, 1)$, $B(1, 9, 1)$, and $C(-1, 5, 0)$ are three points in space.

a Show that OABC is a parallelogram. **b** Hence find the area of OABC.

6 Calculate the area of parallelogram ABCD given $A(-1, 2, 2)$, $B(2, -1, 4)$, and $C(0, 1, 0)$.

7 ABCD is a parallelogram with vertices $A(-1, 3, 2)$, $B(2, 0, 4)$, and $C(-1, -2, 5)$. Find:

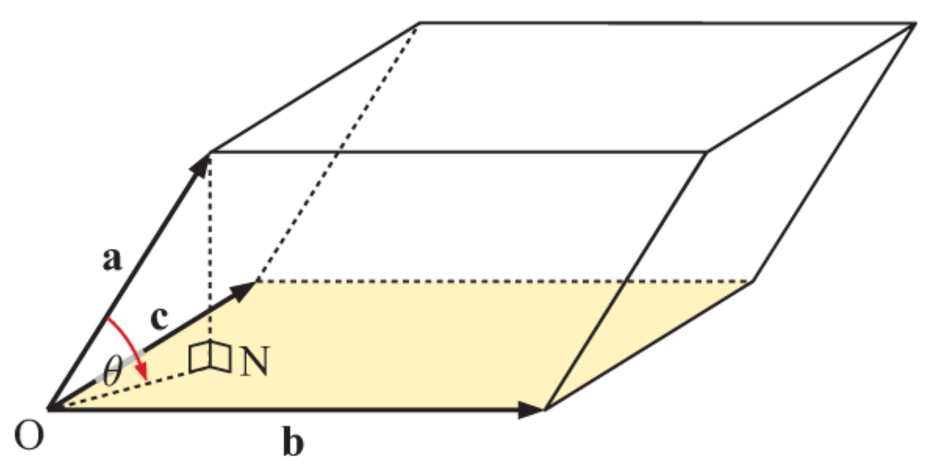
a the coordinates of D **b** the area of the parallelogram.

8 The quadrilateral PQRS has coordinates $P(-2, 0, 3)$, $Q(4, -2, 4)$, $R(-1, -6, 4)$, and $S(-7, -4, 3)$.

a Show that PQRS is a rhombus. **b** Find the area of PQRS.

9 The 3-dimensional shape alongside is a **parallelepiped** with defining vectors **a**, **b**, and **c**.

Let θ be the angle between **a** and the base plane.



a Find:

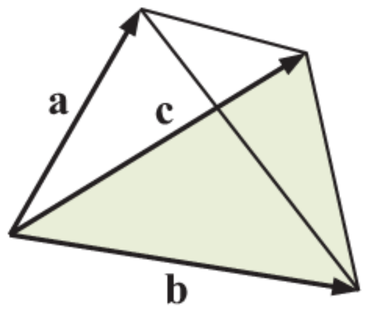
i the area of the base plane

ii the perpendicular height of the parallelepiped.

b Hence show that the volume of the parallelepiped is $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$ units^3.

c $A(3, 0, 0)$, $B(0, 1, 0)$, and $C(1, 2, 3)$ are the vertices of a parallelepiped which are adjacent to another vertex at $O(0, 0, 0)$. Find the volume of the parallelepiped.

10 A **tetrahedron** or **triangular-based pyramid** alongside has defining vectors **a**, **b**, and **c**.



a Show that the volume of the tetrahedron is $\frac{1}{6}|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$ units^3.

b Find the volume of the tetrahedron with vertices $P(0, 0, 1)$, $Q(2, 3, 0)$, $R(-1, 2, 1)$, and $S(1, -2, 4)$.

DISCUSSION

Discuss the following statement:

"Scalar product is a measure of the *parallelism* of two vectors, whereas vector product is a measure of the *perpendicularity* of two vectors."

REVIEW SET 12A

1 Using a scale of $1\text{ cm} \equiv 10$ units, sketch a vector to represent:

a an aeroplane taking off at an angle of $8°$ to a runway with a speed of 60 m s^{-1}

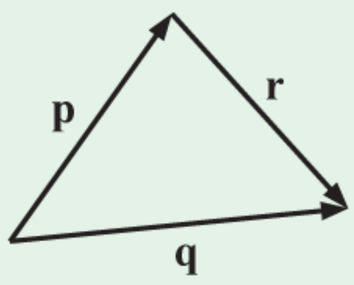
b a displacement of 45 m in a north-easterly direction.

2 Simplify:

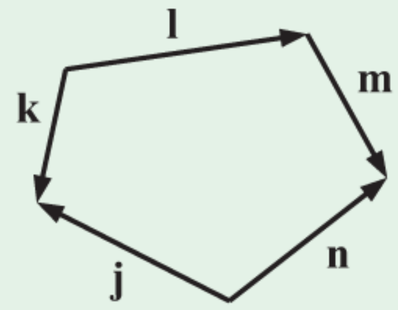
a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$

3 Construct a vector equation for:

a

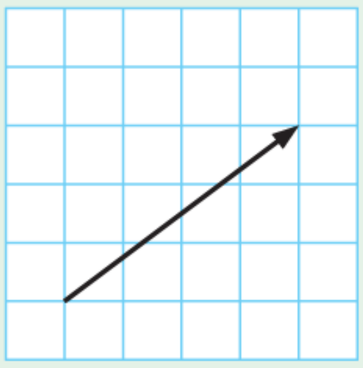


b

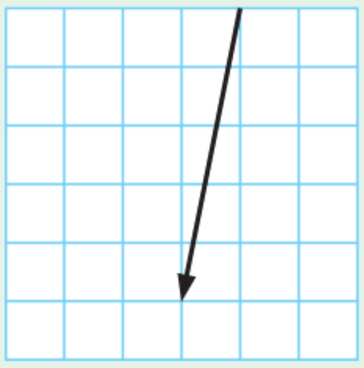


4 Write the illustrated vectors in component form and in unit vector form:

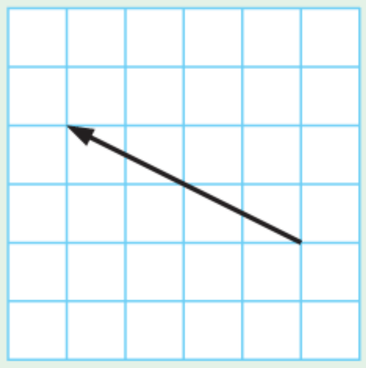
a



b



c

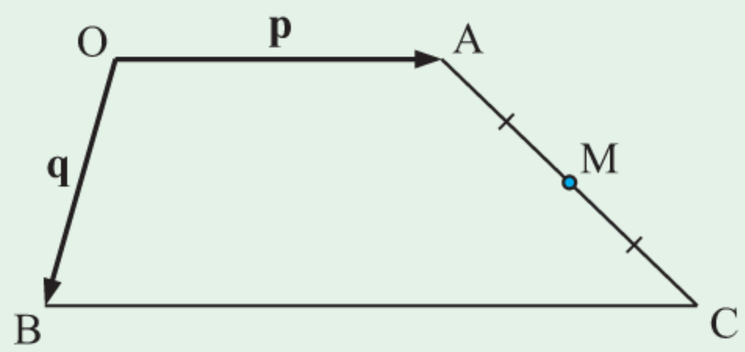


5 Illustrate P and find its distance from the origin O:

a P$(1, 3, 2)$ **b** P$(2, -1, 3)$

6 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, find $\overrightarrow{SP}$.

7



[BC] is parallel to [OA] and is twice its length.
Find, in terms of **p** and **q**, vector expressions for:

a $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$.

8 Given $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$, find:

a $\mathbf{a} - \mathbf{b}$ **b** $|\mathbf{a}|$ **c** $2\mathbf{a} + 3\mathbf{b}$

d $-\frac{1}{3}\mathbf{a}$ **e** $|\mathbf{b}|\,\mathbf{a}$ **f** $|\mathbf{b} - 2\mathbf{a}|$

9 Find scalars a, b, and c such that $3\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + a\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} b \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$.

10 The points A, B, and C have position vectors $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $5\mathbf{i} - \mathbf{k}$, and $-\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ respectively.

a M is the midpoint of BC. Find the magnitude of $\overrightarrow{AM}$.

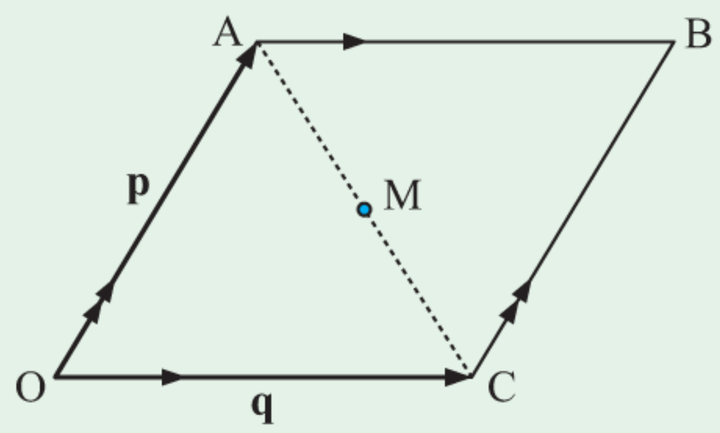
b Point D is such that $\overrightarrow{AC} = \overrightarrow{BD}$. Find the position vector of D.

11 Find m and n if $\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix}$ are parallel vectors.

12 Prove that $P(-6, 8, 2)$, $Q(4, 6, 8)$, and $R(19, 3, 17)$ are collinear.

13 In this question you may *not* assume any diagonal properties of parallelograms.

OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{p}$ and $\overrightarrow{OC} = \mathbf{q}$. M is the midpoint of [AC].



a Find in terms of $\mathbf{p}$ and $\mathbf{q}$:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{OM}$

b Hence show that O, M, and B are collinear, and that M is the midpoint of [OB].

14 [AB] and [CD] are diameters of a circle with centre O.

a If $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{OB} = \mathbf{r}$, find:

i $\overrightarrow{DB}$ in terms of $\mathbf{q}$ and $\mathbf{r}$ **ii** $\overrightarrow{AC}$ in terms of $\mathbf{q}$ and $\mathbf{r}$.

b What can be deduced about [DB] and [AC]?

15 Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

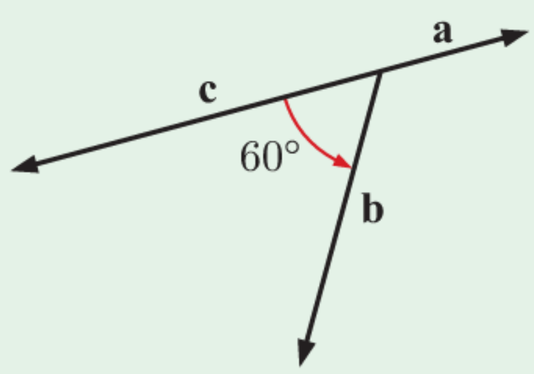
16 If $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r})$

17 Suppose $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$, and $|\mathbf{c}| = 5$.

Find: **a** $\mathbf{a} \bullet \mathbf{b}$

b $\mathbf{b} \bullet \mathbf{c}$

c $\mathbf{a} \bullet \mathbf{c}$



18 If $\mathbf{u} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$, find:

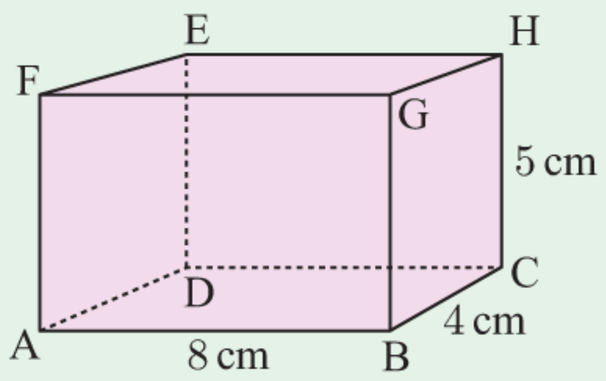
a $\mathbf{u} \bullet \mathbf{v}$ **b** the angle between $\mathbf{u}$ and $\mathbf{v}$.

19 Find the angle between the vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

20 Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular.

21 Find a unit vector which is parallel to $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$ and perpendicular to $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

22 Use vector methods to find the measure of $G\hat{A}C$:



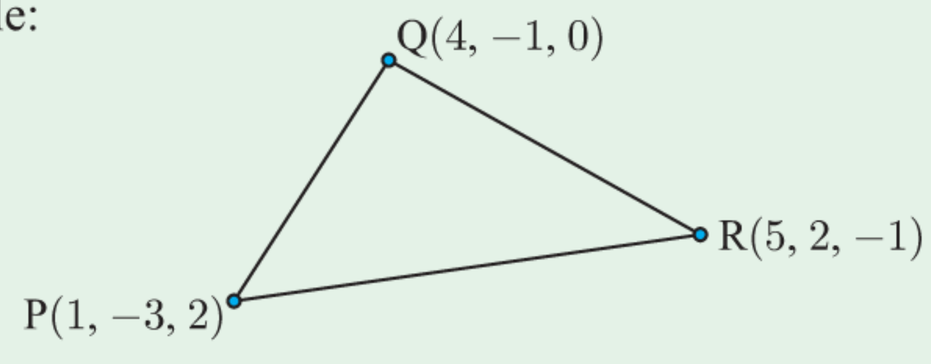
23 Given $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, find:

a $\mathbf{a} \times \mathbf{b}$ **b** $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ **c** $\mathbf{b} \times \mathbf{c}$

24 Find *two* vectors of length 3 units which are perpendicular to both $-\mathbf{i} + 3\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

25 For any two vectors $\mathbf{a}$ and $\mathbf{b}$, explain why $|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})| = |\mathbf{a}|\,|\mathbf{a} \times \mathbf{b}|$.

26 Find the area of this triangle:

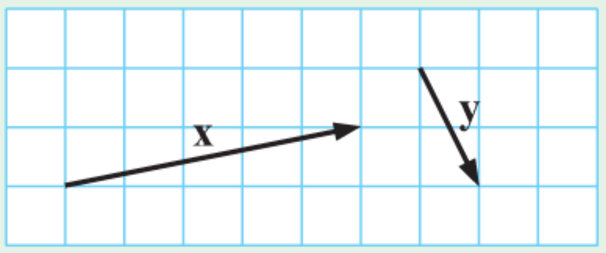


27 A, B, and C are three points with position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively. Find a formula for S, the total surface area of the tetrahedron OABC.

REVIEW SET 12B

1 Copy the given vectors and find geometrically:

a $\mathbf{x} + \mathbf{y}$ **b** $\mathbf{y} - 2\mathbf{x}$

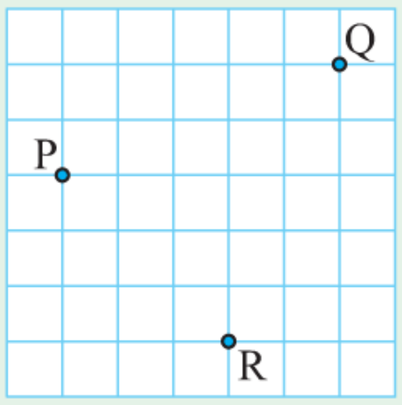


2 Find a single vector which is equal to:

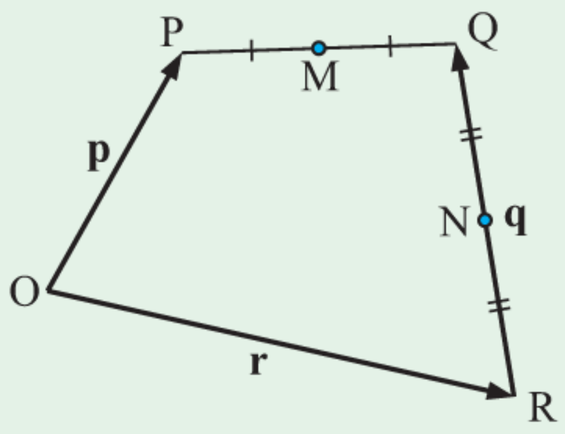
a $\overrightarrow{PR} + \overrightarrow{RQ}$ **b** $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR}$

3 Find in component form and in unit vector form:

a $\overrightarrow{PQ}$ **b** $\overrightarrow{RP}$



4



In the figure alongside, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$, and $\overrightarrow{RQ} = \mathbf{q}$. M and N are the midpoints of [PQ] and [QR] respectively. Find, in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$:

a $\overrightarrow{OQ}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{ON}$ **d** $\overrightarrow{MN}$

5 For $\mathbf{m} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, and $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$, find:

a $\mathbf{m} - \mathbf{n} + \mathbf{p}$ **b** $2\mathbf{n} - 3\mathbf{p}$ **c** $|\mathbf{m} + \mathbf{p}|$

6 If $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$, find $\overrightarrow{CB}$.

7 Given P$(2, -5, 6)$ and Q$(-1, 7, 9)$, find:

a the displacement vector of Q relative to P **b** the distance from P to Q

c the distance from P to the X-axis.

8 Find k given that $k\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - k\mathbf{k}$ is a unit vector.

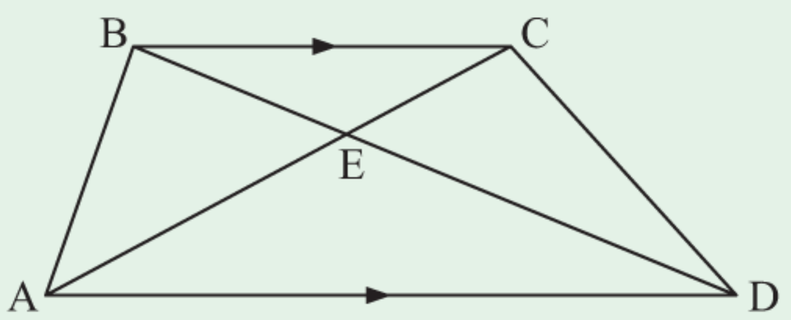
9 Find a and b if J$(-4, 1, 3)$, K$(2, -2, 0)$, and L$(a, b, 2)$ are collinear.

10 Find *two* vectors of length 4 units which are parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

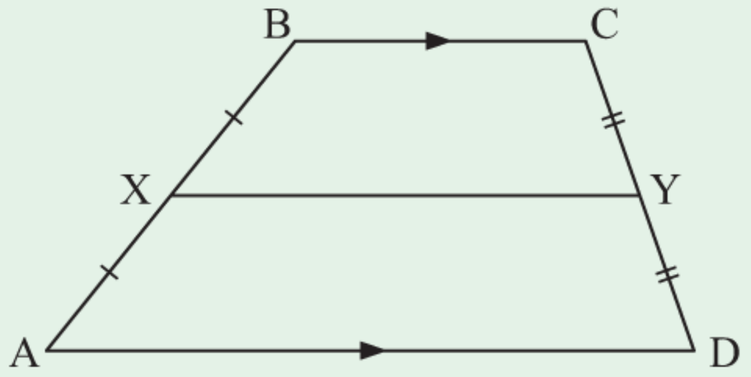
11 In the diagram alongside, $AD = k\,BC$, $k \in \mathbb{R}$.
Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$.

a Write $\overrightarrow{CD}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and k.

b Prove that $\overrightarrow{ED} = k\overrightarrow{BE}$.



12 ABCD is a trapezium in which [BC] and [AD] are parallel. X and Y are the midpoints of [AB] and [CD] respectively. If $\overrightarrow{XB} = \mathbf{p}$, $\overrightarrow{BC} = \mathbf{q}$, and $\overrightarrow{CY} = \mathbf{r}$:



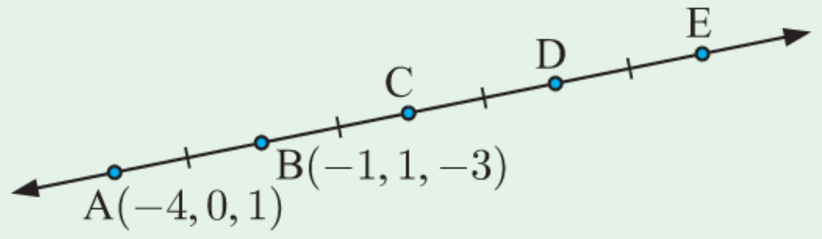
a Show that $\overrightarrow{AD} = 2\mathbf{p} + \mathbf{q} + 2\mathbf{r}$.

b If $\overrightarrow{AD} = k\,\overrightarrow{BC}$, find another vector expression for $\overrightarrow{AD}$.

c Deduce that $\mathbf{p} + \mathbf{r} = \left(\frac{k-1}{2}\right)\mathbf{q}$ and that $\overrightarrow{XY} = \left(\frac{k+1}{2}\right)\mathbf{q}$.

d What conclusion can be drawn from **c** concerning *the line joining the midpoints of the non-parallel sides of a trapezium*?

13 Find the coordinates of C, D, and E.



14 Find the coordinates of point X which is 12 units from $(-2, 1, -5)$, in the direction $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

15 Suppose $|\mathbf{v}| = 3$ and $|\mathbf{w}| = 2$. If $\mathbf{v}$ is parallel to $\mathbf{w}$, what values might $\mathbf{v} \bullet \mathbf{w}$ take?

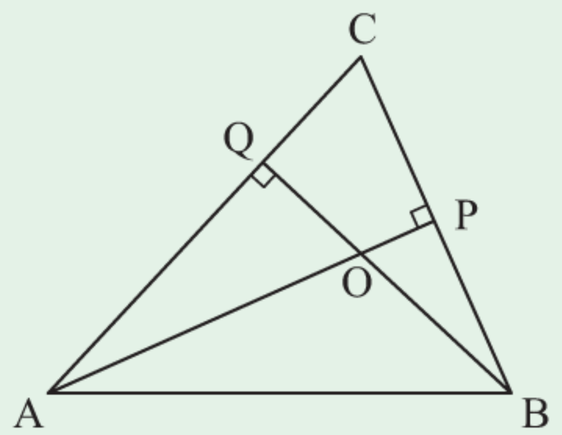
16 Using $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, verify that
$\mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$.

17 Suppose $\overrightarrow{OM} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\overrightarrow{MP} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, $\overrightarrow{MP} \bullet \overrightarrow{PT} = 0$, and $|\overrightarrow{MP}| = |\overrightarrow{PT}|$.

Write down the two possible position vectors $\overrightarrow{OT}$.

18 [AP] and [BQ] are altitudes of triangle ABC.

Let $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OB} = \mathbf{q}$, and $\overrightarrow{OC} = \mathbf{r}$.



a Find vector expressions for $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.

b Using the property $\mathbf{a} \bullet (\mathbf{b} - \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{c}$, deduce that $\mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r}$.

c Hence prove that [OC] is perpendicular to [AB].

19 P$(-1, 2, 3)$ and Q$(4, 0, -1)$ are two points in space. Find:

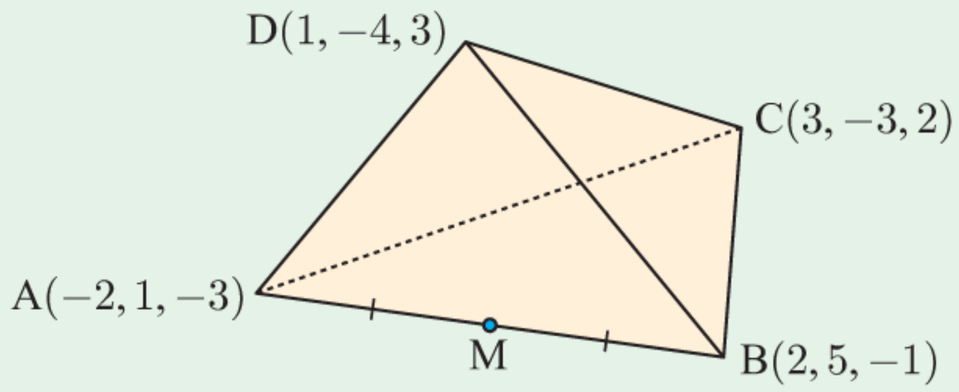
a $\overrightarrow{PQ}$ **b** the angle that $\overrightarrow{PQ}$ makes with the X-axis.

20 Given $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{q} = -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, find the angle between $\mathbf{p}$ and $\mathbf{q}$.

21 Find t given $\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix}$ are perpendicular vectors.

22 Find the angle between $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

23 Find the measure of $D\widehat{M}C$.



24 Find all vectors perpendicular to both $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

25 Given $|\mathbf{u}| = 3$, $|\mathbf{v}| = 5$, and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$, find the possible values of $\mathbf{u} \bullet \mathbf{v}$.

26 Three distinct points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively. Prove that A, B, and C are collinear $\Leftrightarrow$ $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$.

27 Point A has position vector $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and point B has position vector $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$.

a Find $\mathbf{a} \times \mathbf{b}$.

b Point C is 6 units from the origin O, in the direction opposite to that of $\mathbf{a} \times \mathbf{b}$. Find the coordinates of C.

c Find the area of the triangle OAB.

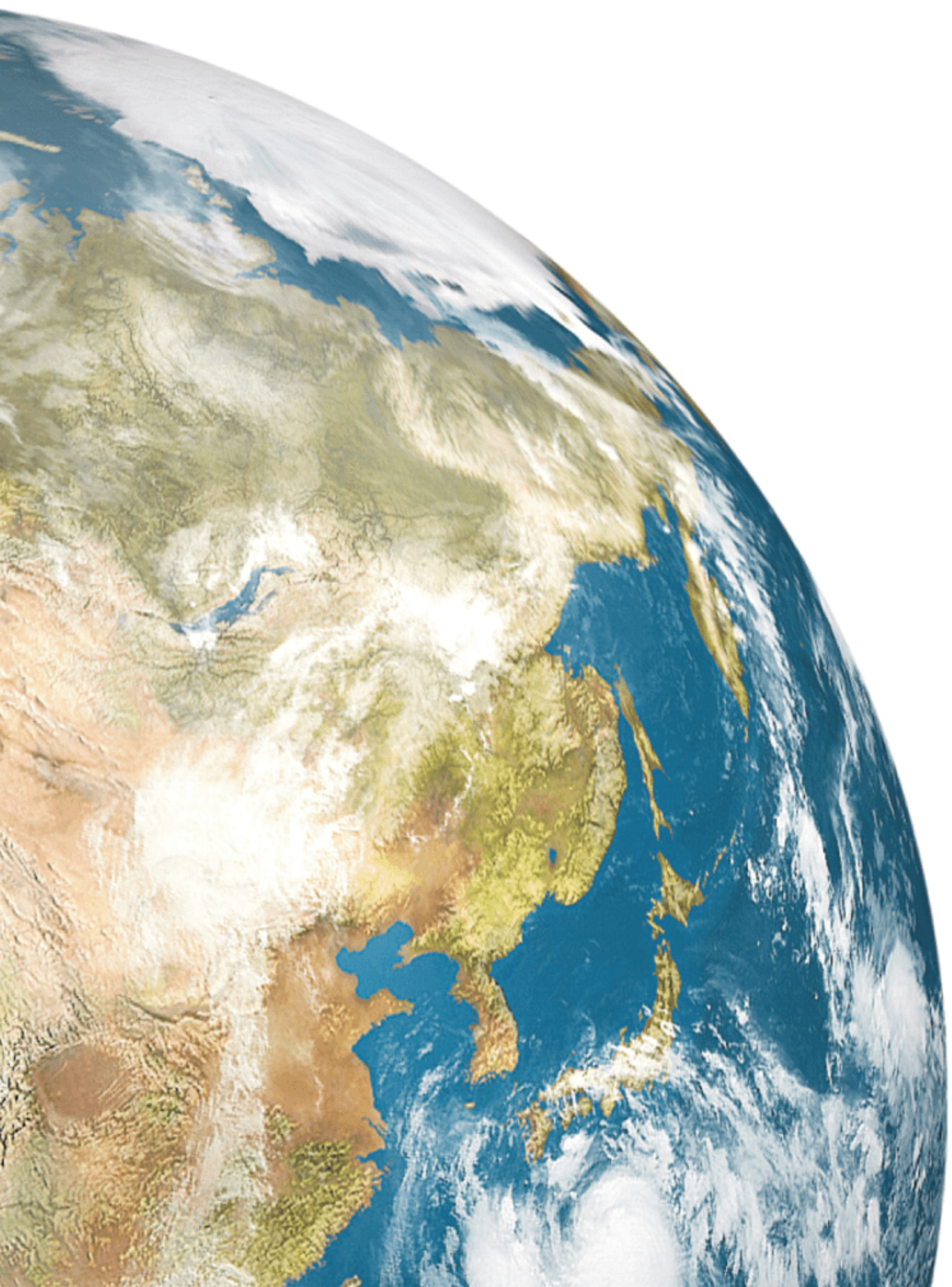
d Hence find the volume of the tetrahedron OABC.

Chapter 13

Vector applications

Contents:

A Lines in 2 and 3 dimensions
B The angle between two lines
C Constant velocity problems
D The shortest distance from a point to a line
E Intersecting lines
F Relationships between lines
G Planes
H Angles in space
I Intersecting planes



OPENING PROBLEM

A **plane** is a flat surface that extends forever, and has zero thickness.

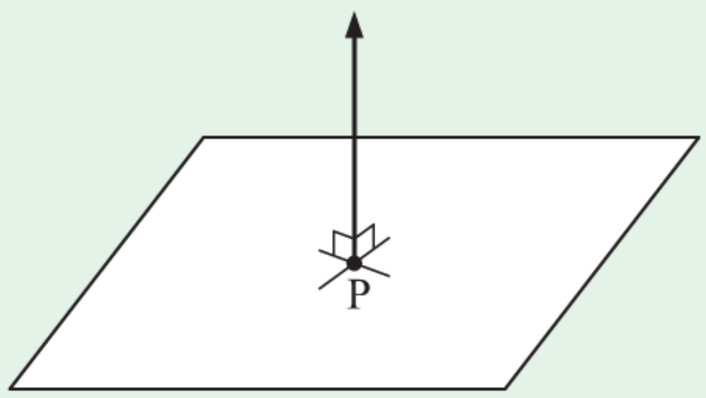
Things to think about:

a Given that a plane is 2-dimensional, does it make sense to talk about a plane having *direction*?

b Suppose you are at a point P on a plane.

i In how many different directions could you move from P and still be on the plane?

ii Is there a direction which is perpendicular to *all* of the directions you considered in **i**? If so, how would you find that direction?



c Can we use a point and a direction vector to specify a *plane*?

d If there are two planes in space, do they necessarily meet?

There are many applications of vectors in geometry. Some of these applications we do not use in 2-dimensional planar geometry because we have other tools.

In 3-dimensional space, however, vector methods are very efficient, in particular when we consider the relationships between lines and planes.

A LINES IN 2 AND 3 DIMENSIONS

In both 2-dimensional and 3-dimensional geometry we can determine the **equation of a line** using its **direction** and any **fixed point** on the line.

In 2 dimensions we are dealing with a **line in a plane**. In 3 dimensions we are dealing with a **line in space**.

Suppose a line passes through a fixed point A with position vector **a**, and the line is parallel to the vector **b**.

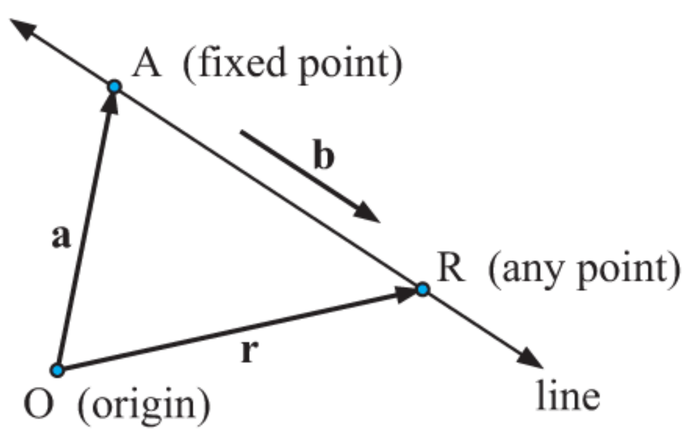
Consider a point R on the line with position vector **r**.

By vector addition, $\mathbf{r} = \mathbf{a} + \overrightarrow{\text{AR}}$.

Since $\overrightarrow{\text{AR}}$ is parallel to **b**,

$\overrightarrow{\text{AR}} = \lambda\mathbf{b}$ for some scalar $\lambda \in \mathbb{R}$

$\therefore \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$



DEMO

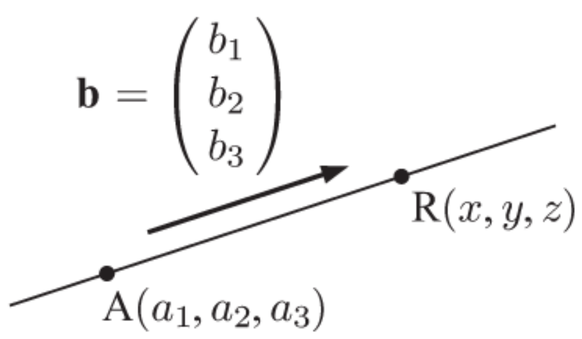


So, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ is the **vector equation** of the line.

Suppose $\text{R}(x, y, z)$ is any point on the line,

$\text{A}(a_1, a_2, a_3)$ is the known or fixed point on the line,

and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **direction vector** of the line.



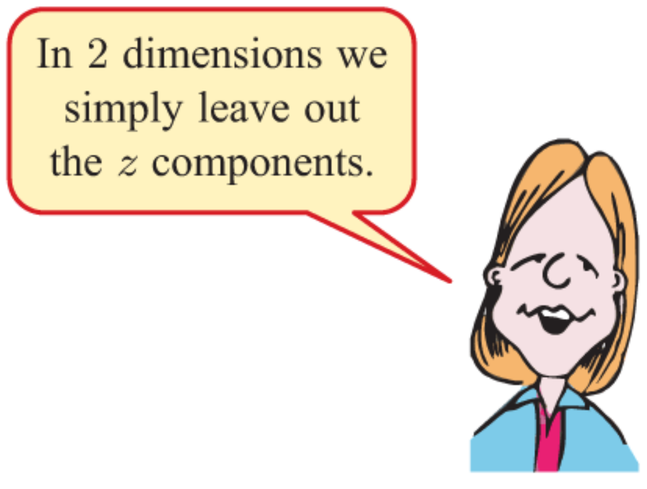
- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **vector equation** of the line.

- The **parametric equations** of the line are:

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases} \quad \text{where } \lambda \in \mathbb{R} \text{ is the } \textbf{parameter}.$$

- By equating λ values, we obtain the **Cartesian equation** of the line $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$.

In 2 dimensions, the Cartesian form can be written in the familiar form $ax + by = d$.



DISCUSSION

In 2 dimensions we talk about the gradient of a line.

- If a line in 2 dimensions has direction vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, what is its gradient?
- Does it make sense to talk about the gradient of a line in space?

Example 1 **Self Tutor**

A line in space passes through $(1, -2, 3)$ in the direction $4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

Describe the line using:

a a vector equation **b** parametric equations **c** a Cartesian equation.

a The vector equation is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

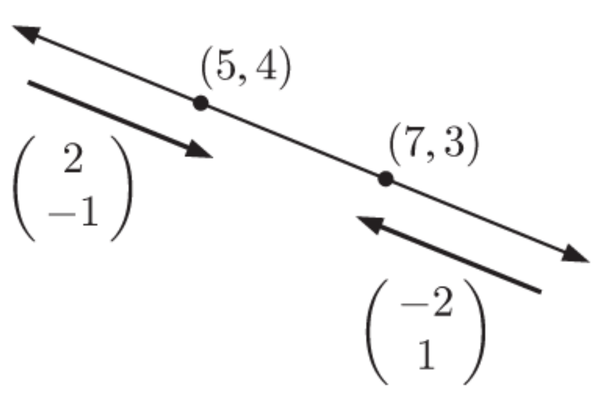
b The parametric equations are: $x = 1 + 4\lambda, \quad y = -2 + 5\lambda, \quad z = 3 - 6\lambda, \quad \lambda \in \mathbb{R}$.

c Equating λ values, the Cartesian equations are: $\dfrac{x-1}{4} = \dfrac{y+2}{5} = \dfrac{z-3}{-6}$.

NON-UNIQUENESS OF THE VECTOR EQUATION OF A LINE

Consider the line passing through $(5, 4)$ and $(7, 3)$. When writing the equation of the line, we could use either point to give the position vector $\mathbf{a}$.

Similarly, we could use the direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, but we could also use $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ or any non-zero scalar multiple of these vectors.



We could thus write the equation of the line as

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{or} \quad \mathbf{r} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R} \quad \text{and so on.}$$

Notice how we use different parameters λ and μ when we write these equations. This is because the parameters are clearly not the same: when $\lambda = 0$, we have the point $(5, 4)$
when $\mu = 0$, we have the point $(7, 3)$.

In fact, the parameters are related by $\mu = 1 - \lambda$.

Example 2 **Self Tutor**

Find parametric equations for the line through $A(2, -1, 4)$ and $B(-1, 0, 2)$.

We require a direction vector for the line, either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{AB} = \begin{pmatrix} -1-2 \\ 0-(-1) \\ 2-4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

Using point A, the equations are:

$$x = 2 - 3\lambda, \quad y = -1 + \lambda, \quad z = 4 - 2\lambda, \quad \lambda \in \mathbb{R}$$

or using point B, the equations are:

$$x = -1 - 3\mu, \quad y = \mu, \quad z = 2 - 2\mu, \quad \mu \in \mathbb{R}.$$

EXERCISE 13A

1 Describe each of the following lines using:

i a vector equation **ii** parametric equations **iii** a Cartesian equation.

a a line with direction $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ which passes through $(1, 4)$

b a line passing through $(5, 2)$ which is perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

c a line parallel to $3\mathbf{i} + 7\mathbf{j}$ which cuts the x-axis at -6

d a line perpendicular to $3\mathbf{i} - \mathbf{j}$ which cuts the y-axis at 2

e a line passing through $(3, 0)$ and $(7, 2)$

f a line passing through $(-2, 5)$ and $(4, -6)$.

2 A line passes through $(4, -3)$ with direction vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

a Write parametric equations for the line using the parameter λ.

b Find the points on the line for which $\lambda = 0, 1, 2, -1$, and -3.

3 Line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$.

a Locate the point on the line corresponding to $\lambda = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

4 Describe each of the following lines using:

i a vector equation **ii** parametric equations **iii** a Cartesian equation.

a a line parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ which passes through $(1, 3, -7)$

b a line which passes through $(0, 1, 2)$ with direction vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

c a line parallel to the X-axis which passes through $(-2, 2, 1)$

d a line parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ which passes through $(0, 2, -1)$

e a line perpendicular to the XY-plane which passes through $(3, 2, -1)$.

5 Find the vector equation of the line which passes through:

a A$(1, 2, 1)$ and B$(-1, 3, 2)$ **b** C$(0, 1, 3)$ and D$(3, 1, -1)$

c E$(1, 2, 5)$ and F$(1, -1, 5)$ **d** G$(0, 1, -1)$ and H$(5, -1, 3)$.

6 Find the direction vector of the line:

a $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ **b** $x = 5 - t, \quad y = 1 + t, \quad z = 2 - 3t, \quad t \in \mathbb{R}$

c $\dfrac{x-2}{3} = \dfrac{y+1}{2} = z - 1$ **d** $\dfrac{1-x}{2} = \dfrac{y}{4} = \dfrac{z-3}{3}$

7 Find the coordinates of the point where the line with parametric equations $x = 1 - \lambda$, $y = 3 + \lambda$, and $z = 3 - 2\lambda$, $\lambda \in \mathbb{R}$ meets:

a the YZ-plane $x = 0$ **b** the XZ-plane $y = 0$ **c** the XY-plane $z = 0$.

8 The parametric equations of a line are $x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$ for $\lambda \in \mathbb{R}$.

a Find the coordinates of the point on the line corresponding to $\lambda = 0$.

b Find the direction vector of the line.

c Write the Cartesian equation for the line.

9 Find points on the line with parametric equations $x = 2 - \lambda$, $y = 3 + 2\lambda$, and $z = 1 + \lambda$, $\lambda \in \mathbb{R}$ which are $5\sqrt{3}$ units from the point $(1, 0, -2)$.

10 A line has parametric equations $x = 1 + \lambda$, $y = 2 - \lambda$, $z = 3 + \lambda$.

a Find the distance from P$(1, 1, 2)$ to a general point X on the line.

b Use quadratic theory to find the value of λ which minimises the distance PX.

c Show that when the distance PX is minimised, $\overrightarrow{\text{PX}}$ is perpendicular to the line.

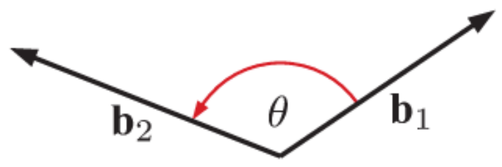
11 Line L_1 has equation $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$. Line L_2 passes through $(2, 0, 3)$ and $(-2, 1, 3)$.

a Find the equation of L_2.

b The line L_3 is perpendicular to both L_1 and L_2, and cuts the Z-axis at -2. Find the equation of L_3 in Cartesian form.

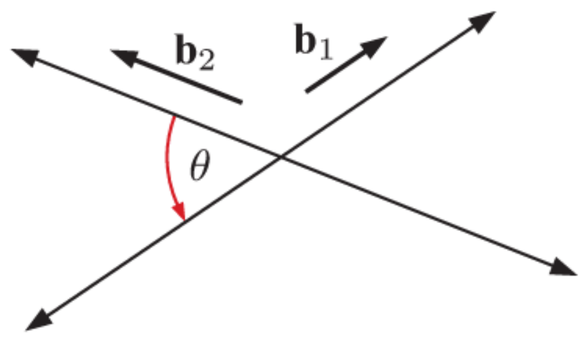
B THE ANGLE BETWEEN TWO LINES

In **Chapter 12** we saw that the angle between two vectors is measured in the range $0° \leqslant \theta \leqslant 180°$, and found using the formula $\cos\theta = \dfrac{\mathbf{b}_1 \bullet \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.



Since lines continue infinitely in both directions, we talk about the *acute* angle between them. For an acute angle, $\cos\theta > 0$, so we use the formula

$$\cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$



where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the given lines.

Example 3 — Self Tutor

Find the angle between the lines L_1: $x = 2 - 3t$, $y = -1 + t$ and L_2: $x = 1 + 2s$, $y = -4 + 3s$, where $s, t \in \mathbb{R}$

L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Now $\cos\theta = \dfrac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|} = \dfrac{|-6+3|}{\sqrt{10}\sqrt{13}} \approx 0.2631$

$\therefore \ \theta \approx 74.7°$

$\therefore$ the angle between L_1 and L_2 is about $74.7°$.

PARALLEL AND PERPENDICULAR LINES

Consider two lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{b}_2$.

- $\mathbf{r}_1$ and $\mathbf{r}_2$ are **parallel** if $\mathbf{b}_1 = k\mathbf{b}_2$ for some scalar k. We write $\mathbf{r}_1 \parallel \mathbf{r}_2$.
- $\mathbf{r}_1$ and $\mathbf{r}_2$ are **perpendicular** if $\mathbf{b}_1 \bullet \mathbf{b}_2 = 0$. We write $\mathbf{r}_1 \perp \mathbf{r}_2$.

Example 4 — Self Tutor

Find the angle between the lines:

$$L_1: \ \mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{and} \quad L_2: \ \frac{1-x}{3} = y = \frac{2-z}{2}.$$

L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$.

Since $\mathbf{b}_1 = \mathbf{b}_2$, the lines are parallel so the angle between them is $0°$.

EXERCISE 13B

1 Find the angle between the lines L_1: $x = -4 + 12t$, $y = 3 + 5t$, $t \in \mathbb{R}$
and L_2: $x = 3s$, $y = -6 - 4s$, $s \in \mathbb{R}$.

2 Show that the lines L_1: $x = 2 + 5p$, $y = 19 - 3p$, $z = 9 + 2p$, $p \in \mathbb{R}$
and L_2: $x = 3 + 4r$, $y = 7 + 10r$, $z = -3 + 5r$, $r \in \mathbb{R}$ are perpendicular.

3 The line L_1 passes through $(-6, 3)$ and is parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.
The line L_2 has direction $5\mathbf{i} + 4\mathbf{j}$ and cuts the y-axis at 8.
Find the angle between the lines.

4 **a** Find the angle between the lines:

$$L_1: \quad \frac{x-8}{3} = \frac{9-y}{16} = \frac{z-10}{7} \qquad \text{and} \qquad L_2: \quad \mathbf{r}_2 = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}, \quad t \in \mathbb{R}.$$

b A third line L_3 is perpendicular to L_1 and has direction vector $-3\mathbf{j} + a\mathbf{k}$.

i Find the value of a.

ii Find the angle between L_2 and L_3.

5 The line L_1 passes through $(-1, 5, 2)$ and $(0, 2, -2)$, and the line L_2 is parallel to $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.
Find the angle between the lines.

6 The line L_1 has equation $\dfrac{x-2}{2} = \dfrac{y+3}{3} = 1 - z$. The line L_2 meets L_1 on the X-axis and passes through $(5, -3, 1)$. Find the angle between the lines.

Example 5 — Self Tutor

Find the measure of the angle between the lines $2x + y = 5$ and $3x - 2y = 8$.

$2x + y = 5$ has gradient $-\frac{2}{1}$ $\therefore$ it has direction vector $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$3x - 2y = 8$ has gradient $\frac{3}{2}$ $\therefore$ it has direction vector $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

If the angle between the lines is θ, then

$$\cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|} = \frac{|(1 \times 2) + (-2 \times 3)|}{\sqrt{1+4}\sqrt{4+9}} = \frac{4}{\sqrt{5}\sqrt{13}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{4}{\sqrt{65}}\right) \approx 60.3^\circ$$

$\therefore$ the angle between the lines is about 60.3°.

If a line has gradient $\frac{b}{a}$, it has direction vector $\begin{pmatrix} a \\ b \end{pmatrix}$.



7 Find the measure of the angle between the lines:

a $x - y = 3$ and $3x + 2y = 11$

b $y = x + 2$ and $y = 1 - 3x$

c $y + x = 7$ and $x - 3y + 2 = 0$

d $y = 2 - x$ and $x - 2y = 7$

C CONSTANT VELOCITY PROBLEMS

An object moving with a constant velocity will travel in a straight line. To model the position using vectors:

- the **velocity vector** of the motion gives the direction vector of the line
- **time** is the parameter
- the **initial position** of the object gives a fixed point on the line.

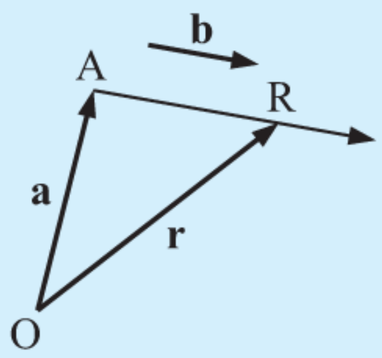
Make sure the units of distance, time, and velocity correspond.



If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\mathbf{b}|$.



Example 6 **Self Tutor**

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position.

b Plot the path of the object for $t = 0, 1, 2, 3$.

c Find the velocity vector of the object.

d Find the object's speed.

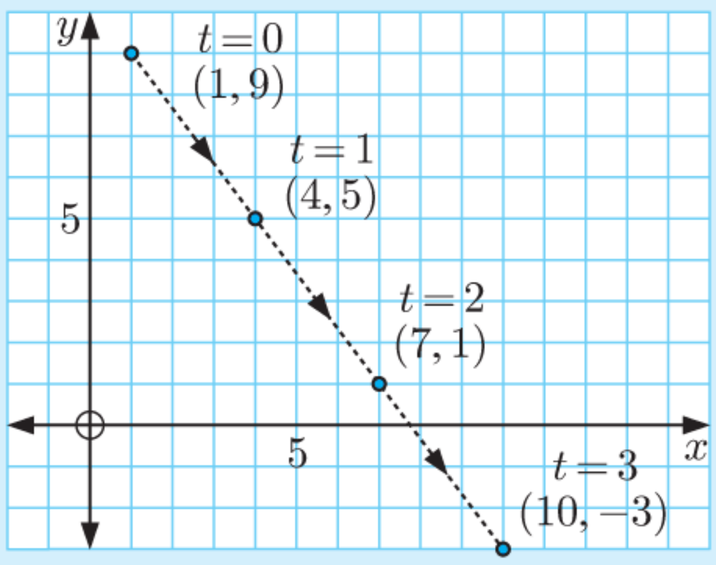
e If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.



a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$\therefore$ the object is at $(1, 9)$.

b



c The velocity vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2}$

$= 5$ m s^{-1}.

c Previously, the speed was 5 m s^{-1} and the velocity vector was $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\therefore$ the new velocity vector is $\frac{30}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

Velocity is a vector. Speed is a scalar.



Example 7 **Self Tutor**

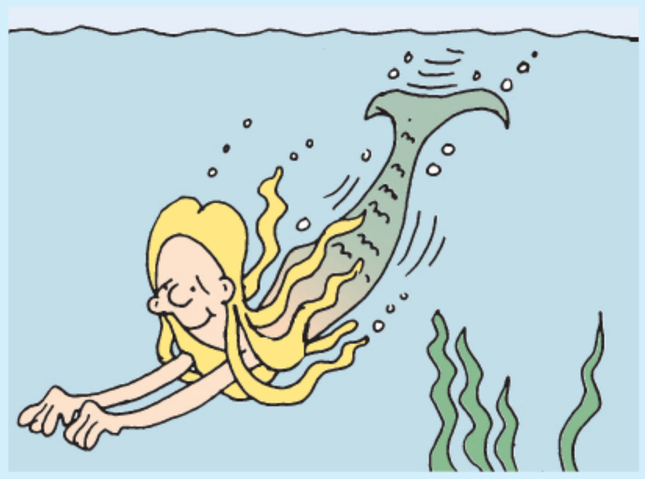
A mermaid is initially at $(15, 8, -1)$. She swims with velocity vector $3\mathbf{i} - 5\mathbf{j} - \frac{9}{2}\mathbf{k}$ metres per second.

Find:

a the position of the mermaid after t seconds

b the speed of the mermaid

c the time when the mermaid reaches depth 28 m.



a $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 8 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -5 \\ -\frac{9}{2} \end{pmatrix} = \begin{pmatrix} 15+3t \\ 8-5t \\ -1-\frac{9}{2}t \end{pmatrix}, \quad t \in \mathbb{R}$$

After t seconds the mermaid is at $(15+3t,\ 8-5t,\ -1-\frac{9}{2}t)$.

b The speed of the mermaid

$$= \sqrt{3^2 + (-5)^2 + (-\tfrac{9}{2})^2}$$
$$= \sqrt{54\tfrac{1}{4}}$$
$$\approx 7.37 \text{ m s}^{-1}$$

c The mermaid reaches depth 28 m when

$$z(t) = -1 - \tfrac{9}{2}t = -28$$
$$\therefore \ \tfrac{9}{2}t = 27$$
$$\therefore \ t = 6 \text{ seconds}$$

EXERCISE 13C

1 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

a Find the initial position of P.

b Plot the path of P for $t = 0, 1, 2, 3$.

c Find the velocity vector of P.

d Find the speed of P.

e A second particle is travelling in the opposite direction, but with speed 8 cm s^{-1}. Find its velocity vector.

2 Each of the following vector equations represents the path of a moving object. t is measured in seconds, $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$

b $x = 3 + 2t,\ y = -t,\ z = 4 - 2t$

3 **a** Find the vector equation of a boat initially at $(2, 3)$, which travels with velocity vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$. The grid units are kilometres, and the time is in hours.

b Locate the boat's position after 90 minutes.

c How long will it take for the boat to reach the point $(5, -0.75)$?

4 A remote controlled toy car is initially at the point $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.

a Write an expression for the position vector of the car at any time $t \geqslant 0$.

b Hence find the position vector of the car at time $t = 2.5$.

c Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.

d Plot the car's position at times $t = 0, 0.5, 1, 1.5, 2, 2.5$.

5 Find the velocity vector of:

a a speed boat moving parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with speed 150 km h^{-1}

b a jogger moving parallel to $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ with speed 7.8 km h^{-1}

c a ferry moving parallel to $6\mathbf{i} + 7\mathbf{j}$ with speed 25 km h^{-1}

d a hot air balloon moving parallel to $4\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ with speed 33 km h^{-1}

e a swooping eagle moving in the direction $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$ with speed 90 km h^{-1}.

6 Yacht A moves according to $x_A(t) = 4 + t$, $y_A(t) = 5 - 2t$, $t \geqslant 0$. Yacht B moves according to $x_B(t) = 1 + 2t$, $y_B(t) = -8 + t$, $t \geqslant 0$. The distance units are kilometres, and the time units are hours.

a Find the initial position of each yacht. **b** Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Show, using vector algebra, that the paths of the yachts are at right angles to each other.

7 At 1:34 pm, submarine P fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ from the point $(-5, 4)$.

a minutes later submarine Q fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ from the point $(15, 7)$.

Distances are measured in kilometres, and time is in minutes.

a Show that the position of P's torpedo can be written as $P(x_1(t), y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.

b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $Q(x_2(t), y_2(t))$ where $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$.

d Find the angle between the paths of the torpedoes.

e Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo and at what time did the explosion occur?

8 A cable car leaves from $(10, 3, 0)$. It moves in the direction $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ at 4.5 m s^{-1}.

a Find the velocity vector of the cable car.

b Find the position of the cable car after 30 seconds.

c The destination of the cable car has X-coordinate 550. Find the length of the cable car ride.

d At what angle to the horizontal does the cable car travel?



9 A helicopter at $A(6, 9, 3)$ moves with constant velocity. 10 minutes later it is at $B(3, 10, 2.5)$. Distances are in kilometres.

a Find $\overrightarrow{AB}$. **b** Find the helicopter's speed.

c Determine the vector equation which represents the path of the helicopter.

d The helicopter is travelling directly towards its helipad, which has Z-coordinate 0. Find the total time taken for the helicopter to land.

e At what angle to the horizontal is the helicopter flying?

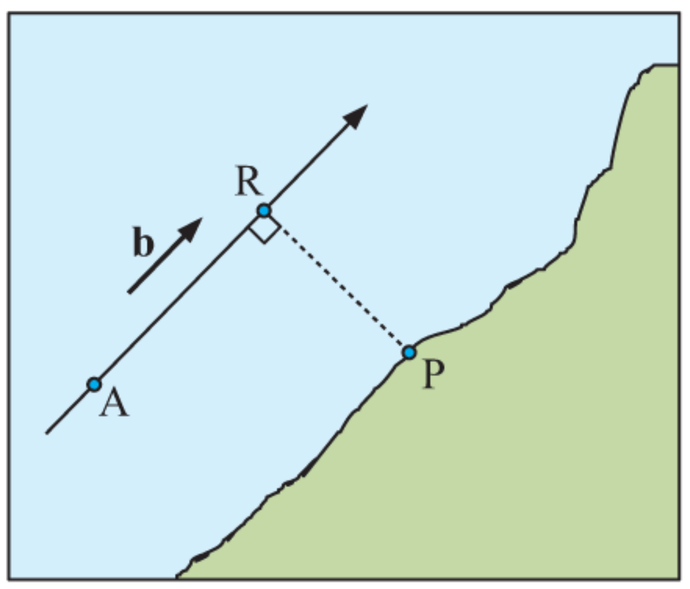
D THE SHORTEST DISTANCE FROM A POINT TO A LINE

A ship R sails through point A in the direction **b** and continues past a port P.

The ship is *closest* to the port when [PR] is perpendicular to [AR].

$\therefore \quad \overrightarrow{PR} \bullet \mathbf{b} = 0$

In this situation, point R is called the **foot of the perpendicular from P to the line**.



Example 8 **Self Tutor**

A line has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$. Find the shortest distance from the point $P(5, -1)$ to the line.

Let N be the point on the line closest to P.

N has coordinates $(1 + 3t, 2 - t)$ for some t, and $\overrightarrow{PN}$ is $\begin{pmatrix} 1 + 3t - 5 \\ 2 - t - (-1) \end{pmatrix} = \begin{pmatrix} 3t - 4 \\ 3 - t \end{pmatrix}$.

The distance between P and the line is minimised when $\overrightarrow{PN} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$.

$\therefore \quad \begin{pmatrix} 3t - 4 \\ 3 - t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$

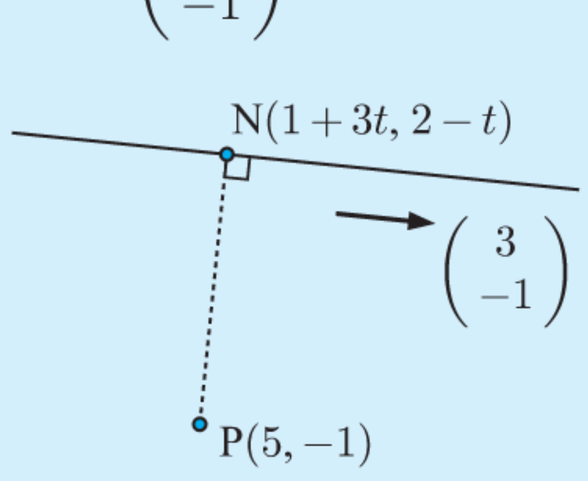
$\therefore \quad 3(3t - 4) - (3 - t) = 0$

$\therefore \quad 9t - 12 - 3 + t = 0$

$\therefore \quad 10t = 15$

$\therefore \quad t = \frac{15}{10} = \frac{3}{2}$

Thus $\overrightarrow{PN} = \begin{pmatrix} \frac{9}{2} - 4 \\ 3 - \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



$\therefore \quad \overrightarrow{PN} = \frac{1}{2}\sqrt{1^2 + 3^2} = \frac{\sqrt{10}}{2}$ units

$\therefore$ the shortest distance from the point P to the line is $\frac{\sqrt{10}}{2}$ units.

EXERCISE 13D

1 Find the shortest distance from:

a P$(3, 2)$ to the line with parametric equations $x = 2 + t,\ y = 3 + 2t,\ t \in \mathbb{R}$

b Q$(-1, 1)$ to the line with parametric equations $x = t,\ y = 1 - t,\ t \in \mathbb{R}$

c R$(-3, -1)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ s \in \mathbb{R}$

d S$(5, -2)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -7 \end{pmatrix},\ t \in \mathbb{R}$.

2 An ocean liner is at $(6, -6)$, cruising at 10 km h^{-1} in the direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
A fishing boat is anchored at $(0, 0)$. Distances are in kilometres.

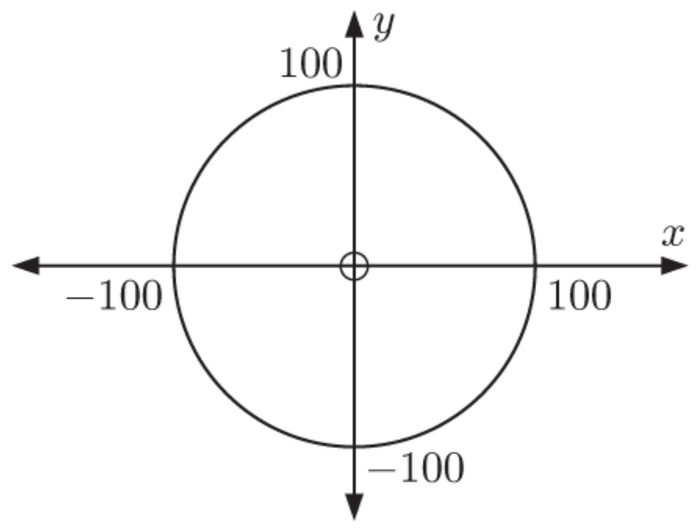
a Find, in terms of **i** and **j**, the initial position vector of the liner from the fishing boat.

b Write an expression for the position vector of the liner at any time t hours after it has sailed from $(6, -6)$.

c When will the liner be due east of the fishing boat?

d Find the time and position of the liner when it is nearest to the fishing boat.

3 Let **i** represent a displacement 1 km due east
and **j** represent a displacement 1 km due north.
The control tower of an airport is at $(0, 0)$. Aircraft within 100 km of $(0, 0)$ are visible on the radar screen at the control tower.
At 12:00 noon an aircraft is 200 km east and 100 km north of the control tower. It is flying parallel to the vector $\mathbf{b} = -3\mathbf{i} - \mathbf{j}$ with speed $40\sqrt{10}$ km h^{-1}.



a Write down the velocity vector of the aircraft.

b Write a vector equation for the path of the aircraft using t to represent the time in hours that has elapsed since 12:00 noon.

c Find the position of the aircraft at 1:00 pm.

d Show that the aircraft first becomes visible on the radar screen at 1:00 pm.

e Find the time when the aircraft is closest to the control tower, and find the distance between the aircraft and the control tower at this time.

f At what time will the aircraft disappear from the radar screen?

Example 9 — Self Tutor

Consider the point P$(-1, 2, 3)$ and the line with parametric equations $x = 1 + 2t,\ y = -4 + 3t,\ z = 3 + t,\ t \in \mathbb{R}$.

a Find the coordinates of the foot of the perpendicular from P to the line.

b Find the shortest distance from the point to the line.

a The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let $A(1+2t,\ -4+3t,\ 3+t)$ be any point on the given line.

$$\therefore\ \overrightarrow{PA} = \begin{pmatrix} 1+2t-(-1) \\ -4+3t-2 \\ 3+t-3 \end{pmatrix} = \begin{pmatrix} 2+2t \\ -6+3t \\ t \end{pmatrix}$$

If A is the closest point on the line to P, then $\overrightarrow{PA} \perp \mathbf{b}$.

$$\therefore\ \overrightarrow{PA} \bullet \mathbf{b} = 0$$

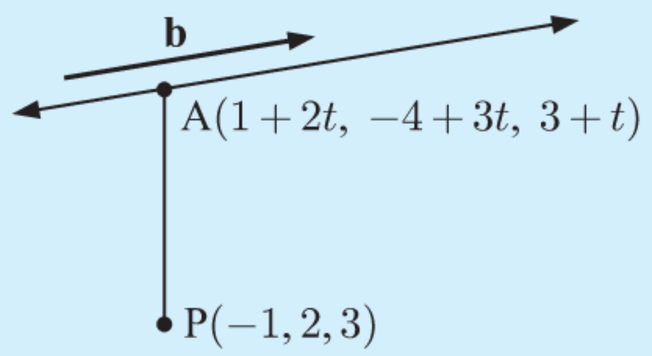
$$\therefore\ \begin{pmatrix} 2+2t \\ -6+3t \\ t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore\ 2(2+2t)+3(-6+3t)+1(t) = 0$$

$$\therefore\ 4+4t-18+9t+t = 0$$

$$\therefore\ 14t = 14$$

$$\therefore\ t = 1$$



Substituting $t = 1$ into the parametric equations, the foot of the perpendicular is $(3, -1, 4)$.

b Using $t = 1$, $\overrightarrow{PA} = \begin{pmatrix} 2+2 \\ -6+3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

$$\therefore\ |\overrightarrow{PA}| = \sqrt{4^2+(-3)^2+1^2} = \sqrt{26} \text{ units}$$

So, the shortest distance from P to the line is $\sqrt{26}$ units.

4 **a** Find the coordinates of the foot of the perpendicular from $(3, 0, -1)$ to the line with parametric equations $x = 2+3t,\ y = -1+2t,\ z = 4+t,\ t \in \mathbb{R}$.

b Hence find the shortest distance from the line to the point.

5 **a** Find the coordinates of the foot of the perpendicular from $(1, 1, 3)$ to the line with vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},\ t \in \mathbb{R}$.

b Hence find the shortest distance from the point to the line.

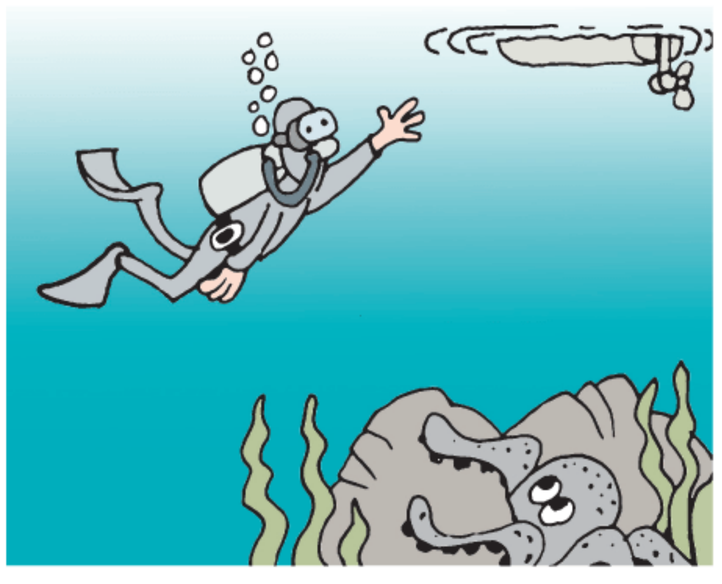
6 A diver swims from $(12, 25, -20)$ back to his boat at $(7, -15, 0)$, at a speed of 0.9 m s^{-1}.

a Find the velocity vector of the diver.

b An octopus watches the diver from its home at $(12, -8, -5)$.

i At what time is the diver closest to the octopus?

ii Find the shortest distance from the diver to the octopus.



E INTERSECTING LINES

Vector equations of two intersecting lines can be **solved simultaneously** to find the point where the lines meet.

Example 10 **Self Tutor**

Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \ s \in \mathbb{R}.$

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}.$

Find the point where the two lines meet.

The lines meet where $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore \ -2 + 3s = 15 - 4t$ and $1 + 2s = 5 + t$

$\therefore \ 3s + 4t = 17$ .... (1) and $2s - t = 4$ .... (2)

$$\begin{array}{rll} 3s + 4t = 17 & & \{(1)\} \\ 8s - 4t = 16 & & \{(2) \times 4\} \\ \hline \therefore \ 11s = 33 & & \\ \therefore \ s = 3 & & \end{array}$$



Using (2), $2(3) - t = 4$

$\therefore \ t = 2$

Using line 1, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Checking in line 2, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$ ✓

$\therefore$ the lines meet at $(7, 7)$.

EXERCISE 13E

1 Triangle ABC is formed by three lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, (AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and

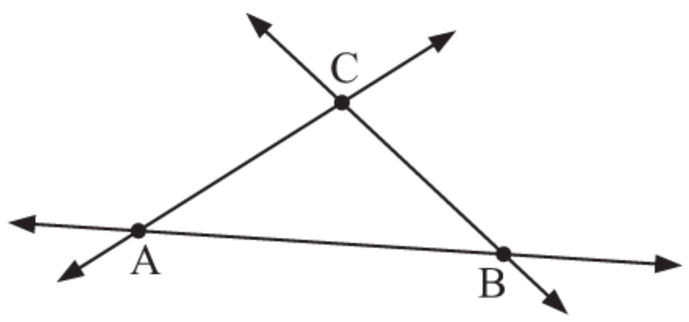
(BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, where r, s, and t are scalars.

a Draw the three lines accurately on a grid.

b Hence find the coordinates of A, B, and C.

c Prove that triangle ABC is isosceles.

d Use vector methods to check your answers to **b**.



2 The parallelogram ABCD is defined by four lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + r\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, (AD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

(CD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + t\begin{pmatrix} -7 \\ -3 \end{pmatrix}$, (CB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + u\begin{pmatrix} -1 \\ -2 \end{pmatrix}$,

where r, s, t, and u are scalars.

a Draw an accurate sketch of the four lines and the parallelogram formed by them.

b Hence find the coordinates of A, B, C, and D.

c Use vector methods to check your answers to **b**.

3 The isosceles triangle ABC is defined by three lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix}$, and

(AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where r, s and t are scalars.

a Use vector methods to find the coordinates of A, B, and C.

b Find all side lengths of the triangle, and hence determine which two sides are equal in length.

4 (QP) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 14 \\ 10 \end{pmatrix}$, (QR) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 17 \\ -9 \end{pmatrix}$, and

(PR) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t\begin{pmatrix} 5 \\ -7 \end{pmatrix}$, where r, s, and t are scalars.

a Use vector methods to find the coordinates of P, Q, and R.

b Find vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and evaluate $\overrightarrow{PQ} \bullet \overrightarrow{PR}$.

c Hence find the size of $\widehat{QPR}$.

d Find the area of triangle PQR.

5 (AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix}$,

(CD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix}$, and (AD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$,

where r, s, t, and u are scalars.

a Use vector methods to find the coordinates of A, B, C, and D.

b Find:

i $\overrightarrow{AC}$ and $\overrightarrow{DB}$ **ii** $|\overrightarrow{AC}|$ and $|\overrightarrow{DB}|$ **iii** $\overrightarrow{AC} \bullet \overrightarrow{DB}$

iv the midpoints of [AC] and [DB].

c What do the answers to **b** tell you about quadrilateral ABCD?

THEORY OF KNOWLEDGE

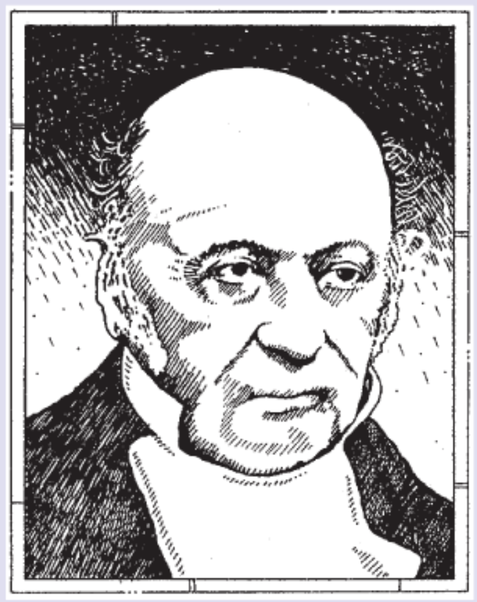
In his 1827 book entitled *The Barycentric Calculus*, German mathematician **August Möbius** demonstrated the use of directed line segments in projective geometry. He discussed the addition and scalar multiplication of directed line segments, but he did not name them vectors.

Gauss, **Grassmann**, **Laplace**, and **Lagrange** all contributed to build the field we know as vector analysis today.

An important contribution was made in 1843 by **Sir William Hamilton**, who defined a *quaternion* as the quotient of two directed line segments.

The American mathematician **Josiah Willard Gibbs** (1839 - 1903) began teaching a course on vector analysis at Yale University in 1879. He published *Elements of Vector Analysis* in two halves, and in 1901 his work was summarised by his student **Edwin Wilson** in the book *Vector Analysis*.

Oliver Heaviside (1850 - 1925) developed vector analysis separately from Gibbs, receiving a copy of Gibbs' work later, in 1888. Heaviside wrote:



Sir William Hamilton

"... the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius."

1 Are geometry and algebra two separate domains of knowledge?

2 Given that Gibbs and Heaviside developed vector analysis independently, who can claim to be the founder of this field?

3 Does mathematics *evolve*?

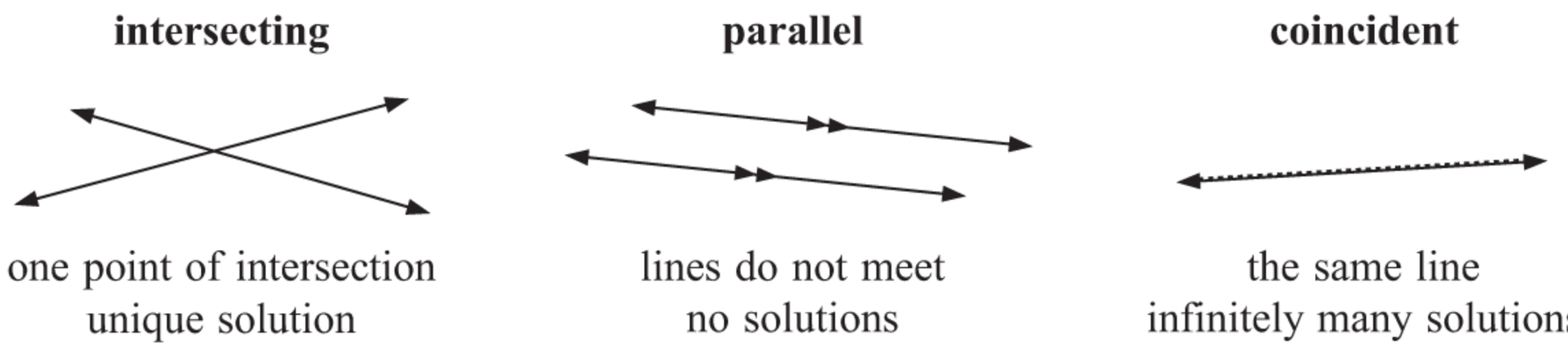
F RELATIONSHIPS BETWEEN LINES

We have just seen how the intersection of two lines in parametric form can be found by vector methods. We have also seen how to determine whether two lines are parallel.

We now consider finding the possible relationships between lines in Cartesian form. We will consider lines in both 2 and 3 dimensions.

LINE CLASSIFICATION IN 2 DIMENSIONS

There are three possible relationships between two lines in a plane:



In order to classify a pair of lines in 2 dimensions, we attempt to solve the equations of the lines simultaneously using row reduction.

Example 11 **Self Tutor**

Reduce the system $\begin{cases} 3x + 5y = 9 \\ 2x - 3y = -13 \end{cases}$ using row reduction. Hence describe the relationship between the pair of lines.

In augmented matrix form, the system is

$$\left(\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & -3 & -13 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & \frac{5}{3} & 3 \\ 0 & 19 & 57 \end{array}\right) \quad \begin{array}{r} \frac{1}{3}R_1 \to R_2 \\ 2R_1 - 3R_2 \to R_2 \end{array}$$

$$\left\{\begin{array}{ccc} 6 & 10 & 18 \\ -6 & 9 & 39 \\ \hline 0 & 19 & 57 \end{array}\right\}$$

$$\sim \left(\begin{array}{cc|c} 1 & \frac{5}{3} & 3 \\ 0 & 1 & 3 \end{array}\right) \quad \frac{1}{19}R_2 \to R_2$$

From row 2, $y = 3$

Substituting into the first equation, $x + \frac{5}{3}(3) = 3$

$$\therefore \quad x = -2$$

So, the lines intersect at the point $(-2, 3)$.

Example 12 **Self Tutor**

Use row operations to solve $\begin{cases} x + 3y = 5 \\ 4x + 12y = k \end{cases}$ where k is a constant.

Give a geometric interpretation of the solutions.

In augmented matrix form, the system is:

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 4 & 12 & k \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & k-20 \end{array}\right) \quad R_2 - 4R_1 \to R_2$$

$$\left\{\begin{array}{ccc} 4 & 12 & k \\ -4 & -12 & -20 \\ \hline 0 & 0 & k-20 \end{array}\right\}$$

The second line reads $0x + 0y = k - 20$

Case 1: If $k \neq 20$ we have $0 =$ a non-zero number, which is impossible. There are no solutions, so the equations represent parallel lines.

Case 2: If $k = 20$ we have $0 = 0$, which is true for all x and y.

This means that the lines are coincident, and we can obtain the solutions from $x + 3y = 5$ alone.

Letting $y = t$, $x = 5 - 3t$ for all values of t.

$\therefore$ there are infinitely many solutions of the form $x = 5 - 3t$, $y = t$, $t \in \mathbb{R}$.

EXERCISE 13F.1

1 By inspection, classify the following pairs of equations as either intersecting, parallel, or coincident lines:

a $\begin{cases} x - 3y = 2 \\ 3x + y = 8 \end{cases}$ **b** $\begin{cases} x + y = 7 \\ 3x + 3y = 1 \end{cases}$ **c** $\begin{cases} 4x - y = 8 \\ y = 2 \end{cases}$

d $\begin{cases} x - 2y = 4 \\ 2x - 4y = 8 \end{cases}$ **e** $\begin{cases} 5x - 11y = 2 \\ 6x + y = 8 \end{cases}$ **f** $\begin{cases} 3x - 4y = 5 \\ -3x + 4y = 2 \end{cases}$

2 Reduce the system $\begin{cases} 4x + 5y = 15 \\ 3x - 2y = 17 \end{cases}$ using row reduction. Hence describe the relationship between the pair of lines.

3 Consider the equation pair $\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases}$.

a Explain why there are infinitely many solutions, giving geometric evidence.

b Explain why the second equation can be ignored when finding all solutions.

c Give all solutions in the form:

i $x = t,\ y = ...$ **ii** $y = s,\ x = ...$

4 **a** Use row reduction on the system $\begin{cases} 2x + 3y = 5 \\ 2x + 3y = 11 \end{cases}$ to reduce the system to $\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right)$.

b What does the second row indicate? Explain this result geometrically.

5 **a** Use row reduction on the system $\begin{cases} 2x + 3y = 5 \\ 4x + 6y = 10 \end{cases}$ to reduce the system to $\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right)$.

b Explain this result geometrically.

6 **a** By using augmented matrices, show that $\begin{cases} 3x - y = 2 \\ 6x - 2y = 4 \end{cases}$ has infinitely many solutions of the form $x = t,\ y = 3t - 2$.

b Discuss the solutions to $\begin{cases} 3x - y = 2 \\ 6x - 2y = k \end{cases}$ where k can take any real value.

7 Consider $\begin{cases} 3x - y = 8 \\ 6x - 2y = k \end{cases}$ where k can take any real value.

a Use row operations to reduce the system to the form: $\left(\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 0 & .... \end{array}\right)$.

b For what value(s) of k is there infinitely many solutions? Find the solutions in this case.

c For what value(s) of k does the system have no solutions? Explain this result geometrically.

8 Consider the system $\begin{cases} 4x + 8y = 1 \\ 2x - ay = 11 \end{cases}$ where a can take any real value.

a Use row operations to reduce the system to row echelon form.

b For what values of a does the system have a unique solution?

Show that the unique solution is $x = \dfrac{a + 88}{4a + 16},\ y = \dfrac{-21}{2a + 8}$ for these values of a.

c What is the solution in all other cases? Interpret the situation geometrically.

9 Consider the system $\begin{cases} mx + 2y = 6 \\ 2x + my = 6 \end{cases}$ where m can take any real value.

a Find the value(s) of m for which the system has a unique solution, and state the form of the unique solution. Interpret this solution geometrically.

b Discuss the solutions in the other *two* cases.

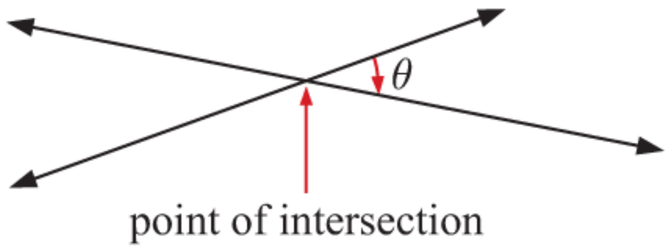
LINE CLASSIFICATION IN 3 DIMENSIONS

Lines in space are **coplanar** if they lie in the same plane. In this case they may be **intersecting**, **parallel**, or **coincident**.

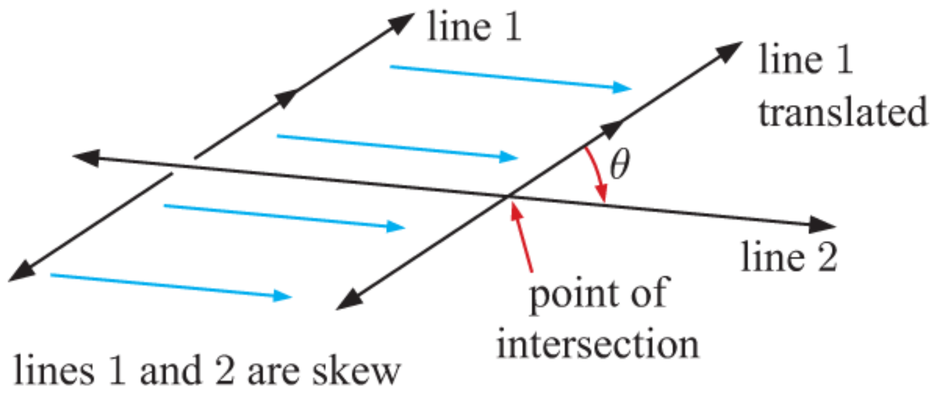
If the lines are not coplanar then they are **skew**.

We calculate the angle between two lines in space using the scalar product of the direction vectors of the lines.

- If the lines are **parallel**, the angle between them is $0°$.
- If the lines are **intersecting**, the angle between them is θ, as shown.



- If the lines are **skew**, we suppose one line is translated to intersect with the other. The angle between the original lines is defined as the angle between the intersecting lines, which is the angle θ.



Example 13 **Self Tutor**

Line 1 has equations $x = -1 + 2s,\ y = 1 - 2s,\ z = 1 + 4s,\ s \in \mathbb{R}$.

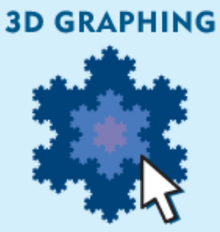
Line 2 has equations $x = 1 - t,\ y = t,\ z = 3 - 2t,\ t \in \mathbb{R}$.

Line 3 has equations $x = 1 + 2u,\ y = -1 - u,\ z = 4 + 3u,\ u \in \mathbb{R}$.

a Show that line 1 and line 2 are parallel but not coincident.

b Show that line 2 and line 3 intersect and find the angle between them.

c Show that line 1 and line 3 are skew.



a Line 1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$ with direction vector $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$.

Line 2 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ with direction vector $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

Since $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, the lines are parallel.

If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then $\mathbf{a} \parallel \mathbf{b}$.



When $s = 0$, the point on line 1 is $(-1, 1, 1)$.

For line 2, $y = t$.

$\therefore$ the unique point on line 2 with y-coordinate 1 is the point where $t = 1$. This point is $(0, 1, 1)$.

Since $(0, 1, 1) \neq (-1, 1, 1)$, the lines are not coincident.

b Equating x, y, and z values in lines 2 and 3 gives

$$\begin{aligned} &1-t=1+2u, \quad t=-1-u \quad \text{and} \quad 3-2t=4+3u \\ &\therefore\ t=-2u, \quad t=-1-u \quad \text{and} \quad 3u+2t=-1 \quad \ldots.\ (1) \end{aligned}$$

Solving these we get $-2u=-1-u$

$$\therefore\ -u=-1$$

$$\therefore\ u=1 \text{ and so } t=-2$$

Checking in (1): $3u+2t=3(1)+2(-2)=3-4=-1$ ✓

Using $u=1$, lines 2 and 3 intersect at $(1+2(1),\ -1-(1),\ 4+3(1))$ which is $(3,\ -2,\ 7)$.

Direction vectors for lines 2 and 3 are $\mathbf{a}=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ respectively.

If θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta=\frac{|\mathbf{a}\bullet\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}=\frac{|-2-1-6|}{\sqrt{1+1+4}\sqrt{4+1+9}}=\frac{9}{\sqrt{84}}$$

$$\therefore\ \theta\approx 10.89^\circ$$

$\therefore$ the angle between lines 2 and 3 is about 10.9°.

c Equating x, y, and z values in lines 1 and 3 gives

$$\begin{aligned} &-1+2s=1+2u \qquad 1-2s=-1-u \quad \text{and} \quad 1+4s=4+3u \\ &\therefore\ 2s-2u=2 \qquad \therefore\ -2s+u=-2 \quad \text{and} \quad 4s-3u=3 \quad \ldots.\ (2) \end{aligned}$$

Solving these we get $2s-2u=2$

$$-2s+u=-2$$

Adding, $-u=0$ and so $s=1$

Checking in (2): $4s-3u=4(1)-3(0)=4\neq 3$ ✗

There is no simultaneous solution to all 3 equations, so the lines do not intersect.

The direction vector for line 1 is $\begin{pmatrix}2\\-2\\4\end{pmatrix}$ and $\begin{pmatrix}2\\-2\\4\end{pmatrix}\neq k\begin{pmatrix}2\\-1\\3\end{pmatrix}$ for any $k\in\mathbb{R}$.

$\therefore$ lines 1 and 3 are not parallel.

Since the lines are not parallel and do not intersect, they are skew.

EXERCISE 13F.2

1 Classify the following line pairs as either parallel, intersecting, coincident, or skew. Where possible, state the point of intersection and find the measure of the acute angle between them.

a $x=1+2t,\ y=2-t,\ z=3+t$ and $x=-2+3s,\ y=3-s,\ z=1+2s$

b $x=-1+2\lambda,\ y=2-12\lambda,\ z=4+12\lambda$ and $x=4\mu-3,\ y=3\mu+2,\ z=-\mu-1$

c $x=6t,\ y=3+8t,\ z=-1+2t$ and $x=2+3s,\ y=4s,\ z=1+s$

d $x=2-y=z+2$ and $x=1+3s,\ y=-2-2s,\ z=2s+\frac{1}{2}$

e $x=1+\lambda,\ y=2-\lambda,\ z=3+2\lambda$ and $x=2+3\mu,\ y=3-2\mu,\ z=\mu-5$

f $x=1-2t,\ y=8+t,\ z=5$ and $x=2+4s,\ y=-1-2s,\ z=3$

g $x=1+2\lambda,\ y=-\lambda,\ z=1+3\lambda$ and $x=3-4\mu,\ y=-1+2\mu,\ z=4-6\mu$

2 Discuss the relationship between:

Line 1: $\mathbf{r}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

Line 2: $x = 1 - 2\mu, \quad y = 4\mu, \quad z = -1 + 2\mu$

Line 3: $x = \dfrac{y-1}{2} = z - 1$

INVESTIGATION 1 MOTION OF PARTICLES IN SPACE

In this Investigation we consider the motion of two particles in space, and whether they will collide.

What to do:

1 Consider the lines L_1: $\begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$ and L_2: $\begin{pmatrix} 1 \\ -4 \\ 14 \end{pmatrix} + u \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$, $u \in \mathbb{R}$.

Show that these lines intersect, and give the coordinates of the point of intersection.

2 Suppose we have two particles moving in space.

Particle A's position after t seconds is given by

$$x_A(t) = 5 - 2t, \quad y_A(t) = 4 - 3t, \quad z_A(t) = 4 + 4t.$$

Particle B's position after t seconds is given by

$$x_B(t) = 1, \quad y_B(t) = -4 - 2t, \quad z_B(t) = 14 + 2t.$$

All distance units are metres.

a Find the initial position of each particle.

b Find the velocity vector of each particle.

c Will the particles collide? Explain your answer.

3 Suppose D is the distance between particles A and B at time t.

a Write an equation connecting D and t.

b Hence find the minimum distance between the particles, and the time when this occurs.

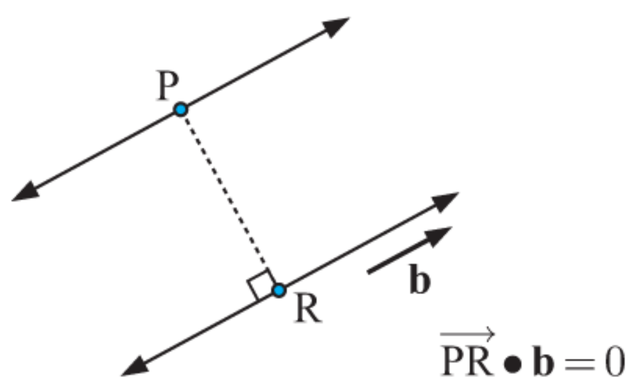
THE SHORTEST DISTANCE BETWEEN TWO LINES

We have seen that two lines in space may be *intersecting*, *parallel*, *coincident*, or *skew*.

If two lines in space intersect or are coincident, then clearly the shortest distance between them will be zero.

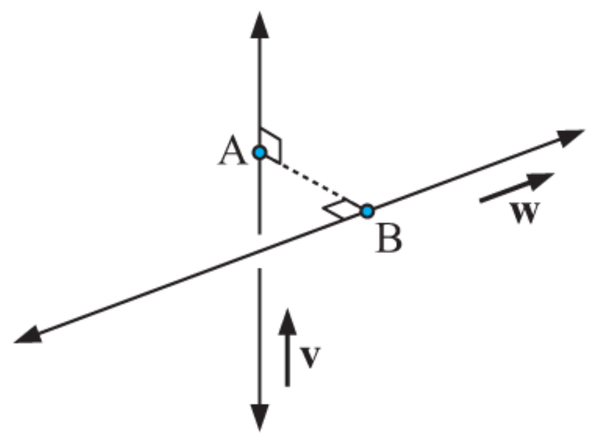
THE SHORTEST DISTANCE BETWEEN PARALLEL LINES

If the lines are **parallel**, we choose any point on one of the lines and find the shortest distance to the other line. We saw this in **Section D**.



THE SHORTEST DISTANCE BETWEEN SKEW LINES

If the lines are **skew**, we need to find two points A and B, one on each line, so that $\overrightarrow{AB}$ is perpendicular to *both* skew lines. So, if the direction vectors of the lines are $\mathbf{v}$ and $\mathbf{w}$ respectively, then $\overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}$.



We write $\overrightarrow{AB}$ in terms of the parameters for the two lines, then use $\overrightarrow{AB} = k(\mathbf{v} \times \mathbf{w})$ to write a set of three equations in three unknowns. We solve these simultaneously to find the points A and B.

The shortest distance between the lines is $|\overrightarrow{AB}|$.

Example 14 **Self Tutor**

Find the shortest distance between the skew lines $x = t,\ y = 1 - t,\ z = 2 + t$ and $x = 3 - s,\ y = -1 + 2s,\ z = 4 - s$, where $s, t \in \mathbb{R}$.

The lines have direction vectors $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

so $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + \mathbf{k}$

Let A and B be points on the skew lines such that $|\overrightarrow{AB}|$ is the shortest distance between them.

$\therefore$ A is $(t,\ 1 - t,\ 2 + t)$ and B is $(3 - s,\ -1 + 2s,\ 4 - s)$ for some $s, t \in \mathbb{R}$.

Now $\overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}$, so $\begin{pmatrix} 3 - s - t \\ -1 + 2s - 1 + t \\ 4 - s - 2 - t \end{pmatrix} = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ for some $k \in \mathbb{R}$.

$\therefore\ 3 - s - t = -k$ .... (1)

$-2 + 2s + t = 0$ .... (2)

$2 - s - t = k$ .... (3)

Adding (1) and (3) gives $5 - 2s - 2t = 0$

$\therefore\ 2s + 2t = 5$

and rearranging (2) gives $-2s - t = -2$

Adding, $t = 3$

Using (2), $-2 + 2s + 3 = 0$ and so $s = -\frac{1}{2}$.

$\therefore\ \overrightarrow{AB} = \begin{pmatrix} 3 - (-\frac{1}{2}) - 3 \\ -2 + 2(-\frac{1}{2}) + 3 \\ 2 - (-\frac{1}{2}) - 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$

and the shortest distance between the lines $= |\overrightarrow{AB}| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2}$

$= \frac{1}{\sqrt{2}}$ units.

EXERCISE 13F.3

1 Find the shortest distance between the skew lines:

a $x = 1 + 2t,\ y = -t,\ z = 2 + 3t,\ t \in \mathbb{R}$ and $x = y = z$

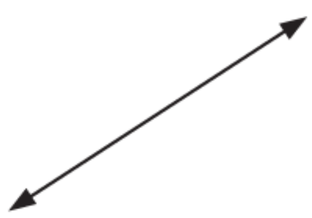
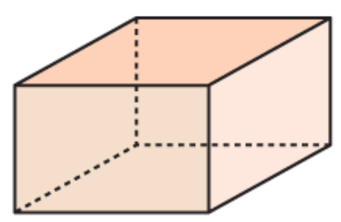
b $x = 1 - t,\ y = 1 + t,\ z = 3 - t$ and $x = 2 + s,\ y = 1 - 2s,\ z = s$, where $s, t \in \mathbb{R}$.

2 Find the shortest distance between each pair of lines in **Exercise 13F.2** question **1**.

G PLANES

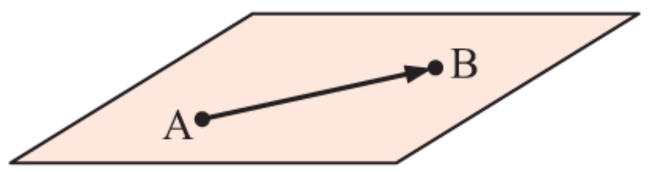
A **plane** is a flat surface that extends forever, and has zero thickness.

It fits into a scheme which starts with a point and which builds to three dimensional space.

point	**line**	**plane**	**space**
0 dimensions	1 dimension	2 dimensions	3 dimensions
•			

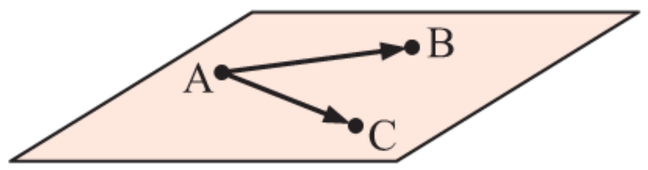
$\longrightarrow$

Suppose two points A and B lie on a plane. We say that AB is *parallel* to the plane.



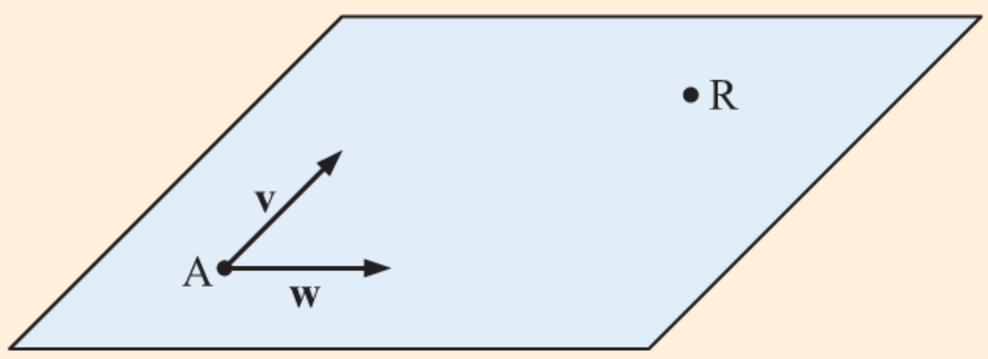
A vector is **parallel** to a plane if, by travelling along the vector, we remain on the plane.

In the diagram alongside, we can see two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ which are both parallel to the plane, but *not* parallel to each other.



INVESTIGATION 2 LINEAR COMBINATIONS

Suppose we are at a fixed point A with position vector **a** on a plane. Also suppose two non-parallel vectors **v** and **w** are both parallel to the plane.



In this Investigation we consider travelling from A to another point R using a **linear combination** of **v** and **w**, which means adding scalar multiples of **v** and **w** to the position vector of A. If the position vector of R is **r**, then $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ for some $s, t \in \mathbb{R}$.

What to do:

1 Explain why:

a the vector $\mathbf{a} + s\mathbf{v}$ lies on the plane for any $s \in \mathbb{R}$

b the vector $\mathbf{a} + s\mathbf{v} + t\mathbf{w}$ lies on the plane for any $s, t \in \mathbb{R}$

c it is important that $\mathbf{v}$ and $\mathbf{w}$ are non-parallel.

2 Consider two lines on the plane:

Line 1 has vector equation $\mathbf{a} + s\mathbf{v}$, $s \in \mathbb{R}$.

Line 2 has vector equation $\mathbf{r} - t\mathbf{w}$, $t \in \mathbb{R}$.

a Draw a vector diagram to illustrate this situation.

b Write a vector equation for the intersection of the two lines.

c Explain how the lines and the vector equation relate to our problem of moving from A to R using a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

d When we solve the vector equation to find s and t, explain why there will be a unique solution.

3 **a** Can *any* point on the plane be reached from A using a linear combination of $\mathbf{v}$ and $\mathbf{w}$?

b Can any point *not* on the plane be reached from A using a linear combination of $\mathbf{v}$ and $\mathbf{w}$?

THE VECTOR EQUATION OF A PLANE

From the **Investigation**, you should have found that any point on a plane can be reached from any other using a unique linear combination of any two non-parallel vectors which are parallel to the plane.

$\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ is the **vector equation of a plane**

where $\mathbf{r}$ is the position vector of any point on the plane,

$\mathbf{a}$ is the position vector of a known point on the plane

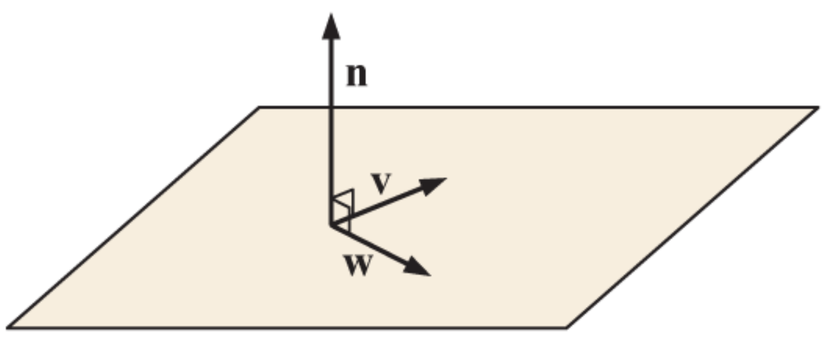
$\mathbf{v}$ and $\mathbf{w}$ are any two non-parallel vectors that are parallel to the plane

$s, t \in \mathbb{R}$ are two independent parameters.

THE NORMAL OF A PLANE

A vector is **normal** to a plane if it is perpendicular to all vectors which are parallel to the plane.

For example, if $\mathbf{v}$ and $\mathbf{w}$ are any two non-parallel vectors on a plane, then $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is normal to the plane.

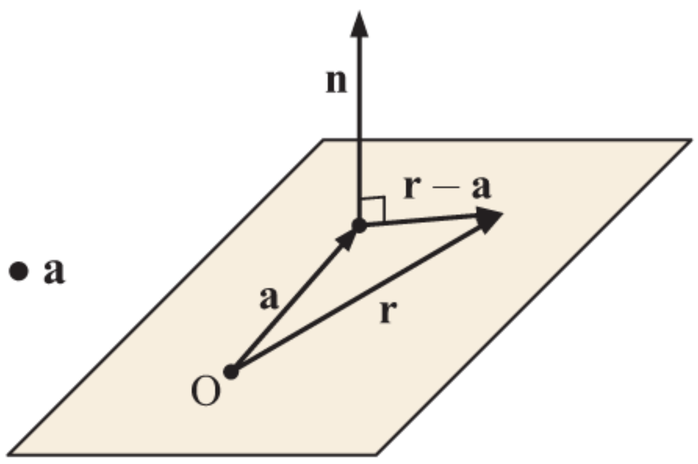


We can also write the vector equation of a plane using a normal vector.

Suppose $\mathbf{r}$ is the position vector of any point on a plane
and $\mathbf{a}$ is the position vector of a known point on the plane.

Since $\mathbf{n}$ is perpendicular to any vector in the plane, $\mathbf{n} \bullet (\mathbf{r} - \mathbf{a}) = 0$

$\therefore \quad \mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet \mathbf{a}$



The normal equation of a plane is $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$.

THE CARTESIAN EQUATION OF A PLANE

We can use this form of the vector equation of a plane to derive a Cartesian form:

If a plane has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and passes through (X, Y, Z) then it has equation $ax + by + cz = aX + bY + cZ = d$, where d is a constant.

This is the **Cartesian equation of the plane**.

Example 15 **Self Tutor**

Find the equation of the plane with normal vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and which passes through the point $(-1, 2, 4)$.

$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and the point $(-1, 2, 4)$ lies on the plane.

$\therefore$ the equation is $x + 2y + 3z = (-1) + 2(2) + 3(4)$

which is $x + 2y + 3z = 15$

EXERCISE 13G

1 Find the Cartesian equation of the plane:

a with normal vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and which passes through $(-1, 2, 4)$

b perpendicular to the line connecting $A(2, 3, 1)$ and $B(5, 7, 2)$, and which passes through A.

2 State the normal vector to the plane with equation:

a $2x + 3y - z = 8$ **b** $3x - y = 11$ **c** $z = 2$ **d** $x = 0$

3 Find the Cartesian equation of the:

a XZ-plane

b plane perpendicular to the Z-axis, which passes through $(2, -1, 4)$.

Example 16 **Self Tutor**

Find the equation of the plane through $A(-1, 2, 0)$, $B(3, 1, 1)$, and $C(1, 0, 3)$:

a in vector form **b** in Cartesian form.

a $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are two non-parallel vectors in the plane.

Using C as the known (fixed) point on the plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

b The normal vector $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 2 & -2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & -1 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$

$$= -1\mathbf{i} - 10\mathbf{j} - 6\mathbf{k}$$

$$= -(\mathbf{i} + 10\mathbf{j} + 6\mathbf{k})$$

$\therefore$ the vector $\begin{pmatrix} 1 \\ 10 \\ 6 \end{pmatrix}$ is also normal to the plane.

Thus the plane has equation

$$x + 10y + 6z = 1(-1) + 10(2) + 6(0) \quad \{\text{using point A}\}$$

$$\therefore x + 10y + 6z = 19$$



4 Find the equation of the plane through A, B, and C, giving your answer in:

i vector form **ii** Cartesian form.

a $A(0, 2, 6)$, $B(1, 3, 2)$, and $C(-1, 2, 4)$ **b** $A(3, 1, 2)$, $B(0, 4, 0)$, and $C(0, 0, 1)$

c $A(2, 0, 3)$, $B(0, -1, 2)$, and $C(4, -3, 0)$.

5 A plane has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, $s, t \in \mathbb{R}$. Write the equation of the plane in Cartesian form.

6 Find the Cartesian equation of the plane containing $A(3, 2, 1)$ and the line $x = 1 + t$, $y = 2 - t$, $z = 3 + 2t$, $t \in \mathbb{R}$.

7 Find the Cartesian equation of the plane through $A(1, 2, 3)$ and $B(0, -1, 2)$ which is parallel to:

a the X-axis **b** the Y-axis **c** the Z-axis.

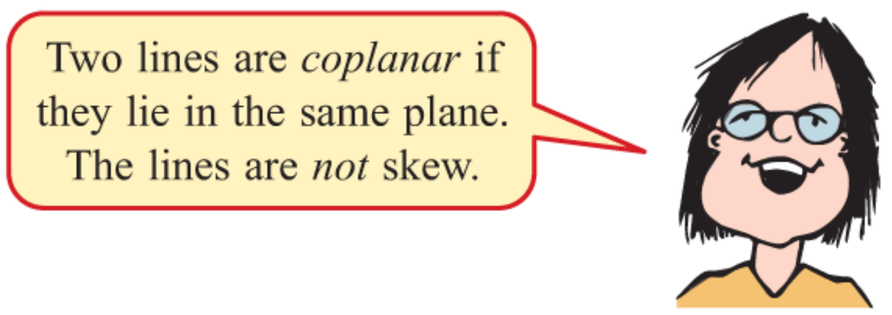
8 Find the vector equation of the line:

a which is normal to the plane $x - 3y + 4z = 8$ and passes through $(1, -2, 0)$

b which is normal to the plane $x - y - 2z = 11$ and passes through $(3, 4, -1)$.

9 Does the line passing through $(3, 4, -1)$ and which is normal to $x + 4y - z = -2$, intersect any of the coordinate axes?

10 Show that the lines $x - 1 = \dfrac{y-2}{2} = z + 3$ and $x + 1 = y - 3 = 2z + 5$ are coplanar, and find the Cartesian equation of the plane which contains them.



11 The equations of two lines are:

$$L_1: \quad \mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{and} \quad L_2: \quad x = \frac{y-5}{2} = \frac{-z-1}{2}.$$

a Find the point of intersection of L_1 and L_2.

b Find the equation of the plane that contains L_1 and L_2.

12 A$(1, 2, k)$ lies on the plane $x + 2y - 2z = 8$. (AB) is normal to the plane, and B is 6 units from it.

a Find the value of k.

b Find the coordinates of B.

Example 17 — Self Tutor

Find the parametric equations of the line through A$(-1, 2, 3)$ and B$(2, 0, -3)$.

Hence find where this line meets the plane with equation $x - 2y + 3z = 26$.

$\overrightarrow{\text{AB}} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$ so (AB) has parametric equations

$$x = -1 + 3t, \quad y = 2 - 2t, \quad z = 3 - 6t, \quad t \in \mathbb{R} \quad \ldots. \ (*)$$

This line meets the plane $x - 2y + 3z = 26$ where

$$-1 + 3t - 2(2 - 2t) + 3(3 - 6t) = 26$$
$$\therefore \ 4 - 11t = 26$$
$$\therefore \ -11t = 22$$
$$\therefore \ t = -2$$

Substituting $t = -2$ into $(*)$, the line meets the plane at $(-7, 6, 15)$.

13 Find the parametric equations of the line through A$(2, -1, 3)$ and B$(1, 2, 0)$.
Hence find where this line meets the plane with equation $x + 2y - z = 21$.

14 **a** Find the parametric equations of the line through P$(1, -2, 4)$ and Q$(2, 0, -1)$.

b Hence find where this line meets:

i the YZ-plane

ii the plane with equation $y + z = 2$

iii the line with Cartesian equation $\dfrac{x-3}{2} = \dfrac{y+2}{3} = \dfrac{z-30}{-1}$.

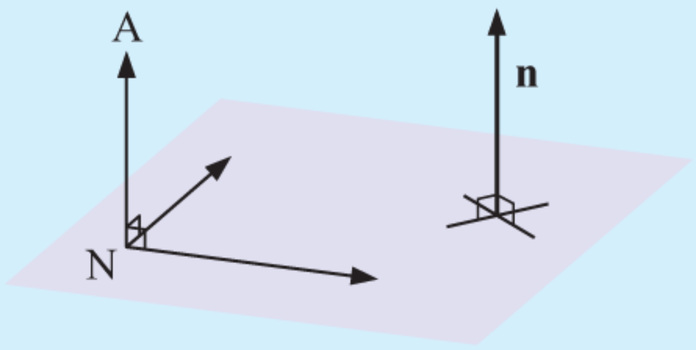
15 A plane P has equation $2x - 5y + z = 10$. A line L is normal to P, and passes through $(1, 5, 3)$.

a Find the parametric equations of L.

b Find the point at which L meets P.

Example 18 **Self Tutor**

Suppose N is the foot of the normal from $A(2, -1, 3)$ to the plane $x - y + 2z = 27$. Find the coordinates of N, and hence find the shortest distance from A to the plane.



$x - y + 2z = 27$ has normal vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ the parametric equations of (AN) are:

$$x = 2 + t, \quad y = -1 - t, \quad z = 3 + 2t, \quad t \in \mathbb{R}.$$

This line meets the plane $x - y + 2z = 27$ where

$$\begin{aligned} 2 + t - (-1 - t) + 2(3 + 2t) &= 27 \\ \therefore \quad 2 + t + 1 + t + 6 + 4t &= 27 \\ \therefore \quad 6t + 9 &= 27 \\ \therefore \quad 6t &= 18 \\ \therefore \quad t &= 3 \end{aligned}$$

Since N lies on the plane, its coordinates must satisfy the equation of the plane.



$\therefore$ N has coordinates $(5, -4, 9)$.

The shortest distance, $|\overrightarrow{AN}| = \sqrt{(5-2)^2 + (-4-(-1))^2 + (9-3)^2}$

$= \sqrt{54}$ units.

16 Find the foot of the normal from A to the given plane, and hence find the shortest distance from A to the plane:

a $A(1, 0, 2)$, $2x + y - 2z + 11 = 0$

b $A(2, -1, 3)$, $x - y + 3z = -10$

c $A(1, -4, -3)$, $4x - y - 2z = 8$

17 Find the coordinates of the mirror image of $A(3, 1, 2)$ when reflected in the plane $x + 2y + z = 1$.

18 A plane P has equation $\mathbf{r} = 2\mathbf{i} - 4\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ where $s, t \in \mathbb{R}$.

a Write the equation of P in Cartesian form.

b Find the coordinates of the foot of the normal from $A(5, 8, 5)$ to P.

19 In each of the following cases, find the foot of the normal from A to the given plane, and hence find the shortest distance from A to the plane:

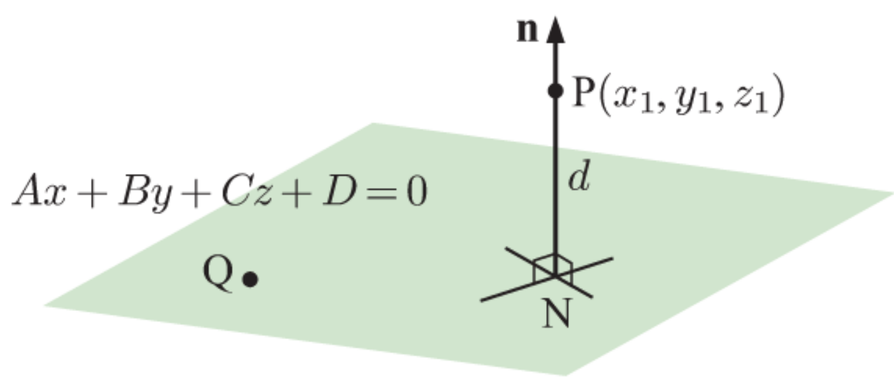
a $A(3, 2, 1)$, $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$, $\lambda, \mu \in \mathbb{R}$

b $A(1, 0, -2)$, $\mathbf{r} = \mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k})$, $\lambda, \mu \in \mathbb{R}$

20 Suppose $\mathbf{v}$ and $\mathbf{w}$ are non-parallel vectors which lie in the same plane. The normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w}$.

Prove that $\mathbf{n} \perp (s\mathbf{v} + t\mathbf{w})$ for all $s, t \in \mathbb{R}$ except when $s = t = 0$.

21 Suppose Q is any point in the plane $Ax + By + Cz + D = 0$. Let d be the distance from $P(x_1, y_1, z_1)$ to the given plane.



a Explain why $d = \dfrac{|\overrightarrow{QP} \bullet \mathbf{n}|}{|\mathbf{n}|}$.

b Hence show that $d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

c Check your answers to question **16** using the formula in **b**.

d Find the distance from:

i $(0, 0, 0)$ to $x + 2y - z = 10$ **ii** $(1, -3, 2)$ to $x + y - z = 2$.

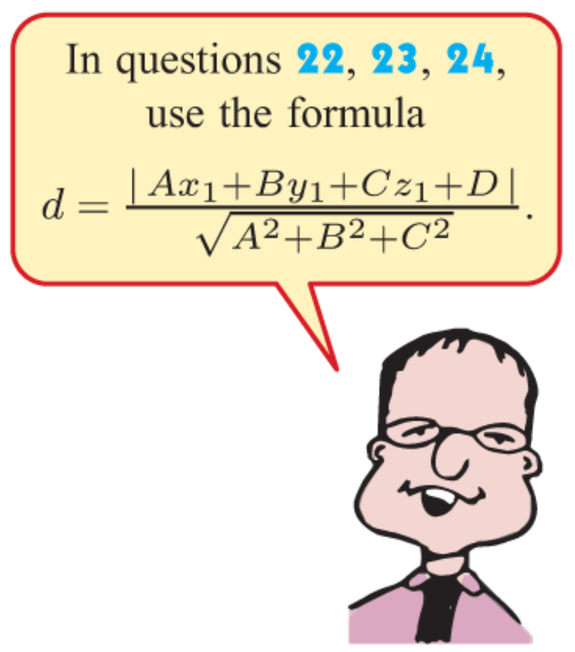
22 Find the distance between the parallel planes:

a $x + y + 2z = 4$ and $2x + 2y + 4z + 11 = 0$

b $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$.

23 Show that the line $x = 2 + t$, $y = -1 + 2t$, $z = -3t$ is parallel to the plane $11x - 4y + z = 0$, and find its distance from the plane.

24 Find the Cartesian equations of the two planes which are parallel to $2x - y + 2z = 5$ and 2 units from it.



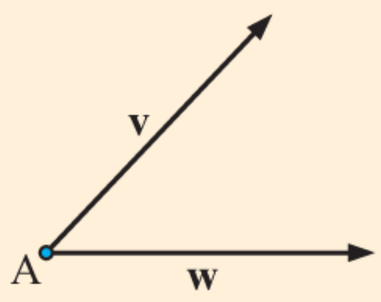
INVESTIGATION 3 REGIONS OF A PLANE

We have previously seen that the position of any point on a plane can be described by $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ where $\mathbf{a}$ is the position vector of a fixed point on the plane, and $s\mathbf{v} + t\mathbf{w}$ is a linear combination of any two non-parallel vectors which are parallel to the plane.

We will now investigate *regions* of a plane by restricting the possible values of the parameters s and t.

What to do:

1



a Copy the diagram alongside. Shade the region which includes all points $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ where $\mathbf{a}$ is the position vector of point A, $0 \leqslant s \leqslant 1$, and $0 \leqslant t \leqslant 1$.

b Describe the shape of the region.

c What would the shape of the region be for $0 \leqslant s \leqslant 2$ and $0 \leqslant t \leqslant 1$?

2 Suppose $\mathbf{v}$ and $\mathbf{w}$ are perpendicular and that $|\mathbf{v}| = |\mathbf{w}|$. Using a sketch to help you if necessary, describe the shape of the region formed on the plane $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ for:

a $0 \leqslant s \leqslant 1$ and $0 \leqslant t \leqslant 1$ **b** $0 \leqslant s \leqslant 2$ and $0 \leqslant t \leqslant 1$

3 Consider the range of values $s^2 + t^2 \leqslant 1$. Describe the shape of the region formed on the plane $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ if $\mathbf{v}$ and $\mathbf{w}$ are:

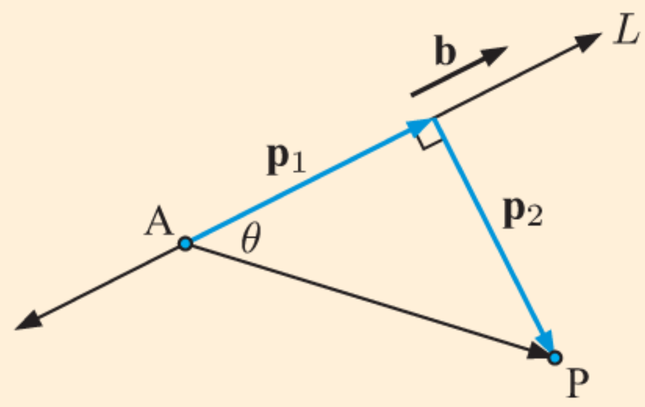
a perpendicular **b** neither parallel nor perpendicular.

INVESTIGATION 4 SHORTEST DISTANCE FORMULAE

The process of finding the shortest distance from a point to a line or plane, or between two skew lines, can often be long or difficult. In this Investigation, we will consider an alternate approach to these types of problems.

What to do:

1 Suppose the line L: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ passes through a point A. Let P be a point that does *not* lie on L.



$\overrightarrow{\text{AP}}$ can be *decomposed* into a vector $\mathbf{p}_1$ which is parallel to L, and a vector $\mathbf{p}_2$ which is perpendicular to L as shown.

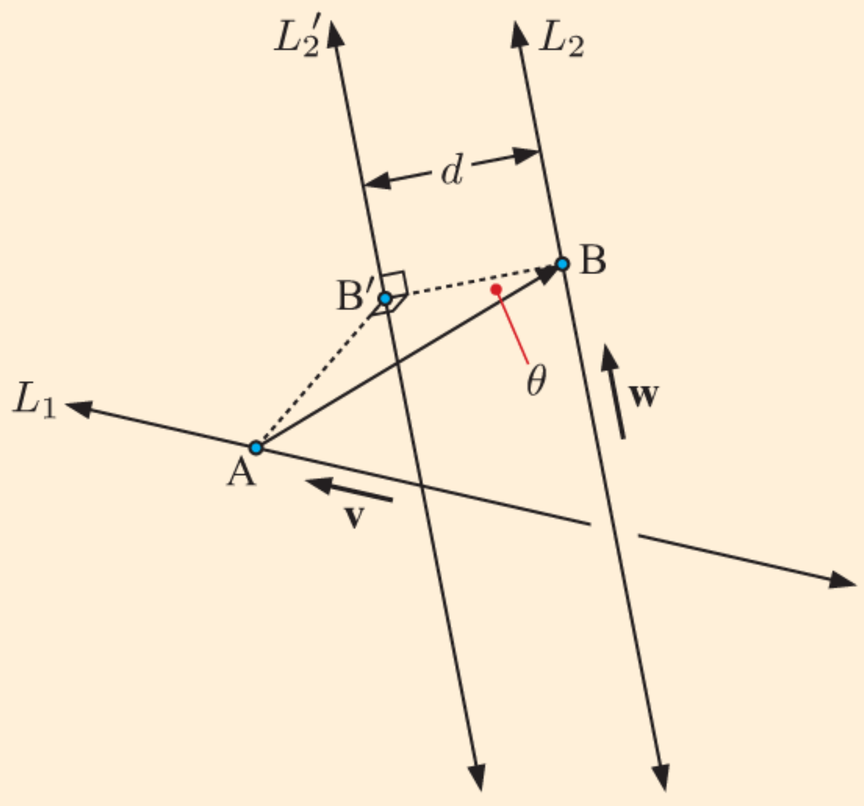
a Explain why the shortest distance from P to L is given by $|\mathbf{p}_2|$.

b Hence show that the shortest distance from P to L is given by $\dfrac{|\overrightarrow{\text{AP}} \times \mathbf{b}|}{|\mathbf{b}|}$.

c Explain why the expression is only useful in 3-dimensional space.

2 Consider the skew lines L_1: $\mathbf{r}_1 = \mathbf{a} + \lambda\mathbf{v}$ and L_2: $\mathbf{r}_2 = \mathbf{b} + \mu\mathbf{w}$, $\lambda, \mu \in \mathbb{R}$. Let A and B be points on L_1 and L_2 respectively.

Suppose L_2 is translated onto a line L_2' which intersects with L_1, so L_2' and L_1 lie on the same plane and B′ is the foot of the perpendicular from B to this plane.



a Explain why $\overrightarrow{\text{B}'\text{B}}$ is a scalar multiple of $\mathbf{v} \times \mathbf{w}$.

b Hence show that the shortest distance between L_1 and L_2 is given by

$$d = \frac{|\overrightarrow{\text{AB}} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|}.$$

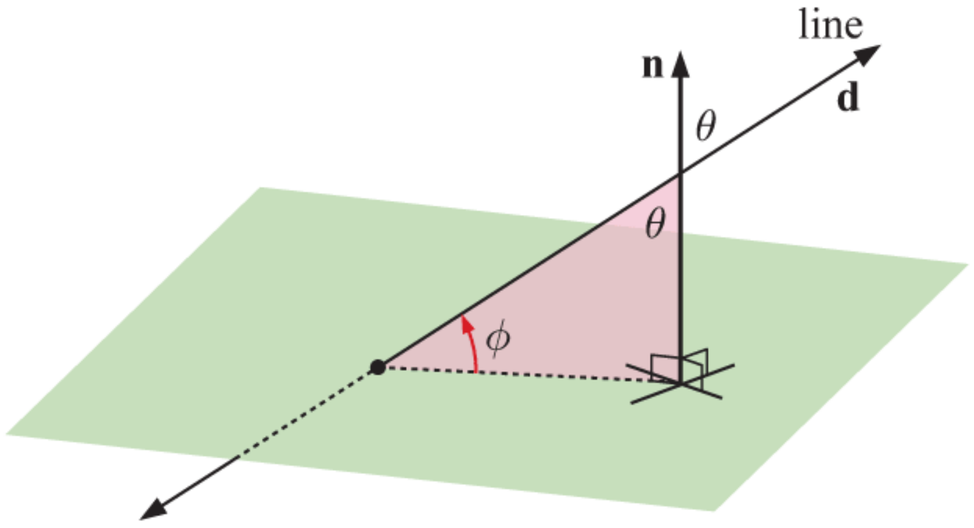
c Explain why the above expression cannot be used to find the distance between two non-coincident parallel lines.

d Check your answers to **Exercise 13F.3** using the expression you derived in **b**.

H ANGLES IN SPACE

THE ANGLE BETWEEN A LINE AND A PLANE

Suppose a line has direction vector **d** and a plane has normal vector **n**. We allow **n** to intersect the line, making an angle of θ with it. The angle between the line and the plane is the acute angle ϕ.



Now $\sin\phi = \cos\theta = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}$

$$\therefore \quad \phi = \sin^{-1}\left(\frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}\right)$$

Example 19 **Self Tutor**

Find the angle between the plane $x + 2y - z = 8$ and the line with equations $x = t,\ y = 1 - t,\ z = 3 + 2t,\ t \in \mathbb{R}$.

$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

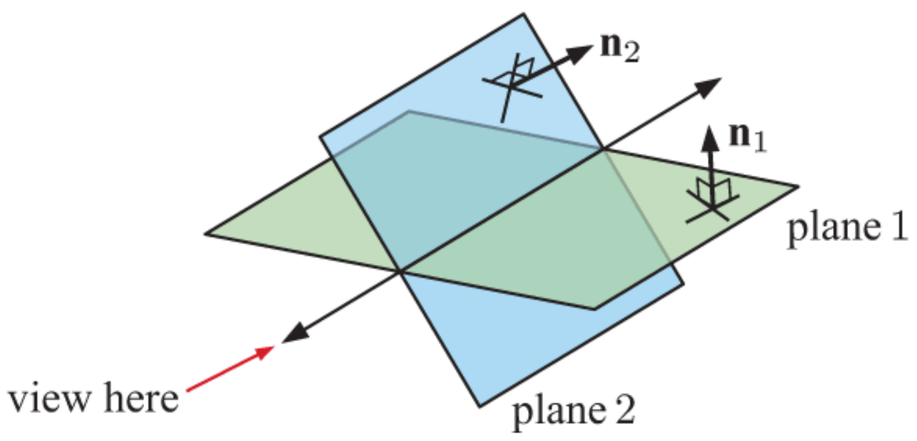
The angle between the plane and the line is

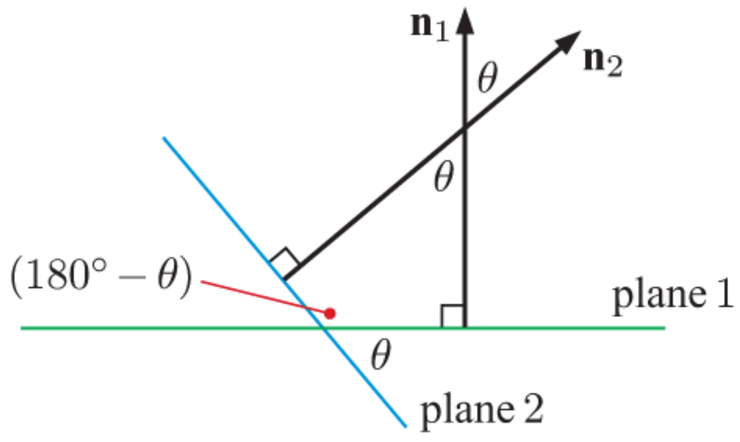
$$\phi = \sin^{-1}\left(\frac{|1 - 2 - 2|}{\sqrt{1+4+1}\sqrt{1+1+4}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{6}\sqrt{6}}\right)$$

$$= \sin^{-1}\left(\tfrac{1}{2}\right) = 30°$$

THE ANGLE BETWEEN TWO PLANES



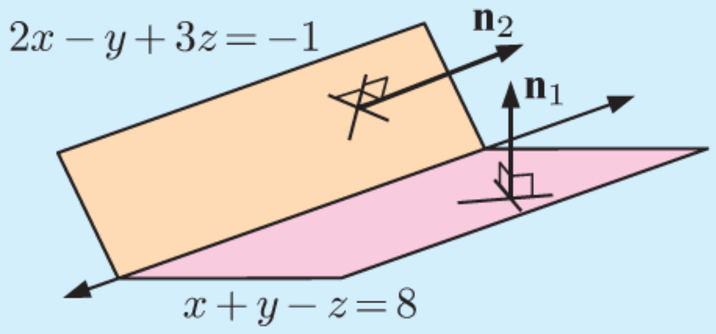


The cosine of the acute angle between the two normals is $\cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$.

If two planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and θ is the acute angle between them, then $\theta = \cos^{-1}\left(\dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$.

Example 20 **Self Tutor**

Find the angle between the planes with equations $x+y-z=8$ and $2x-y+3z=-1$.



$x+y-z=8$ has normal vector $\mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$2x-y+3z=-1$ has normal vector $\mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

The acute angle between the planes is $\theta = \cos^{-1}\left(\dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$

$$= \cos^{-1}\left(\frac{|2-1-3|}{\sqrt{1+1+1}\sqrt{4+1+9}}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$$

$$\approx 72.0^\circ$$

EXERCISE 13H

1 Find the angle between:

a the plane $x-y+z=5$ and the line $\dfrac{x-1}{4} = \dfrac{y+1}{3} = z+2$

b the plane $2x-y+z=8$ and the line $x=t+1,\ y=-1+3t,\ z=t,\ t \in \mathbb{R}$

c the plane $3x+4y-z=-4$ and the line $x-4=3-y=2(z+1)$

d the plane $\mathbf{r}_p = 2\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(3\mathbf{i}-4\mathbf{j}-\mathbf{k})+\mu(\mathbf{i}+\mathbf{j}-2\mathbf{k}),\ \lambda, \mu \in \mathbb{R}$ and the line $\mathbf{r}_l = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}+t(\mathbf{i}-\mathbf{j}+\mathbf{k}),\ t \in \mathbb{R}$.

2 **a** Show that the line $x=1+3t,\ y=4-t,\ z=2+4t,\ t \in \mathbb{R}$ is parallel to the plane $2x+2y-z=5$.

b Find the distance between the line and the plane.

3 Find the angle between the plane $3x-y+4z=-7$ and the:

a X-axis **b** Y-axis **c** Z-axis.

4 The line L passes through $\mathrm{A}(-3,\ 2,\ 5)$ and $\mathrm{B}(0,\ 2,\ -1)$, and the plane π_1 has equation $4x-2y+5z=16$.

a Find the angle between L and π_1.

b Find the coordinates of C, the point at which π_1 cuts the Y-axis.

c The plane π_2 is perpendicular to (AC) and passes through B.

i Find the equation of π_2. **ii** Find the angle between L and π_2.

5 Find the acute angle between the planes with equations:

a $2x-y+z=3$, $x+3y+2z=8$ **b** $x-y+3z=2$, $3x+y-z=-5$ **c** $3x-y+z=-11$, $2x+4y-z=2$

d $\mathbf{r}_1 = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}-\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k})+\mu(2\mathbf{i}-4\mathbf{j}+3\mathbf{k}),\ \lambda, \mu \in \mathbb{R}$
$\mathbf{r}_2 = \mathbf{i}+\mathbf{j}-\mathbf{k}-\lambda(2\mathbf{i}+\mathbf{j}+\mathbf{k})+\mu(\mathbf{i}+\mathbf{j}+\mathbf{k}),\ \lambda, \mu \in \mathbb{R}$

e $3x - 4y + z = -2$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

6 The planes $x + 5y - 3z = 8$ and $2x + 2y + kz = 9$ are perpendicular. Find the value of k.

7 The plane π_1 contains lines L_1 and L_2 which intersect.

L_1 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$, and

L_2 has equation $\mathbf{r} = \begin{pmatrix} 8 \\ -3 \\ k \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, $s \in \mathbb{R}$.

a Find k and the point of intersection.

b Find, in Cartesian form, the equation of π_1.

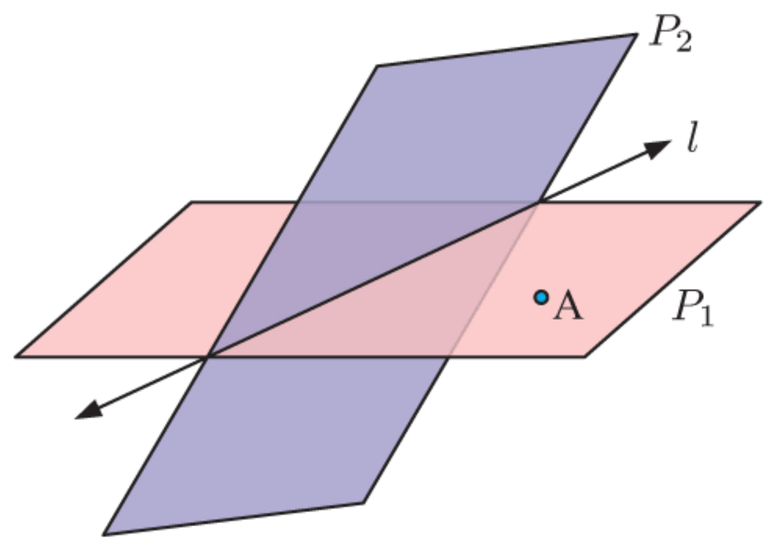
c The plane π_2 has equation $2y + 11z = -7$. Show that π_1 and π_2 are perpendicular.

d Find the angle between:

i L_1 and L_2 **ii** L_1 and π_2 **iii** L_2 and π_2.

e Find the sum of the angles in **d**. Draw a diagram to help explain your result.

8 P_1 has equation $2x - y + 3z = 7$, and P_2 has equation $x - 4y + z = 23$.



a Find the angle between P_1 and P_2.

b A$(-2, 4, k)$ lies on P_1. Find the value of k.

c A line through A, normal to P_1, cuts P_2 at B. Find the coordinates of B.

d l is the line of intersection of P_1 and P_2.

i Show that $(-4, -6, 3)$ lies on l.

ii Explain why l has direction $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$.

e The plane P_3 contains l and the midpoint of [AB]. Find the equation of P_3.

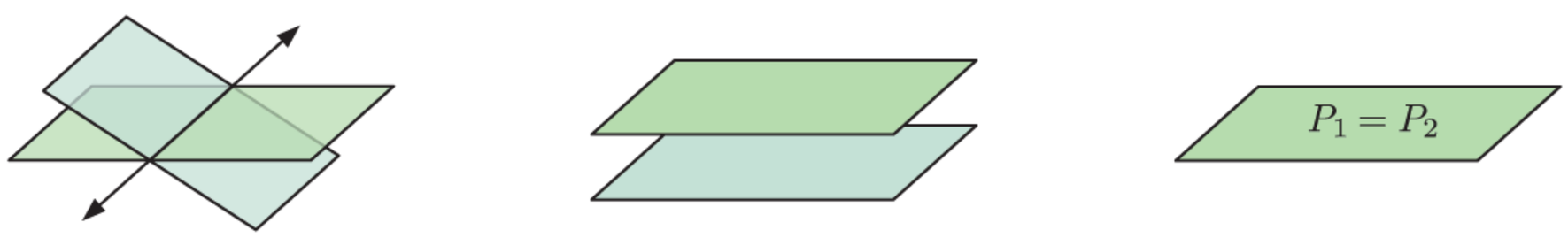
f Find the angle between: **i** P_1 and P_3 **ii** P_3 and P_2.

g Find the sum of the angles in **f**. Comment on your answer.

I INTERSECTING PLANES

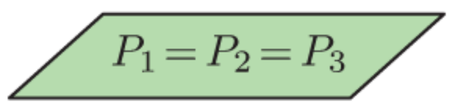
- **Two planes** in space could have any of the following three arrangements:

(1) intersecting **(2)** parallel **(3)** coincident

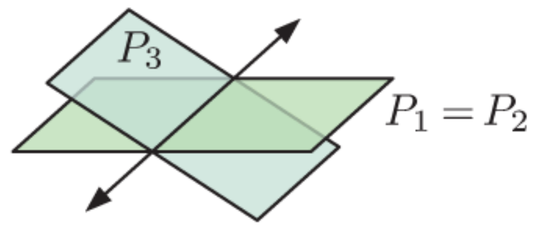


- **Three planes** in space could have any of the following eight arrangements:

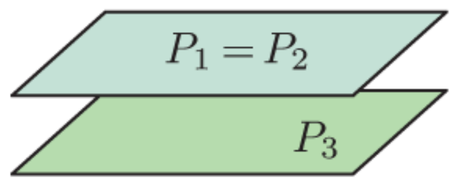
(1) all coincident



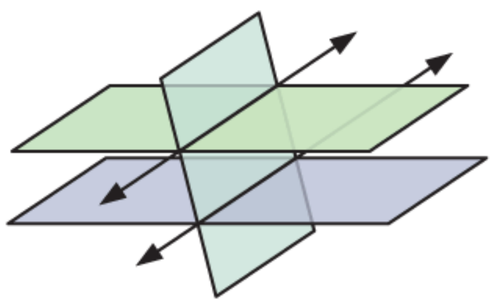
(2) two coincident and one intersecting



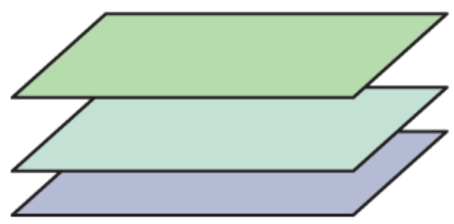
(3) two coincident and one parallel



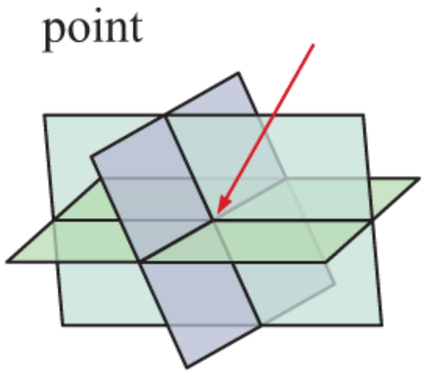
(4) two parallel and one intersecting



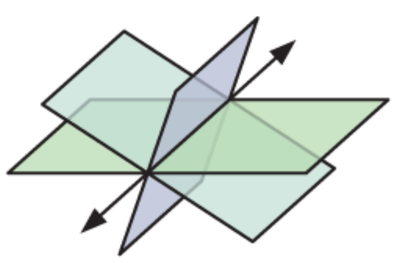
(5) all three parallel



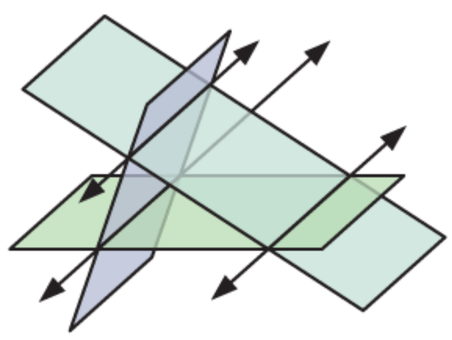
(6) all meet at the one point



(7) all meet in a common line



(8) the line of intersection of any two is parallel to the third plane.



Click on the icon to explore these cases in three dimensions.

DEMO



To determine the relationship between planes, we need to solve a system of linear equations. We do this using row operations.

Example 21 — Self Tutor

Solve the following systems of linear equations. In each case give a geometric interpretation of the system.

a $\begin{cases} x + 3y - z = 15 \\ 2x + y + z = 7 \\ x - y - 2z = 0 \end{cases}$ **b** $\begin{cases} 2x - y + z = 5 \\ x + y - z = 2 \\ 3x - 3y + 3z = 8 \end{cases}$

a In augmented matrix form, the system is:

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & -5 & 3 & -23 \\ 0 & -4 & -1 & -15 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & 1 & -\frac{3}{5} & \frac{23}{5} \\ 0 & 0 & -17 & 17 \end{array}\right) \quad \begin{array}{l} -\frac{1}{5}R_2 \to R_2 \\ 5R_3 - 4R_2 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & 1 & 1 & 7 \\ -2 & -6 & 2 & -30 \\ \hline 0 & -5 & 3 & -23 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 1 & -1 & -2 & 0 \\ -1 & -3 & 1 & -15 \\ \hline 0 & -4 & -1 & -15 \end{array}\right\}$$

$$\left\{\begin{array}{cccc} 0 & -20 & -5 & -75 \\ 0 & 20 & -12 & 92 \\ \hline 0 & 0 & -17 & 17 \end{array}\right\}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & 1 & -\frac{3}{5} & \frac{23}{5} \\ 0 & 0 & 1 & -1 \end{array}\right) \quad -\tfrac{1}{17}R_3 \to R_3$$

The last row gives $z = -1$

Using the final row 2, $y - \frac{3}{5}(-1) = \frac{23}{5}$

$\therefore \quad y = 4$

Using the final row 1, $x + 3(4) - (-1) = 15$

$\therefore \quad x = 2$

Thus we have a unique solution $x = 2,\ y = 4,\ z = -1$.

The system represents three planes that meet at the single point $(2,\ 4,\ -1)$. {case **(6)**}

b In augmented matrix form, the system is:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 5 \\ 3 & -3 & 3 & 8 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & 1 \\ 0 & -6 & 6 & 2 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} -\frac{1}{3}R_2 \to R_2 \\ R_3 - 2R_2 \to R_3 \end{array}$$

$$\left\{\begin{array}{cccc} 2 & -1 & 1 & 5 \\ -2 & -2 & 2 & -4 \\ \hline 0 & -3 & 3 & 1 \end{array}\right\} \qquad \left\{\begin{array}{cccc} 3 & -3 & 3 & 8 \\ -3 & -3 & 3 & -6 \\ \hline 0 & -6 & 6 & 2 \end{array}\right\} \qquad \left\{\begin{array}{cccc} 0 & -6 & 6 & 2 \\ 0 & 6 & -6 & -2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right\}$$

The row of zeros indicates infinitely many solutions.

If we let $z = t$ in row 2, then $y - t = -\frac{1}{3}$

$\therefore \quad y = -\frac{1}{3} + t$

Thus in row 1, $x + (-\frac{1}{3} + t) - t = 2$

$\therefore \quad x - \frac{1}{3} = 2$ and so $x = \frac{7}{3}$

$\therefore$ the solutions have the form: $x = \frac{7}{3},\ y = -\frac{1}{3} + t,\ z = t,$ where $t \in \mathbb{R}$.

The system represents three planes that intersect in a line. {case **(7)**}

EXERCISE 13I

1 **a** How many geometric interpretations are possible when solving $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$ simultaneously?

b Under what conditions will the planes in **a** be:

i parallel **ii** coincident?

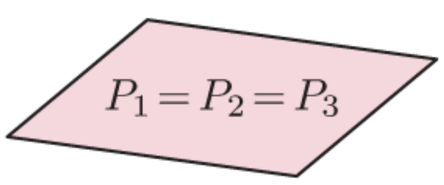
c Solve the following using elementary row operations, and interpret each system of equations geometrically:

i $\begin{cases} x - 3y + 2z = 8 \\ 3x - 9y + 2z = 4 \end{cases}$ **ii** $\begin{cases} 2x + y + z = 5 \\ x - y + z = 3 \end{cases}$ **iii** $\begin{cases} x + 2y - 3z = 6 \\ 3x + 6y - 9z = 18 \end{cases}$

2 Discuss the possible solutions of the following systems where k is a real number, and interpret the solutions geometrically.

a $\begin{cases} x+2y-z=6 \\ 2x+4y+kz=12 \end{cases}$ **b** $\begin{cases} x-y+3z=8 \\ 2x-2y+6z=k \end{cases}$

3 For each of the eight possible geometric scenarios of three planes in space, comment on the solutions to the corresponding system of linear equations.

For example, 

has infinitely many solutions where x, y, and z are expressed in terms of two parameters s and t.

4 Solve the following systems using row operations. State the geometric meaning of each solution.

a $\begin{cases} x+y-z=-5 \\ x-y+2z=11 \\ 4x+y-5z=-18 \end{cases}$ **b** $\begin{cases} x-y+2z=1 \\ 2x+y-z=8 \\ 5x-2y+5z=11 \end{cases}$ **c** $\begin{cases} x+2y-z=8 \\ 2x-y-z=5 \\ 3x-4y-z=2 \end{cases}$

d $\begin{cases} x-y+z=8 \\ 2x-2y+2z=11 \\ x+3y-z=-2 \end{cases}$ **e** $\begin{cases} x+y-2z=1 \\ x-y+z=4 \\ 3x+3y-6z=3 \end{cases}$ **f** $\begin{cases} x-y-z=5 \\ x+y+z=1 \\ 5x-y+2z=17 \end{cases}$

5 **a** Solve the system of equations $\begin{cases} 2x-y+4z=5 \\ x+y-2z=-3 \\ -x-4y+10z=6 \end{cases}$.
Give a geometric interpretation of your solution.

b Find the equation of l, the line of intersection of the planes $2x-y+4z=5$ and $x+y-2z=-3$.

c Verify that l is parallel to the plane $-x-4y+10z=6$.

Example 22 **Self Tutor**

Consider the system $\begin{cases} x+y+3z=2 \\ 3x-y+2z=5 \\ 5x+y+8z=a \end{cases}$ where a could take any real value.

a Use row operations to reduce the system to row echelon form.

b For what value(s) of a does the system have no solutions? Interpret the result geometrically.

c For what value(s) of a does the system have infinitely many solutions? Find the solutions in this case, and interpret them geometrically.

a In augmented matrix form, the system is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & -1 & 2 & 5 \\ 5 & 1 & 8 & a \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -4 & -7 & -1 \\ 0 & -4 & -7 & a-10 \end{array}\right) \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 - 5R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & \frac{7}{4} & \frac{1}{4} \\ 0 & 0 & 0 & a-9 \end{array}\right) \begin{array}{l} \\ -\frac{1}{4}R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{array}$$

b If $a \neq 9$, we have an inconsistent system. There are no solutions.
In this case, the intersection of any two planes is parallel to the third plane. {case **(8)**}

c If $a = 9$, the last row is all zeros, indicating infinitely many solutions.

Letting $z = t$ in row 2 gives

$$y + \tfrac{7}{4}t = \tfrac{1}{4}$$
$$\therefore \quad y = \frac{1-7t}{4}$$

Using row 1, $x + \left(\dfrac{1-7t}{4}\right) + 3t = 2$

$$\therefore \quad x + \frac{5t}{4} = \frac{7}{4}$$
$$\therefore \quad x = \frac{7-5t}{4}$$

We have infinitely many solutions of the form:

$$x = \frac{7-5t}{4}, \quad y = \frac{1-7t}{4}, \quad z = t, \quad \text{where} \quad t \in \mathbb{R}.$$

In this case we have three planes which meet in a line. {case **(7)**}

6 Solve the system of equations $\begin{cases} x - y + 3z = 1 \\ 2x - 3y - z = 3 \\ 3x - 5y - 5z = k \end{cases}$ where k is a real number.

State the geometrical meaning of the different solutions.

7 Solve the system of equations $\begin{cases} x - y + 2z = 1 \\ 3x + y + 2z = 0 \\ 5x + 3y + az = 2 \end{cases}$ where a is a real number.

State the geometrical meaning of the different solutions.

8 Consider the system of equations $\begin{cases} x + 2y + 3z = 4 \\ -x - 2y + 2z = 1 \\ 3x + 6y + kz = 5 \end{cases}$ where k is a real number.

a Reduce the system to row echelon form using row operations.

b Show that when $k \neq 2$, the system is inconsistent. For which of theses value(s) of k are two of the planes parallel? Give a geometric interpretation for the remaining values of k for which the system is inconsistent.

c Find the value(s) of k for which the system has infinitely many solutions. Interpret this result geometrically.

9 Reduce the system of equations: $\begin{cases} 2x + y - z = 3 \\ mx - 2y + z = 1 \\ x + 2y + mz = -1 \end{cases}$ to a form in which the solutions may be determined for all real values of m.

a Show that the system has no solutions for one value of m. Interpret this result geometrically.

b Show that there are infinitely many solutions for another value of m. Interpret this result.

c **i** For what values of m does the system have a unique solution? Interpret this result.

ii Show that the unique solution is $x = \dfrac{7}{m+5}$, $y = \dfrac{3(m-2)}{m+5}$, $z = \dfrac{-7}{m+5}$.

10 **a** Reduce this system to row echelon form: $\begin{cases} 2x - y + 3z = 4 \\ 2x + y + (a+3)z = 10 - a \\ 4x + 6y + (a^2+6)z = a^2 \end{cases}$

b Find the value(s) of a for which the system has no solutions. Interpret this result geometrically.

c Find the value(s) of a for which the system has infinitely many solutions, and find the form of these solutions. Interpret this result geometrically.

d Find the value(s) of a for which the system has a unique solution, and find the solution in the case where $a = 2$. Interpret this result geometrically.

11 Consider the system of equations $\begin{cases} x + 3y + 3z = a - 1 \\ 2x - y + z = 7 \\ 3x - 5y + az = 16 \end{cases}$ where a could take any real value.

a Reduce the system to row echelon form using row operations.

b Show that if $a = -1$ the system has infinitely many solutions, and find their form. Interpret this result geometrically.

c If $a \neq -1$, find the unique solution in terms of a. Interpret this result geometrically.

12 Find if and where the following planes meet:

P_1: $\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j} + \lambda(3\mathbf{i} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$, $\lambda, \mu \in \mathbb{R}$

P_2: $\mathbf{r}_2 = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k} + r(2\mathbf{i} - \mathbf{k}) + s(\mathbf{i} + \mathbf{j})$, $r, s \in \mathbb{R}$

P_3: $\mathbf{r}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - u\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, $t, u \in \mathbb{R}$

DISCUSSION

Click on the icon to find a printable summary for the equations of lines and planes.

SUMMARY



Discuss the similarities and differences between the equations.

REVIEW SET 13A

1 For the line that passes through $(-6, 3)$ with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, write down the corresponding:

a vector equation **b** parametric equations **c** Cartesian equation.

2 $(-3, m)$ lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$.
Find m.

3 Consider A$(2, -1, 3)$ and B$(0, 1, -1)$.

a Find the vector equation of the line through A and B.

b Hence find the coordinates of C on (AB) which is 2 units from A.

4 **a** Consider two unit vectors **a** and **b**. Prove that the vector $\mathbf{a} + \mathbf{b}$ bisects the angle between vector **a** and vector **b**.

b Consider the points $\text{H}(9, 5, -5)$, $\text{J}(7, 3, -4)$, and $\text{K}(1, 0, 2)$.

Find the vector equation of the line L that passes through J and bisects $\widehat{\text{HJK}}$.

c Find the coordinates of the point where L meets (HK).

5 Triangle ABC is formed by three lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and

(AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where s, t, and u are scalars.

a Use vector methods to find the coordinates of A, B, and C.

b Find $|\overrightarrow{\text{AB}}|$, $|\overrightarrow{\text{BC}}|$, and $|\overrightarrow{\text{AC}}|$.

c Classify triangle ABC.

6 A yacht is sailing with constant speed $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll. Distances are in kilometres.



a Find, in terms of **i** and **j**, the:

i initial position vector of the yacht

ii velocity vector of the yacht

iii position vector of the yacht after t hours, $t \geqslant 0$.

b Find the time when the yacht is closest to the beacon.

c If there is a reef of radius 8 km around the atoll, will the yacht hit the reef?

7 Submarine X23 is at $(2, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm.

Submarine Y18 is at $(11, 3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23. Distance units are kilometres, and time units are minutes.

a Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.

b Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.

c At what time does the interception occur?

d What was the direction and speed of the intercepting torpedo?

e Find the acute angle between the paths of the torpedoes.

8 Find the coordinates of the foot of the perpendicular from $\text{Q}(-1, 2, 3)$ to the line $2 - x = y - 3 = -\frac{1}{2}z$.

9 **a** Find the Cartesian equation of the plane through $\text{A}(-1, 0, 2)$, $\text{B}(0, -1, 1)$, and $\text{C}(1, 2, -1)$.

b Find the equation of the line, in parametric form, which passes through the origin and is normal to the plane in **a**.

c Find the point where the line in **b** intersects the plane in **a**.

10 Consider the lines L_1: $\dfrac{x-8}{3} = \dfrac{y+9}{-16} = \dfrac{z-10}{7}$

and L_2: $x = 15+3t,\quad y = 29+8t,\quad z = 5-5t,\quad t \in \mathbb{R}$

a Show that the lines are skew.

b Find the acute angle between them.

c Find the shortest distance between them.

11 Suppose A is $(3,\ 2,\ -1)$ and B is $(-1,\ 2,\ 4)$.

a Write down a vector equation of the line through A and B.

b Find the Cartesian equation of the plane through B with normal $\overrightarrow{AB}$.

c Find *two* points on (AB) which are $2\sqrt{41}$ units from A.

12 For $C(-3,\ 2,\ -1)$ and $D(0,\ 1,\ -4)$, find the coordinates of the point where the line passing through C and D meets the plane with equation $2x - y + z = 3$.

13 **a** Find if possible the point where the line through $L(1,\ 0,\ 1)$ and $M(-1,\ 2,\ -1)$ meets the plane with equation $x - 2y - 3z = 14$.

b Find the shortest distance from L to the plane.

14 Given the point $A(-1,\ 3,\ 2)$, the plane $2x - y + 2z = 8$, and the line defined by $x = 7 - 2t,\ y = -6 + t,\ z = 1 + 5t,\ t \in \mathbb{R}$, find:

a the distance from A to the plane

b the coordinates of the point on the plane nearest to A

c the shortest distance from A to the line.

15 Given $A(-1,\ 2,\ 3)$, $B(1,\ 0,\ -1)$, and $C(1,\ 3,\ 0)$, find:

a the normal vector to the plane containing A, B, and C

b D, the fourth vertex of parallelogram ACBD

c the area of parallelogram ACBD

d the coordinates of the foot of the perpendicular from C to the line (AB).

16 Lines L_1 and L_2 are defined by

$$L_1:\ \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad L_2:\ \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

a Find the coordinates of A, the point of intersection of the lines.

b Show that the point $B(0,\ -3,\ 2)$ lies on the line L_2.

c Find the vector equation of the line (BC) given that $C(3,\ -2,\ -2)$ lies on L_1.

d Find the Cartesian equation of the plane containing A, B, and C.

e Find the area of triangle ABC.

f Show that the point $D(9,\ -4,\ 2)$ lies on the normal to the plane passing through C.

17 Consider $A(-1,\ 2,\ 3)$, $B(2,\ 0,\ -1)$, and $C(-3,\ 2,\ -4)$.

a Find the Cartesian equation of the plane defined by A, B, and C.

b Find the measure of $C\widehat{A}B$.

c $D(r,\ 1,\ -r)$ is a point such that $B\widehat{D}C$ is a right angle. Find r.

18 Suppose P_1 is the plane $2x - y - 2z = 9$ and P_2 is the plane $x + y + 2z = 1$.
L is the line with parametric equations $x = t,\ y = 2t - 1,\ z = 3 - t,\ t \in \mathbb{R}$.
Find the acute angle between:

a L and P_1 **b** P_1 and P_2.

19 The three planes P_1, P_2, and P_3 are mutually perpendicular. P_1 has equation $ax + 3y - z = 2$, P_2 has equation $bx + 2z = 5$, and P_3 has equation $cx - 5y - 3z = 20$, where a, b, and c are positive constants.

a Find a, b, and c. **b** Find the point of intersection of the planes.

c Find the equation of L, the line of intersection of P_1 and P_2.

d Show that P$(2, 2, -6)$ lies on P_3.

e Find the equation of P_4, the plane containing L and P.

f Find the angle between P_1 and P_4.

20 Suppose $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

a Find $\mathbf{p} \times \mathbf{q}$.

b Find m if $\mathbf{p} \times \mathbf{q}$ is perpendicular to the line L with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix}$, $\lambda \in \mathbb{R}$.

c Hence find the Cartesian equation of the plane P containing L which is perpendicular to $\mathbf{p} \times \mathbf{q}$.

d Find t if the point A$(4, t, 2)$ lies on the plane P.

e Suppose B is the point $(6, -3, 5)$. For the value of t found in **d**, find the angle between (AB) and the plane P.

21 Use row operations to solve the system $\begin{cases} x - y + z = 5 \\ 2x + y - z = -1 \\ 7x + 2y + kz = -k \end{cases}$, $k \in \mathbb{R}$. Give geometric interpretations of your results.

22 **a** Show that the plane $2x + y + z = 5$ contains the line
L_1: $x = -2t + 2,\ y = t,\ z = 3t + 1,\ t \in \mathbb{R}$.

b For what values of k does the plane $x + ky + z = 3$ contain L_1?

c *Hence* find the values of p and q for which the system $\begin{cases} 2x + y + z = 5 \\ x - y + z = 3 \\ -2x + py + 2z = q \end{cases}$ has an infinite number of solutions. Clearly explain your reasoning.

REVIEW SET 13B

1 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $t \in \mathbb{R}$.

a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

2 Write down **i** a vector equation **ii** parametric equations for the line passing through:

a $(2, -3, 1)$ with direction $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ **b** $(-1, 6, 3)$ and $(5, -2, 0)$.

3 Find the angle between line L_1 passing through $(0, 3)$ and $(5, -2)$, and line L_2 passing through $(-2, 4)$ and $(-6, 7)$.

4 Trapezium KLMN is formed by the following lines:

(KL) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p\begin{pmatrix} 5 \\ -2 \end{pmatrix}$, (ML) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q\begin{pmatrix} -11 \\ 16 \end{pmatrix}$,

(NK) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, and (MN) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 43 \\ -9 \end{pmatrix} + s\begin{pmatrix} -5 \\ 2 \end{pmatrix}$,

where p, q, r, and s are scalars.

a Which two lines are parallel? Explain your answer.

b Which lines are perpendicular? Explain your answer.

c Use vector methods to find the coordinates of K, L, M, and N.

d Calculate the area of trapezium KLMN.

5 Find the angle between the lines:

L_1: $x = 1 - 4t$, $y = 3t$ and L_2: $x = 2 + 5s$, $y = 5 - 12s$, where $t, s \in \mathbb{R}$.

6 Consider A$(3, -1, 1)$ and B$(0, 2, -2)$.

a Find $|\overrightarrow{AB}|$.

b Show that the line passing through A and B can be described by $\mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$ where $\lambda \in \mathbb{R}$.

c Find the angle between (AB) and the line with equation $\mathbf{r} = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $t \in \mathbb{R}$.

7 Find the velocity vector of an object moving in the direction $3\mathbf{i} - \mathbf{j}$ with speed 20 km h^{-1}.

8 A moving particle has coordinates P$(x(t), y(t), z(t))$ where $x(t) = -4 + 8t$, $y(t) = 3 + 6t$, and $z(t) = 1 - 2t$. The distance units are metres, and $t \geqslant 0$ is the time in seconds. Find the:

a initial position of the particle **b** position of the particle after 4 seconds

c particle's velocity vector **d** speed of the particle.

9 The vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + t\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$

represents the path of a person zip-lining through a forest. t is measured in seconds, $t \geqslant 0$. Distances are measured in metres. Find the:

a initial position **b** velocity vector

c speed of the zip-liner

d time taken to reach the end of the line at $z = 0$

e location of the endpoint of the line

f position of the person when he is closest to his friend watching from $(52, 144, 3)$.



10 Boat A's position is given by $x_A(t) = 3 - t$, $y_A(t) = 2t - 4$, and boat B's position is given by $x_B(t) = 4 - 3t$, $y_B(t) = 3 - 2t$. The distance units are kilometres, and the time units are hours.



a Find the initial position of each boat.

b Find the velocity vector of each boat.

c Find the acute angle between the paths of the boats.

d At what time are the boats closest to each other?

11 Classify the following line pairs as either parallel, intersecting, or skew. Where possible, state the point of intersection and find the measure of the acute angle between them.

a $x = 2 + t$, $y = -1 + 2t$, $z = 3 - t$ and $x = -8 + 4s$, $y = s$, $z = 7 - 2s$

b $x = 3 + t$, $y = 5 - 2t$, $z = -1 + 3t$ and $x = 2 - s$, $y = 1 + 3s$, $z = 4 + s$

c $\dfrac{x-3}{2} = y - 4 = \dfrac{z+1}{-2}$ and $x = -1 + 3t$, $y = 2 + 2t$, $z = 3 - t$

12 **a** Find the Cartesian equation of the plane through A$(-1,\ 2,\ 3)$, B$(1,\ 0,\ -1)$, and C$(0,\ -1,\ 5)$.

b If X is $(3,\ 2,\ 4)$, find the angle that (AX) makes with this plane.

13 Find the coordinates of N, the foot of the normal from A$(1,\ 5,\ 3)$ to the plane with equation $\mathbf{r} = -5\mathbf{i} - \mathbf{j} - 5\mathbf{k} + s(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + t(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, $s, t \in \mathbb{R}$.

14 How far is X$(-1,\ 1,\ 3)$ from the plane $x - 2y - 2z = 8$?

15 The equations of two lines are: L_1: $x = 3t - 4$, $y = t + 2$, $z = 2t - 1$, $t \in \mathbb{R}$

and L_2: $x = \dfrac{y-5}{2} = \dfrac{-z-1}{2}$.

a Find the point of intersection of L_1 and the plane $2x + y - z = 2$.

b Find the point of intersection of L_1 and L_2.

c Find the equation of the plane that contains L_1 and L_2.

16 Show that the line $x - 1 = \dfrac{y+2}{2} = \dfrac{z-3}{4}$ is parallel to the plane $6x + 7y - 5z = 8$ and find the distance between them.

17 P$(2,\ 0,\ 1)$, Q$(3,\ 4,\ -2)$, and R$(-1,\ 3,\ 2)$ are three points in space. Find:

a $\overrightarrow{PQ}$, $|\overrightarrow{PQ}|$, and $\overrightarrow{QR}$

b the parametric equations of (PQ)

c a vector equation of the plane through P, Q, and R.

18 Suppose π_1 is the plane $x - 2y + 4z = 21$, π_2 is the plane $5x - y + 3z = 2$, and L is the line $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$.

a Show that π_1 contains L.

b Find the point at which L intersects π_2.

c Find the angle between:

i L and π_2

ii π_1 and π_2.

19 Line 1 has parametric equation $x = s,\ y = 2 - s,\ z = -2 + s,\ s \in \mathbb{R}$.

Line 2 has vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix},\ t \in \mathbb{R}$.

a Show that lines 1 and 2 are skew.

b Find the shortest distance between them.

c Find the coordinates of the points where the common perpendicular meets the lines 1 and 2.

20 Find the angle between the plane $2x + 2y - z = 3$ and the line
$x = t - 1,\ y = -2t + 4,\ z = -t + 3,\ t \in \mathbb{R}$.

21 Consider the following system of linear equations in which p and q are constants:

$$\begin{cases} x - 2y + 3z = 1 \\ x + py + 2z = 0 \\ -2x + p^2y - 4z = q \end{cases}$$

a Show that this system can be reduced to $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & p+2 & -1 & -1 \\ 0 & 0 & p & p+q \end{array}\right)$.

b What values can p and q take if the system has:

i a unique solution **ii** no solutions **iii** infinite solutions?

Explain the geometric significance of each case.

c Specify the infinite solutions in parametric form.

22 Three planes have the equations given below:

Plane A: $x + 3y + 2z = 5$
Plane B: $2x + y + 9z = 20$
Plane C: $x - y + 6z = 8$

a Show that plane A and plane B intersect in a line L_1.

b Show that plane B and plane C intersect in a line L_2.

c Show that plane A and plane C intersect in a line L_3.

d Show that L_1, L_2, and L_3 are parallel but not coincident.

e What does this mean geometrically?

23 Consider the system $\begin{cases} x - 3y + 2z = -5 \\ 3x + y + (2-k)z = 10 \\ -2x + 6y + kz = 5 \end{cases}$ where k can take any real value.

a Reduce the system to row echelon form.

b For what value of k does the system have no solutions? Interpret this result geometrically.

c **i** For what value(s) of k does the system have a unique solution?

ii Find the unique solution in terms of k, and interpret the result geometrically.

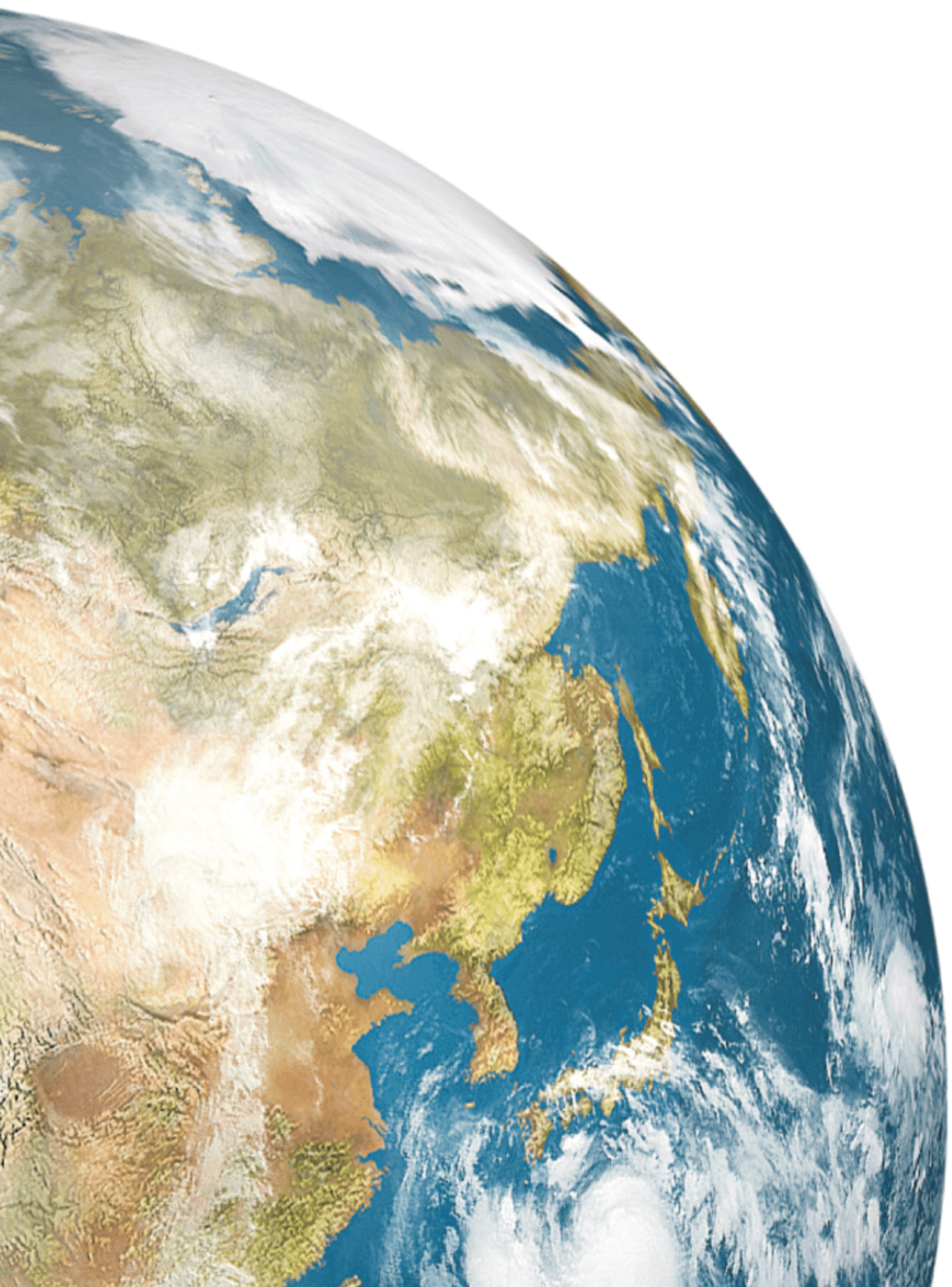
iii Find the unique solution when $k = 1$.

Chapter 14

Complex numbers

Contents:

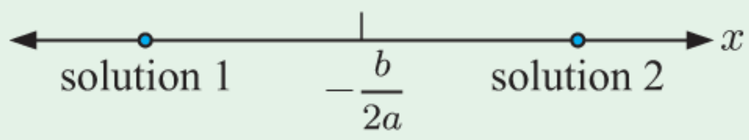
A The complex plane
B Modulus and argument
C Geometry in the complex plane
D Polar form
E Euler's form
F De Moivre's theorem
G Roots of complex numbers



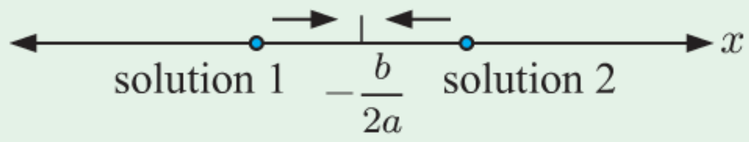
OPENING PROBLEM

Consider a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ with real coefficients.

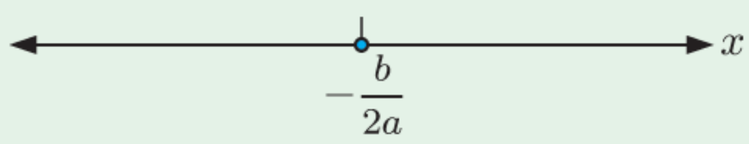
If $\Delta > 0$, there are two distinct real solutions with average value $-\frac{b}{2a}$.



If the value of c is changed so that Δ decreases towards zero, the solutions get closer together.



When $\Delta = 0$, the solutions coincide.



Things to think about:

a Suppose Δ is decreased further so there are two *complex* solutions.
- **i** What is the real component of these solutions?
- **ii** How can we *illustrate* the solutions so we can see both the real and imaginary parts?

b How can we use a *vector* to represent a complex number?

c
- **i** Can you illustrate *all* of the solutions to $x^2 = 1$, $x^3 = 1$, $x^4 = 1$, $x^5 = 1$, and $x^6 = 1$?
- **ii** What do you notice about the *magnitude* of the vectors which represent these solutions?

d
- **i** Can you illustrate *all* of the solutions to $x^3 = 1$, $x^3 = 8$, $x^3 = 27$, $x^3 = 64$, and $x^3 = 125$?
- **ii** What do you notice about the *directions* of the vectors which represent these solutions?

A THE COMPLEX PLANE

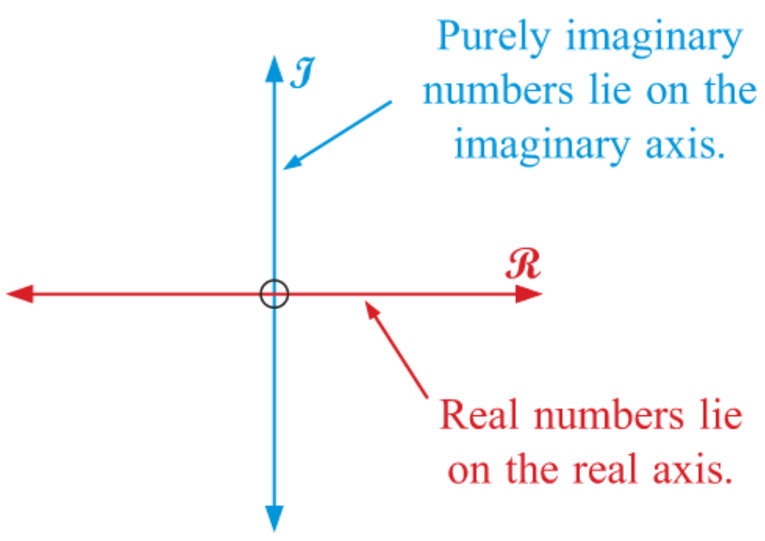
We have seen that a complex number can be written in Cartesian form as $z = a + bi$ where $a = \mathcal{Re}(z)$ and $b = \mathcal{Im}(z)$ are both real numbers.

There is a one-to-one relationship between any complex number $a + bi$ and the point (a, b) in the Cartesian plane.

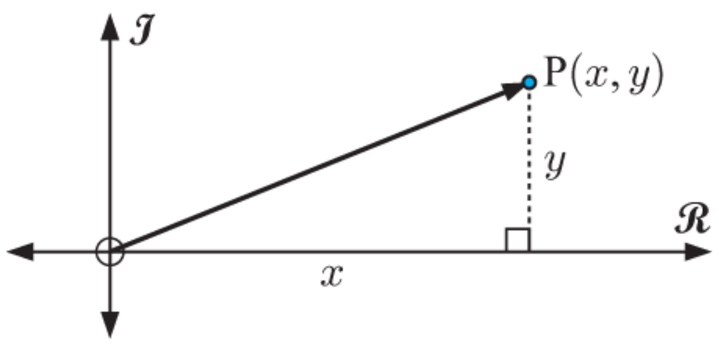
We can therefore plot any complex number on a plane as a unique ordered number pair. We refer to the plane as the **complex plane** or the **Argand plane**.

On the complex plane, the x-axis is called the **real axis** and the y-axis is called the **imaginary axis**.

- All **real** numbers with $b = 0$ lie on the real axis.
- All **purely imaginary** numbers with $a = 0$ lie on the imaginary axis.
- The origin $(0, 0)$ lies on both axes. It corresponds to $z = 0$, a real number.
- Complex numbers that are neither real nor purely imaginary (a and b are both not 0) lie in one of the four quadrants.

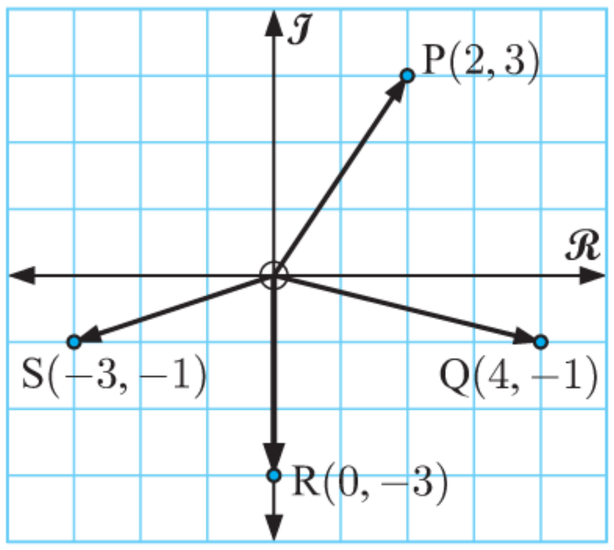


We saw in **Chapter 12** that any point P on the Cartesian plane can be represented by a vector. The position vector for the point P is $\overrightarrow{OP}$. In the same way we can illustrate complex numbers using vectors on the Argand plane. We call this an **Argand diagram**.



$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$

For example:



$\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ represents $2 + 3i$

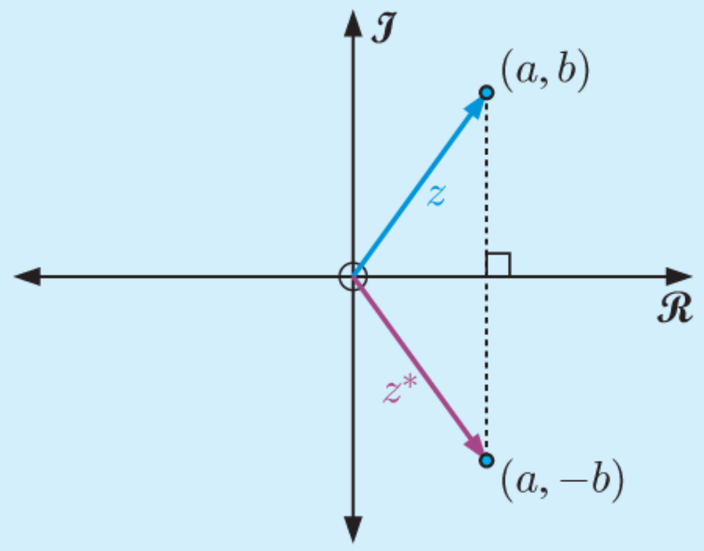
$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ represents $4 - i$

$\overrightarrow{OR} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ represents $-3i$

$\overrightarrow{OS} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ represents $-3 - i$

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

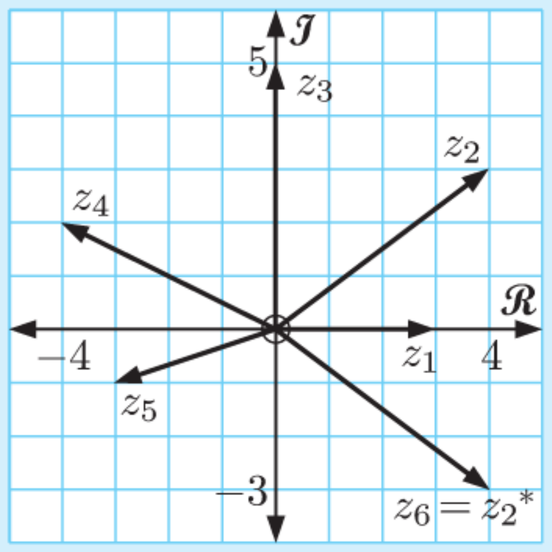
In the complex plane, z^* is the reflection of z in the real axis.



Example 1 **Self Tutor**

Illustrate the positions of the following complex numbers on an Argand diagram:

$z_1 = 3$, $z_2 = 4 + 3i$,
$z_3 = 5i$, $z_4 = -4 + 2i$,
$z_5 = -3 - i$, $z_6 = z_2^*$



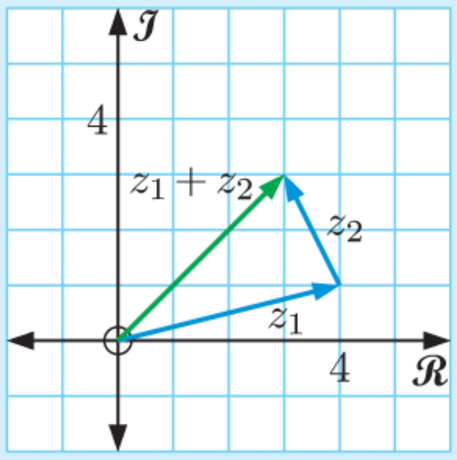
We can apply the vector operations of addition, subtraction, and scalar multiplication to perform these operations with complex numbers.

Example 2 — Self Tutor

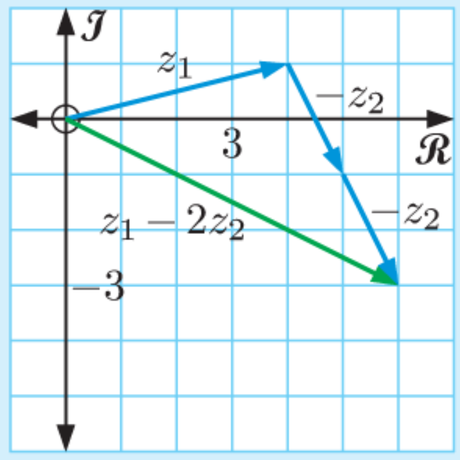
Suppose $z_1 = 4 + i$ and $z_2 = -1 + 2i$. Find using both algebra and vectors:

a $z_1 + z_2$ **b** $z_1 - 2z_2$ **c** $2z_1 + z_2^*$

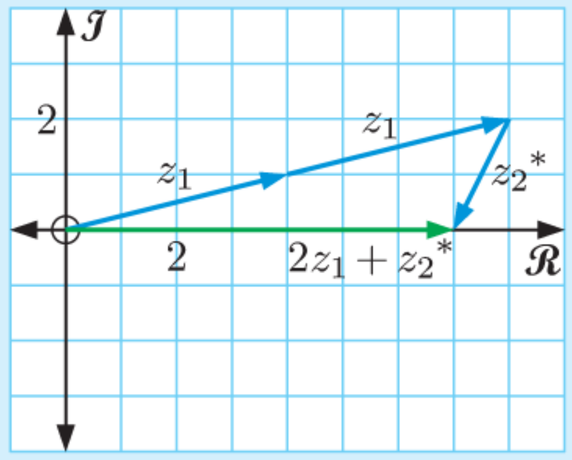
a
$$\begin{aligned} & z_1 + z_2 \\ &= (4 + i) + (-1 + 2i) \\ &= 4 + i - 1 + 2i \\ &= 3 + 3i \end{aligned}$$



b
$$\begin{aligned} & z_1 - 2z_2 \\ &= (4 + i) - 2(-1 + 2i) \\ &= 4 + i + 2 - 4i \\ &= 6 - 3i \end{aligned}$$



c
$$\begin{aligned} & 2z_1 + z_2^* \\ &= 2(4 + i) + (-1 - 2i) \\ &= 8 + 2i - 1 - 2i \\ &= 7 \end{aligned}$$



EXERCISE 14A

1 On an Argand diagram, illustrate the complex numbers:

$$z_1 = 5, \quad z_2 = -1 + 2i, \quad z_3 = -6 - 2i, \quad z_4 = -6i, \quad z_5 = 2 - i, \quad \text{and} \quad z_6 = 4i$$

2 Illustrate on an Argand diagram:

a $z = 3 + 2i$ and $z^* = 3 - 2i$ **b** $z = -2 + 5i$ and $z^* = -2 - 5i$

3 Suppose z is a complex number. Explain, with illustration, how to find geometrically:

a $3z$ **b** $-2z$ **c** $z + 3$ **d** $3i - z$

e $2 - z$ **f** $z^* + i$ **g** $\dfrac{z+2}{3}$ **h** $\dfrac{z-4}{2}$

4 Suppose $z = 1 + 2i$ and $w = 3 - i$. Find using both algebra and vectors:

a $z + w$ **b** $z - w$ **c** $2z - w$ **d** $w - 3z$

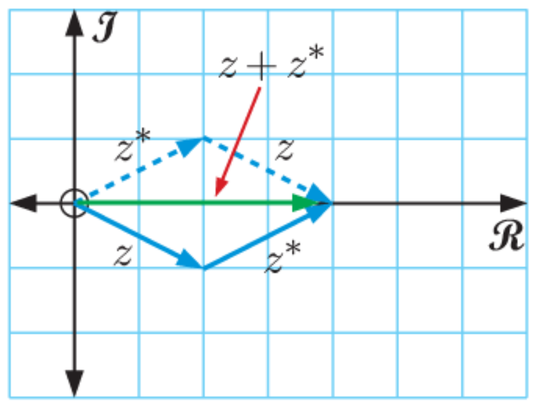
5 Suppose $z_1 = 4 - i$ and $z_2 = 2 + 3i$. Find using both algebra and vectors:

a $z_1 + 1$ **b** $z_1 + 2i$ **c** $z_2 + \frac{1}{2}z_1$ **d** $\dfrac{z_1 + 4}{2}$

6 Suppose $z_1 = -2 + 3i$ and $z_2 = 1 - 2i$. Find using both algebra and vectors:

a $z_1 + z_2^*$ **b** $2z_1^* - z_2$ **c** $z_1^* - 2z_2$ **d** $3z_1 + 2z_2$

7 Suppose $z = 2 - i$. From the diagram we can see that $z + z^* = 4$, which is real.
Explain, with illustration, why $z + z^*$ is always real for any complex number z.



8 Explain, with illustration, why $z - z^*$ is always purely imaginary or zero. What distinguishes these two cases?

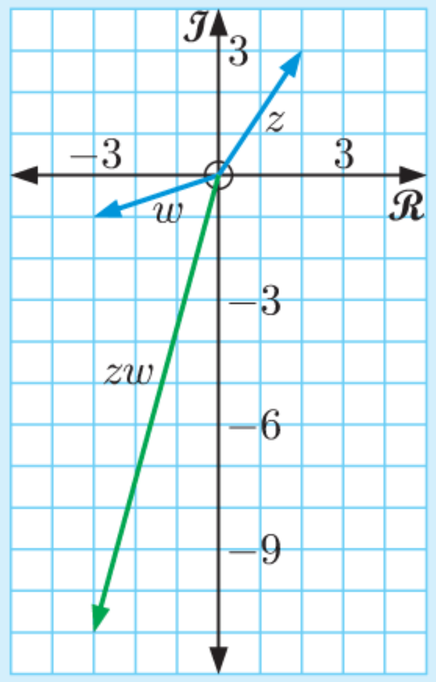
9 Using an Argand diagram, explain why for all complex numbers z and w:

a $(z + w)^* = z^* + w^*$ **b** $(z - w)^* = z^* - w^*$

Example 3 — **Self Tutor**

Let $z = 2 + 3i$ and $w = -3 - i$. Find zw, and illustrate z, w, and zw on the same Argand diagram.

$$\begin{aligned} zw &= (2 + 3i)(-3 - i) \\ &= -6 - 2i - 9i + 3 \\ &= -3 - 11i \end{aligned}$$



10 Find zw, and illustrate z, w, and zw on the same Argand diagram:

a $z = 2 - i$, $w = 1 + 2i$ **b** $z = -1 + i$, $w = 3 - i$

c $z = 3 - 2i$, $w = 3 + 2i$ **d** $z = 2i$, $w = 2 + i$

DISCUSSION

We have seen that for two vectors **a** and **b**, there are two types of vector product:

- the scalar dot product $\mathbf{a} \bullet \mathbf{b}$
- the vector cross product $\mathbf{a} \times \mathbf{b}$.

Are either of these products useful when considering the product of two complex numbers?

Are there any other tools we have seen that can help with the product of two complex numbers?

B MODULUS AND ARGUMENT

Since complex numbers can be represented by vectors in the Argand plane, we can attribute to them both a *magnitude* and a *direction*.

The magnitude of a complex number is called its **modulus**, and its direction is called its **argument**.

MODULUS

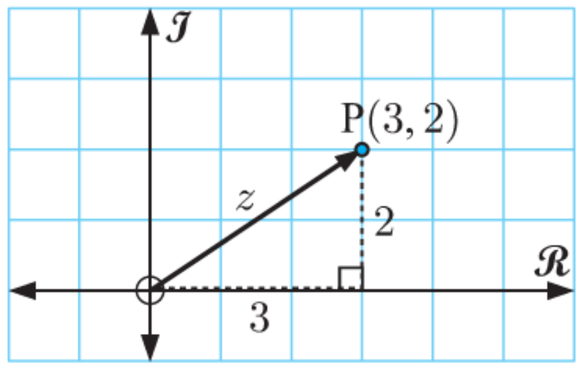
The **modulus** of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number $|z| = \sqrt{a^2 + b^2}$.

Notice that if $z = a + bi$ then $|z|$ gives the distance of the point (a, b) from the origin. This is consistent with the previous definition of modulus that $|x|$ is the distance of real number x from the origin O.

For example, consider the complex number $z = 3 + 2i$.

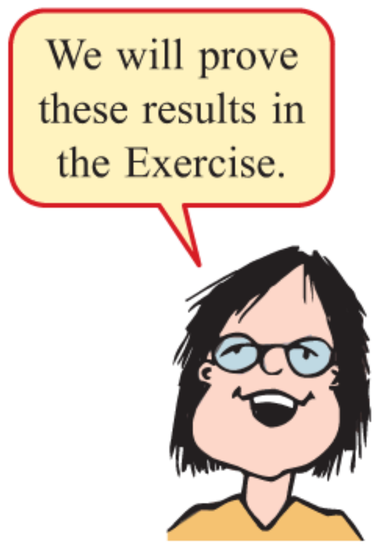
The distance from O to P is its modulus, $|z|$.

$\therefore \quad |z| = \sqrt{3^2 + 2^2}$ {Pythagoras}



PROPERTIES OF MODULUS

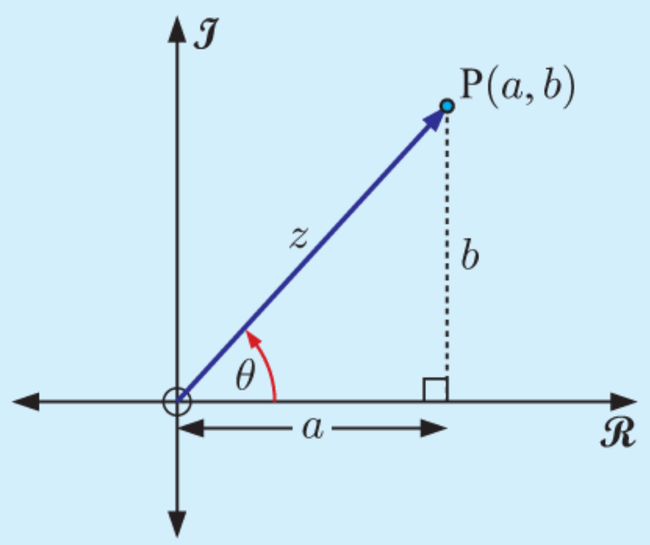
- $|z^*| = |z|$
- $|z|^2 = zz^*$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$
- $|z_1 z_2 z_3 .... z_n| = |z_1||z_2||z_3|....|z_n|$ and $|z^n| = |z|^n$ for $n \in \mathbb{Z}^+$.



ARGUMENT

Suppose the complex number $z = a + bi$ is represented by the vector $\overrightarrow{OP}$.

The **argument** of z, or simply $\arg z$, is the angle θ, where $-\pi < \theta \leqslant \pi$ is measured anticlockwise between the positive real axis and $\overrightarrow{OP}$.



We specify the domain restriction $-\pi < \theta \leqslant \pi$ to avoid confusion with the infinitely many possibilities for θ which are 2π apart. This guarantees that $z \mapsto \arg z = \theta$ is a function.

Notice that:

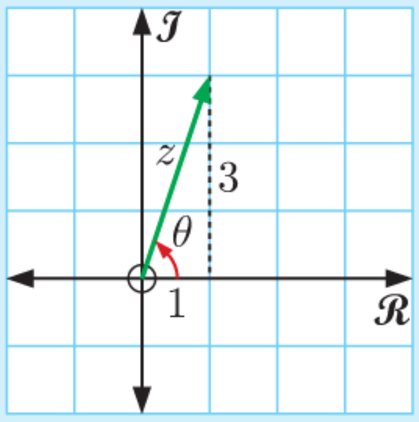
Real numbers (other than zero) have argument 0 or π.
Purely imaginary numbers have argument $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Example 4 — Self Tutor

Find $|z|$ and $\arg z$ for z equal to:

a $1 + 3i$ **b** $-1 + 3i$ **c** $-3 - i$

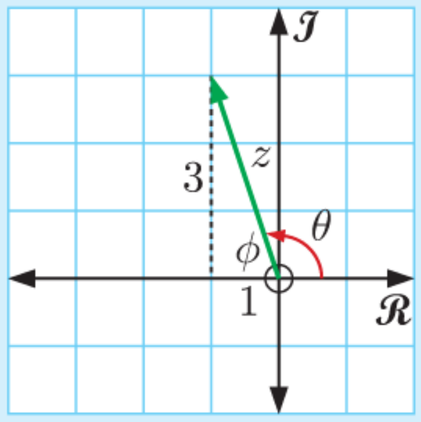
a $|z| = \sqrt{1^2 + 3^2} = \sqrt{10}$



$\tan\theta = \frac{3}{1} = 3$

$\therefore \arg z \approx 1.25$

b $|z| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$



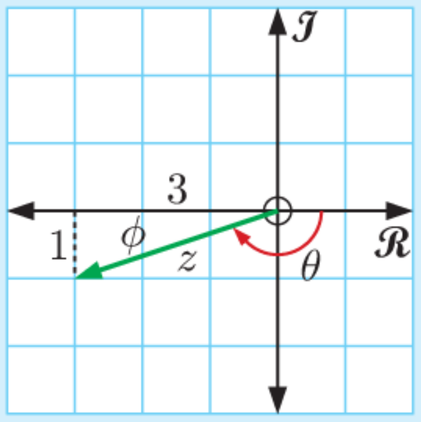
$\tan\phi = \frac{3}{1} = 3$

$\therefore \phi \approx 1.25$

But $\theta = \pi - \phi$

$\therefore \arg z \approx 1.89$

c $|z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$



$\tan\phi = \frac{1}{3}$

$\therefore \phi \approx 0.322$

But $\theta = -\pi + \phi$

$\therefore \arg z \approx -2.82$

EXERCISE 14B

1 Find $|z|$ for z equal to:

a $3-4i$ **b** $5+12i$ **c** $-8+2i$ **d** $3i$ **e** -4

2 Find $|z|$ and $\arg z$ for z equal to:

a $2+i$ **b** $2-i$ **c** $-2+i$ **d** $-2-i$

3 Find $|w|$ and $\arg w$ for w equal to:

a $3+2i$ **b** $-3+2i$ **c** $2-3i$ **d** $-2+3i$

4 Suppose $z=a-i$ where a is real, and $\arg z=-\frac{5\pi}{6}$. Find the exact value of a.

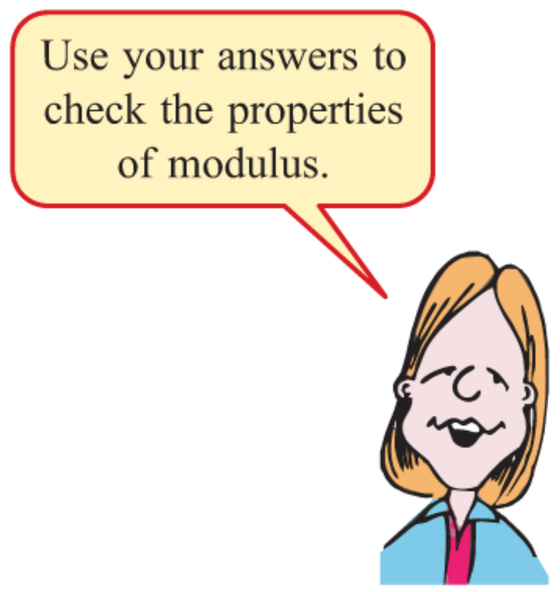
5 If $z=2+i$ and $w=-1+3i$, find:

a $|z|$ **b** $|z^*|$ **c** $|z^*|^2$

d zz^* **e** $|zw|$ **f** $|z||w|$

g $\left|\frac{z}{w}\right|$ **h** $\frac{|z|}{|w|}$ **i** $|z^2|$

j $|z|^2$ **k** $|z^3|$ **l** $|z|^3$



Example 5 — Self Tutor

Prove that $|z_1z_2|=|z_1||z_2|$ for all complex numbers z_1 and z_2.

Let $z_1=a+bi$ and $z_2=c+di$ where a, b, c, and d are real.

$\therefore\quad z_1z_2=(a+bi)(c+di)=[ac-bd]+i[ad+bc]$

Thus
$$\begin{aligned}|z_1z_2| &= \sqrt{(ac-bd)^2+(ad+bc)^2}\\ &= \sqrt{a^2c^2-\cancel{2abcd}+b^2d^2+a^2d^2+\cancel{2abcd}+b^2c^2}\\ &= \sqrt{a^2(c^2+d^2)+b^2(c^2+d^2)}\\ &= \sqrt{(c^2+d^2)(a^2+b^2)}\\ &= \sqrt{a^2+b^2}\times\sqrt{c^2+d^2}\\ &= |z_1||z_2|\end{aligned}$$

6 Prove that for all complex numbers z:

a $|z^*|=|z|$ **b** $|z|^2=zz^*$

7 Find $|z|$ given:

a $z=\cos\theta+i\sin\theta$ **b** $z=r(\cos\theta+i\sin\theta),\ r\in\mathbb{R}$

8 Prove that $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$ for all complex numbers z and w, $w\neq 0$.

9 **a** Prove by mathematical induction that $|z_1 z_2 z_3 .... z_n| = |z_1||z_2||z_3| .... |z_n|$ for all complex numbers $z_1, z_2, z_3, ..., z_n$ and $n \in \mathbb{Z}^+$.

b Hence prove that $|z^n| = |z|^n$ for all $n \in \mathbb{Z}^+$.

c Hence find $\left|z^{30}\right|$ for $z = \frac{3}{4} - i$.

10 Given $|z| = 2$, use the rules $|zw| = |z||w|$ and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ to find:

a $|3z|$ **b** $|-2z|$ **c** $|(2-i)z|$

d $|iz|$ **e** $\left|\frac{1}{z}\right|$ **f** $\left|\frac{z^2}{3i}\right|$

11 Suppose $w = \frac{z+i}{z-i}$ where $z = a + bi$, $a, b \in \mathbb{R}$.

a Write w in the form $X + Yi$ where X and Y involve a and b.

b If $|z| = 1$, find $\mathcal{Re}(w)$.

12 Find the complex number z that satisfies the equation $\frac{50}{z^*} - \frac{10}{z} = 2 + 9i$ given $|z| = 2\sqrt{10}$.

Example 6 **Self Tutor**

Find $|z|$ given that $3|z+1| = |z+9|$ where z is a complex number.

$$3|z+1| = |z+9|$$
$$\therefore\ 9|z+1|^2 = |z+9|^2$$
$$\therefore\ 9(z+1)(z+1)^* = (z+9)(z+9)^* \quad \{\text{as } |z|^2 = zz^*\}$$
$$\therefore\ 9(z+1)(z^*+1) = (z+9)(z^*+9) \quad \{\text{as } (z+w)^* = z^* + w^*\}$$
$$\therefore\ 9zz^* + \cancel{9z} + \cancel{9z^*} + 9 = zz^* + \cancel{9z} + \cancel{9z^*} + 81$$
$$\therefore\ 8zz^* = 72$$
$$\therefore\ zz^* = 9$$
$$\therefore\ |z|^2 = 9$$
$$\therefore\ |z| = 3 \quad \{\text{as } |z| > 0\}$$

13 Suppose z is a complex number. Find $|z|$ if:

a $2|z+4| = |z+16|$ **b** $|z-18| = 3|z-2|$

c $\left|\frac{z+20}{z+5}\right| = 2$ **d** $\left|\frac{z-50}{z-2}\right| = 5$

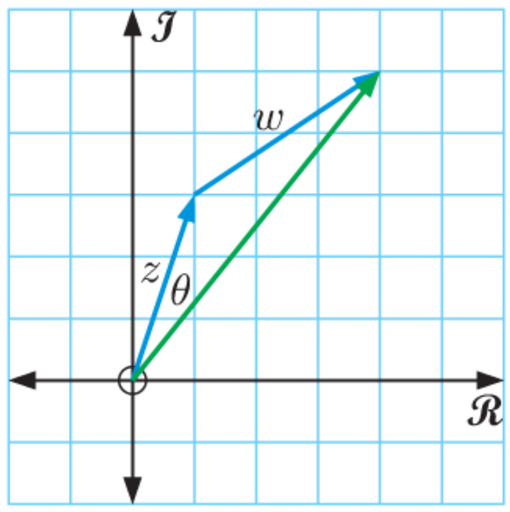
14 **a** Suppose z is a complex number. Find $|z|$ if $3|z-5| = |z-45|$.

b On an Argand plane, sketch the set of complex numbers z which satisfy $3|z-5| = |z-45|$.

15 Suppose z and w are non-zero complex numbers, and $|z+w| = |z-w|$.

Show that $\frac{z}{z^*} = -\frac{w}{w^*}$.

16



Let $z = 1 + 3i$ and $w = 3 + 2i$.

a Find $z + w$.

b Hence find $|z + w|$ and $\arg(z + w)$.

c Use the cosine rule to find θ.

d Hence show that $\arg(z + w) + \theta = \arg z$.

17 Find the Cartesian equation for the locus of points $\mathrm{P}(x, y)$ if $z = x + iy$ and:

a $z^* = -iz$ **b** $\arg(z - i) = \frac{\pi}{6}$ **c** $|z + 3| + |z - 3| = 8$

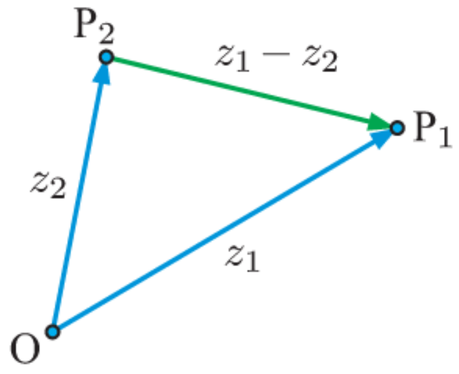
C GEOMETRY IN THE COMPLEX PLANE

When we work in the Cartesian plane, we often need to consider x and y components separately. In the complex plane we have the advantage that both components are recorded in the one number.

DISTANCES

Suppose P_1 and P_2 are two points in the complex plane which correspond to the complex numbers z_1 and z_2.

$$\begin{aligned}\overrightarrow{\mathrm{P_2P_1}} &= \overrightarrow{\mathrm{P_2O}} + \overrightarrow{\mathrm{OP_1}} \\ &= -z_2 + z_1 \\ &= z_1 - z_2 \\ \therefore \quad |z_1 - z_2| &= |\overrightarrow{\mathrm{P_2P_1}}| \\ &= \text{distance between } \mathrm{P}_1 \text{ and } \mathrm{P}_2.\end{aligned}$$



If $z_1 \equiv \overrightarrow{\mathrm{OP_1}}$ and $z_2 \equiv \overrightarrow{\mathrm{OP_2}}$ then $|z_1 - z_2|$ is the distance between the points P_1 and P_2.

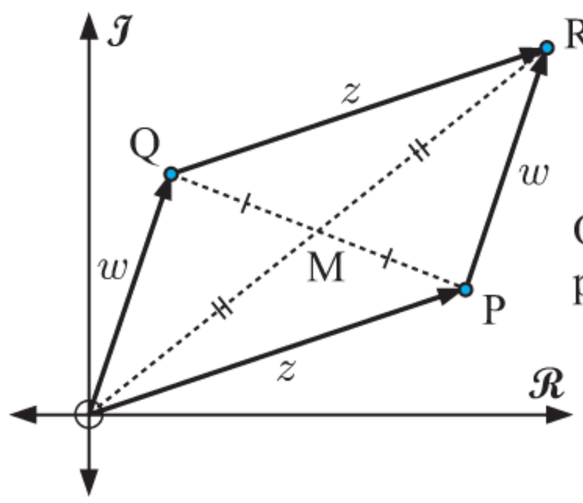
MIDPOINTS

The parallelogram alongside is formed using complex numbers w and z.

The diagonals of the parallelogram are $\overrightarrow{\mathrm{OR}}$ and $\overrightarrow{\mathrm{PQ}}$, and these diagonals bisect each other.

Now $\overrightarrow{\mathrm{OR}} \equiv w + z$

$\therefore \quad \overrightarrow{\mathrm{OM}} \equiv \dfrac{w + z}{2}$.



OPRQ is a parallelogram

Example 7 — Self Tutor

P$(2, 3)$ and Q$(6, 1)$ are two points on the Cartesian plane. Use complex numbers to find:

a distance PQ **b** the midpoint of [PQ].

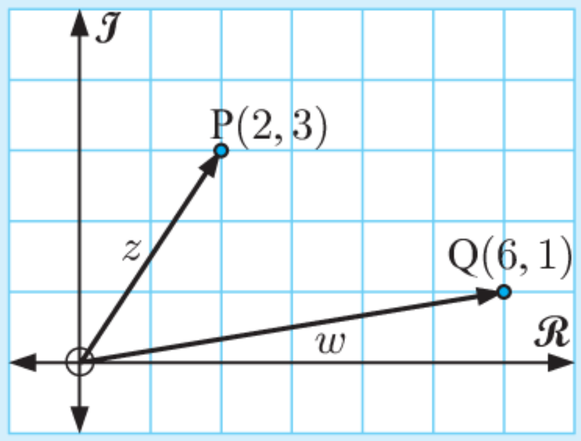
a If $z = 2 + 3i$ and $w = 6 + i$

then $w - z = 6 + i - 2 - 3i$

$= 4 - 2i$

$\therefore \ |w - z| = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

$\therefore$ PQ $= \sqrt{20}$ units



b $\dfrac{z+w}{2} = \dfrac{2+3i+6+i}{2} = 4 + 2i$

$\therefore$ the midpoint of [PQ] is $(4, 2)$.

TRANSFORMATIONS

Example 8 — Self Tutor

Find the transformation which moves z to iz.

If $z = x + iy$, then $iz = i(x + iy)$

$= xi + i^2y$

$= -y + xi$

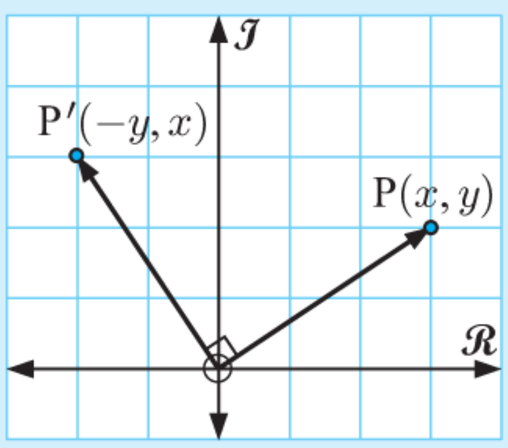
If $z \equiv$ P(x, y) then $iz \equiv$ P$'(-y, x)$

We notice that $|z| = \sqrt{x^2 + y^2}$

and $|iz| = \sqrt{(-y)^2 + x^2}$

$= \sqrt{x^2 + y^2}$

$\therefore$ OP$'$ = OP



So, z moves to iz under an anticlockwise rotation of $\frac{\pi}{2}$ about O.

EXERCISE 14C

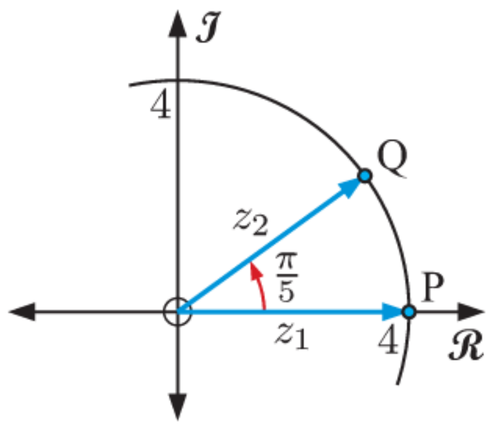
1 Use complex numbers to find **i** distance AB **ii** the midpoint of [AB] for:

a A$(3, 6)$ and B$(-1, 2)$ **b** A$(-4, 7)$ and B$(1, -3)$.

2 z_1 and z_2 correspond to the points P and Q respectively.

a Find the exact value of $|z_1 - z_2|$.

b Find the exact perimeter and area of triangle OPQ.



3 Find the transformation which moves:

a z to z^* **b** z to $-z$ **c** z to $-z^*$ **d** z to $-iz$.

ACTIVITY 1 THE TRIANGLE INEQUALITY

The **triangle inequality** states that for any complex numbers z_1 and z_2, $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

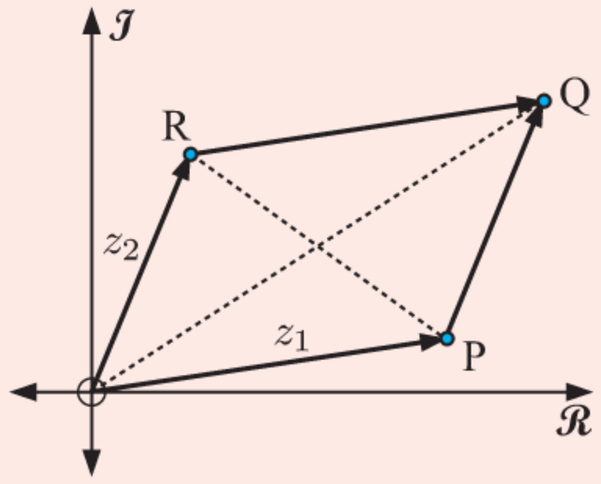
What to do:

1 Verify that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for:

a $z_1 = 3 + 4i, \ z_2 = 6i$ $\longrightarrow$

b $z_1 = 2 - 3i, \ z_2 = 4 + 7i$ $\longrightarrow$

2 OPQR is a parallelogram. OP represents z_1 and OR represents z_2, where z_1 and z_2 are complex numbers.



a Find, in terms of z_1 and z_2:

i $\overrightarrow{OQ}$ **ii** $\overrightarrow{PR}$

b Explain from triangle OPQ, why $|z_1 + z_2| \leqslant |z_1| + |z_2|$.
Discuss the case of equality.

c Explain from triangle OPR, why $|z_1 - z_2| \geqslant |z_2| - |z_1|$.
Discuss the case of equality.

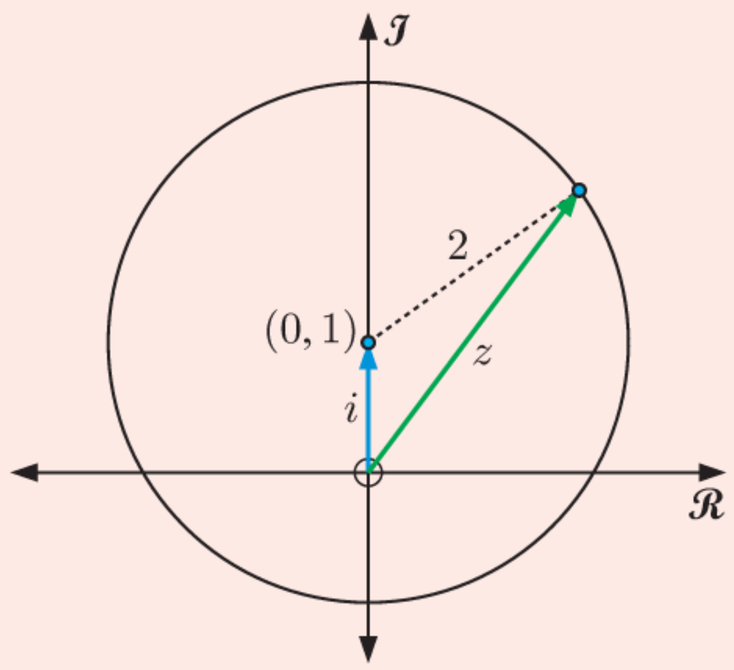
3 Use the principle of mathematical induction to show that $|z_1 + z_2 + ... + z_k| \leqslant |z_1| + |z_2| + ... + |z_k|$ for any integer $k \geqslant 1$.
Discuss the case of equality.

ACTIVITY 2 LOCUS

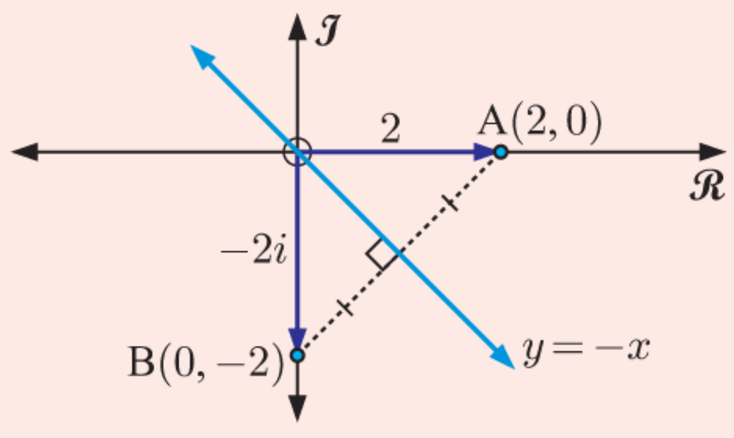
A **locus** is a curve or other figure formed by the set of points which satisfy a particular equation or inequality.

For example:

- Consider the equation $|z - i| = 2$.
  The point representing z is 2 units from the point representing i, so z lies on a circle with centre $(0, 1)$ and radius 2 units.



- Consider the equation $|z - 2| = |z + 2i|$.
  The point representing z is equidistant from $A(2, 0)$ and $B(0, -2)$, so z lies on the **perpendicular bisector** of [AB].
  This is the line $y = -x$.



What to do:

1 **a** Graph all complex numbers which satisfy $|z| = 1$.

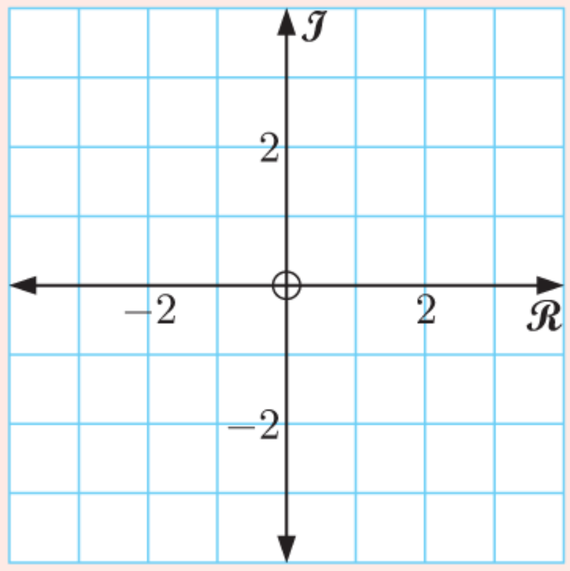
b Write $|z| = 1$ in Cartesian form by letting $z = x+yi$. What does this equation represent?

2 On separate Argand planes, sketch the region corresponding to:

a $|z| = 2$ **b** $|z| \leqslant 1$

c $3 < |z| < 4$ **d** $0 \leqslant \arg z \leqslant \pi$

e $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$ **f** $\arg z = -\frac{\pi}{2}$



3 Illustrate the locus of points corresponding to:

a $|z-4| = 1$ **b** $|z-3i| = 4$ **c** $|z+i| = 2$

d $|z-(2+i)| = 2$ **e** $|z-3-2i| = 5$ **f** $|z+2-i| = 3$

g $|z+1| \leqslant 2$ **h** $|z-1+i| > 2$ **i** $1 < |z-3i| < 2$

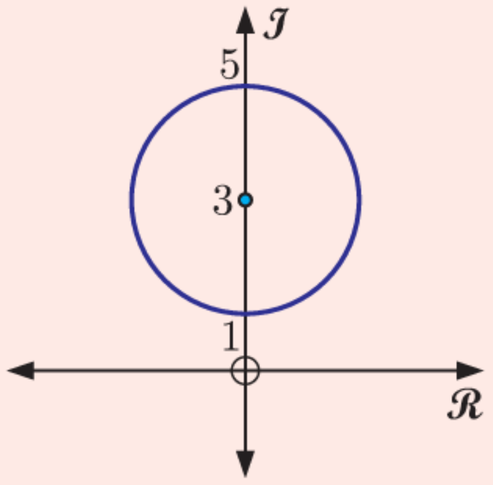
4 Illustrate:

a $\{z \mid |z+i| = 2 \text{ and } \mathcal{Re}(z) \geqslant 0\}$ **b** $\{z \mid |z+2| \leqslant 3 \text{ and } \mathcal{Im}(z) < 0\}$

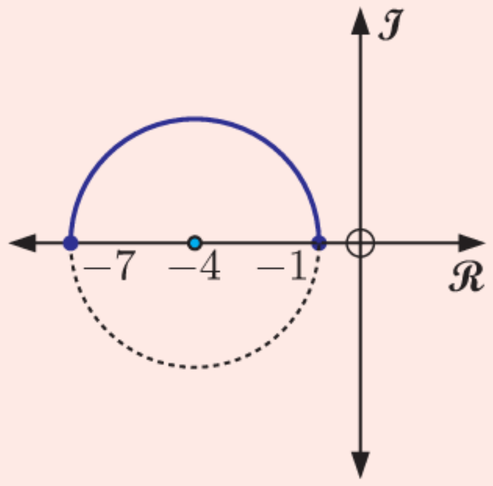
c $\{z \mid |z| \leqslant 4 \text{ and } \frac{\pi}{4} \leqslant \arg z \leqslant \pi\}$

5 Write the solution set of each graph:

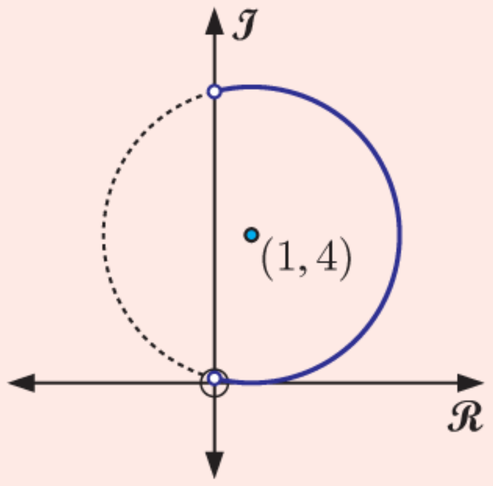
a



b



c



6 Illustrate the locus of points corresponding to:

a $|z+i| = |z-i|$ **b** $|z-1| = |z-3|$ **c** $|z+2i| = |z|$

d $|z-3| = |z+i|$ **e** $|z-3| < |z+5|$ **f** $|z-4i| \geqslant |z+4|$

7 **a** Find the equation, in terms of z, which describes the points on the circle illustrated.

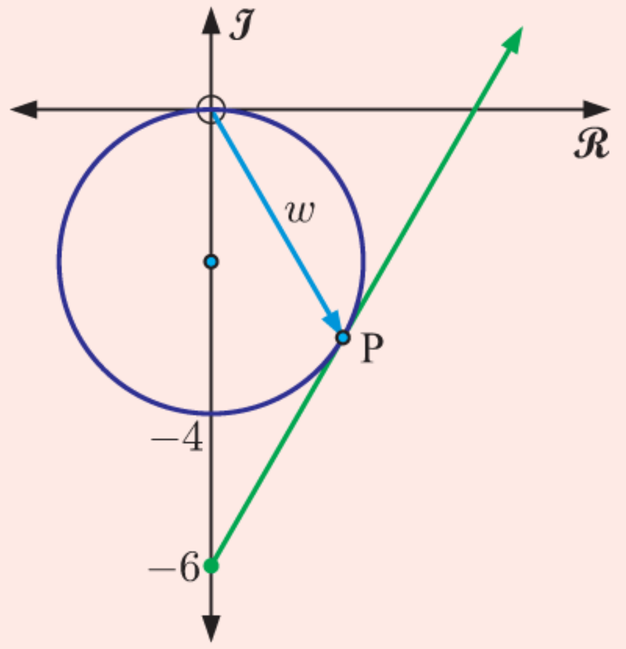
b The tangent to the circle from $(0, -6)$ meets the circle at the point P.

Let $w \equiv \overrightarrow{OP}$.

i Show that $|w| = 2\sqrt{3}$.

ii Find $\arg w$.

iii Hence write w in the form $a + bi$.

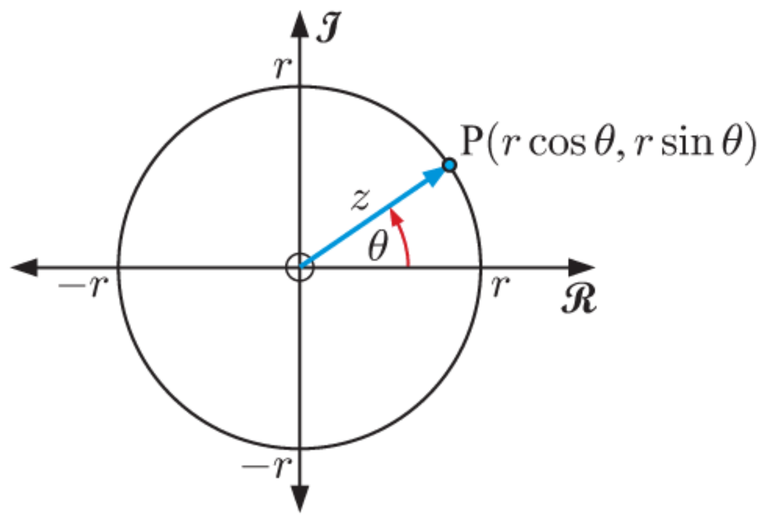


D POLAR FORM

We have seen that the **Cartesian form** of a complex number is $z = a + bi$. However, we can also use a **polar form** which is based on the modulus and argument of z.

Any point P which lies on a circle with centre $O(0, 0)$ and radius r, has Cartesian coordinates $(r\cos\theta, r\sin\theta)$.

$\therefore$ on the Argand plane, the complex number represented by $\overrightarrow{OP}$ is $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$.



We define $\mathbf{cis\,\theta = \cos\theta + i\sin\theta}$

so $z = |z|\operatorname{cis}\theta$.

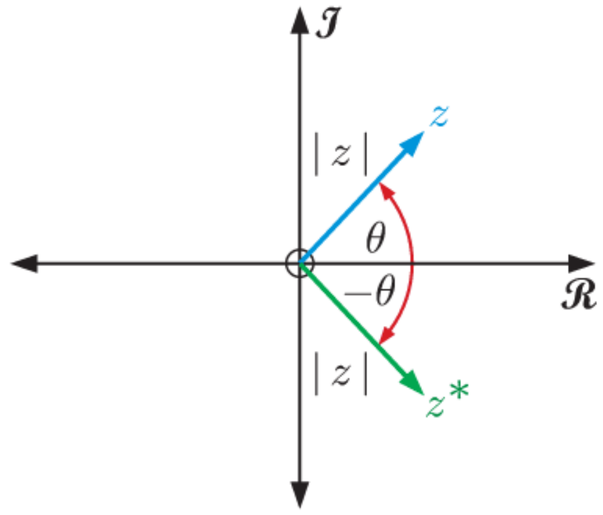
A complex number z has **polar form** $z = |z|\operatorname{cis}\theta$ where $|z|$ is the **modulus** of z, θ is the **argument** of z, and $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Polar form is also called **modulus-argument form**.

We will soon see that polar form is extremely powerful for dealing with multiplication and division of complex numbers, as well as quickly finding powers and roots of numbers.

The conjugate of z has the same length as z, and its argument is $(-\theta)$.

If $z = |z|\operatorname{cis}\theta$, then $z^* = |z|\operatorname{cis}(-\theta)$.

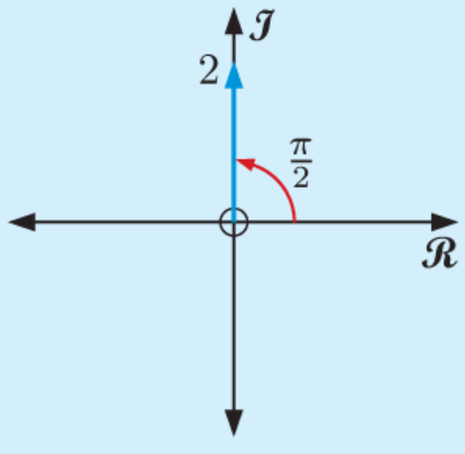


Example 9 — Self Tutor

Write in polar form:

a $2i$ **b** -3 **c** $1 - i$

a

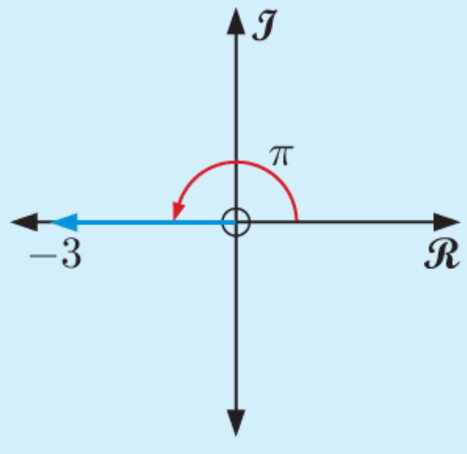


$|2i| = 2$

$\arg(2i) = \frac{\pi}{2}$

$\therefore \quad 2i = 2\operatorname{cis}\frac{\pi}{2}$

b

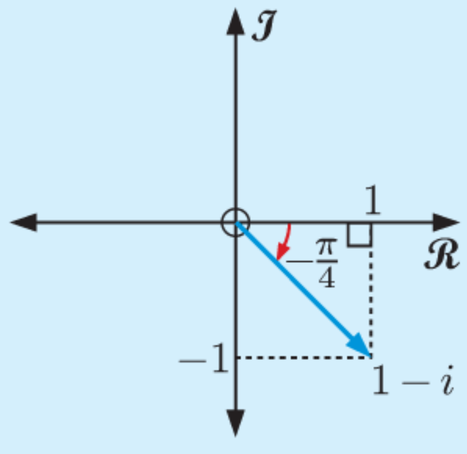


$|-3| = 3$

$\arg(-3) = \pi$

$\therefore \quad -3 = 3\operatorname{cis}\pi$

c



$|1 - i| = \sqrt{1 + 1} = \sqrt{2}$

$\arg(1 - i) = -\frac{\pi}{4}$

$\therefore \quad 1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

EXERCISE 14D.1

1 Write in polar form:

a 4 **b** $4i$ **c** -6 **d** $-3i$

e $1+i$ **f** $2-2i$ **g** $-\sqrt{3}+i$ **h** $2\sqrt{3}+2i$

2 What complex number cannot be written in polar form? Explain your answer.

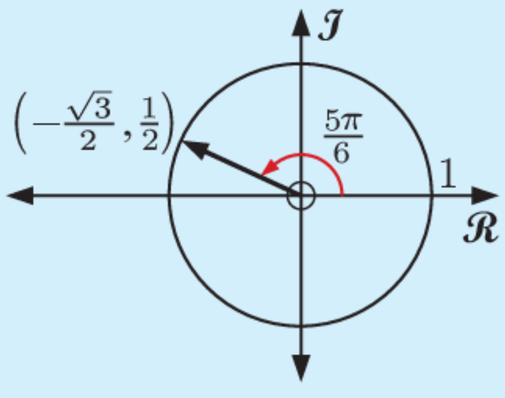
3 Convert $k+ki$ to polar form.

Hint: You must consider $k>0$, $k=0$, and $k<0$ separately.

Example 10 Self Tutor

Convert $\sqrt{3}\,\text{cis}\,\frac{5\pi}{6}$ to Cartesian form.

We expand $\text{cis}\,\frac{5\pi}{6}$ using a unit circle diagram.



$$\sqrt{3}\,\text{cis}\,\tfrac{5\pi}{6}$$
$$=\sqrt{3}\left(\cos\tfrac{5\pi}{6}+i\sin\tfrac{5\pi}{6}\right)$$
$$=\sqrt{3}\left(-\tfrac{\sqrt{3}}{2}+i\times\tfrac{1}{2}\right)$$
$$=-\tfrac{3}{2}+\tfrac{\sqrt{3}}{2}i$$

4 Convert to Cartesian form without using a calculator:

a $2\,\text{cis}\,\frac{\pi}{2}$ **b** $8\,\text{cis}\,\frac{\pi}{4}$ **c** $4\,\text{cis}\,\frac{\pi}{6}$

d $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ **e** $\sqrt{3}\,\text{cis}\,\frac{2\pi}{3}$ **f** $5\,\text{cis}\,\pi$

5 State the value of:

a $\text{cis}\,\frac{3\pi}{4}$ **b** $\text{cis}\,0$ **c** $|\,\text{cis}\,\theta\,|$ for any θ.

6 Use your calculator to convert to Cartesian form:

a $\sqrt{3}\,\text{cis}(2.5187)$ **b** $\sqrt{11}\,\text{cis}\left(-\frac{3\pi}{8}\right)$

c $2.836\,49\,\text{cis}(-2.684\,32)$

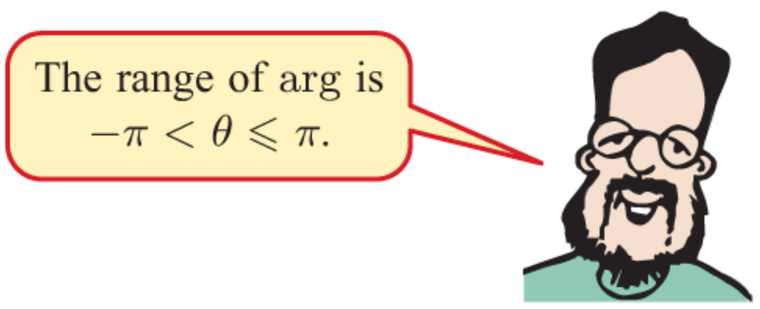


GRAPHICS CALCULATOR INSTRUCTIONS

7 Use your calculator to convert to modulus-argument form:

a $3-4i$ **b** $-5-12i$

c $-11.6814+13.2697i$



8 Add the following using Cartesian form and convert your answer to polar form:

a $3\,\text{cis}\,\frac{\pi}{4}+\text{cis}\left(-\frac{3\pi}{4}\right)$ **b** $2\,\text{cis}\,\frac{2\pi}{3}+5\,\text{cis}\left(-\frac{2\pi}{3}\right)$

9 Show that $\text{cis}\,\alpha\,\text{cis}\,\beta=\text{cis}(\alpha+\beta)$.

PROPERTIES OF $\text{cis}\,\theta$

Three useful properties of $\text{cis}\,\theta$ are:

- $\text{cis}\,\theta \times \text{cis}\,\phi = \text{cis}(\theta + \phi)$
- $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} = \text{cis}(\theta - \phi)$
- $\text{cis}(\theta + k2\pi) = \text{cis}\,\theta \quad \text{for all} \quad k \in \mathbb{Z}.$

Proof:

- $\text{cis}\,\theta \times \text{cis}\,\phi = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
  $= [\cos\theta\cos\phi - \sin\theta\sin\phi] + i[\sin\theta\cos\phi + \cos\theta\sin\phi]$
  $= \cos(\theta + \phi) + i\sin(\theta + \phi)$ {compound angle identities}
  $= \text{cis}(\theta + \phi)$
- $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} = \dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} \times \dfrac{\text{cis}(-\phi)}{\text{cis}(-\phi)}$
  $= \dfrac{\text{cis}(\theta - \phi)}{\text{cis}\,0}$
  $= \text{cis}(\theta - \phi)$ {as $\text{cis}\,0 = 1$}
- $\text{cis}(\theta + 2k\pi)$
  $= \text{cis}\,\theta \times \text{cis}\,k2\pi$
  $= \text{cis}\,\theta \times 1$ {as $\text{cis}\,k2\pi = 1, \ k \in \mathbb{Z}$}
  $= \text{cis}\,\theta$

Example 11 Self Tutor

Use the properties of cis to simplify:

a $\text{cis}\,\frac{\pi}{5}\,\text{cis}\,\frac{3\pi}{10}$ b $\dfrac{\text{cis}\,\frac{\pi}{5}}{\text{cis}\,\frac{7\pi}{10}}$

a
$\text{cis}\,\frac{\pi}{5}\,\text{cis}\,\frac{3\pi}{10}$
$= \text{cis}\left(\frac{\pi}{5} + \frac{3\pi}{10}\right)$
$= \text{cis}\,\frac{5\pi}{10}$
$= \text{cis}\,\frac{\pi}{2}$
$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
$= 0 + i(1)$
$= i$

b
$\dfrac{\text{cis}\,\frac{\pi}{5}}{\text{cis}\,\frac{7\pi}{10}}$
$= \text{cis}\left(\frac{\pi}{5} - \frac{7\pi}{10}\right)$
$= \text{cis}\left(-\frac{5\pi}{10}\right)$
$= \text{cis}\left(-\frac{\pi}{2}\right)$
$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$
$= 0 + i(-1)$
$= -i$

EXERCISE 14D.2

1 Use the properties of cis to simplify the following. Convert your answer to Cartesian form where possible.

a $\text{cis}\,\theta\,\text{cis}\,2\theta$ b $\dfrac{\text{cis}\,3\theta}{\text{cis}\,\theta}$ c $(\text{cis}\,\theta)^3$

d $\text{cis}\,\frac{\pi}{18}\,\text{cis}\,\frac{\pi}{9}$ e $2\,\text{cis}\,\frac{\pi}{12}\,\text{cis}\,\frac{\pi}{6}$ f $2\,\text{cis}\,\frac{2\pi}{5} \times 4\,\text{cis}\,\frac{8\pi}{5}$

g $\dfrac{4\,\text{cis}\,\frac{\pi}{12}}{2\,\text{cis}\,\frac{7\pi}{12}}$ h $\dfrac{\sqrt{32}\,\text{cis}\,\frac{\pi}{8}}{\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right)}$ i $\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{8}\right)^4$

Example 12 — Self Tutor

Simplify $\text{cis}\,\frac{107\pi}{6}$.

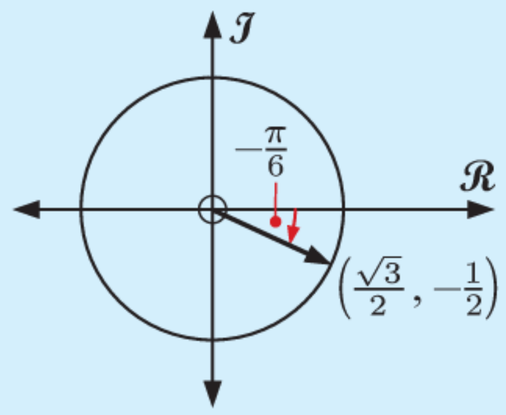
$$\frac{107\pi}{6} = 17\tfrac{5}{6}\pi = 18\pi - \tfrac{\pi}{6}$$

$$\therefore \ \text{cis}\,\tfrac{107\pi}{6} = \text{cis}\left(18\pi - \tfrac{\pi}{6}\right)$$

$$= \text{cis}\left(-\tfrac{\pi}{6}\right) \qquad \{\text{cis}(\theta + k2\pi) = \text{cis}\,\theta\}$$

$$= \cos\left(-\tfrac{\pi}{6}\right) + i\sin\left(-\tfrac{\pi}{6}\right)$$

$$= \tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i$$



2 Use the property $\text{cis}(\theta + k2\pi) = \text{cis}\,\theta$ to evaluate:

a $\text{cis}\,17\pi$ **b** $\text{cis}(-37\pi)$ **c** $\text{cis}\,\frac{91\pi}{3}$

3 If $z = 2\,\text{cis}\,\theta$:

a What is $|z|$ and $\arg z$?

b Write z^* in polar form.

c Write $-z$ in polar form.

d Write $-z^*$ in polar form.

$-2\,\text{cis}\,\theta$ is *not* in polar form since the modulus of a complex number cannot be negative.



Example 13 — Self Tutor

Use the properties of $\text{cis}\,\theta$ to find the transformation which moves z to iz.

Let $z = r\,\text{cis}\,\theta$ and use $i = 1\,\text{cis}\,\frac{\pi}{2}$.

$$\therefore \ iz = \text{cis}\,\tfrac{\pi}{2} \times r\,\text{cis}\,\theta$$

$$= r\,\text{cis}\left(\theta + \tfrac{\pi}{2}\right)$$

$\therefore$ z has been rotated anticlockwise by $\frac{\pi}{2}$ about O.

4 **a** Write $-i$ in polar form.

b Suppose $z = r\,\text{cis}\,\theta$ is any complex number. Write $-iz$ in polar form.

c What transformation maps z to $-iz$?

5 **a** Write -1 in polar form.

b Suppose $z = r\,\text{cis}\,\theta$ is any complex number. Write $-z$ in polar form.

c What transformation maps z to $-z$?

6 Suppose $z = r\,\text{cis}\,\theta$ is a complex number. Use a geometric interpretation of z^* to help write z^* in polar form.

MULTIPLICATION OF COMPLEX NUMBERS

Suppose $z = |z| \operatorname{cis}\theta$ and $w = |w| \operatorname{cis}\phi$

Then $zw = |z| \operatorname{cis}\theta \times |w| \operatorname{cis}\phi$

$= \underbrace{|z||w|}_{\text{non-negative}} \operatorname{cis}(\theta + \phi)$ {property of cis}

$\therefore \ |zw| = |z||w|$ and $\arg zw = \theta + \phi = \arg z + \arg w.$

If a complex number z is multiplied by $r \operatorname{cis}\theta$ then its modulus is *multiplied* by r and its argument is *increased* by θ.

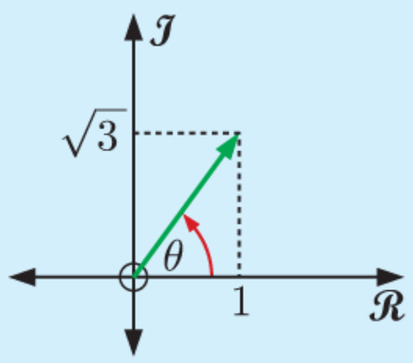
Example 14 Self Tutor

a Write $z = 1 + \sqrt{3}i$ in polar form. **b** Hence multiply z by $2 \operatorname{cis}\frac{\pi}{6}$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $2 \operatorname{cis}\frac{\pi}{6}$?

a



If $z = 1 + \sqrt{3}i$, then $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Now $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore \ \arg z = \frac{\pi}{3}$ $\{\mathcal{Re}(z), \mathcal{Im}(z) > 0\}$

$\therefore \ z = 2 \operatorname{cis}\frac{\pi}{3}$

b $(1 + \sqrt{3}i) \times 2 \operatorname{cis}\frac{\pi}{6} = 2 \operatorname{cis}\frac{\pi}{3} \times 2 \operatorname{cis}\frac{\pi}{6}$

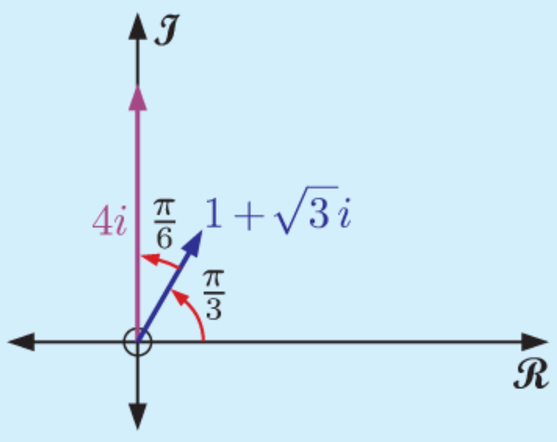
$= 4 \operatorname{cis}\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$

$= 4 \operatorname{cis}\frac{\pi}{2}$

$= 4(0 + 1i)$

$= 4i$

c



d When z was multiplied by $2 \operatorname{cis}\frac{\pi}{6}$, it was dilated with scale factor 2, then rotated anticlockwise through $\frac{\pi}{6}$.

EXERCISE 14D.3

1 **a** Write $z = 2 - 2i$ in polar form. **b** Hence multiply z by $\frac{1}{3} \operatorname{cis}\frac{\pi}{3}$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $\frac{1}{3} \operatorname{cis}\frac{\pi}{3}$?

2 **a** Write $z = -\sqrt{3} + i$ in polar form. **b** Hence multiply z by $\frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $\frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)$?

3 Suppose $z = -1 + i$ and $w = 1 - \sqrt{3}i$.

a Write z and w in polar form.

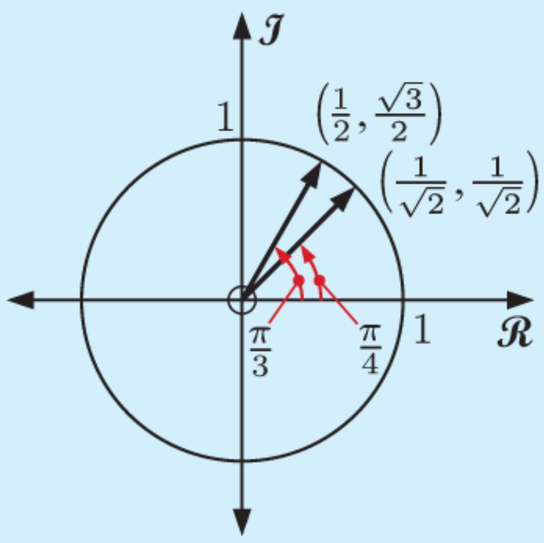
b Hence find zw in polar form.

c Describe the transformation to z when it is multiplied by w.

Example 15 **Self Tutor**

Use complex numbers to write $\cos\frac{7\pi}{12}$ and $\sin\frac{7\pi}{12}$ in simplest surd form.

$$\cos\tfrac{7\pi}{12} + i\sin\tfrac{7\pi}{12}$$
$$= \text{cis}\,\tfrac{7\pi}{12}$$
$$= \text{cis}\left(\tfrac{3\pi}{12} + \tfrac{4\pi}{12}\right)$$
$$= \text{cis}\,\tfrac{\pi}{4} \times \text{cis}\,\tfrac{\pi}{3} \qquad \{\text{cis}(\theta+\phi) = \text{cis}\,\theta \times \text{cis}\,\phi\}$$
$$= \left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right) \times \left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right)$$
$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)$$
$$= \left(\tfrac{1}{2\sqrt{2}} - \tfrac{\sqrt{3}}{2\sqrt{2}}\right) + i\left(\tfrac{\sqrt{3}}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}}\right)$$



Equating real parts: $\cos\frac{7\pi}{12} = \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\sin\frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$

4 Use complex number methods to find, in simplest surd form:

a $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$ **b** $\cos\frac{11\pi}{12}$ and $\sin\frac{11\pi}{12}$

5 Use the principle of mathematical induction to prove that $\arg(z^n) = n\arg z$ for all $n \in \mathbb{Z}^+$.

6 Use polar form to establish:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \text{ and } \arg\left(\frac{z}{w}\right) = \arg z - \arg w, \text{ provided } w \neq 0.$$

Example 16 **Self Tutor**

Suppose $z = \sqrt{2}\,\text{cis}\,\theta$ where θ is obtuse. Find the modulus and argument of:

a $2z$ **b** $\frac{z}{i}$ **c** $(1-i)z$

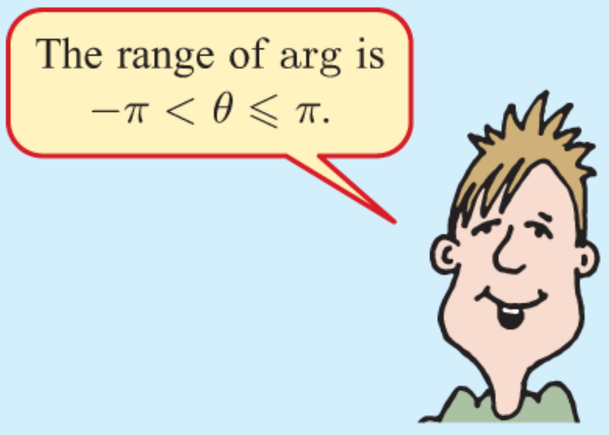
a $2z = 2\sqrt{2}\,\text{cis}\,\theta$

$\therefore |2z| = 2\sqrt{2}$ and $\arg 2z = \theta$

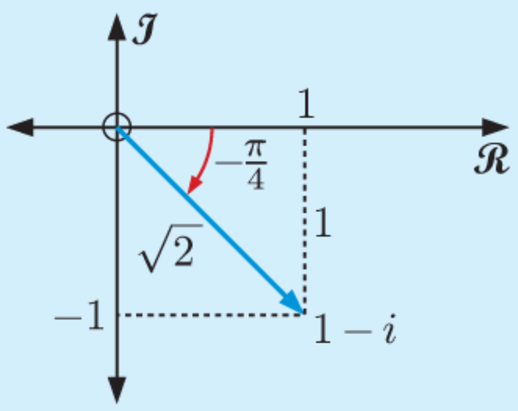
b $i = \text{cis}\,\frac{\pi}{2}$

$$\therefore \frac{z}{i} = \frac{\sqrt{2}\,\text{cis}\,\theta}{\text{cis}\,\frac{\pi}{2}}$$
$$= \sqrt{2}\,\text{cis}\left(\theta - \tfrac{\pi}{2}\right)$$

$\therefore \left|\frac{z}{i}\right| = \sqrt{2}$ and $\arg\left(\frac{z}{i}\right) = \theta - \frac{\pi}{2}$ $\{\text{since } \theta \text{ is obtuse, } -\pi < \theta - \frac{\pi}{2} \leqslant \pi\}$



c



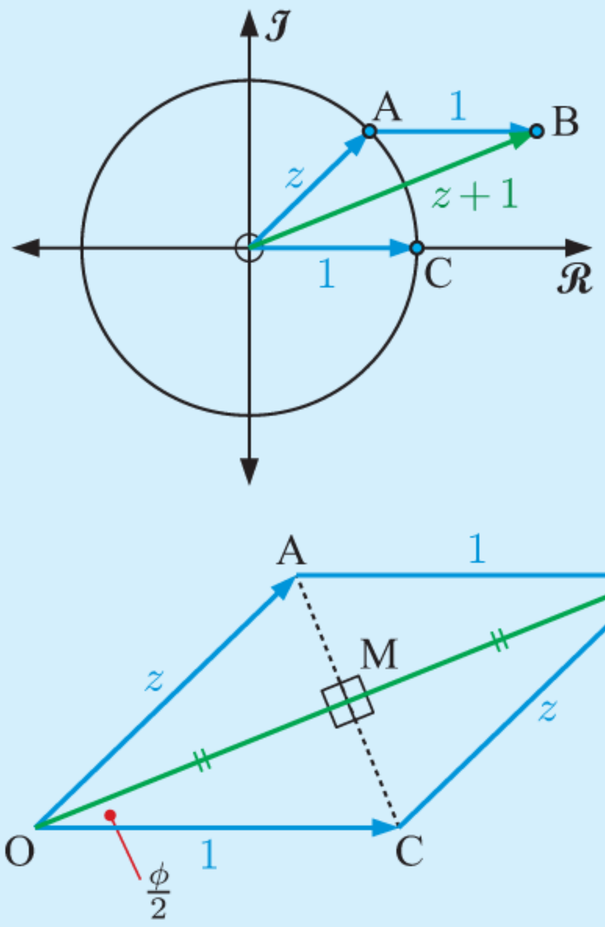
$$1-i = \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)$$
$$\therefore\ (1-i)z = \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right) \times \sqrt{2}\,\text{cis}\,\theta$$
$$= 2\,\text{cis}\left(-\tfrac{\pi}{4} + \theta\right)$$
$$\therefore\ |(1-i)z| = 2 \quad \text{and} \quad \arg((1-i)z) = \theta - \tfrac{\pi}{4}$$

7 Suppose $z = 3\,\text{cis}\,\theta$ where θ is acute. Determine the modulus and argument of:

a $-z$ **b** z^* **c** iz **d** $(1+i)z$ **e** $\frac{z}{i}$ **f** $\frac{z}{1-i}$

Example 17 **Self Tutor**

Suppose $z = \text{cis}\,\phi$ where ϕ is acute. Find the modulus and argument of $z+1$.



$|z| = 1$, so z ends on the unit circle.

$z+1$ is the vector $\overrightarrow{\text{OB}}$ shown.

Now OABC is a rhombus.

$\therefore\ \arg(z+1) = \frac{\phi}{2}$ {diagonals bisect the angles}

Let M be the midpoint of OB.

$\therefore\ \cos\frac{\phi}{2} = \frac{\text{OM}}{1}$

$\therefore\ \text{OM} = \cos\frac{\phi}{2}$

$\therefore\ \text{OB} = 2\cos\frac{\phi}{2}$

$\therefore\ |z+1| = 2\cos\frac{\phi}{2}$

8 **a** If $z = \text{cis}\,\phi$ where ϕ is acute, determine the modulus and argument of $z-1$.

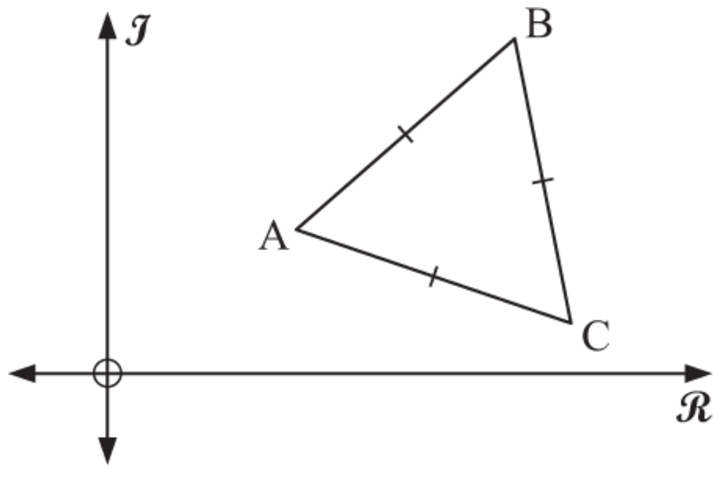
b Hence write $z-1$ and $(z-1)^*$ in polar form.

9 ABC is an equilateral triangle. Suppose z_1 represents OA, z_2 represents $\overrightarrow{\text{OB}}$, and z_3 represents $\overrightarrow{\text{OC}}$.

a Explain what vectors represent $z_2 - z_1$ and $z_3 - z_2$.

b Find $\left|\frac{z_2 - z_1}{z_3 - z_2}\right|$ and $\arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right)$.

c Hence find the value of $\left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3$.



10 Use the sum and product of roots to find the real quadratic equations with roots:

a $2\,\text{cis}\,\frac{2\pi}{3}$ and $2\,\text{cis}\,\frac{4\pi}{3}$ **b** $\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ and $\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$

E EULER'S FORM

HISTORICAL NOTE — EULER'S BEAUTIFUL EQUATION

One of the most remarkable results in mathematics is known as **Euler's beautiful equation** $e^{i\pi} = -1$ named after **Leonhard Euler**.

It is called beautiful because it links together three great constants of mathematics: Euler's constant e, the imaginary number i, and the ratio of a circle's circumference to its diameter π.

Harvard lecturer **Benjamin Pierce** said of $e^{i\pi} = -1$,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

Euler's beautiful equation is a special case of a more general result that Euler proved:

For any $\theta \in \mathbb{R}$, $\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$.

This identity allows us to write any complex number $z = |z|\operatorname{cis}\theta$ in the **Euler form** $z = |z|e^{i\theta}$.

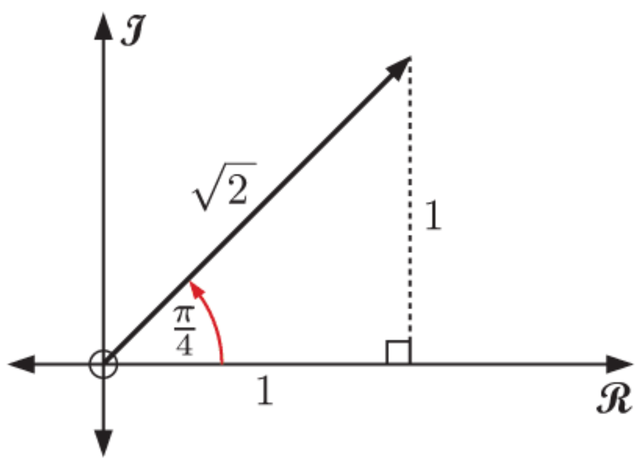
For example, consider $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$,

$\therefore \quad 1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

So, $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ is the **polar form** of $1 + i$

and $\sqrt{2}e^{i\frac{\pi}{4}}$ is the **Euler form** of $1 + i$.



We cannot prove the Euler identity for ourselves until we study calculus. However, in the following **Investigation** we put together our knowledge of exponentials and complex numbers to justify why it is reasonable.

INVESTIGATION — EULER'S FORM

In our previous study of exponentials we saw how the exponential e is related to compound interest. In particular:

- For 100% compound growth with interest calculated in n subintervals, the value after each subinterval is generated using the sequence $1, \left(1+\frac{1}{n}\right), \left(1+\frac{1}{n}\right)^2, ..., \left(1+\frac{1}{n}\right)^n$. We saw that as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e^1$.
- If interest is paid continuously, then the compounded amount after 1 year is the initial amount multiplied by e^R, where R is the interest rate.

We now consider the idea of *imaginary* compound growth.

What to do:

1 Consider the sequence $1, \left(1+\frac{i}{n}\right), \left(1+\frac{i}{n}\right)^2, ..., \left(1+\frac{i}{n}\right)^n$ which has $n+1$ terms. This sequence represents imaginary compound growth with interest calculated in n subintervals.

a For each of the following cases, find the terms of the sequence, plot them on an Argand diagram, then connect them with straight line segments.

i $n = 1$ **ii** $n = 2$ **iii** $n = 3$ **iv** $n = 4$ **v** $n = 5$

b Click on the icon and use the software to plot the terms of the sequence for values of n up to 500.



i What do you notice about the curve being generated as n gets very large?

ii Find $\cos 1^c$ and $\sin 1^c$. **iii** Locate $\left(1 + \frac{i}{500}\right)^{500}$.

iv Copy and complete: As $n \to \infty$, $\left(1 + \frac{i}{n}\right)^n \to \ldots\ldots + \ldots\ldots\ i = e^i$. {Euler}

2 Now consider the sequence $1, \left(1 + \frac{i\theta}{n}\right), \left(1 + \frac{i\theta}{n}\right)^2, \ldots, \left(1 + \frac{i\theta}{n}\right)^n$ which has $n+1$ terms.

Click on the icon to run the software.



a Set $\theta = 2$. Plot the terms of the sequence for values of n up to 500.

i What do you notice about the curve being generated as n gets very large?

ii Find $\cos 2^c$ and $\sin 2^c$.

iii Copy and complete: As $n \to \infty$, $\left(1 + \frac{2i}{n}\right)^n \to \ldots\ldots + \ldots\ldots\ i = e^{2i}$. {Euler}

b Set $\theta = \frac{\pi}{2}$. Plot the terms of the sequence for values of n up to 500.

i Copy and complete:

As $n \to \infty$, $\left(1 + \frac{i\frac{\pi}{2}}{n}\right)^n \to \ldots\ldots + \ldots\ldots\ i = e^{i\frac{\pi}{2}}$. {Euler}

ii Comment on your answer with respect to the result in **Example 13**.

3 Copy and complete:

a The effect of *real* compound growth is to increase the size or modulus of a quantity. So, when we multiply a complex number z by e^r, the vector representing z is ..... with scale factor ..... . This means the ..... of z is multiplied by e^r.

b The effect of *imaginary* compound growth is to rotate the quantity in the complex plane. So, when we multiply a complex number z by $\text{cis}\,\theta = e^{\cdots}$, z is ..... anticlockwise through angle ..... about the origin O. This means the ..... of z is increased by θ.

4 Consider $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$. *Explain* the result when z is multiplied by w.

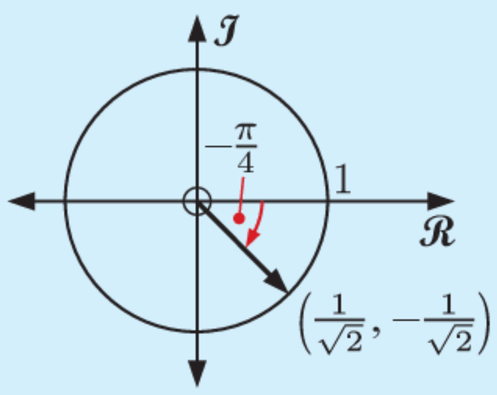
Example 18

Self Tutor

Evaluate: **a** $e^{-i\frac{\pi}{4}}$ **b** i^{-i}

a $e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$



b Now $i = 0 + 1i$
$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
$= e^{i\frac{\pi}{2}}$

$\therefore\ i^{-i} = \left(e^{i\frac{\pi}{2}}\right)^{-i}$
$= e^{-i^2\frac{\pi}{2}}$
$= e^{\frac{\pi}{2}}$

EXERCISE 14E

1 Use $e^{i\theta} = \cos\theta + i\sin\theta$ to write each of the following complex numbers in Cartesian form:

a $e^{i\pi}$ **b** $e^{i\frac{\pi}{3}}$ **c** $e^{-i\frac{\pi}{2}}$

2 Write:

a $-1 + i$ in polar form and Euler form **b** $3\,\text{cis}\left(-\frac{\pi}{6}\right)$ in Cartesian form and Euler form

c $2e^{i\frac{2\pi}{3}}$ in Cartesian form and polar form.

3 Use $\text{cis}\,\theta = e^{i\theta}$ to prove that:

a $\text{cis}\,\theta\,\text{cis}\,\phi = \text{cis}(\theta + \phi)$ **b** $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} = \text{cis}(\theta - \phi)$

4 Suppose $z = \text{cis}\,\theta$, find the argument of:

a $\sqrt{z}$ **b** iz^4 **c** $-iz^{\frac{2}{5}}$ **d** $\dfrac{i}{\sqrt[3]{z}}$

5 Given $z = 2e^{i\theta}$ and $w = 3e^{i\phi}$, find the modulus and argument of:

a $\sqrt{w}$ **b** $\dfrac{z}{w^2}$ **c** $w^3 \times \sqrt{z}$

6 Evaluate:

a e^i **b** 3^i **c** i^i **d** $\left((i^i)^i\right)^2$

7 Prove that for any angle θ:

a $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ **b** $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

F DE MOIVRE'S THEOREM

Polar form enables us to easily calculate powers of complex numbers.

For example, if $z = |z|\,\text{cis}\,\theta$

then $z^2 = |z|\,\text{cis}\,\theta \times |z|\,\text{cis}\,\theta$
$= |z|^2\,\text{cis}(\theta + \theta)$
$= |z|^2\,\text{cis}\,2\theta$

and $z^3 = z^2 z$
$= |z|^2\,\text{cis}\,2\theta \times |z|\,\text{cis}\,\theta$
$= |z|^3\,\text{cis}(2\theta + \theta)$
$= |z|^3\,\text{cis}\,3\theta$

The generalisation of this process is **De Moivre's Theorem**:

$$(|z|\,\text{cis}\,\theta)^n = |z|^n\,\text{cis}\,n\theta \quad \text{for all } n \in \mathbb{Q}.$$

Proof:

We begin by proving the result for $n \in \mathbb{Z}^+$ by the principle of mathematical induction:

P_n is: $(|z|\operatorname{cis}\theta)^n = |z|^n \operatorname{cis} n\theta$ for all $n \in \mathbb{Z}^+$.

(1) If $n = 1$, then $(|z|\operatorname{cis}\theta)^1 = |z|\operatorname{cis}\theta$ $\therefore$ P_1 is true.

(2) If P_k is true, then $(|z|\operatorname{cis}\theta)^k = |z|^k \operatorname{cis} k\theta$

Thus $(|z|\operatorname{cis}\theta)^{k+1} = (|z|\operatorname{cis}\theta)^k \times |z|\operatorname{cis}\theta$ {exponent law}

$= |z|^k \operatorname{cis} k\theta \times |z|\operatorname{cis}\theta$ {using P_k}

$= |z|^{k+1} \operatorname{cis}(k\theta + \theta)$ {exponent law and cis property}

$= |z|^{k+1} \operatorname{cis}(k+1)\theta$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$.

We next observe that $\operatorname{cis}(-n\theta) = \operatorname{cis}(0 - n\theta) = \dfrac{\operatorname{cis} 0}{\operatorname{cis} n\theta}$ $\left\{\dfrac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta - \phi)\right\}$

$\therefore$ $\operatorname{cis}(-n\theta) = \dfrac{1}{\operatorname{cis} n\theta}$ $\{\operatorname{cis} 0 = 1\}$

$= \dfrac{1}{(\operatorname{cis}\theta)^n}$ {for $n \in \mathbb{Z}^+$}

$= (\operatorname{cis}\theta)^{-n}$ so the theorem is true for all $n \in \mathbb{Z}$. .... $(*)$

Finally, we consider $n = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$.

$\left(\operatorname{cis}\frac{\theta}{q}\right)^q = \operatorname{cis}\left(q\left(\frac{\theta}{q}\right)\right) = \operatorname{cis}\theta$ {by $(*)$}

$\therefore$ $(\operatorname{cis}\theta)^{\frac{1}{q}} = \operatorname{cis}\dfrac{\theta}{q}$

$\therefore$ $(\operatorname{cis}\theta)^{\frac{p}{q}} = \left((\operatorname{cis}\theta)^{\frac{1}{q}}\right)^p = \left(\operatorname{cis}\dfrac{\theta}{q}\right)^p = \operatorname{cis}\left(\dfrac{p}{q}\theta\right)$ {by $(*)$}

So, the theorem is true for all $n \in \mathbb{Q}$.

Example 19

Self Tutor

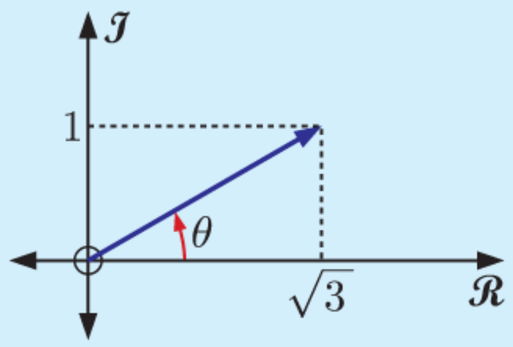
Find the exact value of $(\sqrt{3} + i)^8$ using De Moivre's theorem.

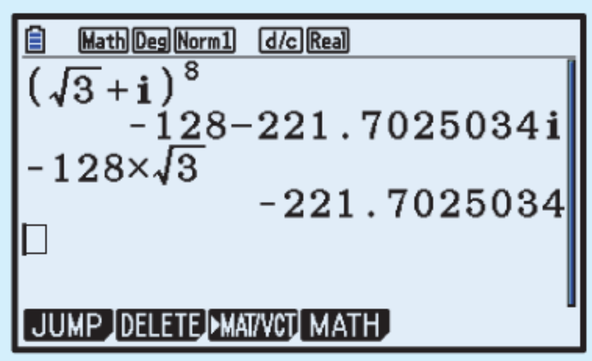
$\sqrt{3} + i$ has modulus $\sqrt{(\sqrt{3})^2 + 1^2} = 2$

Now $\tan\theta = \dfrac{1}{\sqrt{3}}$

$\therefore$ $\arg(\sqrt{3} + i) = \dfrac{\pi}{6}$

$\therefore$ $\sqrt{3} + i = 2\operatorname{cis}\dfrac{\pi}{6}$





$\therefore$ $(\sqrt{3} + i)^8 = \left(2\operatorname{cis}\dfrac{\pi}{6}\right)^8$

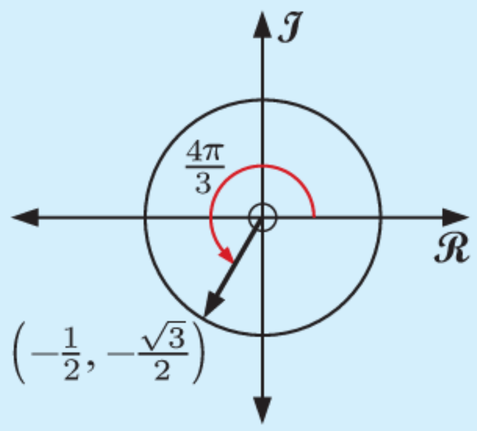
$= 2^8 \operatorname{cis}\dfrac{8\pi}{6}$

$= 2^8 \operatorname{cis}\dfrac{4\pi}{3}$

$= 2^8\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

$= 2^8\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$

$= -128 - 128\sqrt{3}i$



EXERCISE 14F

1 Prove De Moivre's theorem using Euler's form and the laws of exponents.

2 Use De Moivre's theorem to simplify:

a $\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{5}\right)^{10}$ **b** $\left(\text{cis}\,\frac{\pi}{12}\right)^{36}$ **c** $\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{8}\right)^{12}$

d $\sqrt{5\,\text{cis}\,\frac{\pi}{7}}$ **e** $\sqrt[3]{8\,\text{cis}\,\frac{\pi}{2}}$ **f** $\left(8\,\text{cis}\,\frac{\pi}{5}\right)^{\frac{5}{3}}$

3 Use De Moivre's theorem to find the exact value of:

a $(1+i)^{15}$ **b** $\left(1-i\sqrt{3}\right)^{11}$ **c** $\left(\sqrt{2}-i\sqrt{2}\right)^{-19}$

d $(-1+i)^{-11}$ **e** $\left(\sqrt{3}-i\right)^{\frac{1}{2}}$ **f** $\left(2+2i\sqrt{3}\right)^{-\frac{5}{2}}$

Check your answers using technology.

4 Suppose $z = |z|\,\text{cis}\,\theta$ where $-\pi < \theta \leqslant \pi$.

a Use De Moivre's theorem to find $\sqrt{z}$ in terms of $|z|$ and θ.

b What restrictions apply to $\phi = \arg(\sqrt{z})$?

c True or false? "$\sqrt{z}$ has a non-negative real part."

5 Show that $\cos\theta - i\sin\theta = \text{cis}(-\theta)$. Hence simplify $(\cos\theta - i\sin\theta)^{-3}$.

6 **a** Write $z = 1+i$ in polar form and hence write z^n in polar form.

b Find all values of n for which:

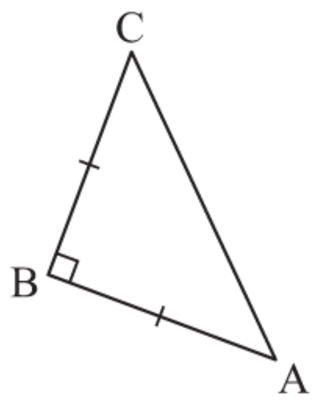
i z^n is real **ii** z^n is purely imaginary.

7 Suppose $|z| = 2$ and $\arg z = \theta$. Determine the modulus and argument of:

a z^3 **b** iz^2 **c** $\frac{1}{z}$ **d** $-\frac{i}{z^2}$

8 If $z = \text{cis}\,\theta$, prove that $\dfrac{z^2-1}{z^2+1} = i\tan\theta$.

9 Points A, B, and C form an isosceles triangle with a right angle at B. Suppose A, B, and C are represented by the complex numbers z_1, z_2, and z_3 respectively.



a Show that $(z_1 - z_2)^2 = -(z_3 - z_2)^2$.

b If ABCD forms a square, what complex number represents the point D? Give your answer in terms of z_1, z_2, and z_3.

Example 20 Self Tutor

By considering $\cos 2\theta + i\sin 2\theta$, deduce the double angle formulae for $\cos 2\theta$ and $\sin 2\theta$.

$$\begin{aligned}
&\cos 2\theta + i\sin 2\theta \\
&= \text{cis}\,2\theta \\
&= [\text{cis}\,\theta]^2 \qquad \{\text{De Moivre's theorem}\} \\
&= [\cos\theta + i\sin\theta]^2 \\
&= [\cos^2\theta - \sin^2\theta] + i\,[2\sin\theta\cos\theta]
\end{aligned}$$

Equating real parts:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Equating imaginary parts:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

10 **a** Use complex number methods to deduce that:

i $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **ii** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b Hence find $\tan 3\theta$ in terms of $\tan\theta$ only.

c Hence solve the equations:

i $4x^3 - 3x = -\frac{1}{\sqrt{2}}$ **ii** $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$

11 Use imaginary i to help find a formula for:

a $\cos 4\theta$ in terms of $\cos\theta$ **b** $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

12 **a** **i** If $z = \text{cis}\,\theta$, prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

ii Hence explain why $z + \frac{1}{z} = 2\cos\theta$.

iii Use the binomial theorem to expand $\left(z + \frac{1}{z}\right)^3$, then simplify your result.

iv Hence show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

v Hence show that the exact value of $\cos^3\left(\frac{13\pi}{12}\right)$ is $\frac{-5\sqrt{2}-3\sqrt{6}}{16}$.

Hint: $\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$

b If $z = \text{cis}\,\theta$, prove that $z^n - \frac{1}{z^n} = 2i\sin n\theta$. Hence show that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

c Hence show that $\sin^3\theta\cos^3\theta = \frac{1}{32}(3\sin 2\theta - \sin 6\theta)$.

13 Let $z = r\,\text{cis}\,\theta$.

a Write $z \times z^2 \times z^3$ in polar form.

b Show that $z \times z^2 \times z^3 \times .... \times z^k = r^{\frac{k(k+1)}{2}}\,\text{cis}\left(\frac{k(k+1)\theta}{2}\right)$.

c Suppose $r = 2$ and $\theta = \frac{\pi}{7}$.

i Find the smallest positive integer k such that $z \times z^2 \times z^3 \times .... \times z^k$ is real, and state the value of $\left|z \times z^2 \times z^3 \times .... \times z^k\right|$ in this case.

ii Show that $z \times z^2 \times z^3 \times .... \times z^k$ is never purely imaginary.

14 Use the binomial expansion of $(1+i)^{2n}$ to prove that:

$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + ... + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\frac{n\pi}{2}\right), \quad n \in \mathbb{Z}^+$.

15 By considering $1 + \text{cis}\,\theta + \text{cis}\,2\theta + \text{cis}\,3\theta + ... + \text{cis}\,n\theta$ as a geometric series, find $\sum_{r=0}^{n}\cos r\theta$.

16 **a** Prove that $1 + \text{cis}\,\theta = 2\cos\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}$.

b Hence determine the sum of the series $\sum_{r=0}^{n}\binom{n}{r}\cos r\theta$.

G ROOTS OF COMPLEX NUMBERS

With De Moivre's theorem to help us, we can now find roots of complex numbers.

SOLVING $z^n = c$

Consider an equation of the form $z^n = c$ where n is a positive integer and c is a complex number.

The **nth roots of the complex number c** are the n solutions of $z^n = c$.

For example, the 4th roots of $2i$ are the four solutions of $z^4 = 2i$.

- There are **exactly** n nth roots of c.
- If $c \in \mathbb{R}$, the complex roots must occur in conjugate pairs.
- If $c \notin \mathbb{R}$, the complex roots do not all occur in conjugate pairs.
- The roots of z^n will all have the same modulus which is $|c|^{\frac{1}{n}}$.
- On an Argand diagram, the roots all lie on a circle with radius $|c|^{\frac{1}{n}}$.
- The roots on the circle $r = |c|^{\frac{1}{n}}$ will be equally spaced around the circle.
  If you join the points corresponding to the roots, you will obtain a regular polygon.

The roots of a complex number may be found by factorisation, but this is usually very difficult. It is therefore desirable to have an alternative method such as the **nth roots method** presented in the following Example.

Example 21 Self Tutor

Find the four 4th roots of 1 by:

a factorisation **b** the "nth roots method".

The 4th roots of 1 are the 4 solutions of $z^4 = 1$.

a By factorisation,

$$z^4 = 1$$
$$\therefore\ z^4 - 1 = 0$$
$$\therefore\ (z^2+1)(z^2-1) = 0$$
$$\therefore\ (z+i)(z-i)(z+1)(z-1) = 0$$
$$\therefore\ z = \pm i \text{ or } \pm 1$$

b By the "nth roots method",

$$z^4 = 1$$
$$\therefore\ z^4 = \text{cis}(0 + k2\pi),\ \ k \in \mathbb{Z}$$
$$\therefore\ z = (\text{cis}\, k2\pi)^{\frac{1}{4}}$$
$$\therefore\ z = \text{cis}\, \tfrac{k2\pi}{4} \quad \{\text{De Moivre}\}$$
$$\therefore\ z = \text{cis}\, \tfrac{k\pi}{2}$$
$$\therefore\ z = \text{cis}\, 0,\ \text{cis}\, \tfrac{\pi}{2},\ \text{cis}\, \pi,\ \text{cis}\, \tfrac{3\pi}{2} \quad \{\text{letting } k = 0, 1, 2, 3\}$$
$$\therefore\ z = 1,\ i,\ -1,\ -i$$

The substitution of $k = 0, 1, 2, 3$ to find the 4 roots could be done using any 4 consecutive integers for k.



EXERCISE 14G.1

1 Find the three cube roots of 1 using:

a factorisation **b** the "nth roots method".

2 Solve for z:

a $z^3 = -8i$ **b** $z^3 = -27i$

3 Find the three cube roots of -1, and display them on an Argand diagram.

4 Solve for z:

a $z^4 = 16$ **b** $z^4 = -16$

5 Find the four fourth roots of $-i$, and display them on an Argand diagram.

6 Solve the following equations, and display the solutions for each on an Argand diagram:

a $z^3 = 2 + 2i$ **b** $z^3 = -2 + 2i$ **c** $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

d $z^4 = \sqrt{3} + i$ **e** $z^5 = -4 - 4i$ **f** $z^3 = -2\sqrt{3} - 2i$

Example 22 — Self Tutor

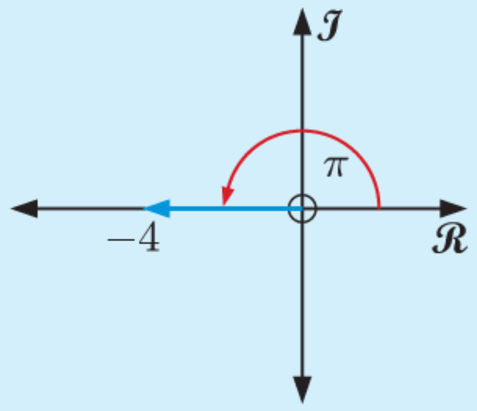
a Find the fourth roots of -4 in the form $a + bi$.

b Hence write $z^4 + 4$ as a product of real quadratic factors.

a The fourth roots of -4 are the solutions of

$z^4 = -4$

$\therefore \ z^4 = 4\,\text{cis}(\pi + k2\pi)$ where $k \in \mathbb{Z}$ {polar form}



$\therefore \ z = [4\,\text{cis}(\pi + k2\pi)]^{\frac{1}{4}}$

$\therefore \ z = 4^{\frac{1}{4}}\,\text{cis}\left(\frac{\pi + k2\pi}{4}\right)$ {De Moivre}

$\therefore \ z = 2^{\frac{1}{2}}\,\text{cis}\,\frac{\pi}{4},\ 2^{\frac{1}{2}}\,\text{cis}\,\frac{3\pi}{4},\ 2^{\frac{1}{2}}\,\text{cis}\,\frac{5\pi}{4},\ 2^{\frac{1}{2}}\,\text{cis}\,\frac{7\pi}{4}$ {letting $k = 0, 1, 2, 3$}

$\therefore \ z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$\therefore \ z = 1 + i,\ -1 + i,\ -1 - i,\ 1 - i$

b The roots $1 \pm i$ have sum $= 2$ and product $= (1 + i)(1 - i) = 2$

$\therefore$ they come from the quadratic factor $z^2 - 2z + 2$.

The roots $-1 \pm i$ have sum $= -2$ and product $= (-1 + i)(-1 - i) = 2$

$\therefore$ they come from the quadratic factor $z^2 + 2z + 2$.

Thus $z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2)$



Observe the connection between polynomial methods and complex number theory.

7 **a** Find the four solutions of $z^4 + 1 = 0$ in the form $a + bi$, and display them on an Argand diagram.

b Hence write $z^4 + 1$ as the product of two real quadratic factors.

8 Consider $z = \dfrac{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2}{\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^5 \left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)^{25}}$

a Using polar form and De Moivre's theorem, find the modulus and argument of z.

b Hence show that z is a cube root of 1.

c Without using a calculator, show that $(1 - 2z)(2z^2 - 1)$ is a real number.

9 **a** Write $-16i$ in polar form.

b Let z be the fourth root of $-16i$ which lies in the second quadrant.
Express z exactly in:

i polar form **ii** Cartesian form.

THE nTH ROOTS OF UNITY

The ***n*th roots of unity** are the solutions of $z^n = 1$.

Example 23 **Self Tutor**

Find the three cube roots of unity and display them on an Argand diagram. If w is the root with smallest positive argument, show that the roots are 1, w, and w^2, and that $1 + w + w^2 = 0$.

The cube roots of unity are the solutions of $z^3 = 1$.

Now $1 = \text{cis}(0 + k2\pi)$ for all $k \in \mathbb{Z}$ {polar form}

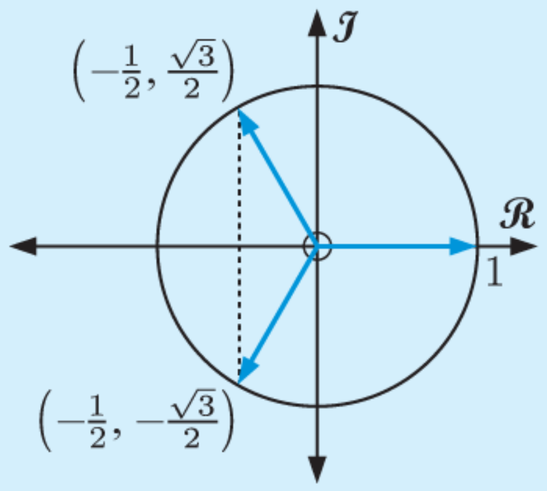
$\therefore\ z^3 = \text{cis}\, k2\pi$

$\therefore\ z = (\text{cis}\, k2\pi)^{\frac{1}{3}}$

$\therefore\ z = \text{cis}\,\frac{k2\pi}{3}$ {De Moivre's theorem}

$\therefore\ z = \text{cis}\, 0,\ \text{cis}\,\frac{2\pi}{3},\ \text{cis}\,\frac{4\pi}{3}$ {letting $k = 0, 1, 2$}

$\therefore\ z = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$



Letting $w = \text{cis}\,\frac{2\pi}{3}$, then $w^2 = \left(\text{cis}\,\frac{2\pi}{3}\right)^2 = \text{cis}\,\frac{4\pi}{3}$

$\therefore$ the roots are 1, w, and w^2, and $1 + w + w^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$

EXERCISE 14G.2

1 In **Example 23** we showed that the cube roots of unity are 1, w, and w^2 where $w = \text{cis}\,\frac{2\pi}{3}$.

a Use this fact to solve the following equations, giving your answers in terms of w:

i $(z + 3)^3 = 1$ **ii** $(z - 1)^3 = 8$ **iii** $(2z - 1)^3 = -1$

b Show by vector addition that $1 + w + w^2 = 0$.

2 In **Example 21** we showed that the four fourth roots of unity are 1, i, -1, and $-i$.

a Show that the four fourth roots of unity can be written in the form 1, w, w^2, and w^3 where $w = \text{cis}\,\frac{\pi}{2}$.

b Hence show that $1 + w + w^2 + w^3 = 0$.

c Check the result in **b** by vector addition.

3 **a** Find the 5 fifth roots of unity and display them on an Argand diagram.

b Letting w be the root with smallest positive argument, show that the roots are 1, w, w^2, w^3, and w^4.

c Simplify $(1 + w + w^2 + w^3 + w^4)(1 - w)$ and hence show that $1 + w + w^2 + w^3 + w^4 = 0$.

d Show by vector addition that $1 + w + w^2 + w^3 + w^4 = 0$.

4 **a** Show that $w = \text{cis}\,\frac{2\pi}{n}$ is the nth root of unity with the smallest positive argument.

b Hence show that:

i the n roots of $z^n = 1$ are 1, w, w^2, w^3, ..., and w^{n-1}

ii $1 + w + w^2 + w^3 + ... + w^{n-1} = 0$

5 Show that for any complex number α, the sum of the n roots of $z^n = \alpha$ is equal to 0.

REVIEW SET 14A

1 On an Argand diagram, illustrate the complex numbers:

a $3 - 2i$ **b** $-1 + 5i$

2 Suppose $z = 3 + 2i$ and $w = -2 + i$. Find using both algebra and vectors:

a $z + w$ **b** $2z - w$ **c** z^* **d** $3w^* - z^*$

3 Find $|z|$ and $\arg z$ for z equal to:

a $5 + 2i$ **b** $-5 + 2i$ **c** $5 - 2i$ **d** $-2 + 5i$

4 Given $|z| = 5$, find:

a $|5z|$ **b** $|-4z^*|$ **c** $|(3 + i)z|$

d $\left|\frac{i}{z}\right|$ **e** $\left|\frac{3}{z^2}\right|$ **f** $\left|\frac{3 + 4i}{z}\right|$

5 Find the transformations which map z onto:

a $2z^*$ **b** $-\frac{1}{2}z$ **c** iz^*

6 If $(x + iy)^n = X + Yi$ where n is a positive integer, show that $X^2 + Y^2 = (x^2 + y^2)^n$.

7 Find the real and imaginary parts of $\left(i - \sqrt{3}\right)^5$.

8 If $z = \text{cis}\,\phi$ and $w = \dfrac{1 + z}{1 + z^*}$, show that $w = \text{cis}\,\phi$ also.

9 Find the cube roots of $-64i$, giving your answers in Cartesian form.

10 If $z = \text{cis}\,\theta$, find the modulus and argument of:

a $(2z)^{-1}$ **b** $1 - z$

11 Write $-1+i\sqrt{3}$ in polar form. Hence find the values of m for which $\left(-1+i\sqrt{3}\right)^m$ is real.

12 Let $z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and $z_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$.

Express $\left(\frac{z_1}{z_2}\right)^3$ in the form $z = a + bi$.

13 If $z = 4+i$ and $w = 2-3i$, find:

a $2w^* - iz$ **b** $|w - z^*|$ **c** $|z^{10}|$ **d** $\arg(w-z)$

14 Find the Cartesian equation for the locus of points $\mathrm{P}(x, y)$ if $z = x+iy$ and:

a $\arg(z-i) = \frac{\pi}{2}$ **b** $\left|\frac{z+2}{z-2}\right| = 2$

15 Write $2-2\sqrt{3}i$ in polar form. Hence find all values of n for which $\left(2-2\sqrt{3}i\right)^n$ is purely imaginary.

16 Determine the cube roots of -27.

17 If $z = 4\,\mathrm{cis}\,\theta$, find the modulus and argument of:

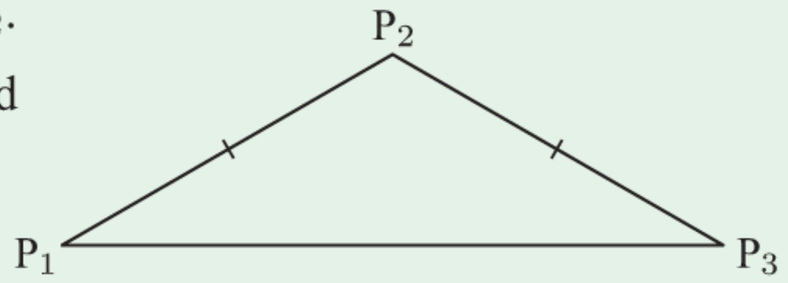
a z^3 **b** $\frac{1}{z}$ **c** iz^*

18 Illustrate the region defined by $\{z \mid 2 \leqslant |z| \leqslant 5 \text{ and } -\frac{\pi}{4} < \arg z \leqslant \frac{\pi}{2}\}$.
Show clearly all included boundary points.

19 Use polar form to deduce that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right) = -\arg z$ for $z \neq 0$.

20 If $z = \mathrm{cis}\,\alpha$, write $1+z$ in polar form. Hence determine the modulus and argument of $1+z$.

21 $\mathrm{P_1P_2P_3}$ is an isosceles triangle where $\mathrm{P_1P_3} = \sqrt{3}\,\mathrm{P_1P_2}$.
O is an origin such that $\overrightarrow{\mathrm{OP_1}} \equiv z_1$, $\overrightarrow{\mathrm{OP_2}} \equiv z_2$, and $\overrightarrow{\mathrm{OP_3}} \equiv z_3$.
Suppose $\arg(z_2 - z_1) = \alpha$.



a Show that $\arg(z_3 - z_2) = \alpha - \frac{\pi}{3}$.

b Find the modulus and argument of $\frac{z_2 - z_1}{z_3 - z_2}$.

22 State the five fifth roots of unity, and hence solve:

a $(2z-1)^5 = 32$ **b** $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z = 0$

c $(z+1)^5 = (z-1)^5$

REVIEW SET 14B

1 Suppose $z = 4-i$. Show, on a single Argand diagram:

a z **b** z^* **c** $z+3i$ **d** $\frac{z+1}{2}$

2 Given $|z| = 4$, use the rules $|zw| = |z||w|$ and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ to find:

a $|3z|$ **b** $|2iz|$ **c** $\left|\frac{i+1}{z}\right|$ **d** $|(3-i)z|$

3 Let $z = 1+i$ and $w = -2-i$. Find zw, and illustrate z, w, and zw on the same Argand diagram.

4 If $z = -1+3i$ and $w = 3+i$, find:

a $|z|$ **b** $|w|$ **c** $|z^*|^2$ **d** zz^*

e $|zw|$ **f** $\left|\frac{z}{w}\right|$ **g** $|z^2|$ **h** $|z^3|$

5 Suppose $z = x+iy$ is a complex number such that $3|z+2| = |z-6|$. Find the Cartesian equation for the locus of points $\mathrm{P}(x, y)$.

6 Prove that $|z-w|^2 + |z+w|^2 = 2\left(|z|^2 + |w|^2\right)$.

7 Write in polar form:

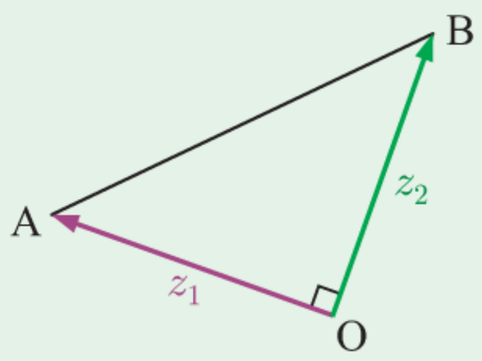
a $-5i$ **b** $2-2i\sqrt{3}$ **c** $k-ki$ where $k < 0$

8 Suppose $z = (1+bi)^2$ where b is real and positive. Find the exact value of b if $\arg z = \frac{\pi}{3}$.

9 **a** Prove that $\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta+\phi)$.

b If $z = 2\sqrt{2}\operatorname{cis}\alpha$, write $(1-i)z$ in polar form. Hence find $\arg[(1-i)z]$.

10 $z_1 \equiv \overrightarrow{\mathrm{OA}}$ and $z_2 \equiv \overrightarrow{\mathrm{OB}}$ represent two sides of a right angled isosceles triangle OAB.



a Determine the modulus and argument of $\frac{z_1^2}{z_2^2}$.

b Hence deduce that $z_1^2 + z_2^2 = 0$.

11 Let $z = \sqrt[4]{a}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $w = \sqrt[4]{b}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$, where $a, b \in \mathbb{R}$ and $a, b > 0$.

Find, in terms of a and b, the real and imaginary parts of $\left(\frac{z}{w}\right)^4$.

12 **a** List the five fifth roots of 1 in terms of w, where $w = \operatorname{cis}\frac{2\pi}{5}$. Display these roots on an Argand diagram.

b By considering the factorisation of $z^5 - 1$ in *two different ways*, show that:
$z^4 + z^3 + z^2 + z + 1 = (z-w)(z-w^2)(z-w^3)(z-w^4)$.

c Hence find the value of $(2-w)(2-w^2)(2-w^3)(2-w^4)$.

13 Write each of the following in the form $(\operatorname{cis}\theta)^n$:

a $\cos 3\theta + i\sin 3\theta$ **b** $\frac{1}{\cos 2\theta + i\sin 2\theta}$ **c** $\cos\theta - i\sin\theta$

14 Determine the fifth roots of $2+2i$.

15 If $z + \frac{1}{z}$ is real, prove that either $|z| = 1$ or z is real.

16 If $z \neq 0$ and $\left|\dfrac{z+1}{z-1}\right| = 1$, prove that z is purely imaginary.

17 Prove that:

a $\arg(z^n) = n \arg z$ for all complex numbers z and rational n

b $\left(\dfrac{z}{w}\right)^* = \dfrac{z^*}{w^*}$ for all z and for all $w \neq 0$.

18 Points A and B represent the numbers $z = 2 - 2i$ and $w = -1 - \sqrt{3}i$ respectively in the complex plane.

a Given the origin O, find AOB in terms of π.

b Calculate the argument of zw in terms of π.

19 If $z = \text{cis}\,\theta$, prove that:

a $|z| = 1$ **b** $z^* = \dfrac{1}{z}$ **c** $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

20 Suppose w is the root of $z^5 = 1$ with the smallest positive argument. Find real quadratic equations with roots:

a w and w^4 **b** $(w + w^4)$ and $(w^2 + w^3)$.

21 If $|z + w| = |z - w|$, prove that $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.

22 The complex number z is a root of the equation $|z| = |z + 4|$.

a Show that the real part of z is -2.

b Let v and w be two possible values of z such that $|z| = 4$ and v is in the second quadrant.

i Sketch the points that represent v and w on an Argand diagram.

ii Show that $\arg v = \frac{2\pi}{3}$.

iii Find $\arg w$.

iv Find $\arg\left(\dfrac{v^m w}{i}\right)$ in terms of m and π.

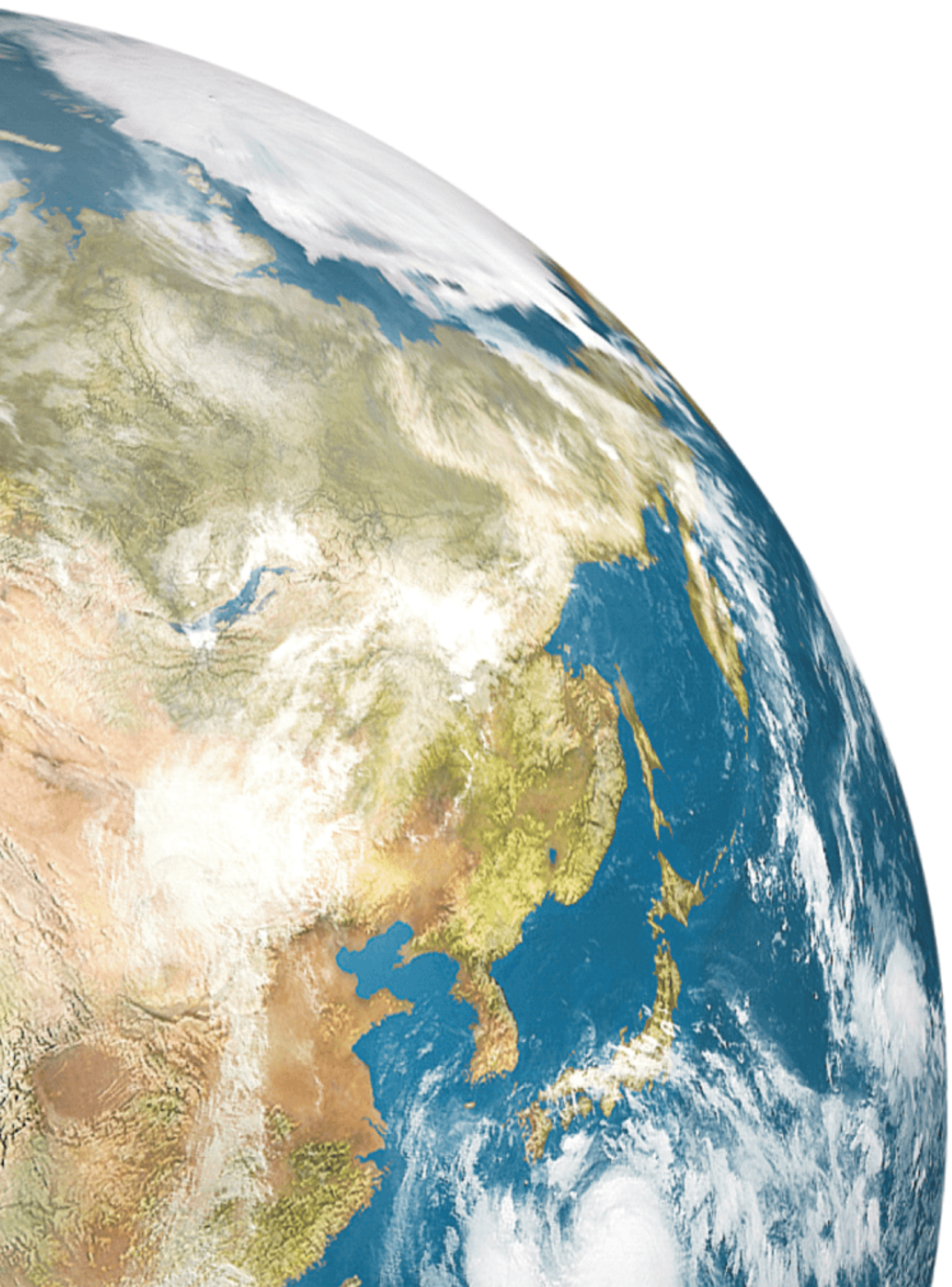
v Hence find a value of m for which $\dfrac{v^m w}{i}$ is a real number.

Chapter 15

Limits

Contents:

A Limits
B The existence of limits
C Limits at infinity
D Trigonometric limits
E Continuity



OPENING PROBLEM

The sequence $0.3,\ 0.33,\ 0.333,\ ....$ can be defined by the general term $x_n = 0.333....3$ where there are n 3s after the decimal point, $n \in \mathbb{Z}^+$.

Things to think about:

1 Copy and complete the table alongside.
In the number $(1 - 3x_n)$, how many 0s are there between the decimal point and the 1?

n	x_n	$3x_n$	$1 - 3x_n$
1			
2			
3			
4			
5			
10			

2 Consider x_{100} which contains 100 3s. In the number $(1 - 3x_{100})$, how many 0s are there between the decimal point and the 1?

3 Suppose we let n get bigger and bigger, so that in fact it gets *infinitely* big.

a In the number $(1 - 3x_n)$, how many 0s are there between the decimal point and the 1?

b How close does x_n get to the value $\frac{1}{3}$?

In the **Opening Problem** we see a **limiting process**. As $n \to \infty$, the sequence approaches the limiting value $\frac{1}{3}$. In fact, the sequence gets indistinguishably close to $\frac{1}{3}$, so that for any number we might choose which is close to $\frac{1}{3}$, we can find a value of n for which the sequence becomes closer and remains closer.

We write $\lim\limits_{n\to\infty} x_n = \frac{1}{3}$ which reads "the limit as n tends to infinity of x_n, is $\frac{1}{3}$".

HISTORICAL NOTE

When Archimedes derived the formula for the surface area of a sphere in the 3rd century BC, he considered the contribution to the surface area from many thin cross-sections of the sphere. As the number of cross-sections increases, the thickness of each decreases, and the approximation to the surface area of the sphere becomes more and more accurate. Essentially, *in the limit* as the number of cross-sections tends to infinity, the surface area tends to $4\pi r^2$. However, in the time of Archimedes, and indeed for almost 2000 years thereafter, the argument was in terms of *infinitesimals*.

In the centuries that followed, mathematicians experimented with more and more complicated limiting processes. Sometimes they obtained correct answers, but often they failed due to the lack of rigour in the study of limits and the treatment of infinity.

In the early 19th century, a more formal approach to limits was developed by mathematicians such as **John Wallis** (1616 - 1703), **Augustin-Louis Cauchy** (1789 - 1857), **Bernard Bolzano** (1781 - 1848), and **Karl Weierstrass** (1815 - 1897). In his book *The Paradoxes of the Infinite*, Bolzano formally defined a **limit**, though this work was not published until after his death.



Bernard Bolzano

A LIMITS

The following definition of a limit is informal but adequate for the purposes of this course:

If $f(x)$ is as close as we like to some real number A for all x sufficiently close to (but not equal to) a, then we say that $f(x)$ has a **limit** of A as x approaches a, and we write $\lim_{x \to a} f(x) = A$.

Notice that the limit is defined for x close to but *not equal to* a. Whether the function f is defined or not at $x = a$ is not important to the definition of the limit of f as x approaches a. What *is* important is the behaviour of the function as x gets *very close to* a.

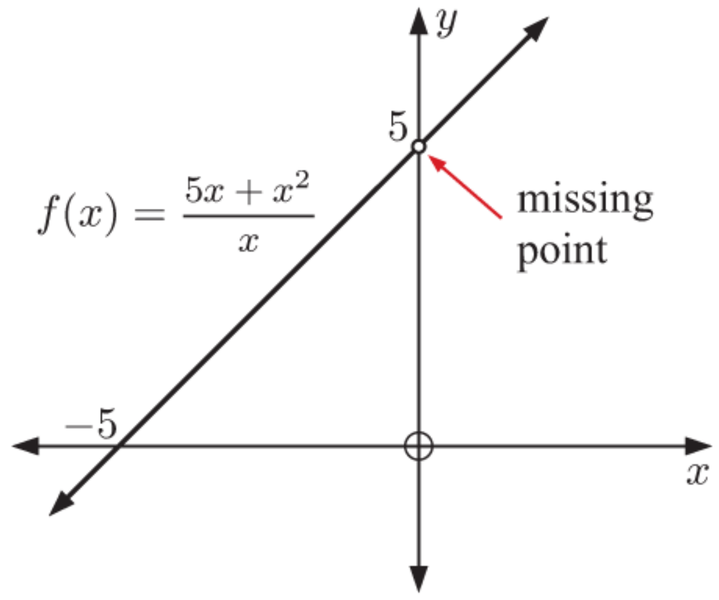
For example, suppose we wish to find the limit of $f(x) = \dfrac{5x + x^2}{x}$ as $x \to 0$.

It is tempting for us to simply substitute $x = 0$ into $f(x)$. However, in doing this, not only do we get the meaningless value of $\frac{0}{0}$, but also we ignore the basic limit definition.

Instead, observe that if $f(x) = \dfrac{5x + x^2}{x} = \dfrac{x(5 + x)}{x}$

$$\text{then} \quad f(x) = \begin{cases} 5 + x & \text{if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$$

The graph of $y = f(x)$ is therefore the straight line $y = x + 5$ with a "hole" at $(0, 5)$. The "hole" is a **point of discontinuity** of the function.



However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \to 0$ from either direction, $f(x) \to 5$.

We write $\lim_{x \to 0} \dfrac{5x + x^2}{x} = 5$ which reads:

"the limit as x approaches 0, of $\dfrac{5x + x^2}{x}$, is 5".

In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

THE LIMIT LAWS

The following **limit laws** will be useful in our study of limits:

Consider functions $f(x)$ and $g(x)$ for which $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, where $a, l, m \in \mathbb{R}$.

- $\lim_{x \to a} (f(x) \pm g(x)) = l \pm m$
- $\lim_{x \to a} f(x)\, g(x) = lm$
- $\lim_{x \to a} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{l}{m}$ provided $m \neq 0$

Example 1 Self Tutor

Evaluate:

a $\lim\limits_{x \to 2} x^2$ **b** $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x}$ **c** $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$\therefore \lim\limits_{x \to 2} x^2 = 4.$

b
$$\lim_{x \to 0} \frac{x^2 + 3x}{x}$$
$$= \lim_{x \to 0} \frac{\cancel{x}(x+3)}{\cancel{x}}$$
$$= \lim_{x \to 0} (x+3) \quad \{\text{since } x \neq 0\}$$
$$= 3 \quad \{\text{as } x \to 0,\ x+3 \to 3\}$$

c
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$
$$= \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$$
$$= \lim_{x \to 3} (x+3) \quad \{\text{since } x \neq 3\}$$
$$= 6 \quad \{\text{as } x \to 3,\ x+3 \to 6\}$$

EXERCISE 15A

1 Evaluate:

a $\lim\limits_{x \to 3} (x + 4)$ **b** $\lim\limits_{x \to -1} (5 - 2x)$ **c** $\lim\limits_{x \to 4} (3x - 1)$

d $\lim\limits_{x \to 2} (5x^2 - 3x + 2)$ **e** $\lim\limits_{h \to 0} h^2(1 - h)$ **f** $\lim\limits_{x \to 0} (x^2 + 5)$

2 Evaluate:

a $\lim\limits_{x \to 0} 5$ **b** $\lim\limits_{h \to 2} 7$ **c** $\lim\limits_{x \to 0} c$, c a constant

3 Evaluate:

a $\lim\limits_{x \to 1} \dfrac{x^2 - 3x}{x}$ **b** $\lim\limits_{h \to 2} \dfrac{h^2 + 5h}{h}$ **c** $\lim\limits_{x \to 0} \dfrac{x - 1}{x + 1}$

4 Explain why $\lim\limits_{x \to 0} \dfrac{x}{x} = 1$, even though $\dfrac{0}{0}$ is meaningless.

5 **a** Copy and complete the tables of values alongside.

b Hence predict the value of $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

c Prove your result by evaluating $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ directly.

x	$\dfrac{x^2 - 4}{x - 2}$
1.9	
1.99	
1.999	
1.9999	
1.999 99	

x	$\dfrac{x^2 - 4}{x - 2}$
2.1	
2.01	
2.001	
2.0001	
2.000 01	

6 Evaluate:

a $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x}$ **b** $\lim\limits_{x \to 0} \dfrac{x^2 + 5x}{x}$ **c** $\lim\limits_{x \to 0} \dfrac{2x^2 - x}{x}$

d $\lim\limits_{h \to 0} \dfrac{2h^2 + 6h}{h}$ **e** $\lim\limits_{h \to 0} \dfrac{3h^2 - 4h}{h}$ **f** $\lim\limits_{h \to 0} \dfrac{h^3 - 8h}{h}$

g $\lim\limits_{x \to 1} \dfrac{x^2 - x}{x - 1}$ **h** $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x - 2}$ **i** $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

7 Let f be a function with domain $\mathbb{R}$, and suppose that $\lim_{x \to a} f(x)$ exists.

Use the limit laws to prove that $\lim_{x \to a} f(x) = l$ if and only if $\lim_{x \to a} (f(x) - l) = 0$.

THEORY OF KNOWLEDGE

The Greek philosopher Zeno of Elea lived in what is now southern Italy, in the 5th century BC. He is most famous for his paradoxes, which were recorded in Aristotle's work *Physics*.

The arrow paradox

"*If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.*"

This argument says that if we choose any particular instant in time, the arrow is motionless. Therefore, how does the arrow actually move?

The dichotomy paradox

"*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*"

If an object is to move a fixed distance then it must travel half that distance. Before it can travel a half the distance, it must travel a half *that* distance. With this process continuing indefinitely, motion is impossible.

Achilles and the tortoise

"*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point from which the pursued started, and so the slower must always hold a lead.*"

According to this principle, the athlete Achilles will never be able to catch the slow tortoise!

1 A paradox is a logical argument that leads to a contradiction or a situation which defies logic or reason. Can a paradox be the truth?

2 Are Zeno's paradoxes really paradoxes?

3 Are the three paradoxes essentially the same?

4 Sheldon argues that between any irrational number and a rational number, there must be another irrational number.

- **a** Can you show that Sheldon is correct?
- **b** Does this mean that rational numbers do not exist?

5 We know from experience that things *do* move, and that Achilles *would* catch the tortoise. Does that mean that logic has failed?

6 What do Zeno's paradoxes have to do with limits?

B THE EXISTENCE OF LIMITS

So far we have considered cases where the function $f(x)$ approaches some finite value A as x approaches a from either side. More formally, we can write this using the one-sided limits $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$.

The limit $\lim\limits_{x\to a} f(x)$ exists and is equal to the finite value A if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = A$.

We say that $f(x)$ **converges** to A as x approaches a.

However, there are many situations where $f(x)$ **diverges** as x approaches a. In these cases, $\lim\limits_{x\to a} f(x)$ does not exist.

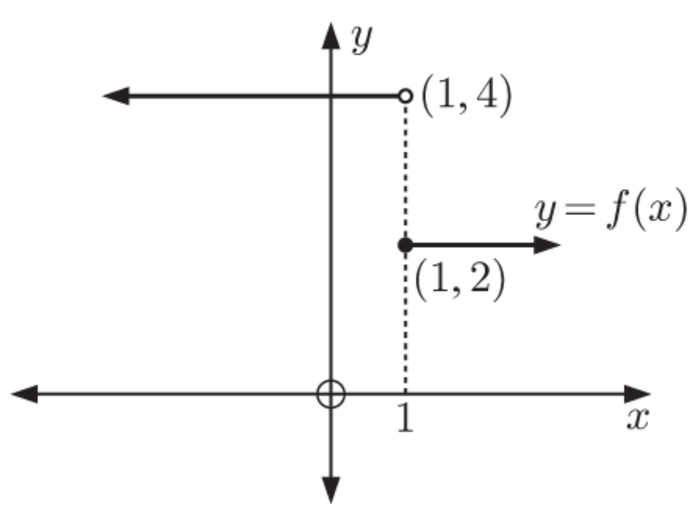
For example:

- Consider $f(x) = \begin{cases} 4, & x < 1 \\ 2, & x \geqslant 1 \end{cases}$ where there is a "jump" at $x = 1$.

  $\lim\limits_{x\to 1^-} f(x) = 4$ and $\lim\limits_{x\to 1^+} f(x) = 2$.

  In this case both $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$ are finite values, but they are not equal.

  $\therefore \lim\limits_{x\to 1} f(x)$ does not exist.

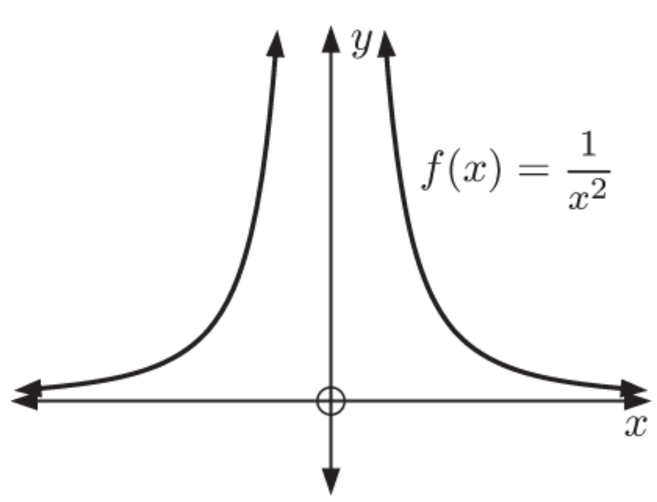


- Consider $f(x) = \dfrac{1}{x^2}$ where there is a "break" at $x = 0$.

  As $x \to 0$, $f(x) \to \infty$.

  Neither $\lim\limits_{x\to 0^-} f(x)$ nor $\lim\limits_{x\to 0^+} f(x)$ exists,

  so $\lim\limits_{x\to 0} f(x)$ does not exist either.



Example 2 **Self Tutor**

Find, if possible:

a $\lim\limits_{x\to 1} \dfrac{1}{x}$ **b** $\lim\limits_{x\to 0} \dfrac{1}{x}$

a As $x \to 1^+$, $\dfrac{1}{x} \to 1$

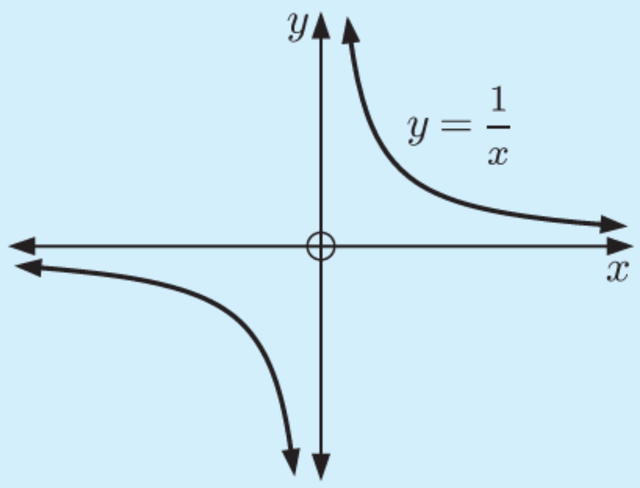
As $x \to 1^-$, $\dfrac{1}{x} \to 1$

$\therefore \lim\limits_{x\to 1} \dfrac{1}{x} = 1$

b As $x \to 0^+$, $\dfrac{1}{x} \to \infty$

As $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

$\therefore \lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist.



We cannot write $\lim\limits_{x\to 0} f(x) = \infty$ because ∞ is not a number.

Rather, $\lim\limits_{x\to 0} f(x)$ does not exist.



EXERCISE 15B

1 **a** Sketch the graph of $y = -\frac{1}{x}$.

b Find, if possible:

i $\lim\limits_{x \to 1} \left(-\frac{1}{x}\right)$ **ii** $\lim\limits_{x \to -1} \left(-\frac{1}{x}\right)$ **iii** $\lim\limits_{x \to 0} \left(-\frac{1}{x}\right)$

2 **a** Sketch the graph of $y = -\frac{1}{(x-3)^2}$.

b Find, if possible:

i $\lim\limits_{x \to 0} \left(-\frac{1}{(x-3)^2}\right)$ **ii** $\lim\limits_{x \to 3} \left(-\frac{1}{(x-3)^2}\right)$ **iii** $\lim\limits_{x \to 5} \left(-\frac{1}{(x-3)^2}\right)$

3 Find, if possible: **a** $\lim\limits_{x \to -1} (3x + 2)$ **b** $\lim\limits_{x \to -2} \dfrac{x^2 + x - 2}{x - 2}$

4 Consider the function $f(x) = \dfrac{x}{x^2 - 2x}$.

a For what values of x is $f(x)$ undefined?

b Sketch the graph of $y = f(x)$.

c Find, if possible: **i** $\lim\limits_{x \to 2} f(x)$ **ii** $\lim\limits_{x \to 1} f(x)$ **iii** $\lim\limits_{x \to 0} f(x)$

5 Let $f(x) = \begin{cases} x^2, & x > 3 \\ 5, & x = 3 \\ 3x, & x < 3. \end{cases}$

a Sketch the graph of $y = f(x)$.

b Evaluate $\lim\limits_{x \to 3^-} f(x)$ and $\lim\limits_{x \to 3^+} f(x)$.

c Does $\lim\limits_{x \to 3} f(x)$ exist? Explain your answer.

6 Does $\lim\limits_{x \to 0} \sqrt{x}$ exist? Explain your answer.

7 Given $f(x) = \begin{cases} (x-1)^3, & x \leqslant 0 \\ 1, & x > 0 \end{cases}$, discuss:

a $\lim\limits_{x \to 0^-} f(x)$ **b** $\lim\limits_{x \to 0^+} f(x)$ **c** $\lim\limits_{x \to 0} f(x)$

8 Given $f(x) = \begin{cases} \sin x, & x < 0 \\ \dfrac{x^2 - 3x + 5}{5}, & x > 0 \end{cases}$, discuss:

a $\lim\limits_{x \to 0^-} f(x)$ **b** $\lim\limits_{x \to 0^+} f(x)$ **c** $\lim\limits_{x \to 0} f(x)$

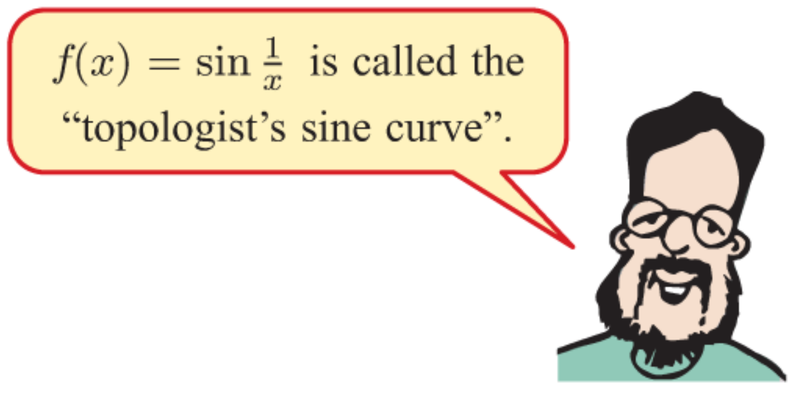
9 Find, if possible:

a $\lim\limits_{x \to 0} \dfrac{\sin x}{e^x}$ **b** $\lim\limits_{x \to \pi} \dfrac{\sin x}{1 - \cos x}$ **c** $\lim\limits_{\theta \to 0} \dfrac{\cos \theta}{\theta}$ **d** $\lim\limits_{x \to 2} \dfrac{\ln x}{\sqrt{2 - x}}$

10 **a** Use technology to help sketch the graph of $f(x) = \sin \frac{1}{x}$ for $-0.5 \leqslant x \leqslant 0.5$.

b Discuss the behaviour of the graph as $x \to 0$.

c Does $\lim\limits_{x \to 0} f(x)$ exist? Explain your answer.



11 Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$

a Find: **i** $f(\frac{1}{2})$ **ii** $f(\sqrt{2})$ **iii** $f(0)$

b Find, if possible: **i** $\lim\limits_{x \to \frac{1}{2}} f(x)$ **ii** $\lim\limits_{x \to \sqrt{2}} f(x)$ **iii** $\lim\limits_{x \to 0} f(x)$

Explain your answers.

12 Suppose $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{x^2}{(1+x^2)^n} = \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \dfrac{x^2}{(1+x^2)^3} + \ldots$.

a Find $f(0)$.

b Explain why, for any $x \neq 0$, $f(x)$ is an infinite geometric series. Hence evaluate $f(x)$ for all $x \neq 0$ for which the series converges.

c Find $\lim\limits_{x \to 0} f(x)$

d Graph $y = f(x)$.

C LIMITS AT INFINITY

We can use the idea of limits to discuss the behaviour of functions for extreme values of x.

We write $x \to \infty$ to mean "x tends to plus infinity" and $x \to -\infty$ to mean "x tends to minus infinity".

Notice that as $x \to \infty$, $1 < x < x^2 < x^3 < \ldots$ and as x gets very large, the value of $\dfrac{1}{x}$ gets very small. In fact, we can make $\dfrac{1}{x}$ as close to 0 as we like by making x large enough. This means that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$, even though $\dfrac{1}{x}$ never actually reaches 0.

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{for all } n > 0.$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0 \quad \text{for all } n \in \mathbb{Z}^+.$$

Example 3 Self Tutor

Evaluate:

a $\lim\limits_{x \to \infty} \dfrac{2x+3}{x-4}$ **b** $\lim\limits_{x \to \infty} \dfrac{x^2-3x+2}{1-x^2}$ **c** $\lim\limits_{x \to \infty} \dfrac{x^2+x+1}{x-2}$

a
$$\lim_{x \to \infty} \frac{2x+3}{x-4}$$
$$= \lim_{x \to \infty} \frac{2+\frac{3}{x}}{1-\frac{4}{x}} \quad \{\text{dividing each term by } x, \text{ since } x \neq 0\}$$
$$= \frac{2}{1} \quad \{\text{as } x \to \infty, \ \tfrac{3}{x} \to 0 \text{ and } \tfrac{4}{x} \to 0\}$$
$$= 2$$

b
$$\lim_{x\to\infty} \frac{x^2-3x+2}{1-x^2}$$
$$= \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{\frac{1}{x^2}-1} \quad \{\text{dividing each term by } x^2, \text{ since } x \neq 0\}$$
$$= \frac{1}{-1} \quad \{\text{as } x\to\infty,\ \frac{3}{x}\to 0,\ \frac{2}{x^2}\to 0, \text{ and } \frac{1}{x^2}\to 0\}$$
$$= -1$$

c
$$\lim_{x\to\infty} \frac{x^2+x+1}{x-2}$$
$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{\frac{1}{x}-\frac{2}{x^2}} \quad \{\text{dividing each term by } x^2, \text{ since } x \neq 0\}$$

As $x\to\infty$, the numerator $\to 1$, but the denominator $\to 0$.

Hence as $x\to\infty$, $\dfrac{x^2+x+1}{x-2}\to\infty$.

$\therefore \lim\limits_{x\to\infty} \dfrac{x^2+x+1}{x-2}$ does not exist.

EXERCISE 15C

1 Evaluate $\lim\limits_{x\to\infty} \dfrac{1}{x^2}$.

2 Evaluate:

a $\lim\limits_{x\to\infty} \dfrac{3x-2}{x+1}$ **b** $\lim\limits_{x\to\infty} \dfrac{1-2x}{3x+2}$ **c** $\lim\limits_{x\to\infty} \dfrac{x}{1-x}$

d $\lim\limits_{x\to\infty} \dfrac{x^2+3}{x^2-1}$ **e** $\lim\limits_{x\to\infty} \dfrac{x^2-2x+4}{4x^2-x-1}$ **f** $\lim\limits_{x\to\infty} \dfrac{x+4}{2x^2+x-1}$

g $\lim\limits_{x\to\infty} \dfrac{x+5}{x^2-x+4}$ **h** $\lim\limits_{x\to\infty} \dfrac{x^2+3x-4}{x-1}$ **i** $\lim\limits_{x\to\infty} \dfrac{2x^2-x+1}{5x+3}$

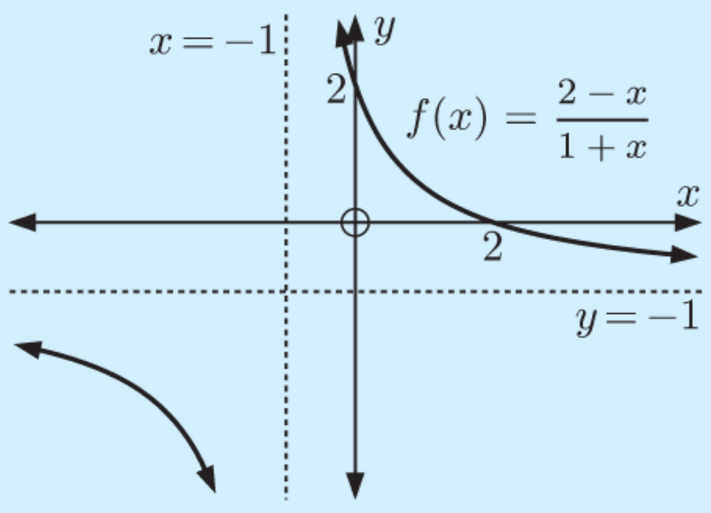
Example 4 Self Tutor

a Discuss the behaviour of $f(x)=\dfrac{2-x}{1+x}$ near its asymptotes, and hence deduce their equations.

b If they exist, state the values of $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

a As $x\to -1^-$, $f(x)\to -\infty$
As $x\to -1^+$, $f(x)\to \infty$
As $x\to -\infty$, $f(x)\to -1^-$
As $x\to \infty$, $f(x)\to -1^+$

The vertical asymptote is $x=-1$.
The horizontal asymptote is $y=-1$.



b $\lim\limits_{x\to-\infty} f(x) = -1$ and $\lim\limits_{x\to\infty} f(x) = -1$

3 For each of the following functions:

i Discuss the behaviour near its asymptotes, and hence deduce their equations.

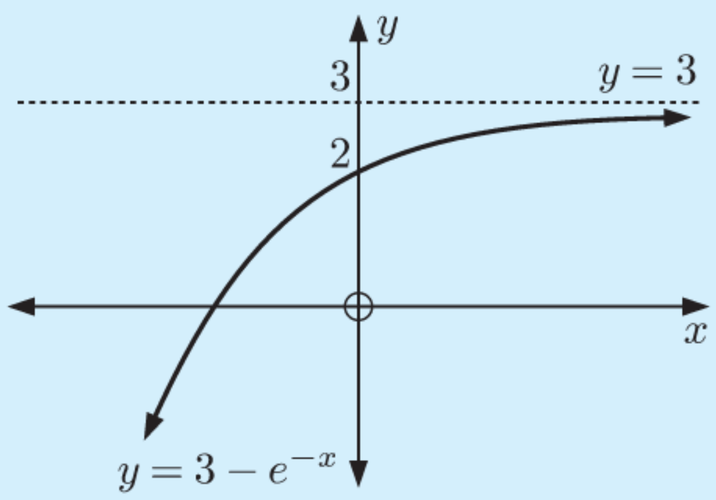
ii If they exist, state the values of $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{3x-2}{x+3}$ **c** $f(x) = \dfrac{1-2x}{3x+2}$ **d** $f(x) = \dfrac{x}{1-x}$

Example 5 **Self Tutor**

Find, if possible:

a $\lim_{x\to-\infty} (3 - e^{-x})$ **b** $\lim_{x\to\infty} (3 - e^{-x})$



a As $x \to -\infty$, $3 - e^{-x} \to -\infty$.

Since $3 - e^{-x}$ does not approach a finite value, $\lim_{x\to-\infty} (3 - e^{-x})$ does not exist.

b As $x \to \infty$, $3 - e^{-x} \to 3^-$

$\therefore \lim_{x\to\infty} (3 - e^{-x}) = 3$

4 **a** Sketch the graph of $y = e^x - 6$.

b Hence discuss the value and geometric interpretation of:

i $\lim_{x\to-\infty} (e^x - 6)$ **ii** $\lim_{x\to\infty} (e^x - 6)$

5 **a** Copy and complete this table of values.

b Hence predict the value of $\lim_{x\to\infty} xe^{-x}$.

c Graph $y = xe^{-x}$ using technology. Does your graph support your prediction in **b**?

x	xe^{-x}
10	
50	
100	
200	

6 Find, if possible:

a $\lim_{x\to-\infty} (2e^{-x} - 3)$ and $\lim_{x\to\infty} (2e^{-x} - 3)$

b $\lim_{x\to-\infty} (e^{-x} + 5e^{\frac{1}{x}})$ and $\lim_{x\to\infty} (e^{-x} + 5e^{\frac{1}{x}})$

7 Find $\lim_{x\to\infty} \sqrt{x^2+x} - x$ by first multiplying by $\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$.

DISCUSSION

- What proportion of quadratic equations have at least one real solution?
- Does it make any sense to compare quantities which are infinitely large?

D TRIGONOMETRIC LIMITS

INVESTIGATION EXAMINING $\frac{\sin\theta}{\theta}$ NEAR $\theta = 0$

In this Investigation we examine $\frac{\sin\theta}{\theta}$ when θ is close to 0 and θ is in radians.

What to do:

1 **a** Graph $y = \frac{\sin x}{x}$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ using a graphics calculator or graphing package.

b Discuss the behaviour of the graph as x approaches 0.

GRAPHING PACKAGE

2 We now prove the result $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

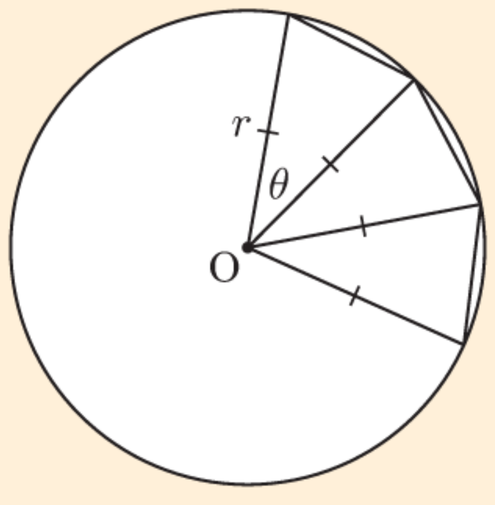
a Show that $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function. What does this mean graphically?

b Suppose $f(\theta)$ is an even function for which $\lim\limits_{\theta\to 0^+} f(\theta) = A$.

c Explain why: **i** $\lim\limits_{\theta\to 0^-} f(\theta) = A$ **ii** $\lim\limits_{\theta\to 0} f(\theta) = A$

d Since $\frac{\sin\theta}{\theta}$ is even, we need only examine $\frac{\sin\theta}{\theta}$ for positive θ.

A circle of radius r contains n congruent isosceles triangles as shown.



i Use the diagram to show that $\lim\limits_{n\to\infty} \frac{n}{2} r^2 \sin\frac{2\pi}{n} = \pi r^2$.

ii Hence show that:

(1) $\lim\limits_{n\to\infty} \dfrac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$ **(2)** $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

From the **Investigation**, you should have found that:

> If θ is in radians, then $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Example 6 Self Tutor

Find $\lim\limits_{\theta\to 0} \frac{\sin 3\theta}{\theta}$.

$$\lim_{\theta\to 0} \frac{\sin 3\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta} \times 3$$
$$= 3 \times \lim_{3\theta\to 0} \frac{\sin 3\theta}{3\theta} \qquad \{\text{as } \theta \to 0,\ 3\theta \to 0 \text{ also}\}$$
$$= 3 \times 1$$
$$= 3$$

EXERCISE 15D

1 Find:

a $\lim\limits_{\theta \to 0} \dfrac{\sin 2\theta}{\theta}$ **b** $\lim\limits_{\theta \to 0} \dfrac{\theta}{\sin \theta}$ **c** $\lim\limits_{\theta \to 0} \dfrac{\tan \theta}{\theta}$

d $\lim\limits_{\theta \to 0} \dfrac{\sin^2 \theta}{\theta}$ **e** $\lim\limits_{\theta \to 0} \dfrac{\sin \theta \sin 4\theta}{\theta^2}$ **f** $\lim\limits_{h \to 0} \dfrac{\sin \frac{h}{2} \cos h}{h}$

g $\lim\limits_{n \to \infty} n \sin \frac{3\pi}{n}$ **h** $\lim\limits_{x \to 0} \dfrac{\sin 7x}{4x}$ **i** $\lim\limits_{x \to 0} \dfrac{x^2 + x}{\sin 2x}$

2 Evaluate, where possible:

a $\lim\limits_{x \to 0} \dfrac{x + \sin x}{x - \sin x}$ by multiplying by $\dfrac{\frac{1}{x}}{\frac{1}{x}}$

b $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h}$ by multiplying by $\dfrac{\cos h + 1}{\cos h + 1}$

c $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ by multiplying by $\dfrac{1 + \cos x}{1 + \cos x}$.

E CONTINUITY

The existence of $f(a)$ and $\lim\limits_{x \to a} f(x)$ are useful for describing the *continuity* of a curve.

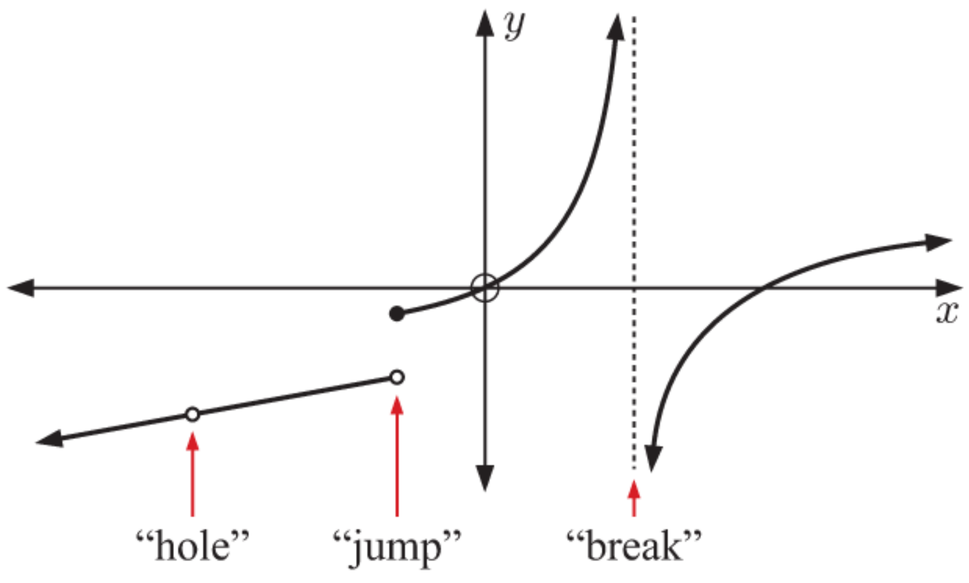
A function f is **continuous at** $x = a$ if $f(a)$ and $\lim\limits_{x \to a} f(x)$ exist and are equal.

f is also continuous at $x = a$ if:

- a is the left endpoint of an interval on which f is defined, and $f(a)$ and $\lim\limits_{x \to a^+} f(x)$ exist and are equal, or
- a is the right endpoint of an interval on which f is defined, and $f(a)$ and $\lim\limits_{x \to a^-} f(x)$ exist and are equal.

If f is *not* continuous at $x = a$, we say that f is *discontinuous* at $x = a$ or *has a discontinuity* at $x = a$.

Graphically, we have seen that isolated points of discontinuity are indicated on the graph of $y = f(x)$ by a "hole", a "jump", or a "break".



If f has a "hole" at $x = a$, then $\lim\limits_{x \to a} f(x)$ exists. The "hole" is a **removable discontinuity** which can be removed by defining a new function based on f but which takes the value $\lim\limits_{x \to a} f(x)$ when $x = a$.

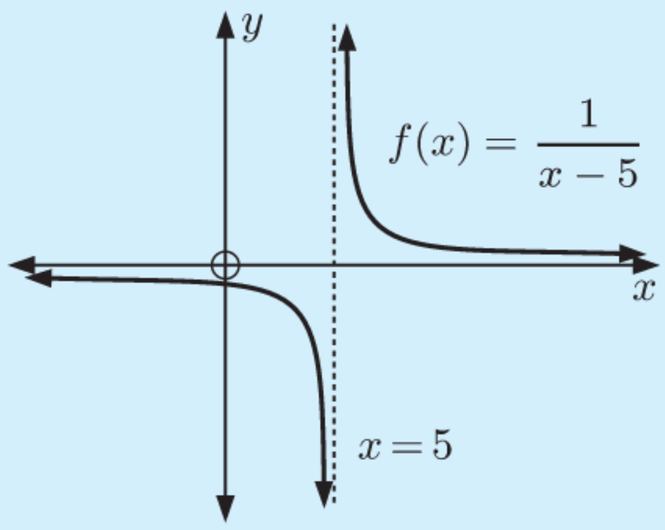
Discontinuities due to "jumps" and "breaks" cannot be removed by simply redefining the value of the function there. They are called **essential discontinuities**.

Example 7 **Self Tutor**

Discuss the continuity of the following functions. If there is a removable discontinuity, describe how it could be removed.

a $f(x) = \dfrac{1}{x-5}$ **b** $f(x) = \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0 \end{cases}$ **c** $f(x) = \begin{cases} x^2, & x \neq 2 \\ 6, & x = 2 \end{cases}$

a

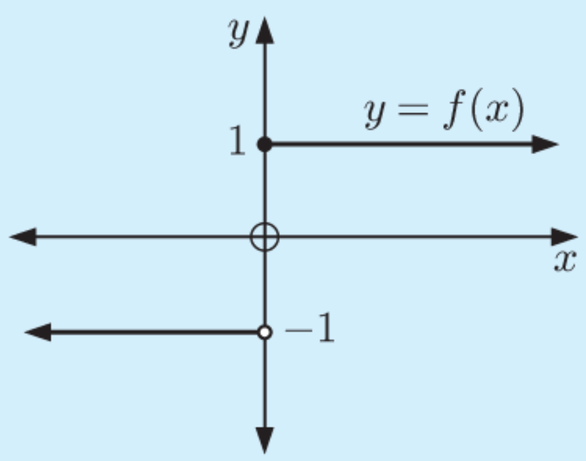


f is not defined at $x = 5$, and $\lim\limits_{x \to 5} f(x)$ does not exist.

f has an essential discontinuity at $x = 5$.

f is continuous for all $x \in \mathbb{R}$, $x \neq 5$.

b

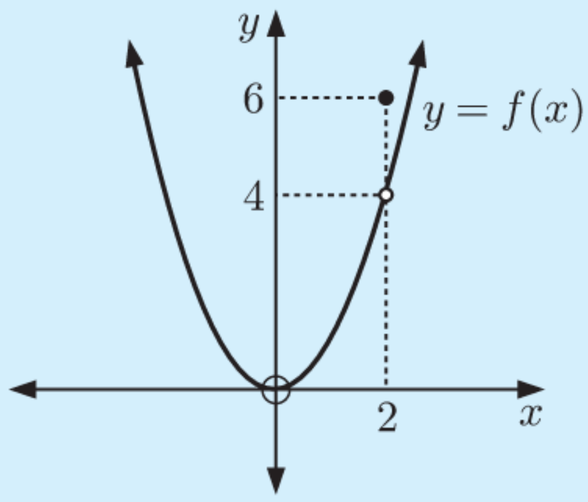


f is defined for all x, but there is a "jump" at $x = 0$.

$\therefore$ $\lim\limits_{x \to 0} f(x)$ does not exist, and f has an essential discontinuity at $x = 0$.

f is continuous for all $x \in \mathbb{R}$, $x \neq 0$.

c



f is defined for all x, but there is a discontinuity at $x = 2$.

$\lim\limits_{x \to 2} f(x) = 4$ but $f(2) = 6$

$\therefore$ there is a removable discontinuity when $x = 2$.

f is continuous for all $x \in \mathbb{R}$, $x \neq 2$.

The discontinuity can be removed by defining a new function

$g(x) = \begin{cases} f(x) = x^2, & x \neq 2 \\ 4, & x = 2 \end{cases}$ which is actually just $g(x) = x^2$.

EXERCISE 15E

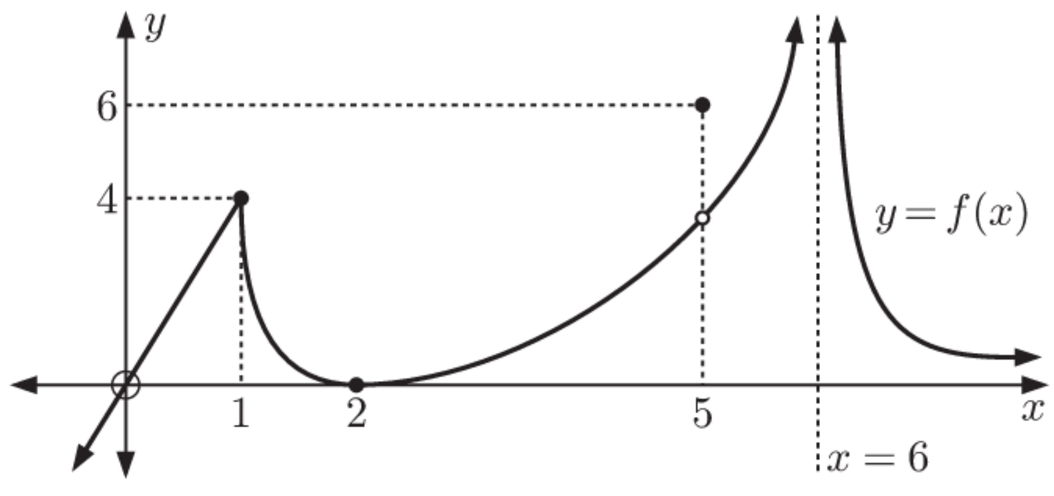
1 The graph of $y = f(x)$ is shown alongside.

State the value(s) of x for which f:

a has an essential discontinuity

b has a removable discontinuity

c is continuous.



2 Let $f(x) = \dfrac{1}{x^2 - 4}$.

a Sketch the graph of $y = f(x)$.

b Discuss the continuity of $f(x)$.

3 Let $f(x) = \sqrt{x}$.

a For what values of a does $\lim\limits_{x \to a} f(x)$ exist?

b For what values of x is $f(x)$ continuous?

4 Discuss the continuity of the following functions. If there is a removable discontinuity, describe how it could be removed.

a $f(x) = \dfrac{3}{x + 2}$

b $f(x) = \dfrac{x^2 + x}{x + 1}$

c $f(x) = \dfrac{x - 4}{x^2 - 2x - 8}$

d $f(x) = \ln(x^2)$

e $f(x) = \begin{cases} x^2 + 2, & x \geqslant 3 \\ 5, & x < 3 \end{cases}$

f $f(x) = \tan x \cos x$

5 Find, if possible, the value(s) of $k \in \mathbb{R}$ for which f is continuous on $\mathbb{R}$. Explain your answers.

a $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ k, & x = 1 \end{cases}$

b $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

c $f(x) = \begin{cases} k + 1, & x < 2 \\ x^2, & x \geqslant 2 \end{cases}$

d $f(x) = \begin{cases} \dfrac{1}{x}, & x > 0 \\ k, & x \leqslant 0 \end{cases}$

e $f(x) = \begin{cases} kx, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$

f $f(x) = \begin{cases} k + x, & x \geqslant 2 \\ |\, k + 2 \,|, & x < 2 \end{cases}$

6 An alternative definition of continuity is:

"f is continuous at $x = a$ if, for any sequence $\{x_n\}$ such that $\lim\limits_{n \to \infty} x_n = a$, then $\lim\limits_{n \to \infty} f(x_n) = f(a)$."

The **Dirichlet function** is defined by $g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

a Consider the sequence $x_n = a + \dfrac{\sqrt{2}}{n}$, $a \in \mathbb{Q}$, $n \in \mathbb{Z}^+$. Show that $g(x)$ is discontinuous at $x = a$.

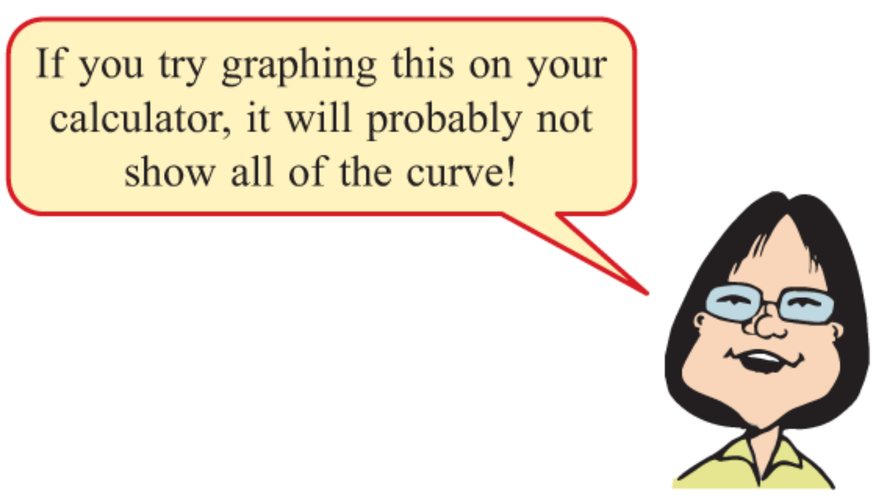
b Now suppose $a \notin \mathbb{Q}$, and that x_n is the decimal expansion of a to n decimal places, $n \in \mathbb{Z}^+$. Show that $g(x)$ is discontinuous at $x = a$.

c Hence discuss the continuity of the Dirichlet function.

7 **a** Use the graphing package to graph $y = x^x$.



b Discuss the continuity of the function.



REVIEW SET 15A

1 Evaluate:

a $\lim\limits_{x\to 1}(6x-7)$ **b** $\lim\limits_{h\to 0}\dfrac{2h^2-h}{h}$ **c** $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}$

2 **a** Sketch the graph of $y=\dfrac{1}{x+2}$.

b Find, if possible:

i $\lim\limits_{x\to 0}\dfrac{1}{x+2}$ **ii** $\lim\limits_{x\to 2}\dfrac{1}{x+2}$ **iii** $\lim\limits_{x\to -2}\dfrac{1}{x+2}$

3 Find:

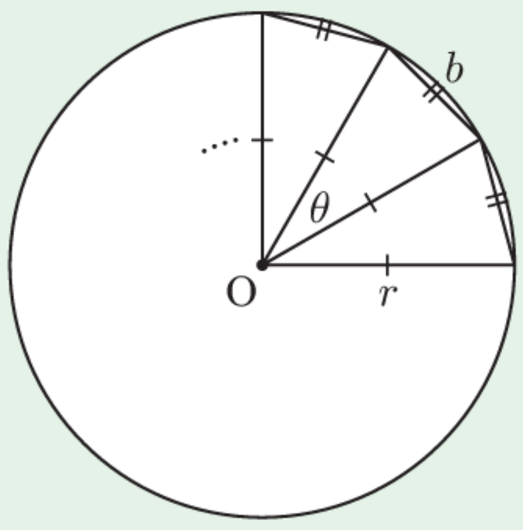
a $\lim\limits_{\theta\to 0}\dfrac{\sin 4\theta}{\theta}$ **b** $\lim\limits_{\theta\to 0}\dfrac{2\theta}{\sin 3\theta}$ **c** $\lim\limits_{n\to\infty} n\sin\frac{\pi}{n}$

4 **a** Sketch the graph of $y=\dfrac{2+x}{x-4}$.

b Discuss the behaviour of the graph near its asymptotes, and hence deduce their equations.

c State the values of $\lim\limits_{x\to -\infty}\dfrac{2+x}{x-4}$ and $\lim\limits_{x\to\infty}\dfrac{2+x}{x-4}$.

5 Consider a circle containing n congruent isosceles triangles with base b as shown alongside.
Let the circle have radius r and circumference C.
By considering $C=\lim\limits_{n\to\infty} n\times b$, show that $C=2\pi r$.



6 Suppose $f(x)=e^{2x}\cos x$.

a Sketch the graph of $y=f(x)$.

b Find, if possible: **i** $\lim\limits_{x\to -\infty} f(x)$ **ii** $\lim\limits_{x\to\infty} f(x)$

7 Discuss the continuity of the following functions. If there is a removable discontinuity, describe how it could be removed.

a $f(x)=e^{-\frac{1}{x}}$ **b** $g(x)=\dfrac{2x-1}{8x^2-10x+3}$

REVIEW SET 15B

1 Evaluate the limits:

a $\lim\limits_{h\to 0}\dfrac{h^3-3h}{h}$ **b** $\lim\limits_{x\to 1}\dfrac{3x^2-3x}{x-1}$ **c** $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{2-x}$

2 Consider the function $f(x)=\dfrac{x+3}{x^2-x-12}$.

a For what values of x is $f(x)$ undefined?

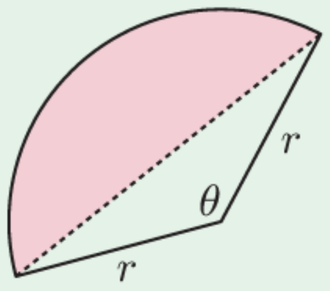
b Find, if possible: **i** $\lim\limits_{x\to -3} f(x)$ **ii** $\lim\limits_{x\to 4} f(x)$

3 **a** Sketch the graph of $y = 2 - e^{x+1}$.

b Hence find, if possible, $\lim_{x \to -\infty} (2 - e^{x+1})$ and $\lim_{x \to \infty} (2 - e^{x+1})$.

c State the equation of the asymptote of $y = 2 - e^{x+1}$.

4



a Show that the area of the shaded segment of a circle of radius r is given by $\frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians.

b Hence explain why $\lim_{\theta \to 0^+} \dfrac{\sin\theta}{\theta} = 1$.

5 Find the value(s) of k that satisfy $\lim_{\theta \to 0} \dfrac{\sin\left(\frac{\theta}{k}\right)}{\theta} = k + 2$.

6 **a** Find the asymptotes of $y = \dfrac{x^2 + 2}{x}$.

b Graph the function $y = \dfrac{x^2 + 2}{x}$ and its asymptotes.

c Find, if possible:

i $\lim_{x \to \infty} \dfrac{x^2 + 2}{x}$ **ii** $\lim_{x \to \infty} \left(\dfrac{x^2 + 2}{x} - x \right)$

d How are the limits in **c** observed on your graph?

7 Let $f(x) = \begin{cases} (x-2)^2 + 1, & x < 4 \\ \sqrt{x}, & x \geqslant 4 \end{cases}$

a Sketch the graph of $y = f(x)$.

b Evaluate $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

c Does f have an essential or removable discontinuity at $x = 4$? Explain your answer.

8 Consider the function $g(x) = \dfrac{\sin x}{\sin 2x}$.

Find all values of x for which g has:

a a removable discontinuity **b** an essential discontinuity.

Chapter 16

Introduction to differential calculus

Contents:

A Rates of change
B Instantaneous rates of change
C The gradient of a tangent
D The derivative function
E Differentiation from first principles
F Differentiability and continuity



OPENING PROBLEM

Suppose we add water to a container at a constant rate. The depth of the water increases over time.

Things to think about:

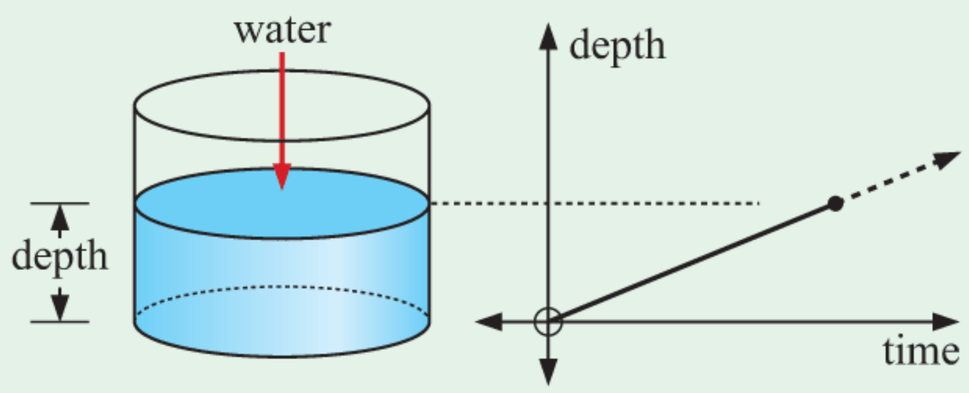
a Think about the graph of the *volume of water added* against *time*.

i Can you explain why this graph is a straight line passing through the origin?

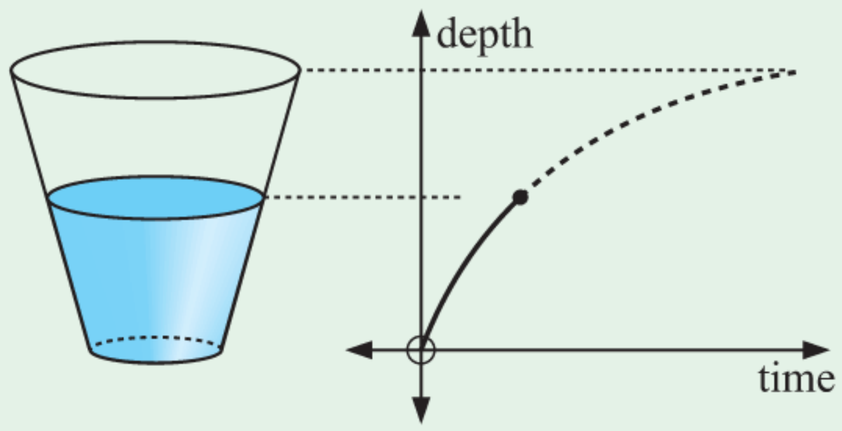
ii What does the *gradient* of the straight line tell us?

b Think about the graph of the *depth of water* against *time*.

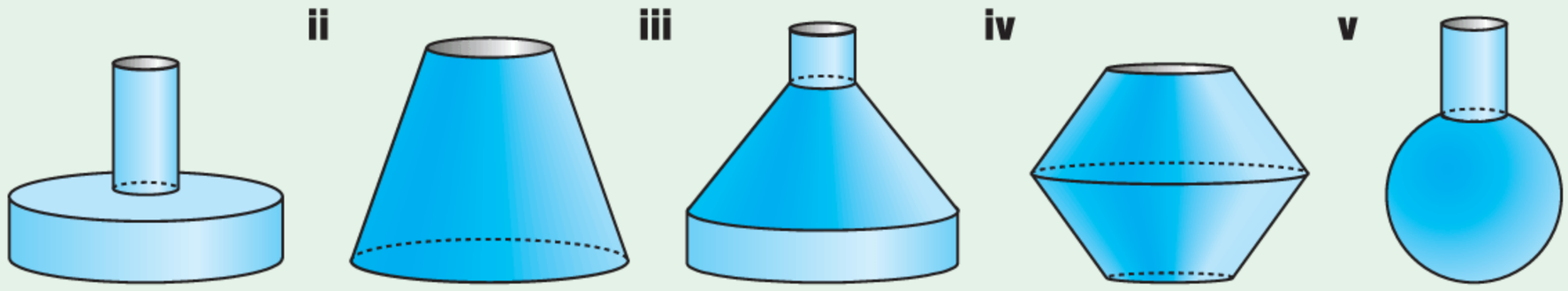
i Can you explain why this graph is a straight line for a cylindrical container?



ii Can you explain why this graph is *not* a straight line for the container shown?



c By examining the shape of each container, can you predict the depth-time graph when water is added at a constant rate?



Use the water filling demonstration to check your answers.

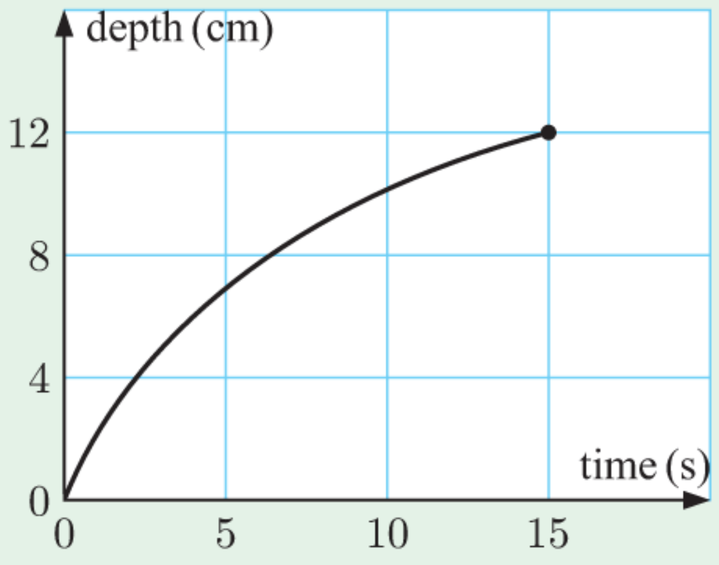
DEMO



d Consider the depth-time graph alongside.

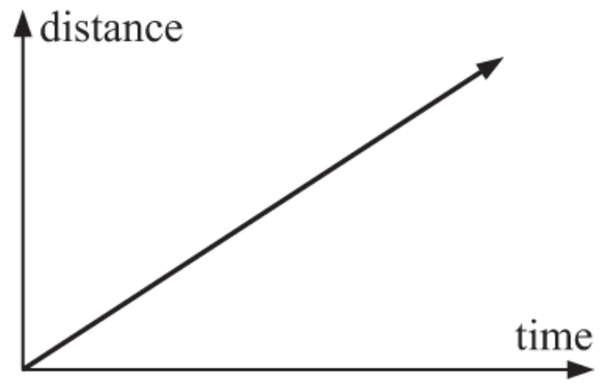
i How can we measure the *average rate* at which the depth increases from $t = 5$ to $t = 10$ seconds?

ii How can we measure the *instantaneous rate* at which the depth is increasing at the instant when $t = 8$ seconds?



If the relationship between two variables is a straight line, the gradient of the line tells us the rate at which one variable changes with respect to the other.

For example, in a travel graph showing distance against time, the gradient of a straight line segment tells us the *speed* of the object.



In the real world, rates such as speed are constantly changing. Their travel graphs are curves rather than straight lines. In order to calculate the instantaneous speed of an object, we need a branch of mathematics called **differential calculus**.

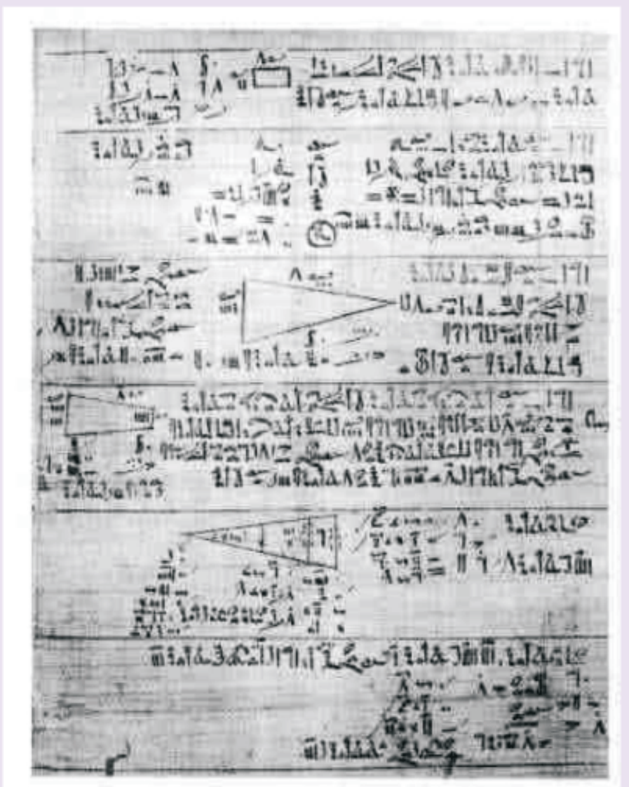
Differential calculus deals with **rates of change**. It has widespread applications in science, engineering, and finance.

HISTORICAL NOTE

The word "calculus" is a Latin word referring to the small pebbles the ancient Romans used for counting.

The first known description of calculus is found on the **Egyptian Moscow papyrus** from about 1850 BC. Here, it was used to calculate areas and volumes.

Ancient Greek mathematicians such as **Democritus** and **Eudoxus** developed these ideas further by dividing objects into an infinite number of sections. This led to the study of infinitesimals used by **Archimedes**, and eventually to the development of modern calculus developed almost 2000 years later by mathematicians such as **Johann Bernoulli** and **Isaac Barrow**.



Egyptian Moscow papyrus

A RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

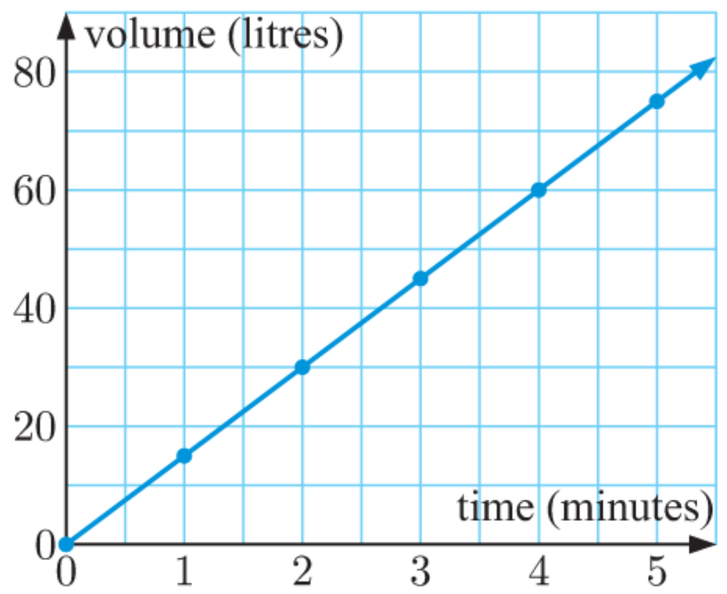
- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 runs per innings.
- Michael Jordan's average basketball scoring rate was 30.1 points per game.
- Rangi's average typing rate is 63 words per minute with an error rate of 2.3 errors per page.

CONSTANT RATES OF CHANGE

Suppose water from a hose is used to fill a swimming pool. The volume of water in the pool is recorded at 1 minute intervals in the table alongside.

Time (minutes)	0	1	2	3	4	5
Volume (litres)	0	15	30	45	60	75
		+15	+15	+15	+15	+15

Notice that the volume of water increases by the same amount each time interval. The rate of change in volume is constant, so the graph of *volume* against *time* is a straight line.



The **gradient** of the line gives the rate of change:

$$\text{rate of flow} = \frac{15-0}{1-0}$$
$$= 15 \text{ litres per minute}$$

EXERCISE 16A.1

1 The table alongside shows the distance travelled by a jogger at 30-second intervals.

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

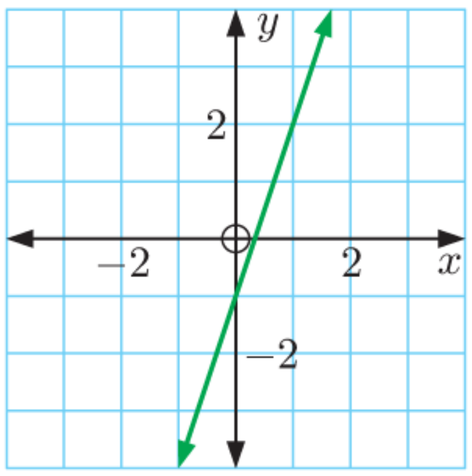
a Is the jogger travelling at a constant speed? Explain your answer.

b Draw the graph of distance against time.

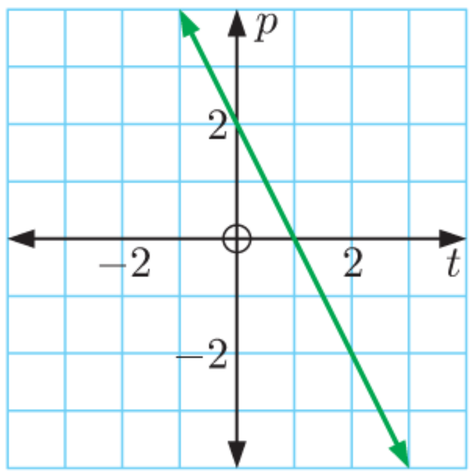
c Find the speed of the jogger in metres per second.

2 Find the rate of change for each function. Do not include units in your answer.

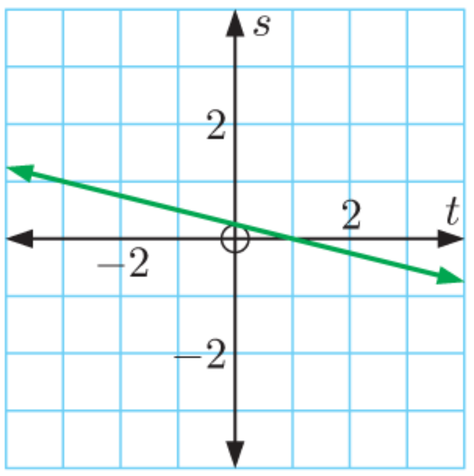
a



b



c

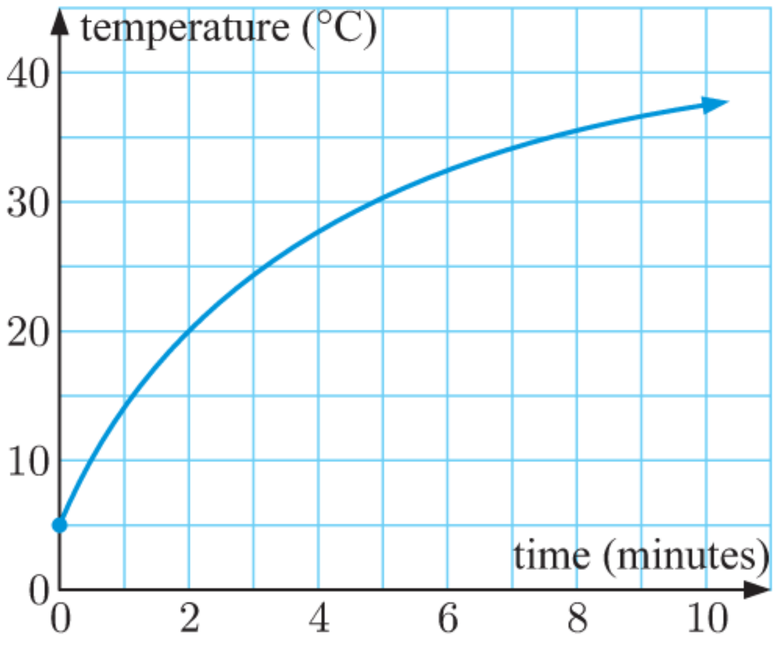


3 Find the rate of change for the function $f(x) = \frac{5}{2}x - 3$.

AVERAGE RATES OF CHANGE

In most real-world situations, rates of change are not constant. They vary over time.

For example, this graph shows the temperature of a glass of water which is left in the sun. The graph is not a straight line, which means the rate of change in temperature is not constant. The temperature increases quickly at first, and then more slowly as time goes by.



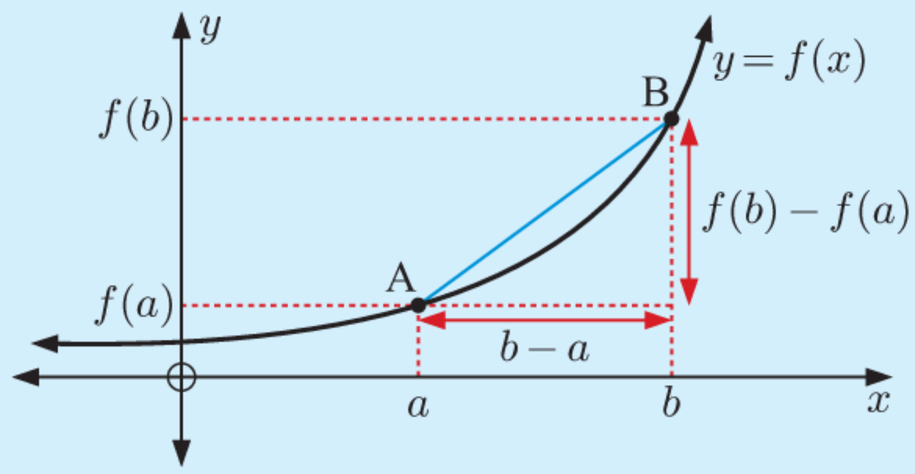
In such cases, we can find an **average rate of change** over a particular time interval.

For example, from time $t = 0$ to $t = 2$ minutes, the temperature increases from 5°C to 20°C. So, the average rate of change is $\frac{20-5}{2-0} = 7.5$°C per minute.

In the context of functions, we say that:

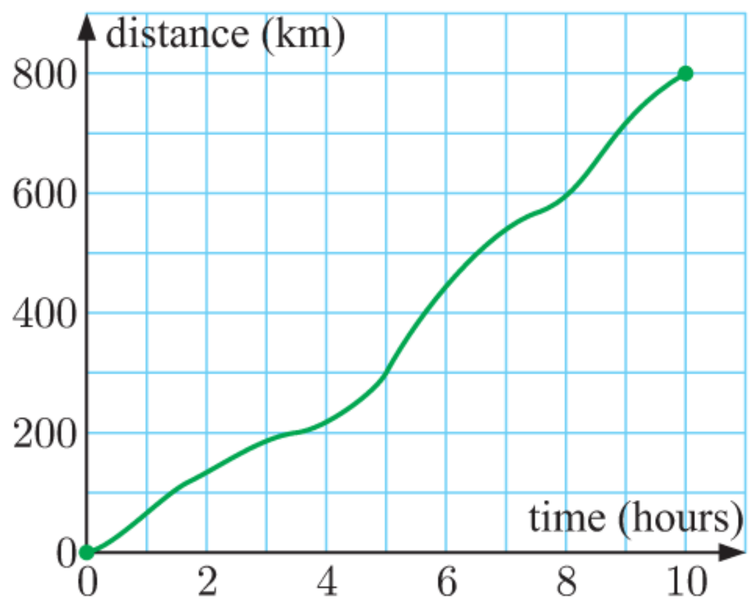
The **average rate of change** in $f(x)$ from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$.

This is the **gradient of the chord [AB]**.



EXERCISE 16A.2

1 Aileen is driving from Amsterdam to Zurich. This graph shows the distance travelled against time.



a Did Aileen travel at constant speed? Explain your answer.

b Find Aileen's average speed for:

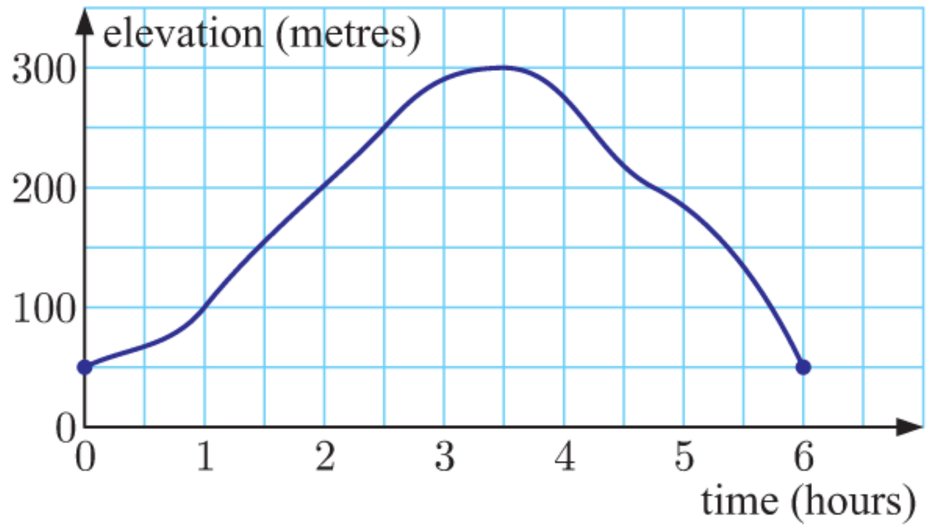
i the first 5 hours **ii** the final 5 hours.

2 Chris went hiking in the mountains. His elevation above sea level is shown on this graph.

Find Chris' average rate of change in elevation from:

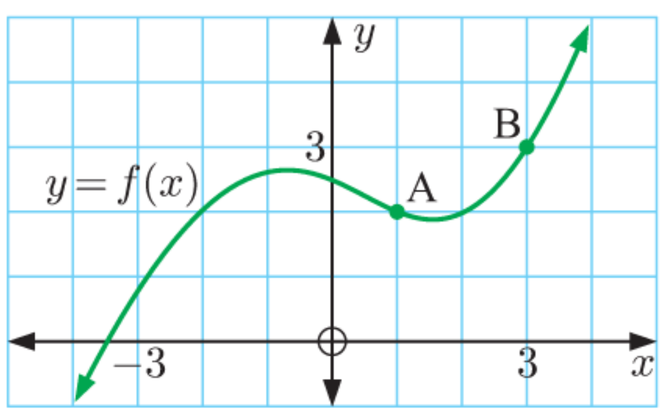
a $t = 1$ hour to $t = 2.5$ hours

b $t = 3.5$ hours to $t = 6$ hours.

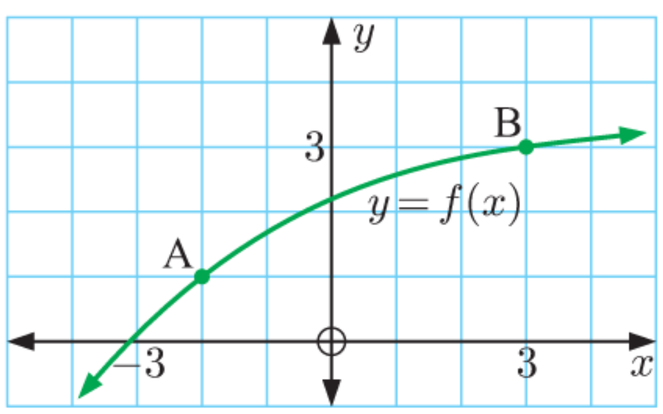


3 For each function, find the average rate of change in $f(x)$ from A to B:

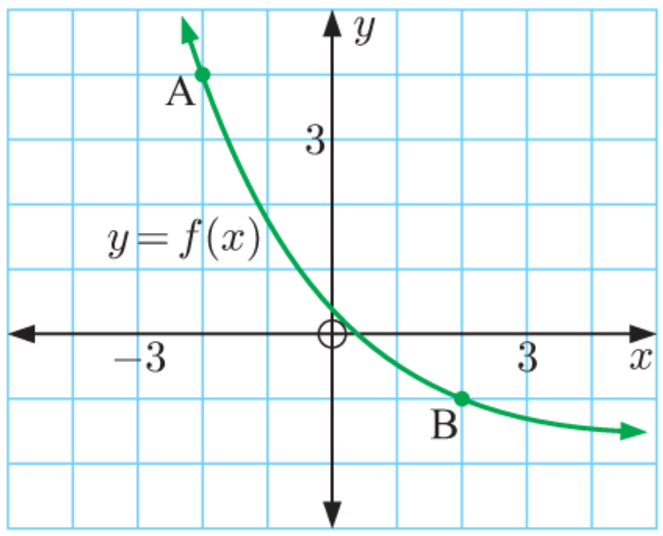
a



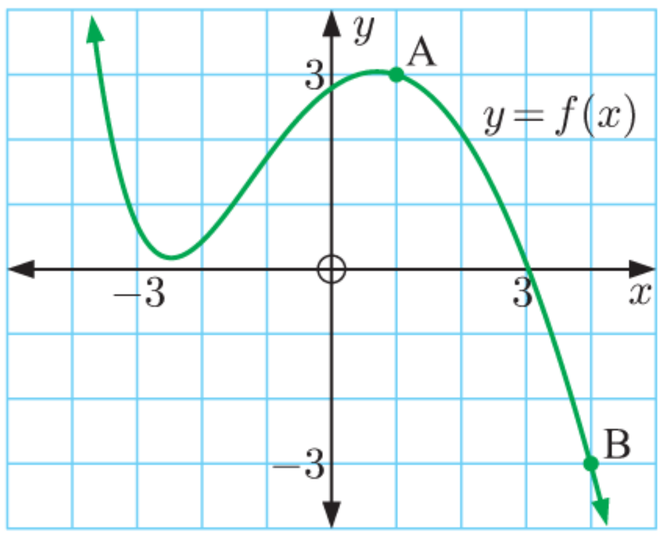
b



c



d

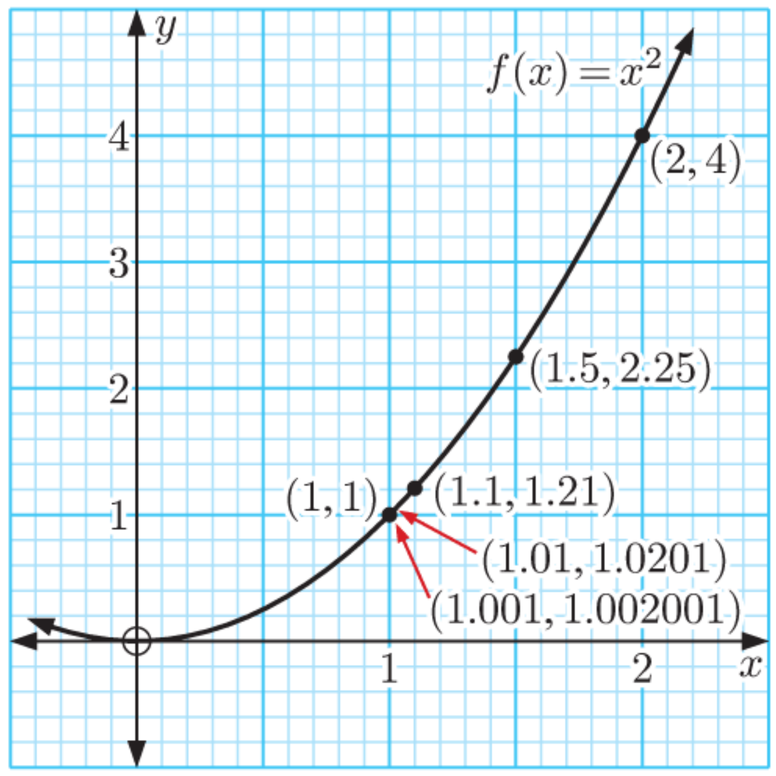


4 Consider the graph of $f(x) = x^2$.

a Find the average rate of change in $f(x)$ from:

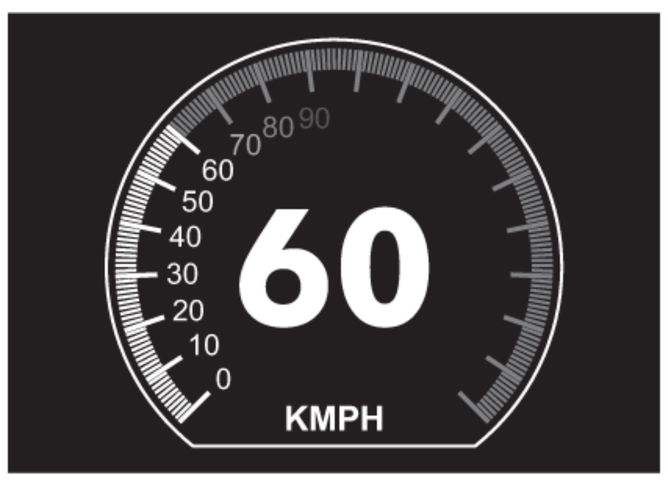
i $x = 1$ to $x = 2$
ii $x = 1$ to $x = 1.5$
iii $x = 1$ to $x = 1.1$
iv $x = 1$ to $x = 1.01$
v $x = 1$ to $x = 1.001$

b Comment on your answers in **a**.



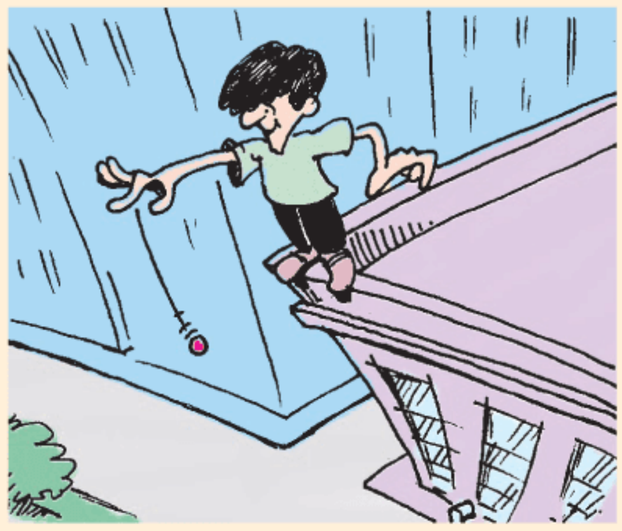
B INSTANTANEOUS RATES OF CHANGE

Suppose the speedometer in a car indicates that you are travelling at 60 km per hour. This is not an average speed, but an *instantaneous speed*. It is the speed at which you are travelling at that particular instant.

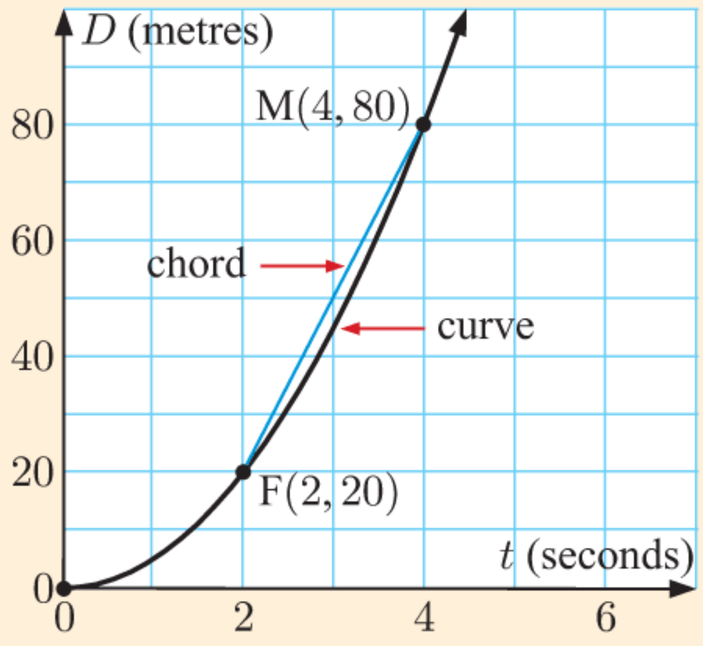


INVESTIGATION 1 INSTANTANEOUS SPEED

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time is obtained.



Consider a fixed point F on the curve when $t = 2$ seconds. We now choose another point M on the curve, and draw the line segment or **chord** [FM] between the two points. To start with, let M be the point when $t = 4$ seconds.



The *average* speed of the ball bearing over the time interval $2 \leqslant t \leqslant 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20)\text{ m}}{(4 - 2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

In this Investigation we will try to measure the *instantaneous* speed of the ball bearing when $t = 2$ seconds.

What to do:

1 Click on the icon to start the demonstration.
The gradient of the chord [FM] is shown. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see that the average speed is 30 m s^{-1}.



2 Click on M and drag it slowly towards F. Copy and complete the table alongside, where M corresponds to the given value of t.

t	gradient of [FM]
4	30
3	
2.5	
2.1	
2.01	

3 Observe what happens as M reaches F. Explain why this is so.

4 What do you suspect is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

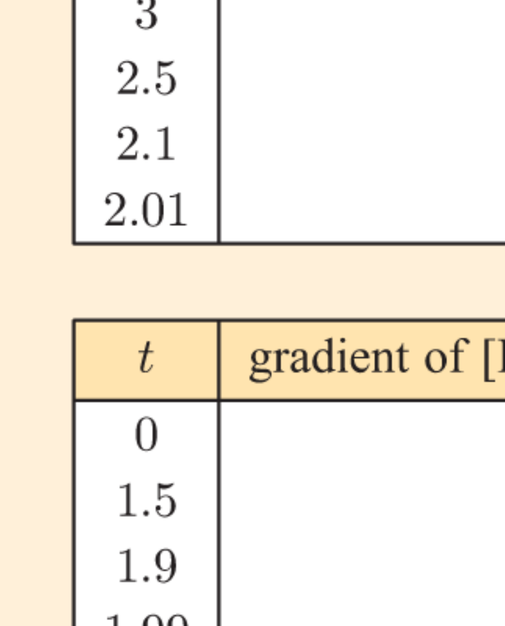
5 Move M to the origin, and then slide it towards F from the left. Copy and complete the table, where M again corresponds to the given value of t.

t	gradient of [FM]
0	
1.5	
1.9	
1.99	

6 Do your results agree with those in **4**?

If A and B are two points on a function, the gradient of the chord [AB] is the average rate of change between these points.

From the **Investigation**, if we let B get closer and closer to A, then the average rate of change from A to B will approach the *instantaneous* rate of change at A.

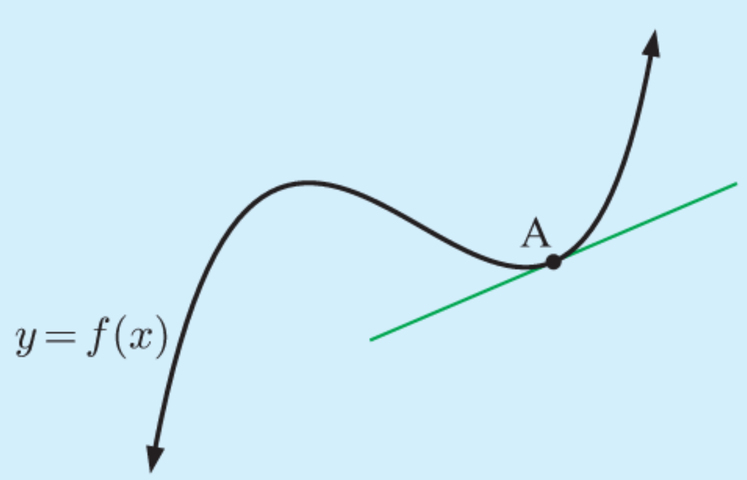


However, as B gets closer to A, the chord [AB] approaches the line which *touches* the curve at A. This line is called the **tangent** to the curve at A.

In particular, as B approaches A, the gradient of [AB] approaches the gradient of the tangent at A.



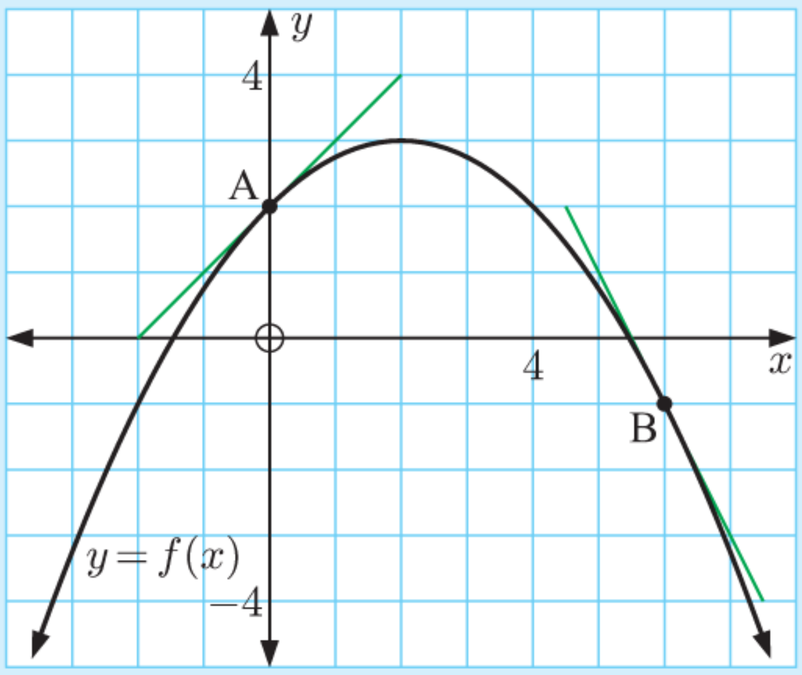
The **instantaneous rate of change** in $f(x)$ at any point A on the curve is the **gradient of the tangent** at A.



Example 1 **Self Tutor**

Use the tangents drawn to find the instantaneous rate of change in $y = f(x)$ at:

a A **b** B



a The tangent at A has gradient 1.
$\therefore$ the instantaneous rate of change at A is 1.

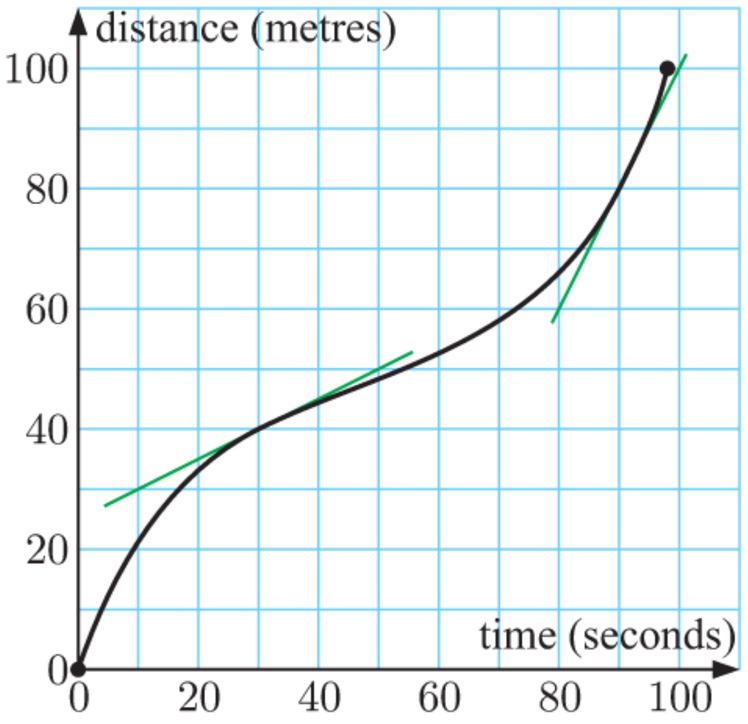
b The tangent at B has gradient -2.
$\therefore$ the instantaneous rate of change at B is -2.

EXERCISE 16B

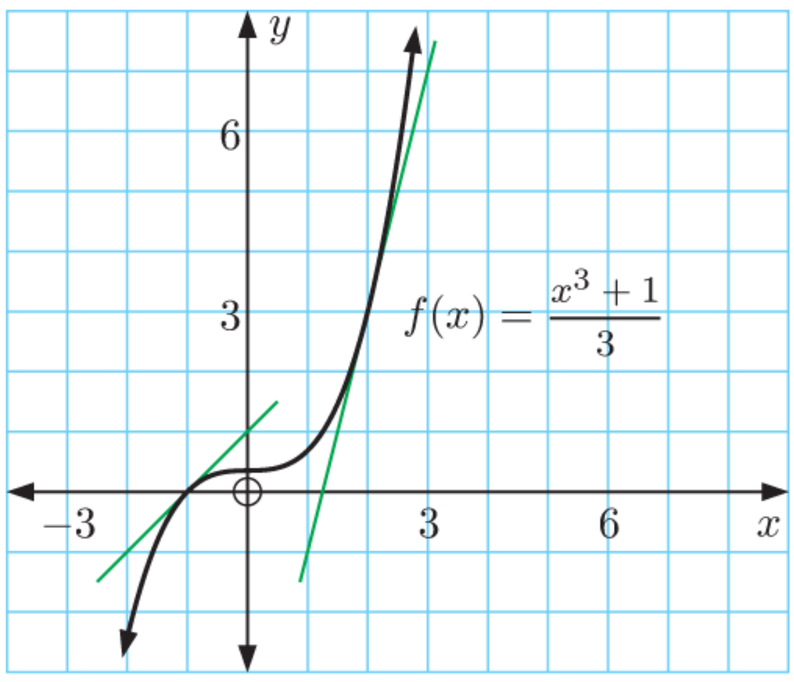
1 This graph shows the distance travelled by a swimmer in a pool.

Use the tangents drawn to find the swimmer's instantaneous speed after:

a 30 seconds **b** 90 seconds.



2



The graph of $f(x) = \dfrac{x^3 + 1}{3}$ is shown alongside.

Use the tangents drawn to find the instantaneous rate of change in $f(x)$ at:

a $x = -1$ **b** $x = 2$.

3 **a** Draw an accurate graph of $y = x^2$ on a fine grid.

b Draw, as accurately as possible, the tangent to $y = x^2$ at $x = -1$.

c Hence find the instantaneous rate of change in $y = x^2$ when $x = -1$.

PRINTABLE GRAPH

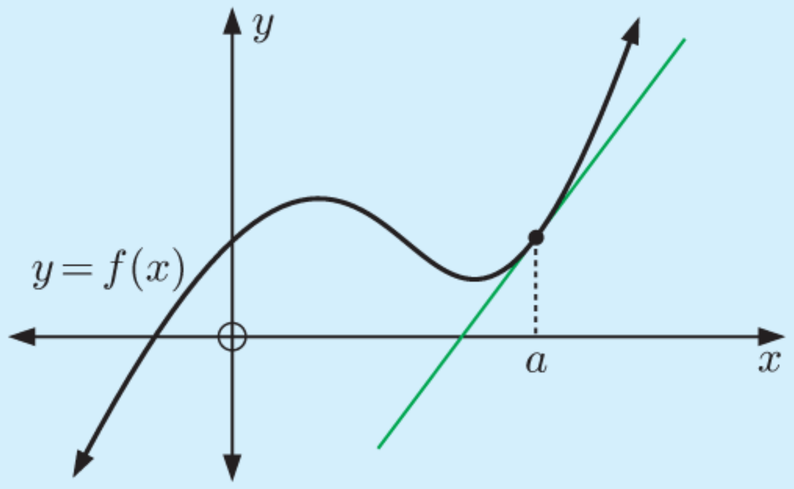
C THE GRADIENT OF A TANGENT

We have seen that for two points $A(a,\ f(a))$ and $B(a+h,\ f(a+h))$ on a function, the gradient of the chord [AB] is $\dfrac{f(a+h)-f(a)}{h}$.

Letting B get infinitely close to A is equivalent to taking the limit as $h \to 0$.

If this limit exists, the **gradient of the tangent** to the curve $y=f(x)$ at the point where $x=a$ is

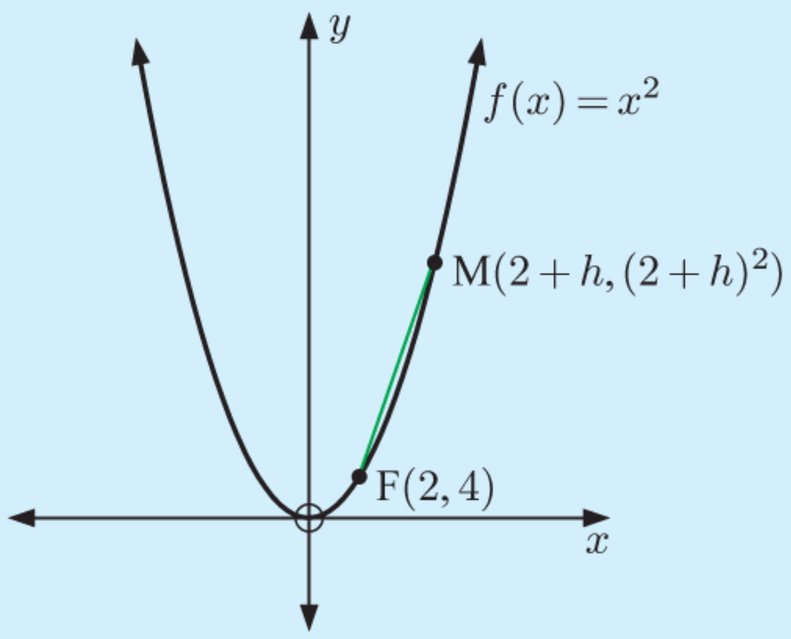
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$



Example 2 **Self Tutor**

Find the gradient of the tangent to $f(x)=x^2$ at the point $(2,\ 4)$.

Let F be the point $(2,\ 4)$. Suppose M has x-coordinate $2+h$ and also lies on the graph, so M is $(2+h,\ (2+h)^2)$.



The gradient of the tangent at F

$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2-4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2-\cancel{4}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (4+h) \qquad \{\text{as } h \neq 0\}$$

$$= 4$$

You can use technology to find the gradient of the tangent to a function at a given point, and hence check your answers.





GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 16C

1 F$(3,\ 9)$ lies on the graph of $f(x)=x^2$. M also lies on the graph, and has x-coordinate $3+h$.

a State the y-coordinate of M.

b Show that the gradient of the line segment [FM] is $6+h$.

c *Hence* find the gradient of [FM] if M has coordinates:

i $(4,\ 16)$ **ii** $(3.5,\ 12.25)$ **iii** $(3.1,\ 9.61)$ **iv** $(3.01,\ 9.0601)$

d Use limits to find the gradient of the tangent to $f(x)=x^2$ at the point $(3,\ 9)$.

2 **a** Find the gradient of the tangent to $f(x) = x^2$ at the point where:

i $x = 1$ **ii** $x = 4$

b Use **a** and other results from this Section to complete the table alongside for $f(x) = x^2$.

c Predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$.

x-coordinate	*Gradient of tangent to* $f(x) = x^2$
1	
2	
3	
4	

3 Find the gradient of the tangent to:

a $f(x) = x^2 + x$ at the point $(2, 6)$ **b** $f(x) = x^3$ at the point where $x = 1$

c $f(x) = \dfrac{4}{x}$ at the point where $x = 2$ **d** $f(x) = x^4$ at the point where $x = 1$.

DISCUSSION

Using technology or otherwise, sketch the graphs of:

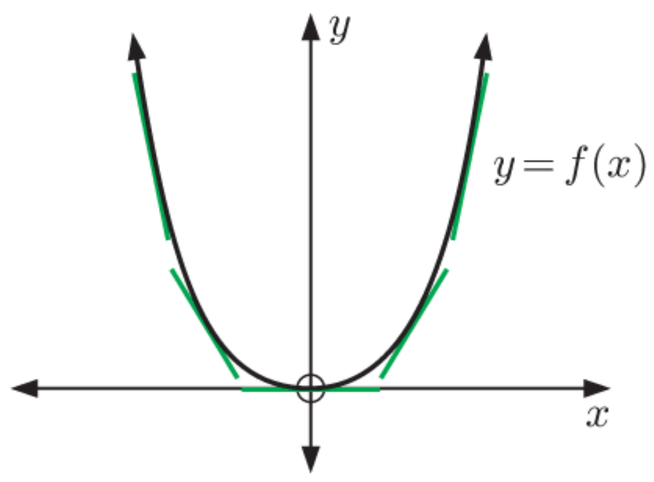
- $y = \dfrac{1}{x^2}$
- $y = \tan x$
- $y = \sqrt[3]{x}$
- $y = |x|$
- $y = \dfrac{x^3 - x^2}{x}$
- $y = \sin \frac{1}{x}$

For a curve $y = f(x)$ and a constant $a \in \mathbb{R}$, list reasons why $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ may not exist.

D THE DERIVATIVE FUNCTION

For a non-linear function $y = f(x)$, the gradient of the tangent changes as we move along the curve.

We can therefore write a **gradient function** which gives the gradient of the tangent for any given value of x.

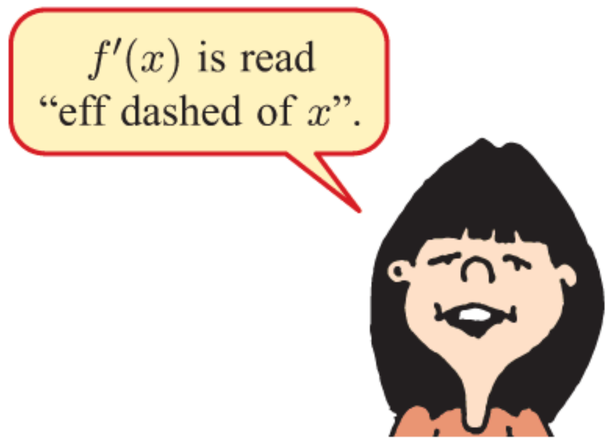


The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

For example, in question **2** of the previous Exercise, you should have observed that for $f(x) = x^2$, the gradient of the tangent is always double the x-coordinate.

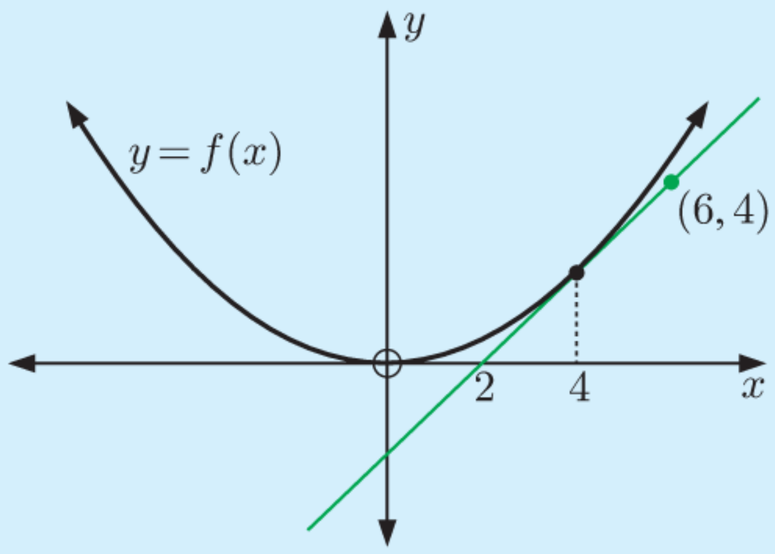
So, for $f(x) = x^2$ we write $f'(x) = 2x$.

Since $f'(3) = 6$, the gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$, is 6.



Example 3 **Self Tutor**

For the given graph, find $f'(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$.

$\therefore \quad f'(4) =$ gradient of the tangent

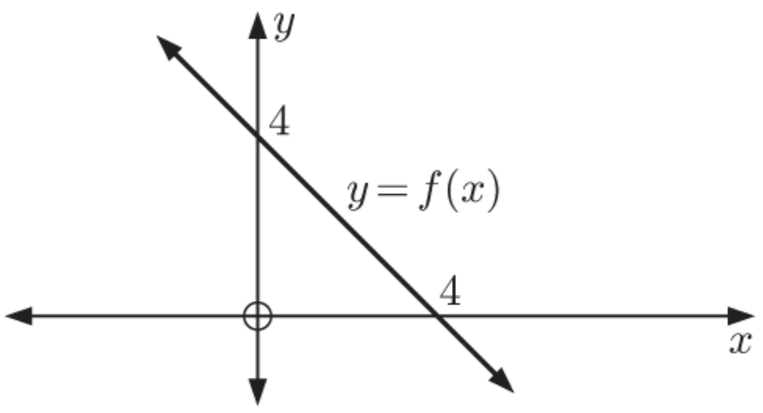
$$= \frac{4-0}{6-2}$$

$$= 1$$

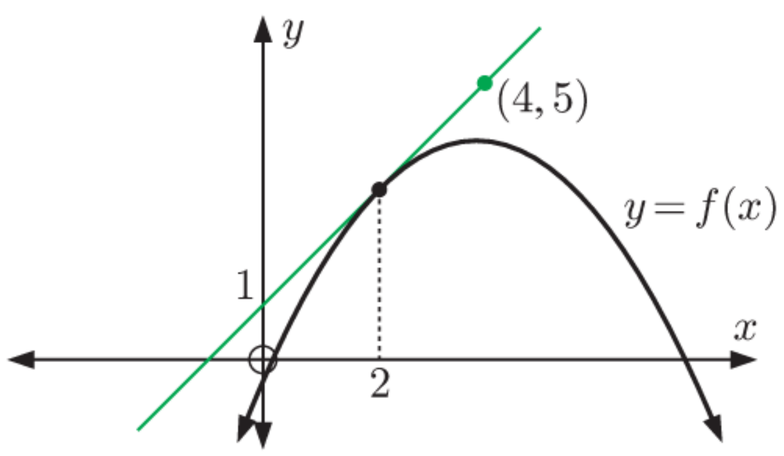
EXERCISE 16D

1 Using the graph below, find:

a $f(0)$ **b** $f'(0)$



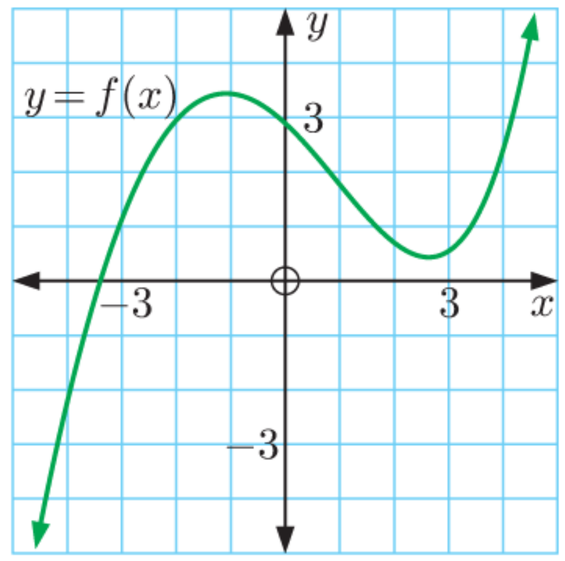
2 Use the graph below to find $f'(2)$.



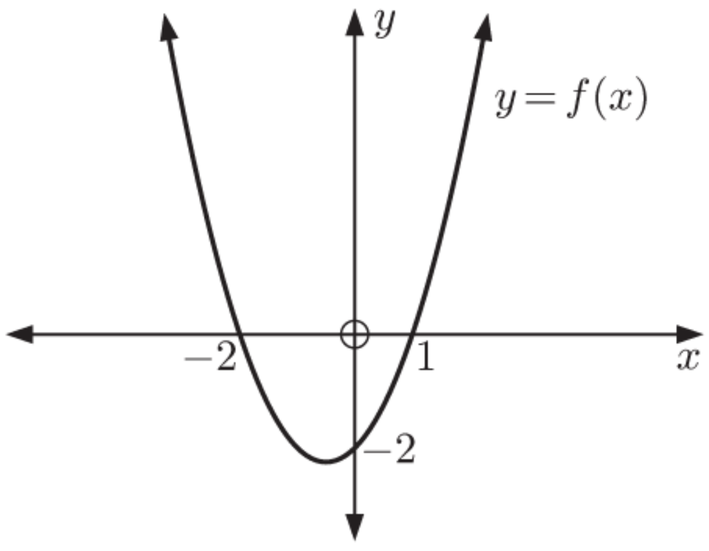
3 For the graph of $y = f(x)$ alongside, decide whether the following are positive or negative:

a $f(3)$ **b** $f'(1)$

c $f(-4)$ **d** $f'(-2)$



4

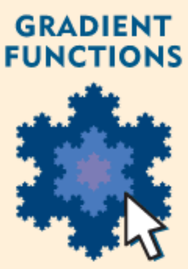


For the graph of $y = f(x)$ alongside, the derivative function is $f'(x) = 2x + 1$.

a Find and interpret: **i** $f'(-2)$ **ii** $f'(0)$

b Copy the graph, and include the information in **a**.

INVESTIGATION 2 GRADIENT FUNCTIONS



The software, accessible using the icon alongside, can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph, we observe the changing gradient of the tangent and hence generate the gradient function $f'(x)$.

What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.

a For each of these functions, what is the gradient?

b Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.

a State the gradient of the function.

b Is the gradient constant for all values of x?

c Use the software to graph the following functions and observe the gradient function $f'(x)$. Hence verify your answer to **b**.

i $f(x) = x - 1$ **ii** $f(x) = 3x + 2$ **iii** $f(x) = -2x + 1$

3 **a** Observe the function $f(x) = x^2$ using the software. What *type* of function is the gradient function $f'(x)$?

b Observe the following quadratic functions using the software:

i $f(x) = x^2 + x - 2$ **ii** $f(x) = 2x^2 - 3$

iii $f(x) = -x^2 + 2x - 1$ **iv** $f(x) = -3x^2 - 3x + 6$

c What *type* of function are each of the gradient functions $f'(x)$ in **b**?

4 **a** Observe the function $f(x) = e^x$ using the software.

b What is the gradient function $f'(x)$?

E DIFFERENTIATION FROM FIRST PRINCIPLES

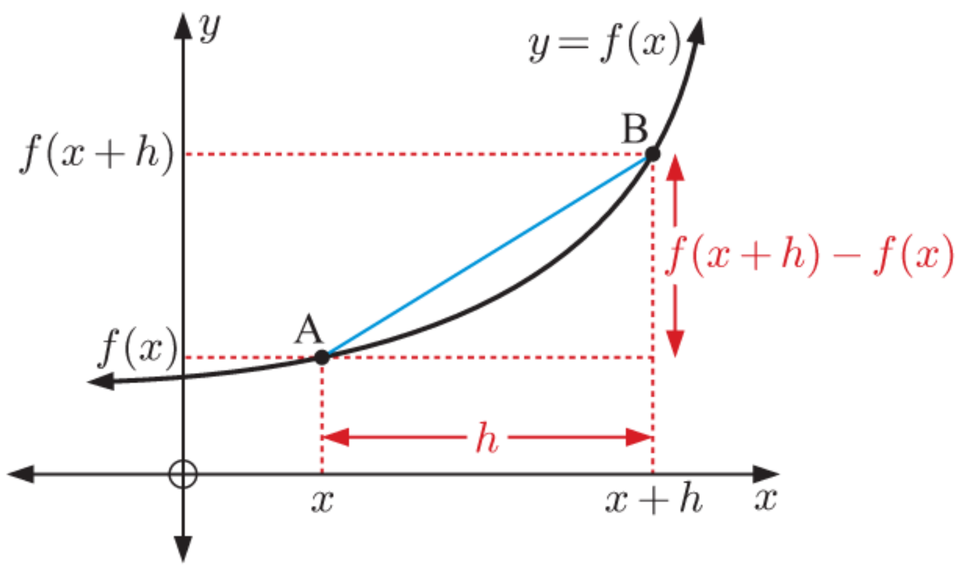
Bringing together our work from the previous two Sections:

If the limit exists, then $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ is the gradient of the tangent to $y = f(x)$ at $x = a$.

We say that $f(x)$ is **differentiable at** $x = a$.

We can use this definition to find the gradient of the tangent to the curve at a general point $(x, f(x))$, and hence define its derivative function.

Let $A(x,\ f(x))$ and $B(x+h,\ f(x+h))$ be two points on the curve.



The chord [AB] has gradient $= \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$

If we let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the general point $(x,\ f(x))$ is $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $f(x)$ is defined as:

$$\boldsymbol{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

The domain of $f'(x)$ is the set of values for which this limit exists.

When we use this limit definition to find a derivative function, we say we are **differentiating from first principles**.

Example 4 **Self Tutor**

Consider the function $f(x) = x^4$.

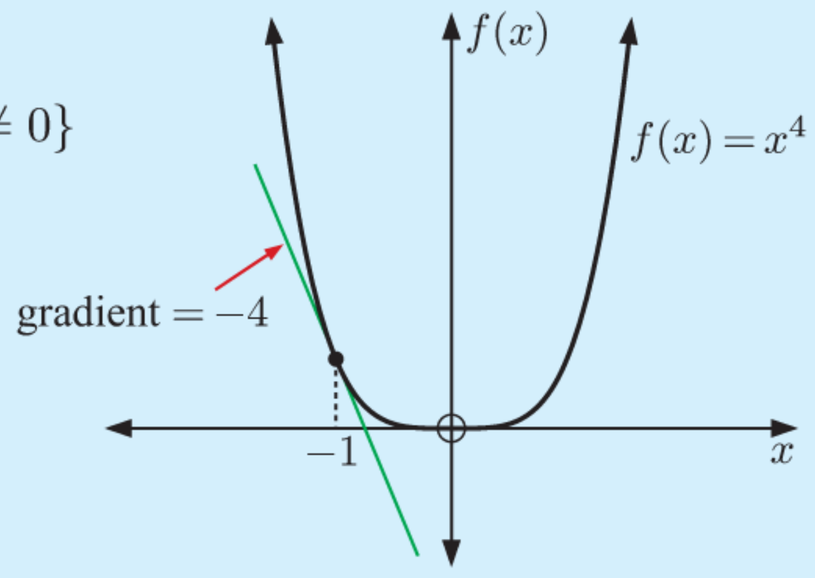
a Use first principles to find the gradient function $f'(x)$.

b Find $f'(-1)$, and interpret your answer.

a
$$\begin{aligned} & f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{x^4}}{h} && \{\text{binomial expansion}\} \\ &= \lim_{h\to 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3)}{\cancel{h}} \\ &= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) && \{\text{since } h \neq 0\} \\ &= 4x^3 \end{aligned}$$

b $f'(-1) = 4(-1)^3$
$= -4$

The tangent to $f(x) = x^4$ at the point where $x = -1$, has gradient -4.



LEIBNIZ NOTATION

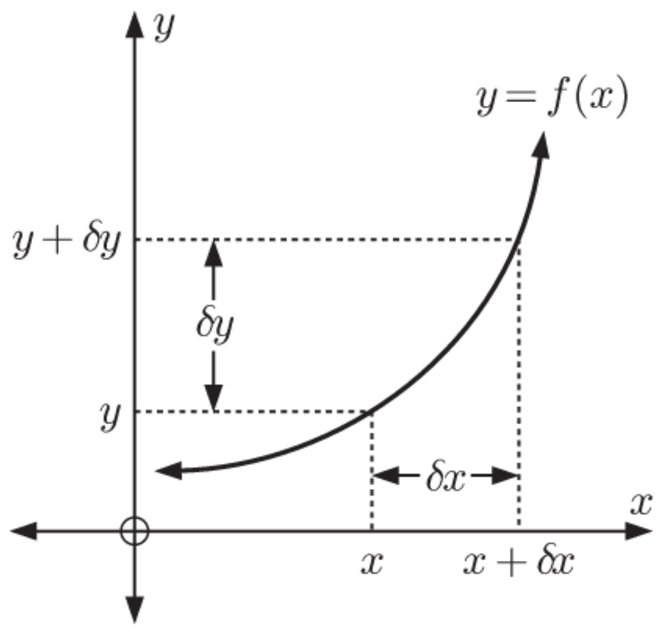
If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

If we are given y in terms of x then we use the **Leibniz notation** $\frac{dy}{dx}$ to represent the derivative.

$\frac{dy}{dx}$ reads "dee y by dee x" or "the derivative of y with respect to x".

$\frac{dy}{dx}$ is *not* a fraction, but it *is* the result of taking the limit of a fraction. If we replace h by δx and $f(x+h)-f(x)$ by δy, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{\delta x\to 0} \frac{\delta y}{\delta x} \\ &= \frac{dy}{dx}. \end{aligned}$$



EXERCISE 16E

1 Find, from first principles, the gradient function of $f(x)$ where $f(x)$ is:

a x **b** 1 **c** x^3

2 Find $f'(x)$ from first principles, given that $f(x)$ is:

a $2x+5$ **b** x^2-3x **c** $-x^2+5x-3$

3 Find $\frac{dy}{dx}$ from first principles given:

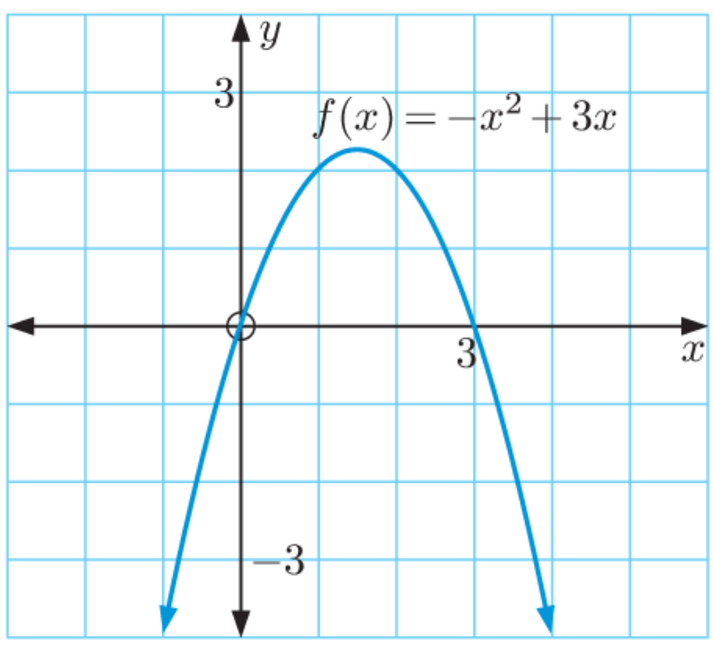
a $y=4-x$ **b** $y=2x^2+x-1$ **c** $y=x^3-2x^2+3$

4 Use the first principles formula $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ to find:

a $f'(2)$ for $f(x)=x^3$ **b** $f'(3)$ for $f(x)=x^4$.

5 **a** Find $f'(x)$ given $f(x)=\frac{1}{x}$. **b** Find $f'(-1)$ and $f'(3)$, and interpret your answers.

6 The graph of $f(x)=-x^2+3x$ is shown alongside.



a Use the graph to estimate the gradient of the tangent to the curve at the point where:

i $x=0$ **ii** $x=2$.

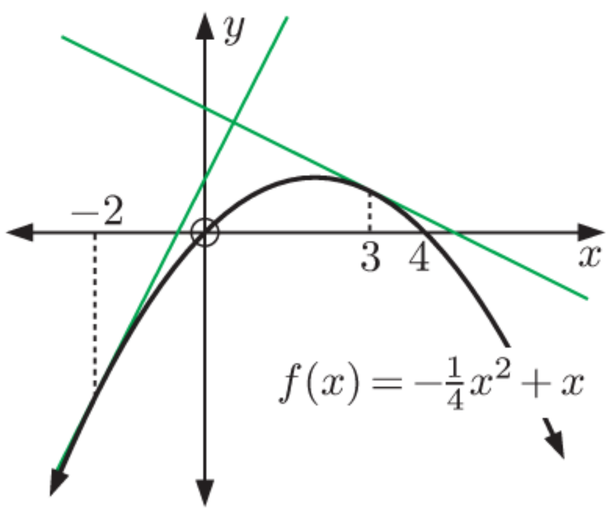
b Find $f'(x)$ from first principles.

c Find $f'(0)$ and $f'(2)$, and hence check your estimates in **a**.

7 **a** Given $y=x^3-3x$, find $\frac{dy}{dx}$ from first principles.

b Hence find the points on the graph at which the tangent has zero gradient.

8



The graph of $f(x) = -\frac{1}{4}x^2 + x$ is shown alongside.

a Find $f'(x)$.

b Hence show that the illustrated tangents are perpendicular.

9 For each function type $f(x)$, find $f'(x)$ from first principles. Hence describe the type of the derivative function.

a the linear function $f(x) = mx + c, \ m \neq 0$

b the quadratic function $f(x) = ax^2 + bx + c, \ a \neq 0$

c the cubic function $f(x) = ax^3 + bx^2 + cx + d, \ a \neq 0$

10 Given $y = \sqrt{x}$, find $\frac{dy}{dx}$ from first principles. Comment on the differentiability of $y = \sqrt{x}$.

11 **a** Use the previous results to complete the table:

$f(x)$	x^4	x^3	x^2	x^1	$x^{\frac{1}{2}}$	x^0	x^{-1}
$f'(x)$							

b Copy and complete: If $f(x) = x^n$, then $f'(x) = ...$

Example 5 — Self Tutor

a Prove that $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0$. **b** Hence find the derivative of $f(x) = \sin x$.

a
$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{\cos h - 1}{h} \times \frac{\cos h + 1}{\cos h + 1}$$
$$= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$
$$= \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$
$$= \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{-\sin h}{\cos h + 1} \qquad \{\text{by the limit laws since both limits exist}\}$$
$$= 1 \times \tfrac{0}{2} = 0$$

b Let $f(x) = \sin x$. If $f'(x)$ exists, then it is given by
$$f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$
$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$
$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \frac{\sin h}{h} \times \cos x \right]$$
$$= \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \qquad \{\text{since } x \text{ is independent of } h\}$$
$$= \sin x \times 0 + \cos x \times 1$$
$$= \cos x$$

12 Use the result $\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} = 0$ and first principles to find the derivative of $f(x) = \cos x$.

13 **a** Show that $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$.

b By letting $S = A + B$ and $D = A - B$, show that $\cos S - \cos D = -2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$.

c Hence find the derivative of $f(x) = \cos x$ from first principles.

F DIFFERENTIABILITY AND CONTINUITY

If f is differentiable at $x = a$, then f is also continuous at $x = a$.

Proof:

$$\lim_{h\to 0} f(a+h) - f(a)$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times h$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h\to 0} h \qquad \{\text{by the limit laws, since both limits exist}\}$$

$$= f'(a) \times 0$$

$$= 0$$

$\therefore \lim\limits_{h\to 0} f(a+h) = f(a)$

Letting $x = a + h$, this is equivalent to $\lim\limits_{x\to a} f(x) = f(a)$.

$\therefore$ f is continuous at $x = a$.

Using the contrapositive:

If a function is not continuous at $x = a$, then it is not differentiable at $x = a$.

The converse of the theorem is *not* true, however:

If a function is continuous at $x = a$, it is *not necessarily* differentiable at $x = a$.

TESTING FOR DIFFERENTIABILITY

To test whether a function is differentiable, we first define:

- the **left-hand derivative** of f at $x = a$ is $f'_-(a) = \lim\limits_{h\to 0^-} \dfrac{f(a+h) - f(a)}{h}$
- the **right-hand derivative** of f at $x = a$ is $f'_+(a) = \lim\limits_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h}$.

A function f with domain D is **differentiable at** $\boldsymbol{x = a}$, $a \in D$, if:

1 f is continuous at $x = a$, and

2 $f'_-(a) = \lim\limits_{h\to 0^-} \dfrac{f(a+h) - f(a)}{h}$ and $f'_+(a) = \lim\limits_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h}$ both exist and are equal.

Example 6 **Self Tutor**

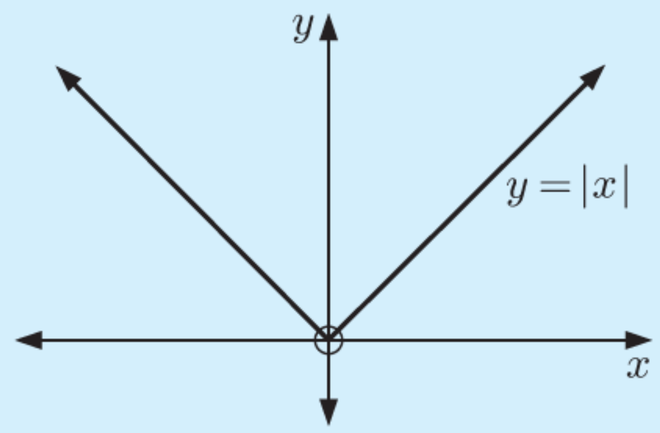
Prove that $f(x) = |x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$ is continuous but not differentiable at $x = 0$.

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0,$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$

$\therefore \lim\limits_{x \to 0} f(x) = 0 = f(0)$

$\therefore$ f is continuous at $x = 0$.



Now $f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$ since the derivative of f exists on the intervals $x < 0$ and $x > 0$.

$\therefore$ $f'_-(0) = -1$ and $f'_+(0) = 1$

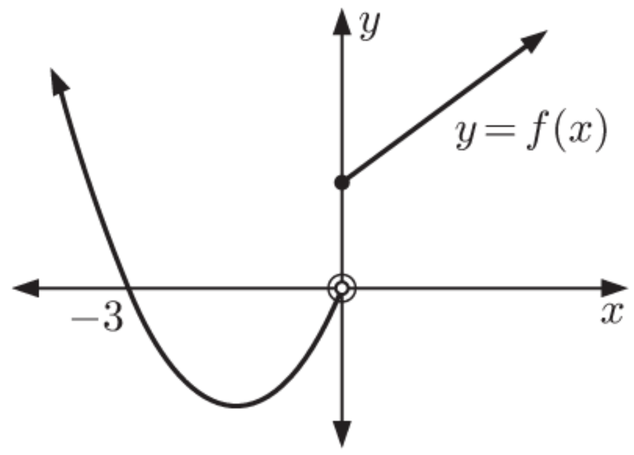
$\therefore$ $f'_-(0) \neq f'_+(0)$

$\therefore$ f is not differentiable at $x = 0$.

EXERCISE 16F

1 Let $f(x) = \begin{cases} x + 2, & x \geqslant 0 \\ x^2 + 3x, & x < 0. \end{cases}$

Explain why $f(x)$ is not differentiable at $x = 0$.



2 Explain why $f(x) = \dfrac{x^2 + x}{x}$ is not differentiable at $x = 0$.

3 Prove that $f(x) = |x - 5| = \begin{cases} x - 5, & x \geqslant 5 \\ 5 - x, & x < 5 \end{cases}$ is continuous but not differentiable at $x = 5$.

4 Let $f(x) = \begin{cases} -x^2 + 5x + 6, & x \geqslant 1 \\ 3x + 10, & x < 1. \end{cases}$

a Sketch the function $y = f(x)$.

b Calculate: **i** $f'_-(1)$ **ii** $f'_+(1)$

c Is f differentiable at $x = 1$? Explain your answer.

5 Let $f(x) = \begin{cases} x^2, & x \leqslant 1 \\ cx + d, & x > 1. \end{cases}$

Find constants $c, d \in \mathbb{R}$ so that $f(x)$ is differentiable at $x = 1$.

6 Let $f(x) = \begin{cases} ax^2, & x \geqslant 2 \\ x + b, & x < 2 \end{cases}$. Find a and b such that $f(x)$ is differentiable at $x = 2$.

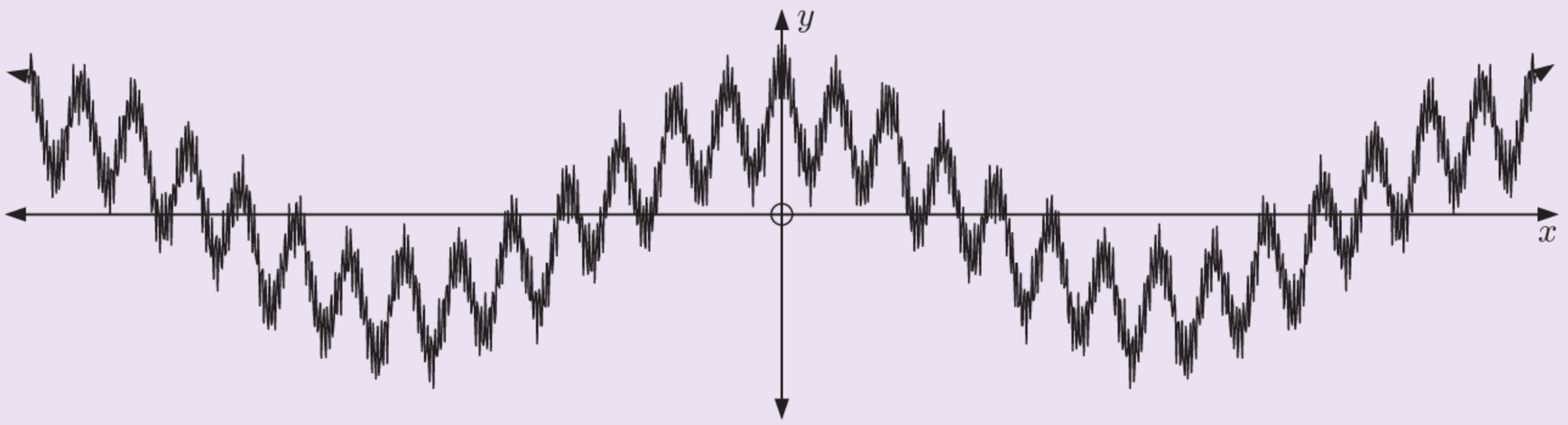
HISTORICAL NOTE

The first formal definition of a limit was given by **Bernard Bolzano** in 1817. It was then worked on by **Karl Weierstrass** (1815 - 1897) who provided the modern (ε, δ)-definition used today.

The **Weierstrass function** now named in his honour was submitted by Weierstrass to the Royal Prussian Academy of Science in 1872. It was the first published example of a function that is continuous everywhere but differentiable nowhere!

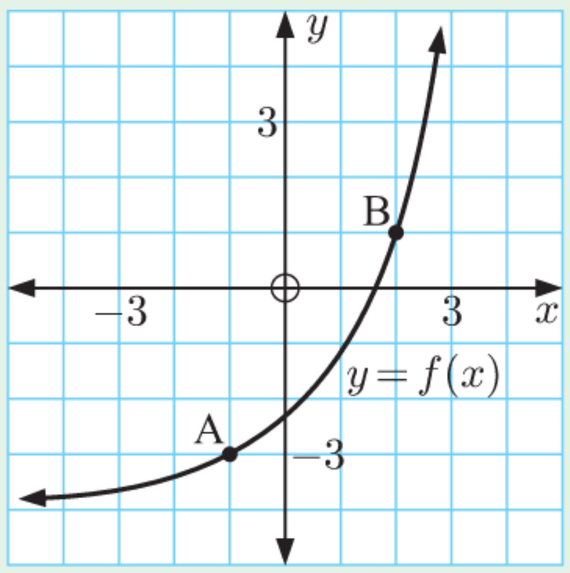
The Weierstrass function is defined as

$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$.

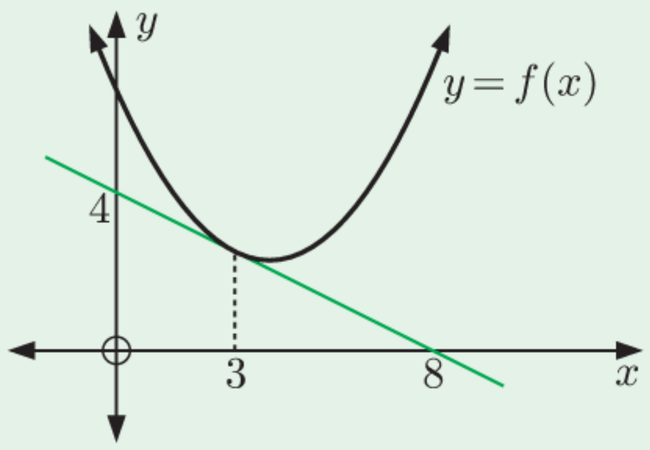


REVIEW SET 16A

1 Find the average rate of change in $f(x)$ from A to B.



2 Use the graph below to find $f'(3)$.



3 Consider $f(x) = 2x^2$.

a Show that $\dfrac{f(x+h) - f(x)}{h} = 4x + 2h$ provided $h \neq 0$.

b Hence evaluate $\dfrac{f(3+h) - f(3)}{h}$ for each value of h in the table.

c Evaluate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$.

Give a geometric interpretation of your result.

h	$\dfrac{f(3+h) - f(3)}{h}$
0.1	
0.01	
0.001	
0.0001	

4 Find, from first principles, the derivative of:

a $f(x) = x^2 + 2x$ **b** $y = 4 - 3x^2$

5 **a** Given $y = 2x^2 - 1$, find $\frac{dy}{dx}$ from first principles.

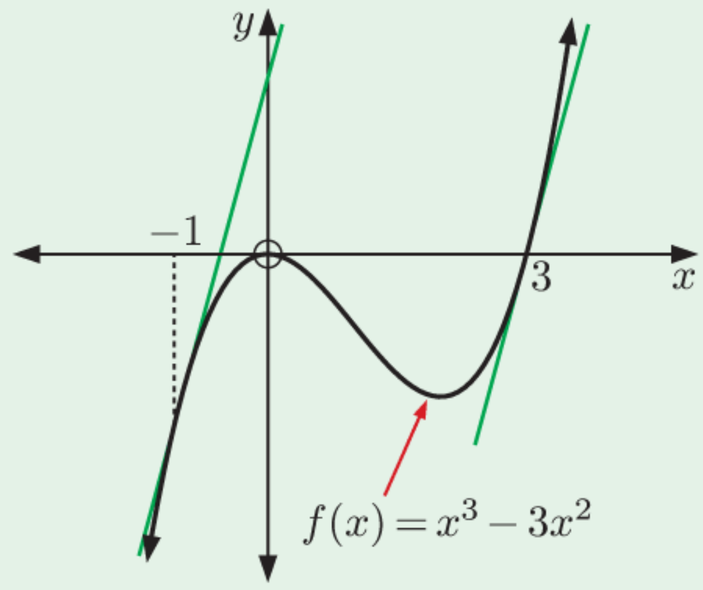
b Hence state the gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$.

c For what value of x is the gradient of the tangent to $y = 2x^2 - 1$ equal to -12?

6 The graph of $f(x) = x^3 - 3x^2$ is shown alongside.

a Find $f'(x)$.

b Hence show that the illustrated tangents are parallel.



7 **a** Given $y = 9x - x^3$, find $\frac{dy}{dx}$ from first principles.

b Hence find the points on the graph at which the tangent has zero gradient.

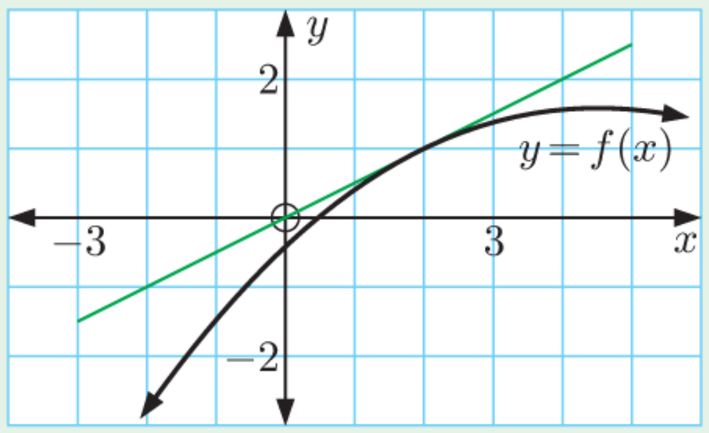
8 Explain why $f(x) = \dfrac{2}{x^2 - 2x}$ is not differentiable at $x = 2$.

9 Let $f(x) = \begin{cases} x + 2, & x \geqslant 3 \\ 11 - 2x, & x < 3 \end{cases}$.

a Sketch the function $y = f(x)$. **b** Is f differentiable at $x = 3$?

REVIEW SET 16B

1

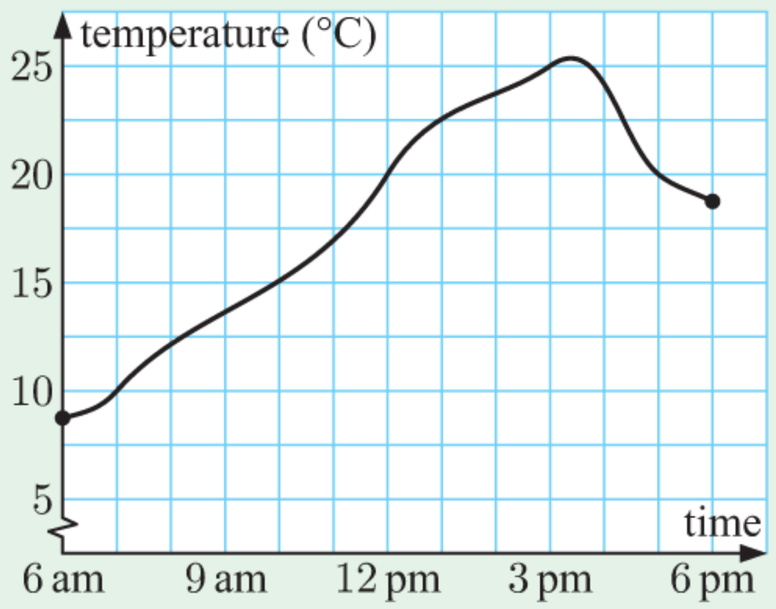


Use the tangent drawn to find the instantaneous rate of change in $f(x)$ at $x = 2$.

2 This graph shows the temperature in Berlin from 6 am to 6 pm on a particular day.

Find the average rate of change in temperature from:

a 7 am to noon **b** 3 pm to 5 pm.



3 **a** Draw an accurate graph of $f(x) = x^2 - 2$.

b Draw, as accurately as possible, the tangent to $f(x) = x^2 - 2$ at $x = 2$.

c Hence find the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$.

d Check your answer using $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

4 **a** Find $f'(x)$ given $f(x) = x^4 - 2x$.

b Find $f'(-2)$ and interpret your answer.

5 **a** Given $y = x^2 + 5x - 2$, find $\dfrac{dy}{dx}$ from first principles.

b Hence find the point on the graph at which the tangent has gradient -3.

6 In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.



a Find the height of the jumper after:

i 1 second **ii** 2 seconds.

b Find $f'(t)$ from first principles.

c Find the speed of the jumper after:

i 1 second **ii** 2 seconds.

7 Consider the function $f(x) = 2x^2 + 2x - 12$.

a Find $f'(x)$.

b Hence find the point where the tangent has gradient -2.

c Draw the graph of $f(x) = 2x^2 + 2x - 12$, and include the information from **b**.

8 **a** Show that $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$.

b By letting $S = A + B$ and $D = A - B$, show that $\sin S - \sin D = 2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$.

c Hence find the derivative of $f(x) = \sin x$ from first principles.

9 Let $f(x) = \begin{cases} \cos x, & x \geqslant \frac{\pi}{2} \\ \sin x, & x < \frac{\pi}{2} \end{cases}$.

a Sketch the function $y = f(x)$.

b Explain why $f(x)$ is not differentiable at $x = \frac{\pi}{2}$.

Chapter 17

Rules of differentiation

Contents:

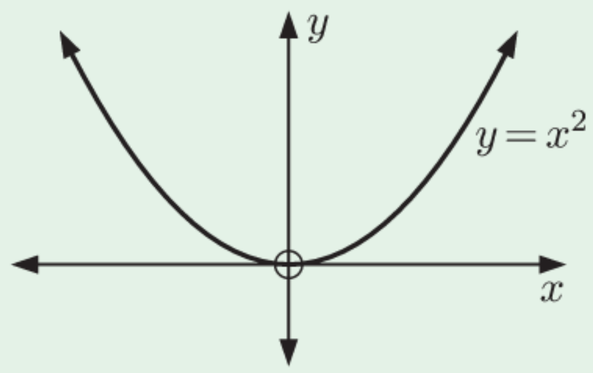
A Simple rules of differentiation
B The chain rule
C The product rule
D The quotient rule
E Derivatives of exponential functions
F Derivatives of logarithmic functions
G Derivatives of trigonometric functions
H Derivatives of inverse trigonometric functions
I Second and higher derivatives
J Implicit differentiation



OPENING PROBLEM

Consider the curve $y = x^2$.

In the previous Chapter we found that the gradient function of this curve is $\frac{dy}{dx} = 2x$.



Things to think about:

a At which point on the graph of $y = x^2$ is the tangent to the curve horizontal? How do we observe this from the derivative function $\frac{dy}{dx}$?

b Consider the transformation of $y = x^2$ onto $y = x^2 + 3$.

- **i** What transformation has taken place?
- **ii** For a given value of x, has the gradient of the tangent to the function changed?
- **iii** What is the gradient function of $y = x^2 + 3$?

c Consider the transformation of $y = x^2$ onto $y = (x - 2)^2$.

- **i** What transformation has taken place?
- **ii** How does the gradient function of $y = (x - 2)^2$ relate to the gradient function of $y = x^2$?
- **iii** Can you write down the gradient function of $y = (x - 2)^2$?

DEMO



d Consider the transformation of $y = x^2$ onto $y = 2x^2$.

- **i** What transformation has taken place?
- **ii** How does the gradient function of $y = 2x^2$ relate to the gradient function of $y = x^2$? If necessary, use the software to help you.

In this Chapter we will discover rules which make it easier to find derivative functions.

A SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

Given a function $f(x)$, we obtain $f'(x)$ by **differentiating with respect to** the variable x.

There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

INVESTIGATION 1 SIMPLE RULES OF DIFFERENTIATION

In **Chapter 16** we used first principles to find the derivative of $f(x) = x^n$ for various values of n.

In this Investigation we generalise this derivative for any $n \in \mathbb{N}$, then consider rules which will allow us to differentiate functions which are the sum or difference of polynomial terms.

What to do:

1 Use the binomial expansion $(x+h)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}h^r$

$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n}h^n$

and the first principles formula $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to find the derivative of $f(x) = x^n$ for $x \in \mathbb{N}$.

2 **a** Differentiate using first principles:

i $4x^2$ **ii** $2x^3$ **iii** $7x^4$

b Copy and complete: "If $f(x) = cx^n$, then $f'(x) = ...$"

3 **a** Use first principles to find $f'(x)$ for:

i $f(x) = x^2 + 3x$ **ii** $f(x) = x^3 - 2x^2$

b Copy and complete: "If $f(x) = u(x) + v(x)$, then $f'(x) = ...$"

The rules you found in the **Investigation** can actually be used much more widely than the cases you just considered.

For example, the rule "if $f(x) = x^n$ then $f'(x) = nx^{n-1}$" is true not just for all $n \in \mathbb{N}$, but actually for all $n \in \mathbb{R}$.

We can summarise the following rules:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$cu(x)$	$cu'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = cu(x)$,
  then $f'(x) = cu'(x)$.

  Proof:

  $f'(x)$
  $= \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
  $= \lim_{h\to 0} \dfrac{cu(x+h)-cu(x)}{h}$
  $= \lim_{h\to 0} c\left[\dfrac{u(x+h)-u(x)}{h}\right]$
  $= c\lim_{h\to 0} \dfrac{u(x+h)-u(x)}{h}$
  $= cu'(x)$

- If $f(x) = u(x)+v(x)$,
  then $f'(x) = u'(x)+v'(x)$

  Proof:

  $f'(x)$
  $= \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
  $= \lim_{h\to 0} \left(\dfrac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right)$
  $= \lim_{h\to 0} \left(\dfrac{u(x+h)-u(x)+v(x+h)-v(x)}{h}\right)$
  $= \lim_{h\to 0} \dfrac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \dfrac{v(x+h)-v(x)}{h}$
  $= u'(x)+v'(x)$

Using the rules we have now developed, we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$f'(x) = 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0$$
$$= 12x^3 + 6x^2 - 10x + 7$$

Example 1 Self Tutor

Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$ **b** $7x - \frac{4}{x} + \frac{3}{x^3}$

a
$$f(x) = 5x^3 + 6x^2 - 3x + 2$$
$$\therefore\ f'(x) = 5(3x^2) + 6(2x) - 3(1)$$
$$= 15x^2 + 12x - 3$$

b
$$f(x) = 7x - \frac{4}{x} + \frac{3}{x^3}$$
$$= 7x - 4x^{-1} + 3x^{-3}$$
$$\therefore\ f'(x) = 7(1) - 4(-1x^{-2}) + 3(-3x^{-4})$$
$$= 7 + 4x^{-2} - 9x^{-4}$$
$$= 7 + \frac{4}{x^2} - \frac{9}{x^4}$$

Example 2 Self Tutor

Find the gradient function for:

a $f(x) = 3\sqrt{x} + \frac{2}{x}$ **b** $g(x) = x^2 - \frac{4}{\sqrt{x}}$

a
$$f(x) = 3\sqrt{x} + \frac{2}{x}$$
$$= 3x^{\frac{1}{2}} + 2x^{-1}$$
$$\therefore\ f'(x) = 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2})$$
$$= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2}$$
$$= \frac{3}{2\sqrt{x}} - \frac{2}{x^2}$$

b
$$g(x) = x^2 - \frac{4}{\sqrt{x}}$$
$$= x^2 - 4x^{-\frac{1}{2}}$$
$$\therefore\ g'(x) = 2x - 4(-\tfrac{1}{2}x^{-\frac{3}{2}})$$
$$= 2x + 2x^{-\frac{3}{2}}$$
$$= 2x + \frac{2}{x\sqrt{x}}$$

EXERCISE 17A

1 Find $f'(x)$ given that $f(x)$ is:

a x^3	**b** x^8	**c** x^{11}	**d** $6x$
e $2x^3$	**f** $7x^2$	**g** $3x^5$	**h** $5x^6$
i $5x - 2$	**j** $x^2 + 3$	**k** $x^2 + x$	**l** $x^2 + 3x - 5$
m $2x^2 + x - 1$	**n** $3x^2 - 7x + 8$	**o** $4 - 2x^2$	**p** $\frac{1}{2}x^4 - 6x^2$
q $x^3 - 4x^2 + 6x$	**r** $2x^3 + x - 1$	**s** $7 - x - 4x^3$	**t** $\frac{1}{5}x^3 - \frac{7}{2}x^2 - 2$

2 Differentiate with respect to x:

a $\frac{1}{x^2}$ **b** $\frac{1}{x^5}$ **c** $\frac{3}{x}$ **d** $\frac{4}{x^3}$

e $-\frac{7}{x^4}$ **f** $2x + \frac{3}{x^2}$ **g** $x^2 - \frac{6}{x}$ **h** $9 - \frac{2}{x^3}$

i $\frac{2}{x^2} + \frac{9}{x^4}$ **j** $3x - \frac{1}{x} + \frac{2}{x^2}$ **k** $5 - \frac{8}{x^2} + \frac{4}{x^3}$ **l** $\frac{1}{5x^2}$

m $4x - \frac{1}{4x}$ **n** $\frac{x^2 - 3}{x}$ **o** $\frac{x^3 + 4}{x}$ **p** $\frac{2x - 5}{x^2}$

3 Find the gradient function for $f(x)$ where $f(x)$ is:

a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $\frac{1}{\sqrt{x}}$ **d** $x^3 - \frac{1}{2}\sqrt{x}$

e $\frac{1}{x^2} + 6\sqrt{x}$ **f** $2x - \sqrt{x}$ **g** $x\sqrt{x}$ **h** $\frac{1}{x\sqrt{x}}$

i $2x^2 - \frac{3}{\sqrt{x}}$ **j** $\frac{\sqrt{x} - 4}{x}$ **k** $\frac{x + 5}{\sqrt{x}}$ **l** $\frac{7 - x^2}{\sqrt{x}}$

m $3x^2 - x\sqrt{x}$ **n** $\frac{4}{x^2\sqrt{x}}$ **o** $2x - \frac{3}{x\sqrt{x}}$ **p** $\frac{x^2 - x + 2}{\sqrt[3]{x}}$

4 Find $\frac{dy}{dx}$ for:

a $y = \pi x^2$ **b** $y = 3x^2 - \frac{8}{x^2}$ **c** $y = 6\sqrt{x} + \frac{5}{x}$

d $y = 4\pi x^3$ **e** $y = 2.5x^3 - 1.4x^2 - 1.3$ **f** $y = 10(x + 1)$

g $y = (x + 1)(x - 2)$ **h** $y = (2x + 1)(3x - 2)$ **i** $y = (5 - x)^2$

j $y = (2x - 1)^2$ **k** $y = x(x + 1)(2x - 5)$ **l** $y = \frac{(x - 3)^2}{\sqrt{x}}$

5 Use a binomial expansion to find the derivative of:

a $f(x) = (1 - x)^3$ **b** $f(x) = \left(3x - \frac{1}{\sqrt{x}}\right)^3$

6 Find:

a $\frac{dy}{dt}$ for $y = \frac{1}{2}t^4 - \frac{1}{3}t$ **b** $\frac{dy}{dt}$ for $y = 7 - \frac{6}{\sqrt{t}}$

c $\frac{dT}{dt}$ for $T = \sqrt[3]{t} - \frac{2}{t^2}$ **d** $\frac{dP}{du}$ for $P = \frac{5}{u} - 10u\sqrt{u}$

Example 3 **Self Tutor**

Find the gradient of the tangent to $y = x^2 - \frac{4}{x}$ at the point where $x = 2$.

$y = x^2 - \frac{4}{x}$
$= x^2 - 4x^{-1}$

$\therefore \frac{dy}{dx} = 2x - 4(-1x^{-2})$
$= 2x + 4x^{-2}$
$= 2x + \frac{4}{x^2}$

When $x = 2$, $\frac{dy}{dx} = 4 + 1 = 5$. So, the tangent has gradient 5.

7 Find the gradient of the tangent to:

a $y = x^2$ at $x = 2$

b $y = x^3 - 5x + 2$ at the point $(3, 14)$

c $y = \frac{8}{x^2}$ at the point $\left(9, \frac{8}{81}\right)$

d $y = 2x^2 - 3x + 7$ at $x = -1$

e $y = 3\sqrt{x}$ at the point $(1, 3)$

f $y = 2x - \frac{5}{x}$ at the point $\left(2, \frac{3}{2}\right)$

g $y = \frac{x^2 - 4}{x^2}$ at the point $\left(4, \frac{3}{4}\right)$

h $y = \frac{x^3 - 4x - 8}{x^2}$ at $x = -1$

8 Find constants b and c such that:

a $f(x) = x^2 + (b+1)x + 2c$, $f(2) = 4$, and $f'(-1) = 2$

b $f(x) = bx + \frac{c}{x}$, $f(3) = 5$, and $f'(1) = 5$.

9 Determine whether $f(x) = \begin{cases} 4x^2 - 3, & x \geqslant 2 \\ x^3 + 2x + 1, & x < 2 \end{cases}$ is differentiable at $x = 2$.

Example 4 **Self Tutor**

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found.
It is also the instantaneous rate of change of y with respect to x.

10 If $y = 4x - \frac{3}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

11 Consider the function $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$.

a State the domain of $f(x)$.

b Find the derivative function $f'(x)$.

c State the domain of $f'(x)$.

d Find $f'(1)$ and interpret your answer.

12 The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds.

a Find $\frac{dS}{dt}$ and interpret its meaning.

b Find the value of $\frac{dS}{dt}$ when $t = 3$, and interpret your answer.

13 The cost of producing x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ pounds. Find the value of $\frac{dC}{dx}$ when $x = 1000$, and interpret its meaning.

14 Suppose that a change δx in the variable x produces a change δy in the variable y.

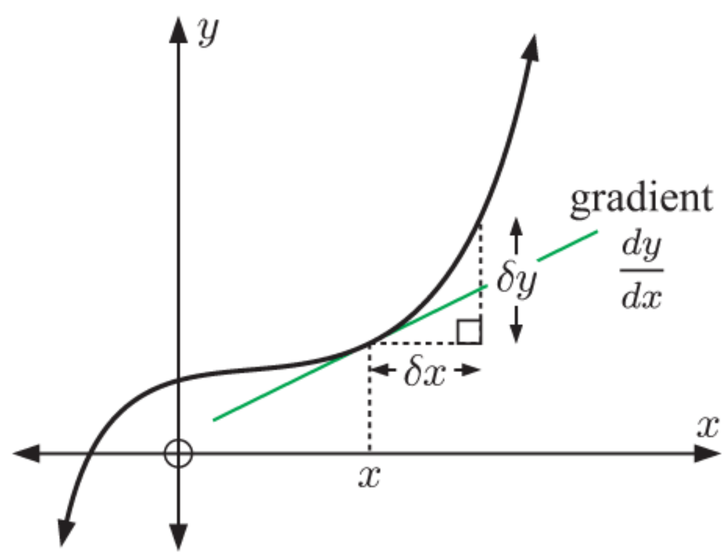
a Use the diagram to explain why $\delta y \approx \frac{dy}{dx} \times \delta x$.

b Explain why the formula only provides accurate estimates for small values of δx.

c Use the formula to estimate the value of:

i 5.01^2 **ii** 2.01^6 **iii** 2.98^3 **iv** 1.95^4

Use your calculator to check your estimates.



B THE CHAIN RULE

In the **Core Topics HL** book we saw that the **composite** of two functions g and f is $(g \circ f)(x)$ or $g(f(x))$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y = (x^2 + 3x)^4$ could be rewritten as $y = u^4$ where $u = x^2 + 3x$, or as $y = g(f(x))$ where $g(x) = x^4$ and $f(x) = x^2 + 3x$.

Example 5 **Self Tutor**

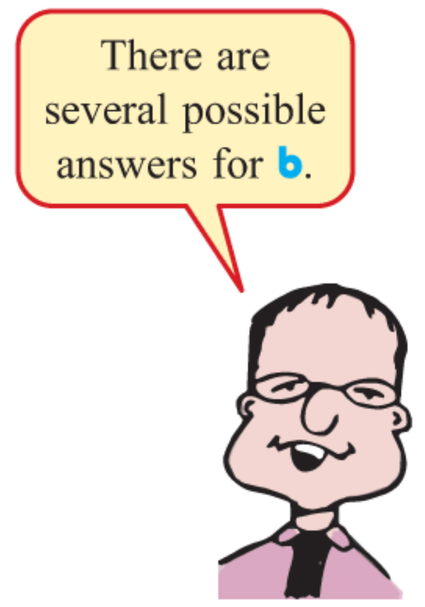
Find:

a $g(f(x))$ if $g(x) = \sqrt{x}$ and $f(x) = 2 - 3x$

b $g(x)$ and $f(x)$ such that $g(f(x)) = \dfrac{1}{x - x^2}$.

a $g(f(x)) = g(2 - 3x)$
$= \sqrt{2 - 3x}$

b If we let $f(x) = x - x^2$ then
$g(f(x)) = \dfrac{1}{f(x)}$
$\therefore \ g(x) = \dfrac{1}{x}$ and $f(x) = x - x^2$



EXERCISE 17B.1

1 Find $g(f(x))$ if:

a $g(x) = x^2$ and $f(x) = 2x + 7$ **b** $g(x) = 2x + 7$ and $f(x) = x^2$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$ **d** $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$ **f** $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $g(f(x))$ is:

a $(3x + 10)^3$ **b** $(7 - 2x)^5$ **c** $\dfrac{1}{2x + 4}$

d $\sqrt{x^2 - 3x}$ **e** $\dfrac{1}{(5x - 1)^4}$ **f** $\dfrac{10}{(3x - x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

INVESTIGATION 2 DIFFERENTIATING COMPOSITE FUNCTIONS

In this Investigation we want to learn how to differentiate composite functions.

Based on the rule "if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x+1)^2$ then $\frac{dy}{dx} = 2(2x+1)^1$. But is this so?

What to do:

1 Expand $y = (2x+1)^2$, and hence find $\frac{dy}{dx}$. How does this compare with $2(2x+1)^1$?

2 Expand $y = (3x+1)^2$, and hence find $\frac{dy}{dx}$. How does this compare with $2(3x+1)^1$?

3 Expand $y = (ax+1)^2$ where a is a constant, and hence find $\frac{dy}{dx}$. How does this compare with $2(ax+1)^1$?

4 Suppose $y = u^2$.

a Find $\frac{dy}{du}$.

b Now suppose $u = ax+1$, so $y = (ax+1)^2$.

i Find $\frac{du}{dx}$.

ii Write $\frac{dy}{du}$ from **a** in terms of x.

iii Hence find $\frac{dy}{du} \times \frac{du}{dx}$.

iv Compare your answer to the result in **3**.

c If $y = u^2$ where u is a function of x, what do you suspect $\frac{dy}{dx}$ will be equal to?

5 Expand $y = (x^2+3x)^2$ and hence find $\frac{dy}{dx}$.

Does your answer agree with the rule you suggested in **4 c**?

6 Consider $y = (2x+1)^3$.

a Expand the brackets and hence find $\frac{dy}{dx}$.

b If we let $u = 2x+1$, then $y = u^3$.

i Find $\frac{du}{dx}$.

ii Find $\frac{dy}{du}$, and write it in terms of x.

iii Hence find $\frac{dy}{du} \times \frac{du}{dx}$.

iv Compare your answer to the result in **a**.

7 Copy and complete: "If y is a function of u, and u is a function of x, then $\frac{dy}{dx} =$ ....."

THE CHAIN RULE

If $y = g(u)$ where $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

This rule is extremely important and allows us to differentiate complicated functions much faster.

For example, for any function $f(x)$:

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$.

Example 6 — Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y = (x^2 - 2x)^4$

b $y = \dfrac{4}{\sqrt{1-2x}}$

a

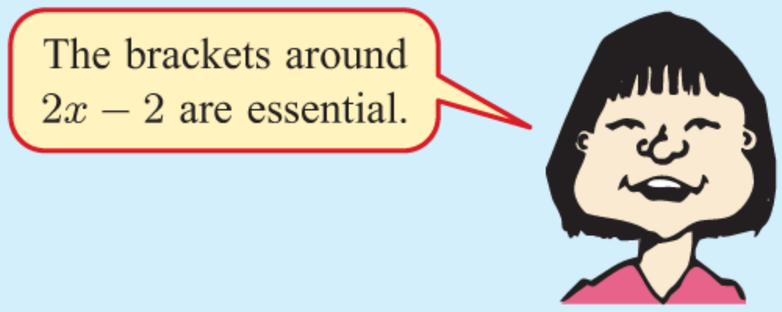
$y = (x^2 - 2x)^4$

$\therefore\ y = u^4$ where $u = x^2 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4u^3(2x-2)$

$= 4(x^2-2x)^3(2x-2)$



b

$y = \dfrac{4}{\sqrt{1-2x}}$

$\therefore\ y = 4u^{-\frac{1}{2}}$ where $u = 1-2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4 \times (-\frac{1}{2}u^{-\frac{3}{2}}) \times (-2)$

$= 4u^{-\frac{3}{2}}$

$= 4(1-2x)^{-\frac{3}{2}}$

EXERCISE 17B.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$ **b** $\sqrt{x^2-3x}$ **c** $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3-x^2}$ **e** $\dfrac{4}{(3-x)^3}$ **f** $\dfrac{10}{x^2-3}$

2 Differentiate $y = (2x+3)^2$ by:

a using the chain rule with $u = 2x+3$

b expanding $y = (2x+3)^2$ then differentiating term-by-term.

3 Find the derivative function $\dfrac{dy}{dx}$ for:

a $y = (4x-5)^2$ **b** $y = \dfrac{1}{5-2x}$ **c** $y = \sqrt{3x-x^2}$

d $y = (1-3x)^4$ **e** $y = 6(5-x)^3$ **f** $y = \sqrt[3]{2x^3-x^2}$

g $y = \dfrac{6}{(5x-4)^2}$ **h** $y = (x^2-5x+8)^5$ **i** $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

4 Find the gradient of the tangent to:

a $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$ **b** $y = (3x+2)^6$ at $x = -1$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$ **d** $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

e $y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$ **f** $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$.

GRAPHING PACKAGE

Check your answers using technology.

5 The gradient function of $f(x) = (2x-b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

6 Suppose $y = \dfrac{a}{\sqrt{1+bx}}$ where a and b are constants. When $x = 3$, $y = 1$ and $\dfrac{dy}{dx} = -\frac{1}{8}$.
Find a and b.

7 Suppose $f(x) = 3\left(ax - \dfrac{b}{x}\right)^3$. Given that $f(\frac{3}{2}) = 3$ and $f'(\frac{3}{2}) = 30$, find a and b.

8 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$, and hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

b Explain why $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

C THE PRODUCT RULE

We have seen the addition rule which allows us to differentiate term-by-term:

$$\text{If } f(x) = u(x) + v(x) \text{ then } f'(x) = u'(x) + v'(x).$$

If we now consider the case $f(x) = u(x)\,v(x)$, we might wonder if $f'(x) = u'(x)\,v'(x)$.

In other words, does the derivative of a product of two functions equal the product of their derivatives?

INVESTIGATION 3 THE PRODUCT RULE

Suppose $u(x)$ and $v(x)$ are two functions of x, and that $f(x) = u(x)\,v(x)$ is the product of these functions.

In this Investigation we attempt to find a rule for determining $f'(x)$.

What to do:

1 Suppose $u(x) = x$ and $v(x) = x$, so $f(x) = x^2$.

a Find $f'(x)$ by direct differentiation. **b** Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

2 Suppose $u(x) = x$ and $v(x) = \sqrt{x}$, so $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$.

a Find $f'(x)$ by direct differentiation. **b** Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

3 Copy and complete the following table, finding $f'(x)$ by direct differentiation:

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x) + u(x)\,v'(x)$
x^2		x	x			
$x^{\frac{3}{2}}$		x	$\sqrt{x}$			
$x(x+1)$		x	$x+1$			
$(x-1)(2-x^2)$		$x-1$	$2-x^2$			

4 Copy and complete: "If $f(x) = u(x)\,v(x)$ then $f'(x) = \ldots\ldots$"

THE PRODUCT RULE

If $f(x) = u(x)\,v(x)$ then $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$.

Alternatively, if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u'v + uv' = \frac{du}{dx}\,v + u\,\frac{dv}{dx}$$

Example 7 — Self Tutor

Find $\frac{dy}{dx}$ if: **a** $y = \sqrt{x}(2x+1)^3$ **b** $y = x^2(x^2-2x)^4$

a $y = \sqrt{x}(2x+1)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (2x+1)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(2x+1)^2 \times 2$ {chain rule}

$= 6(2x+1)^2$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + x^{\frac{1}{2}} \times 6(2x+1)^2$

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2$



b $y = x^2(x^2-2x)^4$ is the product of

$u = x^2$ and $v = (x^2-2x)^4$

$\therefore\ u' = 2x$ and $v' = 4(x^2-2x)^3(2x-2)$ {chain rule}

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(x^2-2x)^4 + x^2 \times 4(x^2-2x)^3(2x-2)$

$= 2x(x^2-2x)^4 + 4x^2(x^2-2x)^3(2x-2)$

EXERCISE 17C

1 Use the product rule to differentiate:

a $f(x) = x(x-1)$ **b** $f(x) = 2x(x+1)$ **c** $f(x) = x^2\sqrt{x+1}$

d $f(x) = (x+3)(x-1)$ **e** $f(x) = x\sqrt{x^2-1}$ **f** $f(x) = x(x+1)^2$

2 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$ **b** $y = 4x(2x+1)^3$ **c** $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$ **e** $y = 5x^2(3x^2-1)^2$ **f** $y = \sqrt{x}(x-x^2)^3$

3 Find the gradient of the tangent to:

a $y = x^4(1-2x)^2$ at $x = -1$ **b** $y = \sqrt{x}(x^2-x+1)^2$ at $x = 4$

c $y = x\sqrt{1-2x}$ at $x = -4$ **d** $y = x^3\sqrt{5-x^2}$ at $x = 1$.

4 Consider $y = \sqrt{x}(3-x)^2$.

a Show that $\frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}}$.

b Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

c State the domain of $\frac{dy}{dx}$. Discuss how it differs from the domain of the original function.

5 Suppose $y = -2x^2(x+4)$. For what values of x does $\frac{dy}{dx} = 10$?

6 Suppose $y = (x+3)(x-2)^2$. For what values of x does $\frac{dy}{dx} = -7$?

7 Find the value of x for which the tangent to $f(x) = ax\sqrt{1-x}$ has gradient:

a 0 **b** a.

8 Find the values of a such that $f(x) = x^2\sqrt{x^2+a}$ and $f'(-2) = -\frac{34}{3}$.

D THE QUOTIENT RULE

Expressions like $\frac{x^2+1}{2x-5}$, $\frac{\sqrt{x}}{1-3x}$, and $\frac{x^3}{(x-x^2)^4}$ are called **quotients** because they represent the division of one expression by another.

Quotient functions have the form $Q(x) = \frac{u(x)}{v(x)}$.

Notice that $u(x) = Q(x)\,v(x)$

$$\therefore \quad u'(x) = Q'(x)\,v(x) + Q(x)\,v'(x) \qquad \{\text{product rule}\}$$

$$\therefore \quad u'(x) - Q(x)\,v'(x) = Q'(x)\,v(x)$$

$$\therefore \quad Q'(x)\,v(x) = u'(x) - \frac{u(x)}{v(x)}\,v'(x)$$

$$\therefore \quad Q'(x)\,v(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)}$$

$$\therefore \quad Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2} \qquad \text{when this exists.}$$

THE QUOTIENT RULE

If $Q(x) = \dfrac{u(x)}{v(x)}$ then $Q'(x) = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$.

Alternatively, if $y = \dfrac{u}{v}$ where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\dfrac{du}{dx}\,v - u\,\dfrac{dv}{dx}}{v^2}$$

Example 8 — Self Tutor

Use the quotient rule to find $\dfrac{dy}{dx}$ if: **a** $y = \dfrac{1+3x}{x^2+1}$ **b** $y = \dfrac{\sqrt{x}}{(1-2x)^2}$

a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore\ u' = 3$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$

$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

If you only need the gradient of a tangent at a given point, you will not need to simplify $\dfrac{dy}{dx}$. In such cases, substitute the value for x into the derivative function immediately.



b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient with

$u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ {chain rule}

$= -4(1-2x)$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{(1-2x)\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^4\ 3}$$

$$= \frac{1 - 2x + 8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

EXERCISE 17D

1 Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{2-x}$ **b** $y = \dfrac{x^2}{2x+1}$ **c** $y = \dfrac{x}{x^2-3}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ **e** $y = \dfrac{x^2-3}{3x-x^2}$ **f** $y = \dfrac{x}{\sqrt{1-3x}}$

2 Find:

a $\dfrac{d}{dx}\left(\dfrac{x+1}{3-x}\right)$ **b** $\dfrac{d}{dx}\left(\dfrac{3x}{x^2-1}\right)$

c $\dfrac{d}{dx}\left(\dfrac{x^3}{2x-1}\right)$ **d** $\dfrac{d}{dx}\left(\dfrac{4x}{\sqrt{x-5}}\right)$

e $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{3-x^2}\right)$ **f** $\dfrac{d}{dx}\left(-\dfrac{x^2}{\sqrt{x^2+3}}\right)$

$\frac{d}{dx}(....)$ reads "the derivative of (....) with respect to x".



3 Find the gradient of the tangent to:

a $y = \dfrac{x}{1-2x}$ at $x = 1$ **b** $y = \dfrac{x^3}{x^2+1}$ at $x = -1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ at $x = 4$ **d** $y = \dfrac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

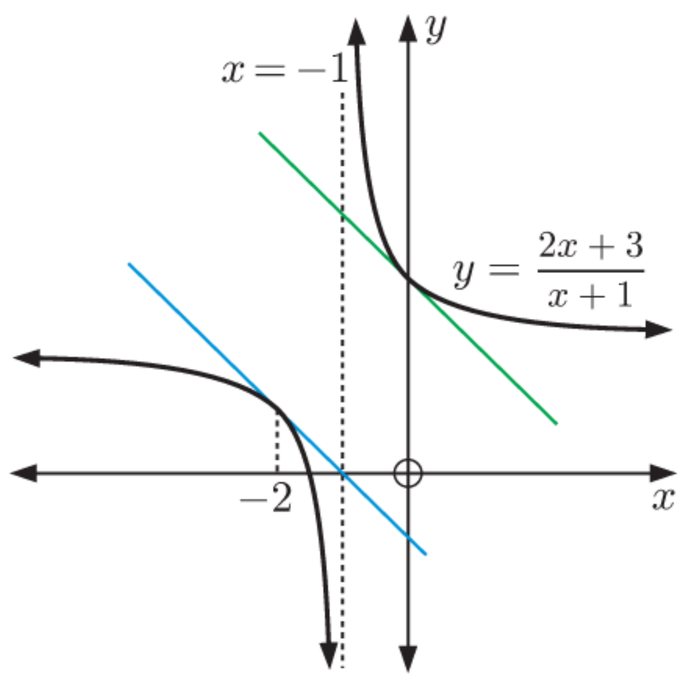
4 Suppose $f(x) = \dfrac{x}{\sqrt{x-1}}$. Find $f'(x)$ using the quotient rule.

Check your answer by writing $f(x) = \dfrac{x-1}{\sqrt{x-1}} + \dfrac{1}{\sqrt{x-1}}$ and then differentiating.

5 Consider the graph of $y = \dfrac{2x+3}{x+1}$ alongside.

a Find $\dfrac{dy}{dx}$.

b Hence show that the illustrated tangents are parallel.



6 **a** If $y = \dfrac{2\sqrt{x}}{1-x}$, show that $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

7 **a** If $y = \dfrac{x^2+6}{2x+1}$, show that $\dfrac{dy}{dx} = \dfrac{2x^2+2x-12}{(2x+1)^2}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

8 **a** If $y = \dfrac{x^2-3x+1}{x+2}$, show that $\dfrac{dy}{dx} = \dfrac{x^2+4x-7}{(x+2)^2}$.

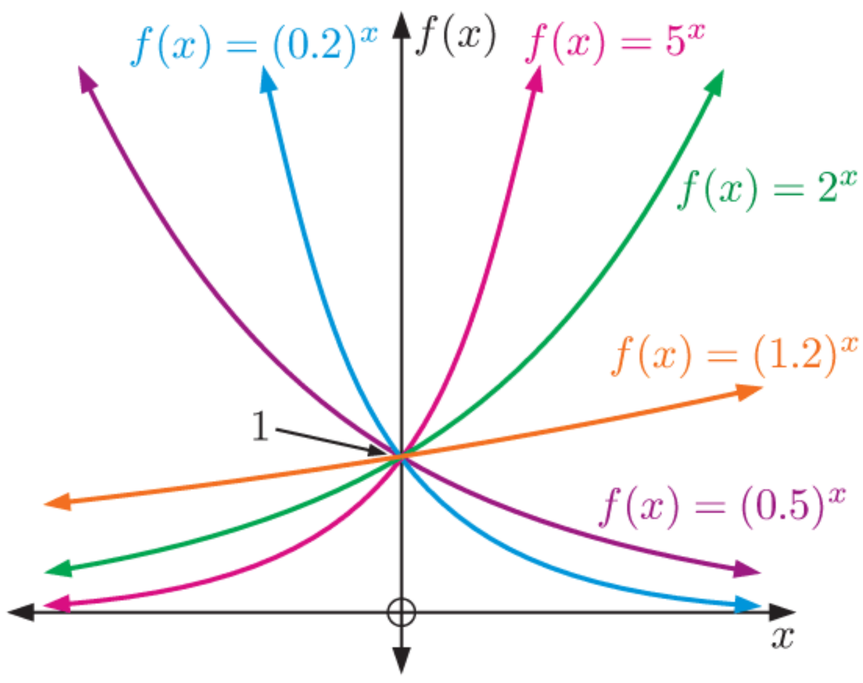
b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

E DERIVATIVES OF EXPONENTIAL FUNCTIONS

We have seen previously that the simplest **exponential functions** have the form $f(x) = b^x$ where b is any positive constant, $b \neq 1$.

The graphs of all members of the exponential family $f(x) = b^x$:

- pass through the point $(0, 1)$
- are asymptotic to the x-axis at one end
- lie above the x-axis for all x.



INVESTIGATION 4 THE DERIVATIVE OF b^x

The purpose of this Investigation is to observe the nature of the derivatives of $f(x) = b^x$ for various values of b.

What to do:

1 Use the software provided to help fill in the table for $y = 2^x$:

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0			
0.5			
1			
1.5			
2			

CALCULUS DEMO



2 Repeat **1** for the following functions:

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = b^x$ for $b > 0$, $b \neq 1$.

From the **Investigation** you should have found that:

If $f(x) = b^x$ then $f'(x) = f'(0) \times b^x$.

Proof:

If $f(x) = b^x$,

then $f'(x) = \lim_{h \to 0} \dfrac{b^{x+h} - b^x}{h}$ {definition of the derivative}

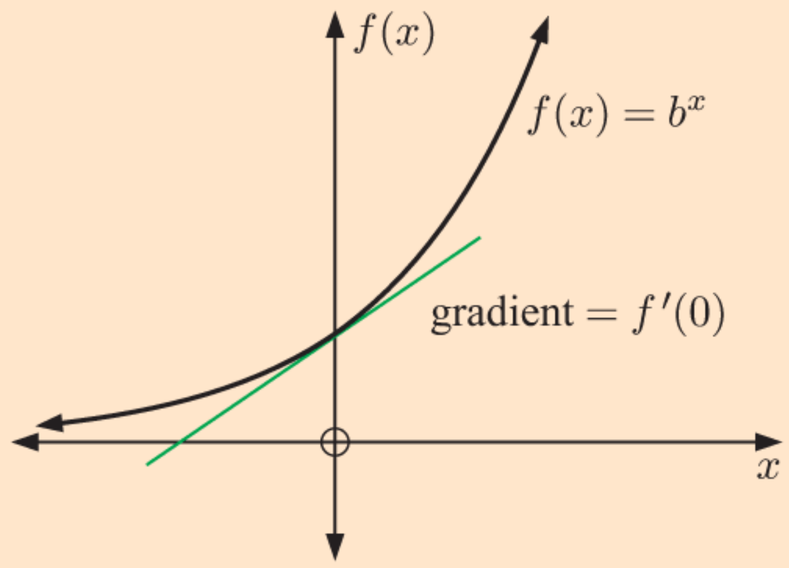
$= \lim_{h \to 0} \dfrac{b^x(b^h - 1)}{h}$

$= b^x \times \left(\lim_{h \to 0} \dfrac{b^h - 1}{h} \right)$ {as b^x is independent of h}

But $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{b^h - 1}{h}$

$\therefore \quad f'(x) = b^x \times f'(0)$



Given this result, if we can find a value of b such that $f'(0) = 1$, then we will have found *a function which is its own derivative*!

INVESTIGATION 5 SOLVING FOR b IF $f(x) = b^x$ AND $f'(x) = b^x$

Click on the icon to graph $f(x) = b^x$ and its derivative function $y = f'(x)$.

Experiment with different values of b until the graphs of $f(x) = b^x$ and $y = f'(x)$ appear the same.

Estimate the corresponding value of b to 3 decimal places.



You should have discovered that $f(x) = f'(x) = b^x$ when $b \approx 2.718$.

To find this value of b more accurately we return to the algebraic approach:

We have already shown that if $f(x) = b^x$ then $f'(x) = b^x \left(\lim_{h \to 0} \dfrac{b^h - 1}{h} \right)$.

So, to find the value of b such that $f'(x) = b^x$, we require $\lim_{h \to 0} \dfrac{b^h - 1}{h} = 1$

$\therefore \quad \lim_{h \to 0} b^h = \lim_{h \to 0} (h+1)$

$\therefore \quad b = \lim_{h \to 0} (1+h)^{\frac{1}{h}}$

Letting $h = \dfrac{1}{n}$, we notice that $h \to 0$ as $n \to \infty$.

$\therefore \quad b = \lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ if this limit exists.

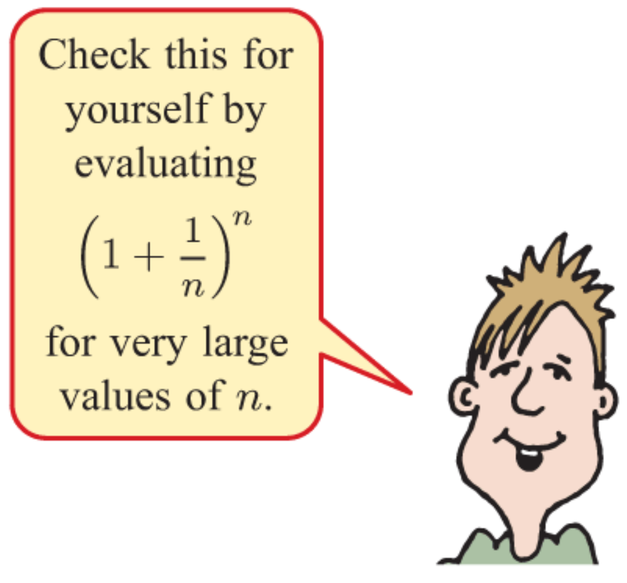
We have in fact already seen this limit previously in an **Investigation** on continuously compounding interest.

We found that as $n \to \infty$,

$$\left(1 + \frac{1}{n}\right)^n \to 2.718\,281\,828\,459\,045\,235\ldots.$$

and this irrational number is the natural exponential e.

We therefore conclude: If $f(x) = e^x$ then $f'(x) = e^x$.



THE DERIVATIVE OF $e^{f(x)}$

The functions e^{-x}, e^{2x+3}, and e^{-x^2} all have the form $e^{f(x)}$.

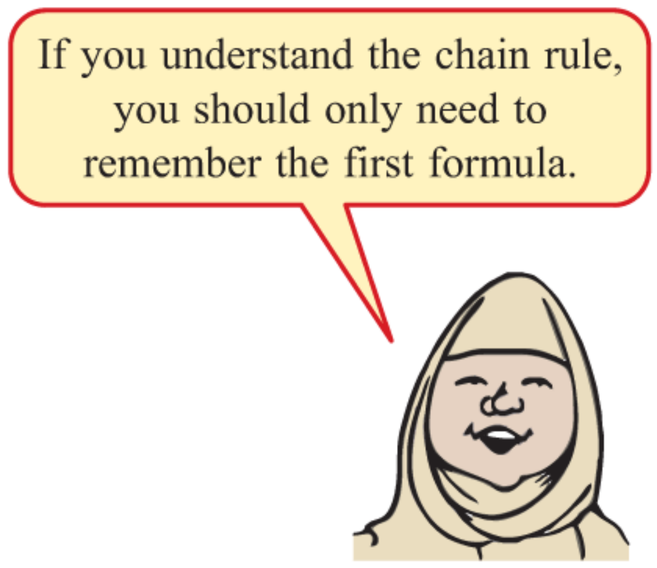
Suppose $y = e^{f(x)} = e^u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= e^u \dfrac{du}{dx}$

$= e^{f(x)} \times f'(x)$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$



Example 9 **Self Tutor**

Find the gradient function for y equal to:

a $2e^x + e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y = 2e^x + e^{-3x}$ then $\dfrac{dy}{dx} = 2e^x + e^{-3x}(-3)$ {addition rule}

$= 2e^x - 3e^{-3x}$

b If $y = x^2e^{-x}$ then $\dfrac{dy}{dx} = 2xe^{-x} + x^2e^{-x}(-1)$ {product rule}

$= 2xe^{-x} - x^2e^{-x}$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}

$= \dfrac{e^{2x}(2x-1)}{x^2}$

EXERCISE 17E.1

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** e^{-2x} **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1 + e^{2x})$ **l** $20(1 - e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $20xe^{-0.5x}$ **h** $\dfrac{e^x + 2}{e^{-x} + 1}$

Example 10 **Self Tutor**

Find the gradient function for y equal to:

a $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a
$$y = (e^x - 1)^3$$
$$= u^3 \quad \text{where} \quad u = e^x - 1$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$= 3u^2 \frac{du}{dx}$$
$$= 3(e^x - 1)^2 \times e^x$$
$$= 3e^x(e^x - 1)^2$$

b
$$y = (2e^{-x} + 1)^{-\frac{1}{2}}$$
$$= u^{-\frac{1}{2}} \quad \text{where} \quad u = 2e^{-x} + 1$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$
$$= -\tfrac{1}{2}u^{-\frac{3}{2}} \frac{du}{dx}$$
$$= -\tfrac{1}{2}(2e^{-x} + 1)^{-\frac{3}{2}} \times 2e^{-x}(-1)$$
$$= e^{-x}(2e^{-x} + 1)^{-\frac{3}{2}}$$

3 Find the gradient function for y equal to:

a $(2 + e^x)^4$ **b** $\sqrt{e^x - 1}$ **c** $(e^x + e^{-x})^{\frac{3}{2}}$ **d** $\dfrac{1}{\sqrt{e^{2x}+2}}$

4 Find the gradient of the tangent to:

a $y = (e^x + 2)^4$ at $x = 0$ **b** $y = \dfrac{1}{2 - e^{-x}}$ at $x = 0$

c $y = \sqrt{e^{2x} + 10}$ at $x = \ln 3$ **d** $y = \dfrac{2 - x}{e^{3x}}$ at $x = 1$.

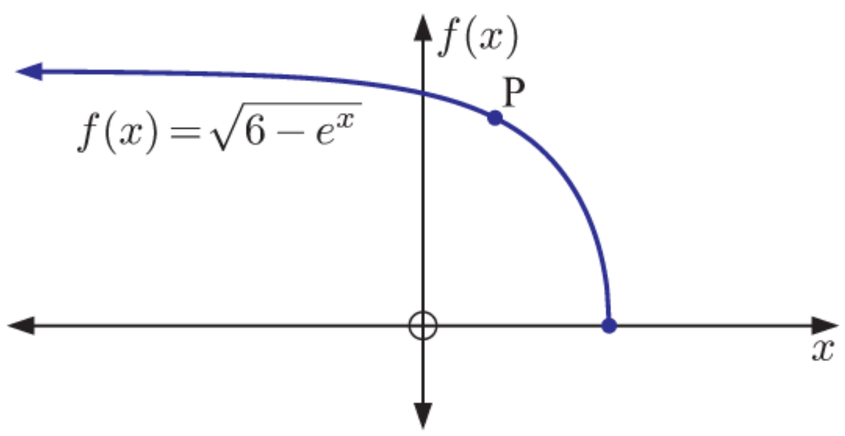
5 The graph of $f(x) = \sqrt{6 - e^x}$ is shown alongside.

a State the domain of the function.

b The point P has y-coordinate 2. Find exactly:

i the coordinates of P

ii the gradient of the tangent at P.



6 Given $f(x) = e^{kx} + x$ and $f'(0) = -8$, find k.

7 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$, find $\dfrac{dy}{dx}$.

b Show that if $y = a^x$ where $a > 0$, $a \neq 1$, then $\dfrac{dy}{dx} = a^x \times \ln a$.

8 The tangent to $f(x) = x^2e^{-x}$ at point P is horizontal. Find the possible coordinates of P.

9 Suppose $S(x) = \frac{1}{2}(e^x - e^{-x})$ and $C(x) = \frac{1}{2}(e^x + e^{-x})$.

a Show that $[C(x)]^2 - [S(x)]^2 = 1$. **b** Show that $\dfrac{d}{dx}[S(x)] = C(x)$.

c Find $\dfrac{d}{dx}[C(x)]$ in terms of $S(x)$.

d If $T(x) = \dfrac{S(x)}{C(x)}$, find $\dfrac{d}{dx}[T(x)]$ in terms of $C(x)$.

THE DERIVATIVE OF a^x

In the previous Exercise you should have shown that:

> If $y = a^x$ where $a > 0$, then $\dfrac{dy}{dx} = a^x \ln a$.

Proof:

Consider $y = a^x$ where $a > 0$

$\therefore \quad y = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$

$\therefore \quad \dfrac{dy}{dx} = e^{(\ln a)x} \times \ln a = a^x \ln a$

THE DERIVATIVE OF $a^{f(x)}$

Suppose $y = a^{f(x)}$

$\therefore \quad y = a^u$ where $u = f(x)$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= a^u \ln a \dfrac{du}{dx}$

$= a^{f(x)} \ln a \times f'(x)$

Function	*Derivative*
a^x	$a^x \ln a$
$a^{f(x)}$	$a^{f(x)} \ln a \times f'(x)$

Example 11 — Self Tutor

Find $\dfrac{dy}{dx}$ for:

a $y = 3^x$ **b** $y = 5^{x^2-x}$

a $y = 3^x$

$\therefore \quad \dfrac{dy}{dx} = 3^x \ln 3$

b $y = 5^{x^2-x}$

$\therefore \quad \dfrac{dy}{dx} = 5^{x^2-x} \ln 5 \times (2x - 1)$

EXERCISE 17E.2

1 Find $\dfrac{dy}{dx}$ for:

a $y = 2^x$ **b** $y = 5^x$ **c** $y = 3^x + 2 \times 7^x + e^x$

d $y = x\,2^x$ **e** $y = \dfrac{2^x}{x}$ **f** $y = \dfrac{x}{3^x}$

2 Find $\dfrac{dy}{dx}$ for:

a $y = 4^{-x}$ **b** $y = 2^{x+3} + 3^{2x}$ **c** $y = 5^{x^3+x}$

d $y = x^3\,6^{-x}$ **e** $y = \dfrac{3x}{2^{5x-1}}$ **f** $y = \dfrac{e^{2x} - 3\sqrt{x}}{5^{x^2}}$

3 Find the gradient of the tangent to:

a $f(x) = 4^x$ at $x = 2$ **b** $f(x) = 3^{-x}$ at $x = -1$

c $f(x) = 3 \times 2^{3x}$ at $y = 10$ **d** $f(x) = x^2\,5^{1-x}$ at $x = 3$.

4 The tangent to $f(x) = 2^x + 4^x$ at point P has gradient $\ln 2$. Find the coordinates of P.

F DERIVATIVES OF LOGARITHMIC FUNCTIONS

INVESTIGATION 6 THE DERIVATIVE OF $\ln x$

What to do:

CALCULUS DEMO

1 Click on the icon to see the graph of $y = \ln x$. Observe the gradient function being drawn as the point moves from left to right along the graph.

2 Predict a formula for the gradient function of $y = \ln x$.

3 Find the gradient of the tangent to $y = \ln x$ for $x = 0.25,\ 0.5,\ 1,\ 2,\ 3,\ 4,$ and 5.
Do your results confirm your prediction in **2**?

From the **Investigation** you should have observed:

PROOF

$$\text{If } \ y = \ln x \ \text{ then } \ \frac{dy}{dx} = \frac{1}{x}, \ \ x > 0.$$

You can obtain a proof for this result by clicking on the icon. However, we will soon be able to prove the result much more easily when we study implicit differentiation.

THE DERIVATIVE OF $\ln f(x)$

Suppose $y = \ln f(x)$

$\therefore\ y = \ln u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$

$= \dfrac{f'(x)}{f(x)}$

Function	*Derivative*
$\ln x$	$\dfrac{1}{x}, \ \ x > 0$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}, \ \ f(x) > 0$

Example 12 Self Tutor

Find the gradient function of:

a $y = \ln(1 - 3x)$ **b** $y = x^3 \ln x$

a $y = \ln(1 - 3x)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-3}{1 - 3x}$

$= \dfrac{3}{3x - 1}$

b $y = x^3 \ln x$

$\therefore\ \dfrac{dy}{dx} = 3x^2 \ln x + x^3\left(\dfrac{1}{x}\right)$ {product rule}

$= 3x^2 \ln x + x^2$

$= x^2(3 \ln x + 1)$

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

For $a > 0,\ b > 0,\ n \in \mathbb{R}$:

$$\ln(ab) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^n) = n \ln a$$

Example 13 **Self Tutor**

Differentiate with respect to x:

a $y = \ln(xe^{-x})$ **b** $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} && \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x && \{\ln e^a = a\} \end{aligned}$$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{x} - 1$

A derivative function will only be valid on *at most* the domain of the original function.



b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] && \left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \end{aligned}$$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{x} - \dfrac{1}{x+2} - \dfrac{1}{x-3}$

EXERCISE 17F.1

1 Find the gradient function of:

a $y = \ln(7x)$ **b** $y = \ln(2x+1)$ **c** $y = \ln(x - x^2)$

d $y = 3 - 2\ln x$ **e** $y = x^2 \ln x$ **f** $y = \dfrac{\ln x}{2x}$

g $y = e^x \ln x$ **h** $y = (\ln x)^2$ **i** $y = \sqrt{\ln x}$

j $y = e^{-x} \ln x$ **k** $y = \sqrt{x}\ln(2x)$ **l** $y = \dfrac{2\sqrt{x}}{\ln x}$

m $y = 3 - 4\ln(1-x)$ **n** $y = x\ln(x^2+1)$ **o** $y = \dfrac{\ln x}{x^2}$

2 **a** Given $f(x) = \ln(kx)$ where $k \neq 0$, find $f'(x)$.

b Use the laws of logarithms to explain why this derivative does not depend on the value of k.

3 Find $\dfrac{dy}{dx}$ for:

a $y = x\ln 5$ **b** $y = \ln(x^3)$ **c** $y = \ln(x^4 + x)$

d $y = \ln(10 - 5x)$ **e** $y = [\ln(2x+1)]^3$ **f** $y = \dfrac{\ln(4x)}{x}$

g $y = \ln\left(\dfrac{1}{x}\right)$ **h** $y = \ln(\ln x)$ **i** $y = \dfrac{1}{\ln x}$

4 Use the laws of logarithms to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$ **b** $y = \ln\left(\dfrac{1}{2x+3}\right)$ **c** $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$ **e** $y = \ln\left(\dfrac{x+3}{x-1}\right)$ **f** $y = \ln\left(\dfrac{x^2}{3-x}\right)$

5 Differentiate with respect to x:

a $f(x) = \ln\left((3x-4)^3\right)$ **b** $f(x) = \ln\left(x(x^2+1)\right)$

c $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$ **d** $f(x) = \ln\left(\dfrac{x^3}{(x+4)(x-1)}\right)$

6 Find the gradient of the tangent to $y = x\ln x$ at the point where $x = e$.

7 Suppose $f(x) = a\ln(bx^2)$ where $f(e) = 3$ and $f'(1) = 6$. Find the constants a and b.

8 Suppose $f(x) = ax\ln(bx)$ where $f(1) = 12$ and $f'(1) = 16$. Find the constants a and b.

9 Find the point(s) at which the tangent to $y = \ln(15 - x^2)$ has gradient 1.

10 The function $f(x) = \begin{cases} \ln(2x+a), & x \geqslant 0 \\ bx, & x < 0 \end{cases}$ is differentiable at $x = 0$.

Find the values of a and b.

THE DERIVATIVE OF $\log_a x$

To differentiate $\log_a x$, we first use the change of base rule to write the expression with natural logarithms.

Let $y = \log_a x = \dfrac{\ln x}{\ln a}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\ln a} \times \frac{1}{x} = \frac{1}{x\ln a}$$

> If $y = \log_a x$ where $a > 0$, then $\dfrac{dy}{dx} = \dfrac{1}{x\ln a}$.

THE DERIVATIVE OF $\log_a f(x)$

Suppose $y = \log_a f(x)$

$\therefore \quad y = \log_a u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$\therefore \quad \frac{dy}{dx} = \frac{1}{u\ln a}\frac{du}{dx} = \frac{f'(x)}{f(x)\ln a}$$

Function	*Derivative*
$\log_a x$	$\dfrac{1}{x\ln a}$
$\log_a f(x)$	$\dfrac{f'(x)}{f(x)\ln a}$

Example 14 Self Tutor

Find $\dfrac{dy}{dx}$ for:

a $y = \log_5 x$ **b** $y = \log_3(x^2 - x)$

a $y = \log_5 x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x\ln 5}$$

b $y = \log_3(x^2 - x)$

$$\therefore \quad \frac{dy}{dx} = \frac{2x-1}{(x^2-x)\ln 3}$$

EXERCISE 17F.2

1 Find $f'(x)$ for:

a $f(x) = \log_2 x$ **b** $f(x) = \log_{10} x$ **c** $f(x) = \log_3 x - \frac{1}{2}\log_5 x$

d $f(x) = x \log_3 x$ **e** $f(x) = \dfrac{\log_5 x}{x^2 - 3x}$ **f** $f(x) = (\log_2 x)^4$

2 Find $\dfrac{dy}{dx}$ for:

a $y = \log_2(3x - 1)$ **b** $y = \log_5(4x^2 + 3x)$ **c** $y = 3^x \log_3(2x)$

d $y = \log_5\left(\dfrac{2}{\sqrt{x}}\right)$ **e** $y = \log_2(\log_3 x)$ **f** $y = \dfrac{\log(4 - x)}{x^2 - 1}$

3 Find the gradient of the tangent to:

a $y = \log_2(x - 3)$ at $x = 5$ **b** $y = x^2 \log_3(2x)$ at $x = \frac{9}{2}$

c $y = \dfrac{\log_2(\sqrt{x})}{e^x}$ at $x = 2$ **d** $y = \frac{1}{2}\log_5(5^x + 4)$ at $y = 1$.

4 Use the laws of logarithms to help differentiate the following with respect to x:

a $y = \log_3\left(\dfrac{1}{x^2}\right)$ **b** $y = \log_2[(x + 5)(x - 1)]$

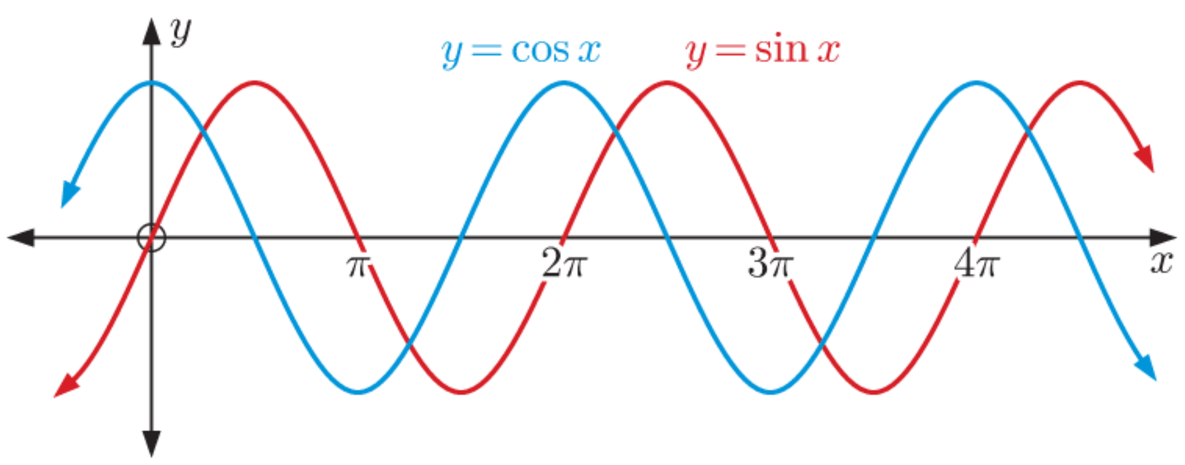
c $y = \log(xe^x)$ **d** $y = \log_5\left[\dfrac{1}{(2x - 1)(x + 2)}\right]$

e $y = \log_2\left(\dfrac{x^2 - x}{x + 3}\right)$ **f** $y = \log_4\left[(\sqrt{x} - 1)^3\right]$

5 Suppose $f(x) = \log_3(ax + b)$ where $f(2) = -1$ and $f'(3) = \dfrac{3}{4 \ln 3}$. Find the constants a and b.

G DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

We have seen previously that the sine and cosine curves naturally arise from circular motion.



INVESTIGATION 7 DERIVATIVES OF $\sin x$ AND $\cos x$

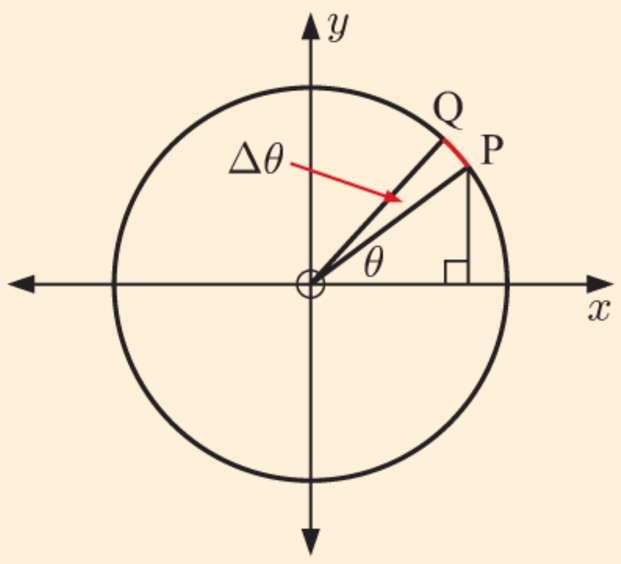
What to do:

1 Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.

2 Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.



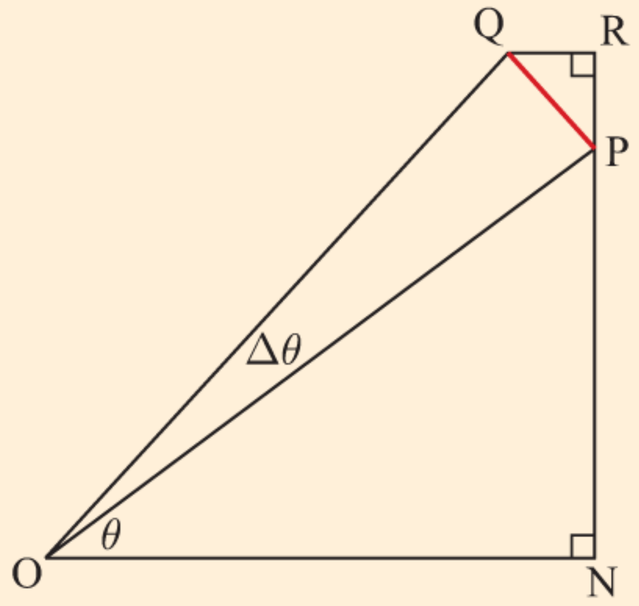
3 Suppose P and Q are points on the unit circle corresponding to angles θ and $\theta + \Delta\theta$ respectively from the positive x-axis.



a Explain why $PR = \sin(\theta + \Delta\theta) - \sin\theta$.

b If P and Q are close together then $\Delta\theta$ is very small. Explain why as Q approaches P:

- **i** the arc PQ resembles line segment [PQ]
- **ii** the length of the line segment $PQ \approx \Delta\theta$
- **iii** $Q\widehat{P}O$ approaches a right angle
- **iv** $Q\widehat{P}R \approx \theta$.



c Use right angled triangle trigonometry in $\triangle QRP$ to show that $\cos\theta \approx \dfrac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}$.

Hence explain why in the limit as $\Delta\theta \to 0$, $\cos\theta = \dfrac{d}{d\theta}(\sin\theta)$.

From the **Investigation** you should have deduced that:

For x in radians:

- If $f(x) = \sin x$ then $f'(x) = \cos x$.
- If $f(x) = \cos x$ then $f'(x) = -\sin x$.

Both of these results were proven in **Chapter 16** from first principles.

THE DERIVATIVE OF $\tan x$

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$

We let $u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$

$\therefore \dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$ {since $\sin^2 x + \cos^2 x = 1$}

$= \sec^2 x$

DERIVATIVES DEMO

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$, AND $\tan[f(x)]$

Suppose $y = \sin[f(x)]$

$\therefore\ y = \sin u$ where $u = f(x)$.

But $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = \cos u \times f'(x)$

$= \cos[f(x)] \times f'(x)$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the following results:

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)]\, f'(x)$

Example 15

Self Tutor

Differentiate with respect to x:

a $x \sin x$ **b** $4\tan^2 3x$

a

$y = x \sin x$

$\therefore\ \dfrac{dy}{dx} = (1)\sin x + (x)\cos x$ {product rule}

$= \sin x + x\cos x$

b

$y = 4\tan^2 3x$

$= 4u^2$ where $u = \tan 3x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = 8u \times \dfrac{du}{dx}$

$= 8\tan 3x \times \dfrac{3}{\cos^2 3x}$

$= \dfrac{24\sin 3x}{\cos^3 3x}$

EXERCISE 17G.1

1 Find $\dfrac{dy}{dx}$ for:

a $y = \sin 2x$ **b** $y = \sin x + \cos x$ **c** $y = \cos 3x - \sin x$

d $y = \sin(x+1)$ **e** $y = \cos(3-2x)$ **f** $y = \tan 5x$

g $y = \sin\frac{x}{2} - 3\cos x$ **h** $y = 4\sin x - \cos 2x$ **i** $y = \frac{1}{2}\cos 6x - 5\sin 4x$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x \cos x$

d $e^{-x}\sin x$ **e** $\ln(\sin x)$ **f** $e^{2x}\tan x$

g $3\tan \pi x$ **h** $\cos\frac{x}{2}$ **i** $3\tan 2x$

j $x\cos x$ **k** $\dfrac{\sin x}{x}$ **l** $x\tan x$

m $3^x \sin 2x$ **n** $\log(\cos 2x)$ **o** $\dfrac{\cos x + \sin 2x}{x^3}$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos\sqrt{x}$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$

e $\cos^3 x$ **f** $\cos x \sin 2x$ **g** $\cos(\cos x)$ **h** $\cos^3 4x$

4 Differentiate $\operatorname{cosec} x = \dfrac{1}{\sin x}$ with respect to x using:

a the quotient rule **b** the chain rule.

5 Differentiate with respect to x:

a $\sec x = \dfrac{1}{\cos x}$ **b** $\cot x = \dfrac{1}{\tan x}$

6 Find the gradient of the tangent to:

a $f(x) = \sin^3 x$ at the point where $x = \frac{2\pi}{3}$

b $f(x) = \cos x \sin x$ at the point where $x = \frac{\pi}{4}$.

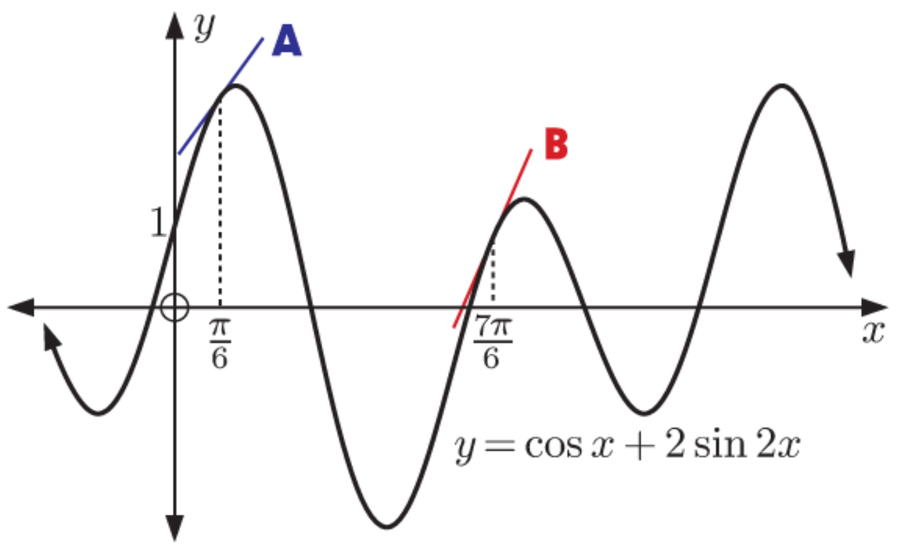
7 Consider the function $f(x) = 2\cos^2 x + 2\sin^2 x + 1$.

a Find $f'(x)$. **b** Explain your answer to **a**.

8 The graph of $y = \cos x + 2\sin 2x$ is shown alongside.

a Which tangent appears to have the steeper gradient?

b Find $\dfrac{dy}{dx}$, and hence check your answer to **a**.



9 Find a and n such that $f(x) = \begin{cases} ax^n, & x < \frac{\pi}{4} \\ \tan^2 x, & \frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2} \end{cases}$ is differentiable at $x = \frac{\pi}{4}$.

10 Consider $f(x) = \begin{cases} \sin x, & x \geqslant 0 \\ x^2 + 5x, & x < 0. \end{cases}$

a Prove that f is continuous but not differentiable at $x = 0$.

b Write down $f'(x)$ as a piecewise defined function.

11 **a** Show that $\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}}$ is a constant by showing that its derivative is zero.

b Find the value of this constant by substituting $\theta = 0$.

c Hence establish Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

THE DERIVATIVES OF cosec x, sec x, AND cot x

In the previous Exercise, you should have found these derivatives for the reciprocal trigonometric functions:

Function	*Derivative*
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

Example 16 **Self Tutor**

Find $\frac{dy}{dx}$ for: **a** $y = \operatorname{cosec} 3x$ **b** $y = \sqrt{\cot \frac{x}{2}}$

a

$y = \operatorname{cosec} 3x$

$\therefore \frac{dy}{dx} = -\operatorname{cosec} 3x \cot 3x \frac{d}{dx}(3x)$

$= -3 \operatorname{cosec} 3x \cot 3x$

b

$y = \left(\cot \frac{x}{2}\right)^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{1}{2}\left(\cot \frac{x}{2}\right)^{-\frac{1}{2}} \times -\operatorname{cosec}^2 \frac{x}{2} \times \frac{1}{2}$

$= \frac{-\operatorname{cosec}^2 \frac{x}{2}}{4\sqrt{\cot \frac{x}{2}}}$

EXERCISE 17G.2

1 Use the chain rule to show that:

a $\frac{d}{dx}(\operatorname{cosec}[f(x)]) = -\operatorname{cosec}[f(x)] \cot[f(x)] f'(x)$

b $\frac{d}{dx}(\cot[f(x)]) = -\operatorname{cosec}^2[f(x)] f'(x)$

2 Differentiate with respect to x:

a $\operatorname{cosec} 4x$ **b** $\sec 2x$ **c** $\cot \frac{x}{3}$

d $2 \operatorname{cosec}^2 2x$ **e** $8 \cot^3 \frac{x}{2}$ **f** $\sqrt{\sec \frac{3x}{2}}$

3 Find $\frac{dy}{dx}$ for:

a $y = x \sec x$ **b** $y = e^x \cot x$ **c** $y = 4 \sec 2x$

d $y = e^{-x} \cot \frac{x}{2}$ **e** $y = x^2 \operatorname{cosec} x$ **f** $y = x\sqrt{\operatorname{cosec} x}$

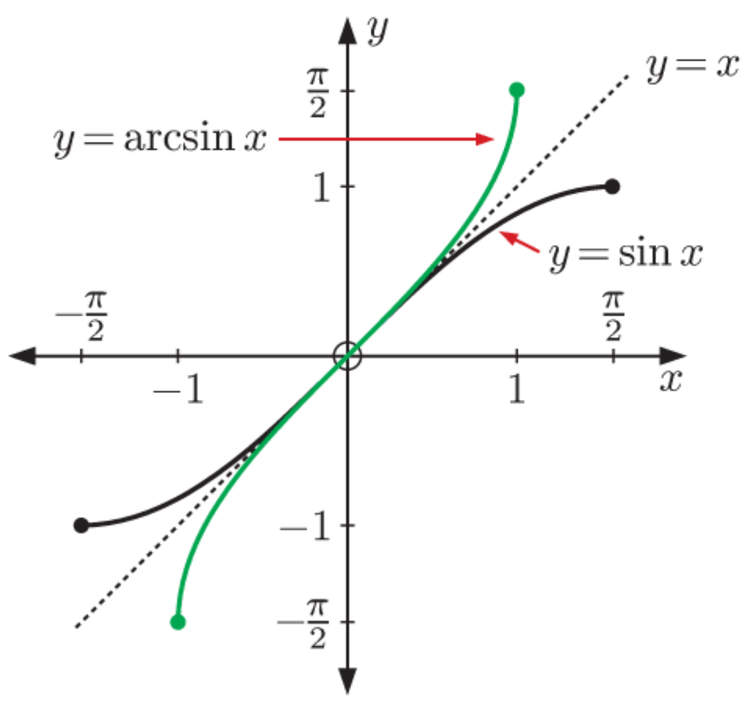
g $y = \ln(\sec x)$ **h** $y = x \operatorname{cosec}(x^2)$ **i** $y = \frac{\cot x}{\sqrt{x}}$

H DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

We have seen that $f(x) = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ is a one-to-one function and has the inverse function $f^{-1}(x) = \arcsin x$.

Remember that arcsin x is the **inverse** of sin x, whereas cosec x is the **reciprocal** of sin x.





Consider the following differentiation of $y = \arcsin x$:

If $y = \arcsin x$ then $x = \sin y$

$$\therefore \quad \frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$$

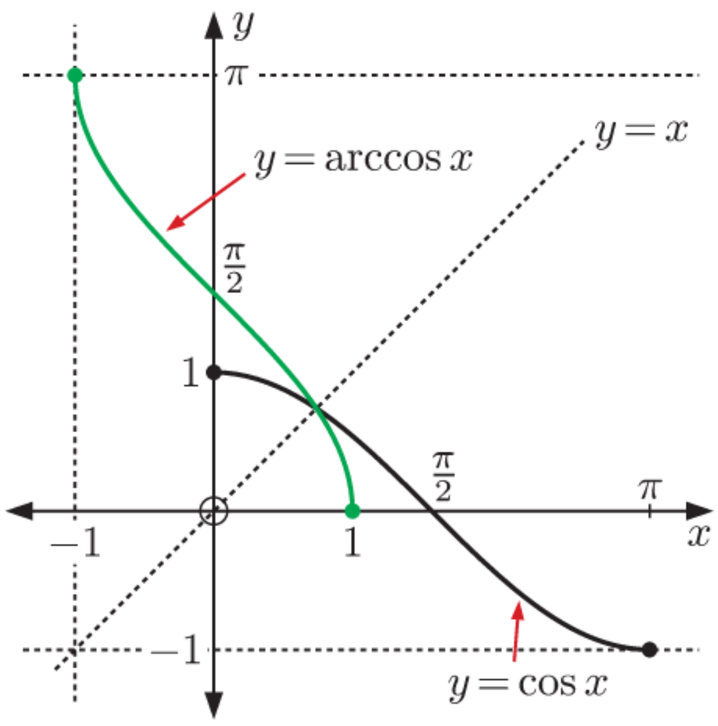
$$\therefore \quad \frac{dx}{dy} = \sqrt{1 - x^2}$$

From the chain rule, $\frac{dy}{dx}\frac{dx}{dy} = \frac{dy}{dy} = 1$, so $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

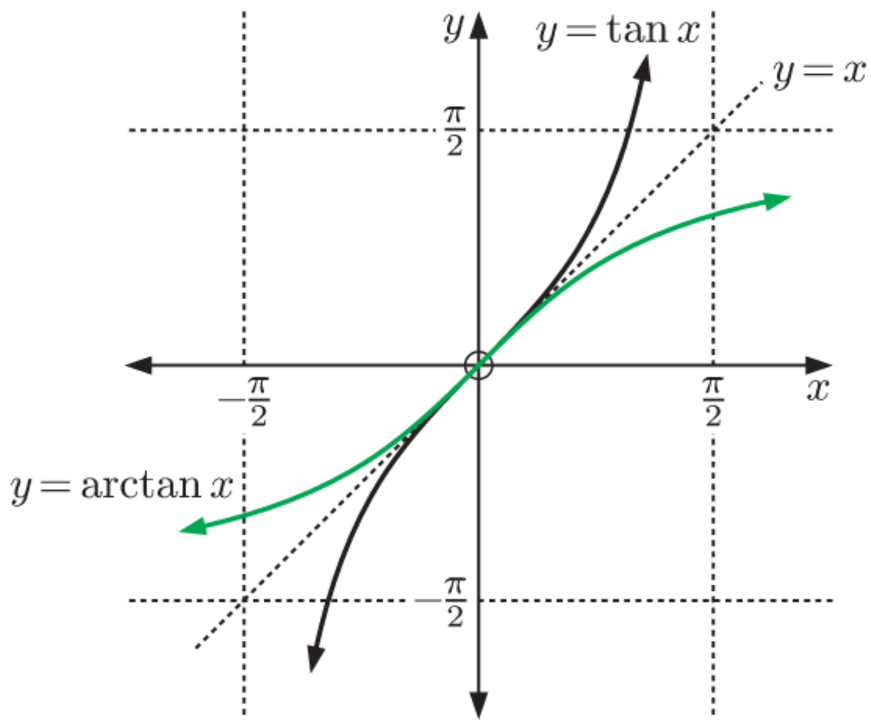
If $y = \arcsin x$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$, $-1 < x < 1$.

$f(x) = \cos x$, $0 \leqslant x \leqslant \pi$ has inverse function $f^{-1}(x) = \arccos x$.



If $y = \arccos x$, then $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1 - x^2}}$, $-1 < x < 1$.

$f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ has inverse function $f^{-1}(x) = \arctan x$.



If $y = \arctan x$, then $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, $x \in \mathbb{R}$.

EXERCISE 17H

1 If $y = \arccos x$, show that $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1 - x^2}}$, $-1 < x < 1$.

2 If $y = \arctan x$, show that $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, $x \in \mathbb{R}$.

3 Use the chain rule to show that:

a $\dfrac{d}{dx}(\arcsin[f(x)]) = \dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$ **b** $\dfrac{d}{dx}(\arctan[f(x)]) = \dfrac{f'(x)}{1 + [f(x)]^2}$

4 Find $\dfrac{dy}{dx}$ for:

a $y = \arctan 2x$ **b** $y = \arccos 3x$ **c** $y = \arcsin \frac{x}{4}$

d $y = \arccos \frac{x}{5}$ **e** $y = \arctan(x^2)$ **f** $y = \arccos(\sin x)$

5 Find $\frac{dy}{dx}$ for:

a $y = x \arcsin x$ **b** $y = e^x \arccos x$ **c** $y = e^{-x} \arctan x$

d $y = x^3 \arcsin x$ **e** $y = (x^2 + 1) \arctan 3x$ **f** $y = 3^x \arccos 2x$

6 Determine the domain of $f(x) = \arccos(1 + x - x^2)$, and find $f'(x)$.

7 The point P has y-coordinate $\frac{\pi}{6}$, and lies on the graph of $f(x) = \arctan \frac{x}{3}$.

a Find the coordinates of P.

b Find the gradient of the tangent to $y = f(x)$ at P.

8 Given $f(x) = \arcsin x + \arccos x$, find $f'(x)$. Explain your answer.

9 Suppose $a \in \mathbb{R}$, $a > 0$ is a constant.

a Prove that if $y = \arcsin \frac{x}{a}$, then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$ for $-a < x < a$.

b If $y = \arctan \frac{x}{a}$, find $\frac{dy}{dx}$.

c If $y = \arccos \frac{x}{a}$, find $\frac{dy}{dx}$.

I SECOND AND HIGHER DERIVATIVES

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$, y'', or $\frac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "f *double dashed* x".

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".



$f''(x)$ is the rate of change of $f'(x)$ with respect to x.

Example 17 **Self Tutor**

Find $f''(x)$ given that $f(x) = x \cos x + \frac{1}{x}$.

$$f(x) = x \cos x + x^{-1}$$

$$\therefore f'(x) = (1) \cos x + (x)(-\sin x) - x^{-2} \quad \{\text{product rule}\}$$

$$= \cos x - x \sin x - x^{-2}$$

$$\therefore f''(x) = -\sin x - (1) \sin x - (x) \cos x + 2x^{-3} \quad \{\text{product rule}\}$$

$$= -2 \sin x - x \cos x + \frac{2}{x^3}$$

We can continue to differentiate to obtain higher derivatives.

The **nth derivative of $f(x)$** is obtained by differentiating $y = f(x)$ n times.

We use $f^{(n)}(x)$ or $\dfrac{d^n y}{dx^n}$ to represent the nth derivative.

Example 18 **Self Tutor**

Given $f(x) = \cos 2x$, show that $f^{(3)}(\frac{\pi}{8}) = 4\sqrt{2}$.

$$
\begin{aligned}
f(x) &= \cos 2x \\
\therefore\ f'(x) &= -2\sin 2x \\
\therefore\ f''(x) &= -4\cos 2x \\
\therefore\ f^{(3)}(x) &= 8\sin 2x \\
\therefore\ f^{(3)}(\tfrac{\pi}{8}) &= 8\sin(2 \times \tfrac{\pi}{8}) \\
&= 8\sin\tfrac{\pi}{4} \\
&= 8 \times \tfrac{1}{\sqrt{2}} \\
&= 4\sqrt{2}
\end{aligned}
$$

EXERCISE 17I

1 Find $f''(x)$ given:

a $f(x) = 3x^2 - 6x + 2$ **b** $f(x) = \dfrac{2}{\sqrt{x}} - 1$ **c** $f(x) = 2x^3 - 3x^2 - x + 5$

d $f(x) = \dfrac{2 - 3x}{x^2}$ **e** $f(x) = (1 - 2x)^3$ **f** $f(x) = \dfrac{x + 2}{2x - 1}$

2 Find $\dfrac{d^2y}{dx^2}$ given:

a $y = x - x^3$ **b** $y = x^2 - \dfrac{5}{x^2}$ **c** $y = 2 - \dfrac{3}{\sqrt{x}}$

d $y = \dfrac{4 - x}{x}$ **e** $y = (x^2 - 3x)^3$ **f** $y = x^2 - x + \dfrac{1}{1 - x}$

g $y = e^{3x} + 2x$ **h** $y = \dfrac{1 - e^{-x}}{x}$ **i** $y = \dfrac{3 - x}{xe^x}$

3 Given $f(x) = x^3 - 2x + 5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f''(2)$ **d** $f^{(3)}(2)$

4 Suppose $y = Ae^{bx}$ where A and b are constants.

Prove by mathematical induction that $\dfrac{d^n y}{dx^n} = b^n y$, for $n \in \mathbb{Z}^+$.

5 Find x such that $f''(x) = 0$, given:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$ **b** $f(x) = \dfrac{x}{x^2 + 2}$

6 Consider the function $f(x) = 2x^3 - x$.
Complete the table alongside by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive (+), negative (−), or zero (0) at each value of x.

x	-1	0	1
$f(x)$	−		
$f'(x)$			
$f''(x)$			

7 Given $f(x) = \frac{2}{3}\sin 3x$, show that $f^{(3)}(\frac{2\pi}{9}) = 9$.

8 Suppose $f(x) = 2\sin^3 x - 3\sin x$.

a Show that $f'(x) = -3\cos x \cos 2x$. **b** Find $f''(x)$.

9 Find $\dfrac{d^2y}{dx^2}$ given:

a $y = -\ln x$ **b** $y = x\ln x$ **c** $y = (\ln x)^2$

10 Given $f(x) = x^2 - \dfrac{1}{x}$, find:

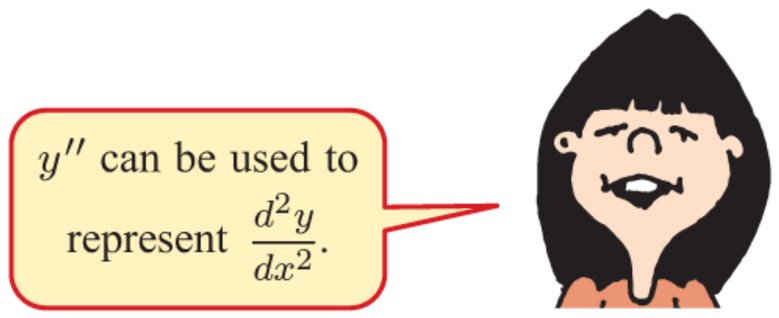
a $f(1)$ **b** $f'(1)$ **c** $f''(1)$ **d** $f^{(3)}(1)$

11 If $y = 2e^{3x} + 5e^{4x}$, show that $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$.

12 If $y = \sin(2x + 3)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

13 If $y = \sin x$, show that $\dfrac{d^4y}{dx^4} = y$.

14 If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$.



15 Let $f(x) = e^{ax}(x + 1)$, $a \in \mathbb{R}$. Prove by mathematical induction that $f^{(n)}(x) = a^{n-1}e^{ax}(a[x + 1] + n)$, $n \in \mathbb{Z}^+$.

16 Let $f(x) = e^{-x}(x + 2)$.

a Find:

i $f'(x)$ **ii** $f''(x)$ **iii** $f'''(x)$ **iv** $f^{(4)}(x)$.

b Conjecture a formula for $f^{(n)}(x)$, $n \in \mathbb{Z}^+$.

c Use the principle of mathematical induction to prove your conjecture.

17 **a** Show that $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

b Let $f(x) = \cos ax$, where $a \in \mathbb{R}$. Use the principle of mathematical induction to prove that $f^{(n)}(x) = (-a)^n \cos\left(ax - \frac{n\pi}{2}\right)$ for all $n \in \mathbb{Z}^+$.

18 Suppose $f(x)$ is an even differentiable function for $x \in \mathbb{R}$. Show that for all $n \in \mathbb{Z}^+$,

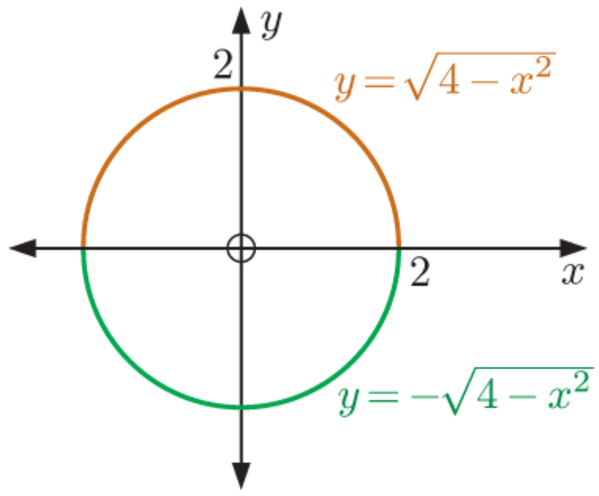
$f^{(n)}(x)$ is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}$.

J IMPLICIT DIFFERENTIATION

For relations such as $y^3+3xy^2-xy+11=0$, it is often difficult or impossible to write y as a function of x. Such relationships between x and y are called **implicit relations**.

Although we cannot write $y=f(x)$ and find a derivative *function* $f'(x)$, we can still find $\frac{dy}{dx}$, the rate of change in y with respect to x.

For example, the circle with centre $(0, 0)$ and radius 2 has equation $x^2+y^2=4$.



One way of finding $\frac{dy}{dx}$ for the circle is to split the relation into two parts.

$$x^2+y^2=4,$$
$$\therefore\ y^2=4-x^2$$
$$\therefore\ y=\pm\sqrt{4-x^2}$$

For the (orange) top half:

$$y=\sqrt{4-x^2}=(4-x^2)^{\frac{1}{2}}$$
$$\therefore\ \frac{dy}{dx}=\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\times(-2x)$$
$$=\frac{-x}{\sqrt{4-x^2}}$$
$$=\frac{-x}{y}$$

For the (green) bottom half:

$$y=-\sqrt{4-x^2}=-(4-x^2)^{\frac{1}{2}}$$
$$\therefore\ \frac{dy}{dx}=-\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\times(-2x)$$
$$=\frac{x}{\sqrt{4-x^2}}$$
$$=\frac{x}{-y}$$

So, in both cases $\frac{dy}{dx}=-\frac{x}{y}$.

However, this method is time-consuming and relies on us being able to rearrange the relation into pieces where y is a function of x.

A more useful method for solving this type of problem is **implicit differentiation**. Using this method we differentiate both sides of the equation with respect to x, applying the chain, product, quotient, and any other rules needed. We then proceed to make $\frac{dy}{dx}$ the subject of the equation.

Using the chain rule, $\frac{d}{dx}(y^n)=\frac{d}{dy}(y^n)\times\frac{dy}{dx}$

$$\therefore\ \boldsymbol{\frac{d}{dx}(y^n)=ny^{n-1}\frac{dy}{dx}}.$$

For example, if $x^2+y^2=4$, then $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(4)$

$$\therefore\ 2x+2y\frac{dy}{dx}=0$$
$$\therefore\ \frac{dy}{dx}=-\frac{x}{y}$$



$x^2+y^2=4$ is a *relation* but not a function.

Example 19 — Self Tutor

If y is a function of x, find: **a** $\dfrac{d}{dx}\left(\dfrac{1}{y}\right)$ **b** $\dfrac{d}{dx}(xy^2)$

a
$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}(y^{-1})$$
$$= -y^{-2}\frac{dy}{dx}$$
$$= -\frac{1}{y^2}\frac{dy}{dx}$$

b
$$\frac{d}{dx}(xy^2) = 1 \times y^2 + x \times 2y\frac{dy}{dx} \quad \{\text{product rule}\}$$
$$= y^2 + 2xy\frac{dy}{dx}$$

Example 20 — Self Tutor

Find $\dfrac{dy}{dx}$ if: **a** $x^2 + y^3 = 8$ **b** $x + x^2y + y^3 = 100$

a
$$x^2 + y^3 = 8$$
$$\therefore \frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(8)$$
$$\therefore 2x + 3y^2\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

b
$$x + x^2y + y^3 = 100$$
$$\therefore \frac{d}{dx}(x) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(100)$$
$$\therefore 1 + \left[2xy + x^2\frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$
$$\therefore (x^2 + 3y^2)\frac{dy}{dx} = -1 - 2xy$$
$$\therefore \frac{dy}{dx} = \frac{-1 - 2xy}{x^2 + 3y^2}$$

When dealing with implicit relationships, $\dfrac{dy}{dx}$ is usually found in terms of x *and* y. Therefore, to find the gradient of the tangent to the relation at a particular point, we need to know both the x- and y-coordinates of the point.

Example 21 — Self Tutor

Find the gradient of the tangent to $x^2 + y^3 = 5$ at the point where $x = 2$.

We first find $\dfrac{dy}{dx}$: $2x + 3y^2\dfrac{dy}{dx} = 0$

$\{\text{implicit differentiation}\}$

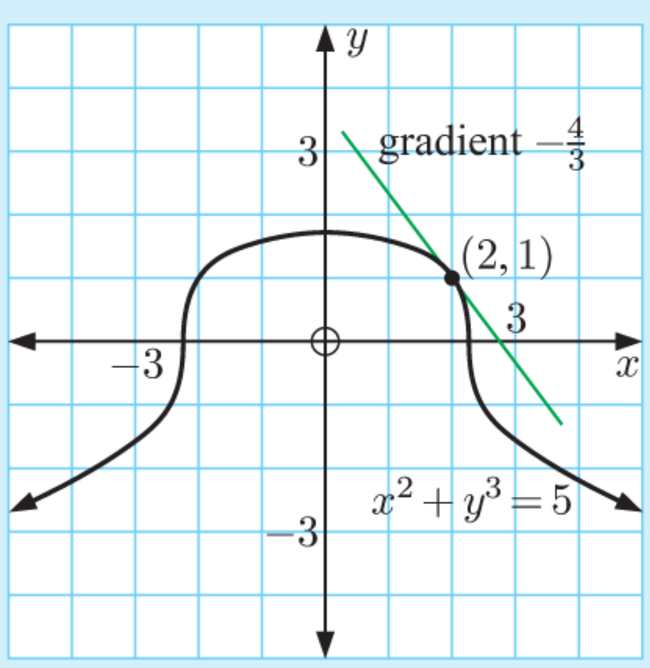
$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

When $x = 2$, $4 + y^3 = 5$

$$\therefore y = 1$$

$\therefore$ at the point $(2, 1)$, $\dfrac{dy}{dx} = \dfrac{-2(2)}{3(1)^2} = -\frac{4}{3}$

$\therefore$ the gradient of the tangent at $x = 2$ is $-\frac{4}{3}$.



EXERCISE 17J

1 If y is a function of x, find:

a $\dfrac{d}{dx}(2y)$ **b** $\dfrac{d}{dx}(-3y)$ **c** $\dfrac{d}{dx}(2y^3)$ **d** $\dfrac{d}{dx}\left(-\dfrac{4}{y}\right)$

e $\dfrac{d}{dx}(y^4)$ **f** $\dfrac{d}{dx}(\sqrt{y})$ **g** $\dfrac{d}{dx}\left(\dfrac{1}{y^2}\right)$ **h** $\dfrac{d}{dx}(xy)$

i $\dfrac{d}{dx}(x^2y)$ **j** $\dfrac{d}{dx}(xy^3)$

2 Find $\dfrac{dy}{dx}$ if:

a $x^2 + y^2 = 25$ **b** $x^2 + 3y^2 = 9$ **c** $y^2 - x^2 = 8$

d $x^2 - y^3 = 10$ **e** $x^2 + xy = 4$ **f** $x^3 - 2xy = 5$

g $xy + \dfrac{2}{x} = 12$ **h** $x + y^2 + 2xy = 7$ **i** $x^3 - 2y^2 + xy^2 = x$

j $\dfrac{x}{y} + y = 0$ **k** $x + \cos y = 1$ **l** $\sin y + xe^y = 2y$

m $2^x + 3^y = 10$ **n** $\ln(xy) = e^y$ **o** $y + x\sec y = \arccos x$

3 Use implicit differentiation to prove that for $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Hint: If $y = \ln x$, then $x = e^y$.

4 Find:

a $\dfrac{dy}{dx}$ if $x^3e^{3y} + 4x^2y^3 = 27e^{-2x}$ **b** $\dfrac{da}{db}$ if $e^{2a}\ln b^2 - a^3b + \ln(ab) = 21$.

5 If $y = x^x$, show that $\dfrac{dy}{dx} = x^x(\ln x + 1)$.

6 Find $\dfrac{d^2y}{dx^2}$ for each of the following implicit relations:

a $x^2 + y^2 = 25$ **b** $x^2 - y^2 = 10$ **c** $x^3 + 2xy = 4$

7 Suppose $3V^2 + 2q = 2Vq$. Find:

a $\dfrac{dV}{dq}$ **b** $\dfrac{d^2q}{dV^2}$

8 Consider the relation $xy^2 - 3x - y = 0$.

a Find $\dfrac{dy}{dx}$.

b Hence find the gradient of the tangent to the relation at:

i $(0, 0)$ **ii** $(2, 2)$

9 Find the gradient of the tangent to:

a $x + y^2 = 5$ at $(1, 2)$ **b** $x^3 - y^2 = -1$ at $(2, -3)$

c $x + y^3 = 4y$ at $y = 1$ **d** $x + y = 8xy$ at $x = \frac{1}{2}$

e $\dfrac{x}{y^2} - x = 2$ at $y = 2$ **f** $x^3 - xy^3 = y - 1$ at $x = 1$.

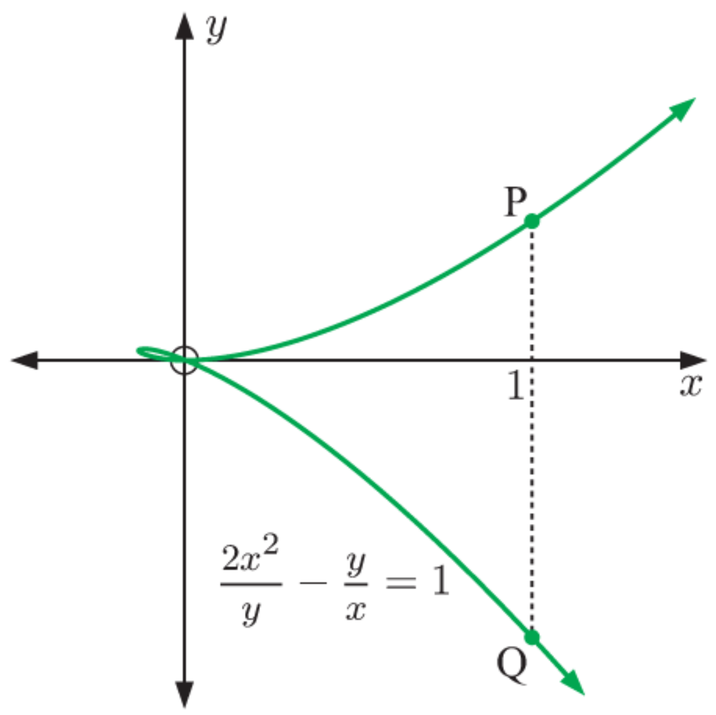
10 The graph of $\frac{2x^2}{y} - \frac{y}{x} = 1$ is shown alongside.

a Find $\frac{dy}{dx}$.

b The points P and Q both have x-coordinate 1.

i Find the coordinates of P and Q.

ii Find the gradient of the tangent to the graph at each point.



11 Consider the relation $x^2 - 3xy + y^2 = 7$.

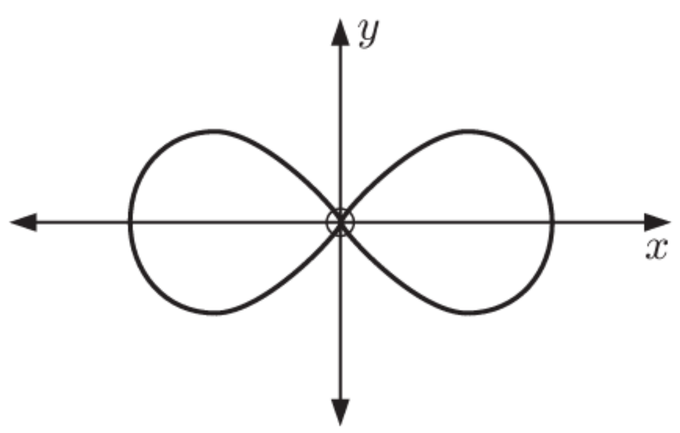
a Find $\frac{dy}{dx}$.

b Hence find the coordinates of all points where the gradient of the tangent to the curve is $\frac{2}{3}$.

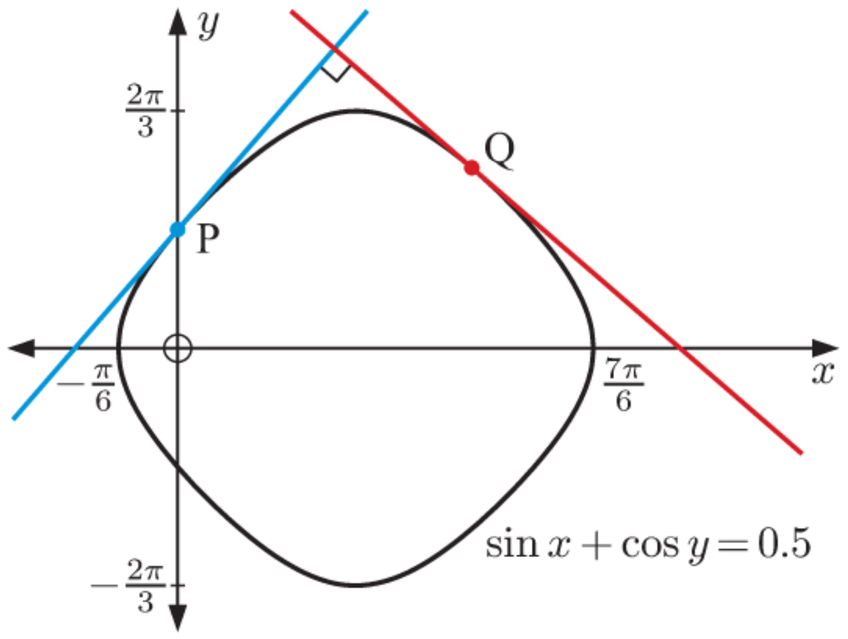
12 The graph shows the relation $(x^2 + y^2)^2 = x^2 - y^2$.

a Find the axes intercepts for this relation.

b Find the coordinates of all points on the graph at which the tangent is horizontal.



13 The relation $\sin x + \cos y = 0.5$ is graphed alongside for $-\frac{\pi}{6} \leqslant x \leqslant \frac{7\pi}{6}$, $-\frac{2\pi}{3} \leqslant y \leqslant \frac{2\pi}{3}$.
P lies on the y-axis, and the tangents at P and Q are perpendicular.
Find the coordinates of Q.



14 An ellipse has equation $x^2 + 3y^2 = 48$. Find the coordinates of the points on the ellipse such that the normal at those points passes through $(0, 2)$.

REVIEW SET 17A

1 Find $f'(x)$ given that $f(x)$ is:

a $5x^3$ **b** $x^6 - 5x$ **c** $7x^2 - \frac{3}{x}$

d $3x - \frac{4}{x^2}$ **e** $2x\sqrt{x}$ **f** $4\sqrt{x} - \frac{1}{\sqrt{x}}$

2 Find $\frac{dy}{dx}$ for:

a $y = 3x^2 - x^4$ **b** $y = \frac{x^3 - x}{x^2}$ **c** $y = x^2\sqrt{x-2}$

3 **a** Find $f'(x)$ given $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

b At what point on the curve $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ does the tangent have gradient 1?

4 Find $\dfrac{dy}{dx}$ if:

a $y = e^{x^3+2}$ **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$ **c** $y = x^3e^{2x}$

5 Find $f'(x)$ for:

a $f(x) = 5^{\sin x}$ **b** $f(x) = \log_2(x^2+4)$ **c** $f(x) = \log_3(4^x)$

6 Find the gradient of the tangent to:

a $f(x) = -x^2 + 4x - 2$ at $(-3, -23)$ **b** $y = (2-3x)^5$ at $x = 1$.

7 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2 + \sqrt{x})^4$ **c** $(x^2+1)(1-x^2)^3$

8 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

9 Differentiate with respect to x:

a $\sin 5x \ln x$ **b** $\sin x \cos 2x$ **c** $e^{-2x}\tan x$

10 Find the gradient of the tangent to $y = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

11 Given $y = 3e^x - e^{-x}$, show that $\dfrac{d^2y}{dx^2} = y$.

12 Consider the function $f(x) = \dfrac{x^2 - 4x - 1}{e^x}$.

a Find $f'(x)$.

b Find the gradient of the tangent to $y = f(x)$ at $x = 1$.

c For what values of x is the tangent to $y = f(x)$ horizontal?

13 Find the derivative with respect to x of:

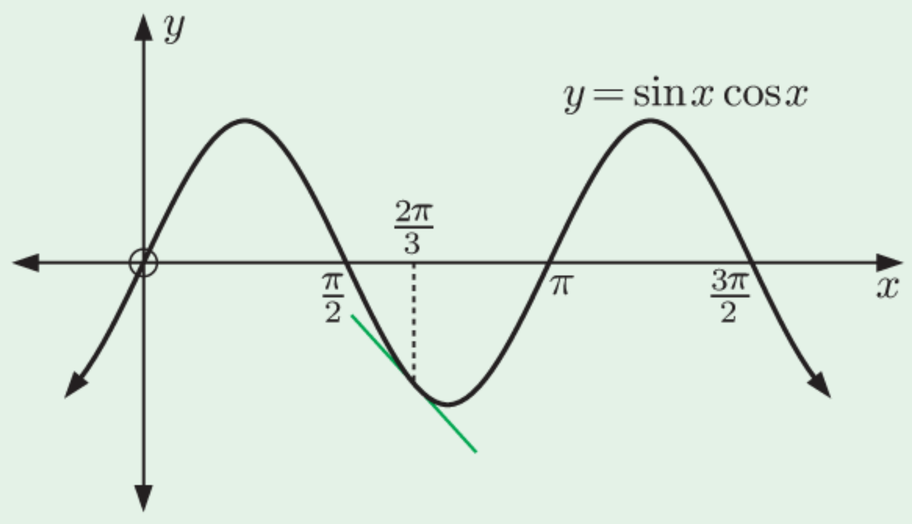
a $f(x) = (x^2+3)^4$ **b** $g(x) = \dfrac{\sqrt{x+5}}{x^2}$ **c** $h(x) = \dfrac{e^{4x}}{1-2x}$

14 The graph of $y = \sin x \cos x$ is shown alongside.

a Find $\dfrac{dy}{dx}$.

b Show that $y = \frac{1}{2}\sin 2x$ has the same derivative.

c Find the gradient of the illustrated tangent.



15 For $f(x) = 2\sin x + \cos 2x$, find:

a $f(\frac{\pi}{2})$ **b** $f'(\frac{\pi}{2})$ **c** $f''(\frac{\pi}{2})$

16 Find the gradient of the tangent to $y = 3\tan\frac{x}{2}$ at $x = \frac{\pi}{3}$.

17 Use the chain rule to show that $\dfrac{d}{dx}(\sec[f(x)]) = \sec[f(x)]\ \tan[f(x)]\ f'(x)$.

18 Find $\frac{dy}{dx}$ for:

a $y = \operatorname{cosec} 4x$ **b** $y = \dfrac{x}{\sqrt{\sec x}}$ **c** $y = e^x \cot 2x$

d $y = \arccos \frac{x}{2}$ **e** $y = x^5 \arcsin 3x$ **f** $y = 2^x \arctan(x - \frac{\pi}{3})$

19 Find $\frac{d^2y}{dx^2}$ for:

a $y = \frac{1}{8}x^4 + \frac{1}{6}x^3 - \frac{1}{4}x^2$ **b** $y = xe^{-x}$

20 For the function $f(x) = \sqrt{x}\cos 4x$:

a Find $f'(x)$ and $f''(x)$.

b Hence find:

i $f'(\frac{\pi}{16})$ **ii** $f''(\frac{\pi}{8})$

21 Consider the curves $y = e^{x-1} + 1$ and $y = 3 - e^{1-x}$.

a Sketch the curves on the same set of axes.

b Find the point of intersection of the two curves.

c Show that the tangents to each curve at this point have the same gradient. Comment on the significance of this result.

22 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ given $x^2 + 2xy + y^2 = 4$.

23 Given $y = (1 - \frac{1}{3}x)^3$, show that $\frac{d^3y}{dx^3} = -\frac{2}{9}$.

24 Given $y = \dfrac{1}{2x+1}$, prove that $\dfrac{d^ny}{dx^n} = \dfrac{(-2)^n n!}{(2x+1)^{n+1}}$ for all $n \in \mathbb{Z}^+$.

25 Let $f(x) = x\,3^x$.

a Find:

i $f'(x)$ **ii** $f''(x)$ **iii** $f^{(3)}(x)$ **iv** $f^{(4)}(x)$.

b Conjecture a formula for $f^{(n)}(x)$, $n \in \mathbb{Z}^+$.

c Use the principle of mathematical induction to prove your conjecture.

26 Find the exact gradient of the tangent to the curve $\arcsin(xy) + y = \frac{2\pi}{3}$ at the point where $y = \frac{\pi}{2}$.

REVIEW SET 17B

1 Find $f'(x)$ given that $f(x)$ is:

a $3x^2 - 7x + 4$ **b** $(x+5)^2$ **c** $2\sqrt{x} - \dfrac{3}{x}$ **d** $6x^2\sqrt{x}$

2 Find $\frac{dy}{dx}$ for:

a $y = 2x^3 - 6x^2 + 7x - 4$ **b** $y = \dfrac{3}{x} - \dfrac{5}{x^3}$ **c** $y = \dfrac{15}{\sqrt[3]{x}}$

3 If $f(x) = 7 + x - 3x^2$, find:

a $f(3)$ **b** $f'(3)$ **c** $f''(3)$

4 Differentiate with respect to x:

a $y = x^3\sqrt{1-x^2}$ **b** $y = \dfrac{x^2 - 3x}{\sqrt{x+1}}$

5 **a** Find $\dfrac{dy}{dx}$ for $y = xe^x$.

b Find all points on the curve $y = xe^x$ where the gradient of the tangent is $2e$.

6 Differentiate with respect to x:

a $f(x) = \ln(e^x + 3)$ **b** $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$ **c** $f(x) = x^{x^2}$

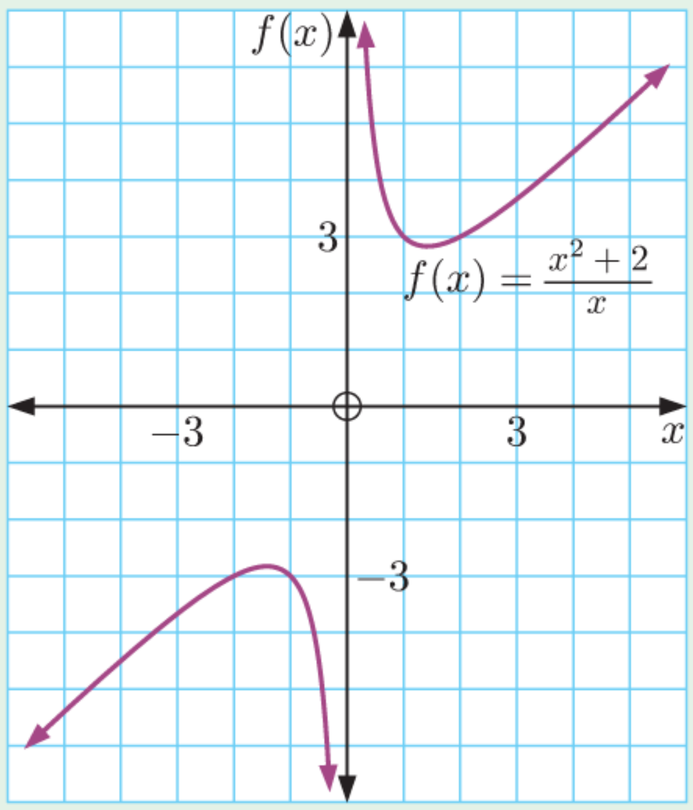
7 The graph of $f(x) = \dfrac{x^2+2}{x}$ is shown alongside.

a Find $f'(x)$.

b Hence find the gradient of the tangent at:

i $x = 1$ **ii** $x = -2$

c Copy the graph, and include the information from **b**.



8 Suppose $y = \left(x - \dfrac{1}{x}\right)^4$. Find $\dfrac{dy}{dx}$ at the point where $x = 1$.

9 Find $\dfrac{dy}{dx}$ if:

a $y = \ln(x^3 - 3x)$ **b** $y = \dfrac{e^x}{x^2}$ **c** $e^{x+y} = \ln(y^2+1)$

10 Find $f'(x)$ if:

a $f(x) = 2^{x^2-5x}$ **b** $f(x) = \log(\tan x)$ **c** $f(x) = \log_3\left(\dfrac{2}{(x+5)(x-4)}\right)$

11 Suppose $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

a Find $f''(x)$. **b** Find x such that $f''(x) = 0$.

12 Differentiate with respect to x:

a $10x - \sin 10x$ **b** $\ln\left(\dfrac{1}{\cos x}\right)$ **c** $\sin 5x \ln(2x)$

13 Find the gradient of the tangent to:

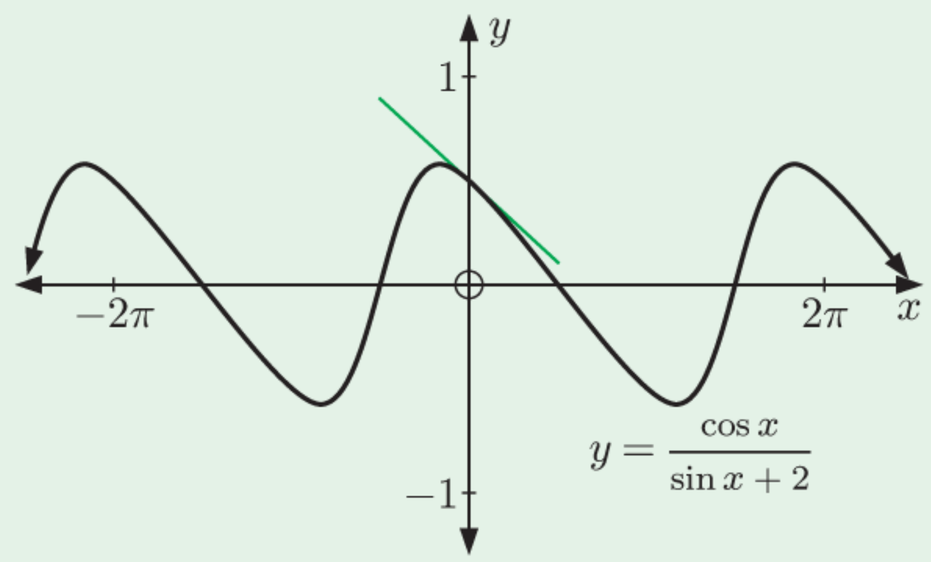
a $y = \dfrac{x^3}{x+1}$ at $x = 2$ **b** $y = \tan 2x$ at $x = \frac{\pi}{6}$.

14 Suppose $f(x) = a\ln(bx)$ where $f(e) = 12$ and $f'(2) = 2$. Find the constants a and b.

15 The graph of $y = \dfrac{\cos x}{\sin x + 2}$ is shown alongside.

a Find the gradient of the illustrated tangent.

b Show that it is impossible to draw a tangent to the graph with gradient $-\frac{1}{2}$.



16 **a** If $y = \dfrac{e^x}{\sqrt{x}}$, show that $\dfrac{dy}{dx} = \dfrac{e^x(2x-1)}{2x\sqrt{x}}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

17 Use the chain rule to show that $\dfrac{d}{dx}(\arccos[f(x)]) = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$.

18 Find $\dfrac{dy}{dx}$ for:

a $y = \cot \frac{x}{4}$ **b** $y = x^2 \sec 3x$ **c** $y = \dfrac{\operatorname{cosec}(e^x)}{e^x}$

d $y = \arcsin 5x$ **e** $y = e^{2x} \arctan 2x$ **f** $y = \arccos(\tan x)$

19 Find $\dfrac{d^2y}{dx^2}$ for:

a $y = \dfrac{3x^2-2}{1-2x}$ **b** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$ **c** $3^x + 2^y = 20$

20 Suppose $y = 3\sin 2x + 2\cos 2x$. Show that $4y + \dfrac{d^2y}{dx^2} = 0$.

21 Let $f(x) = \dfrac{6x}{3+x^2}$. Find the value(s) of x such that:

a $f(x) = -\frac{1}{2}$ **b** $f'(x) = 0$ **c** $f''(x) = 0$

22 Consider $f : x \mapsto -10\sin 2x \cos 2x$, $0 \leqslant x \leqslant \pi$.

a Write an expression for $f(x)$ in the form $k\sin 4x$.

b Solve $f'(x) = 0$, giving exact answers.

23 Given the curve $e^x y - xy^2 = 1$, find:

a $\dfrac{dy}{dx}$ **b** the gradient of the tangent to the curve at $x = 0$.

24 Prove using the principle of mathematical induction that if $y = x^n$, $n \in \mathbb{Z}^+$, then $\dfrac{dy}{dx} = nx^{n-1}$. You may assume the product rule of differentiation.

25 Show that $y = e^{2t}(A\cos t + B\sin t)$ satisfies $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 5y = 0$ for all $A, B \in \mathbb{R}$.

26 Use the product rule for differentiation to prove that:

a If $y = uv$ where u and v are functions of x, then

$$\frac{d^2y}{dx^2} = \left(\frac{d^2u}{dx^2}\right)v + 2\,\frac{du}{dx}\,\frac{dv}{dx} + u\left(\frac{d^2v}{dx^2}\right).$$

b If $y = uvw$ where u, v, and w are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx}\,vw + u\,\frac{dv}{dx}\,w + uv\,\frac{dw}{dx}.$$

27 Consider the function $f(x) = xe^{ax}$, where a is a constant.

a Find $f^{(n)}(x)$ for $n = 1, 2, 3,$ and 4.

b Conjecture a formula for $f^{(n)}(x)$, $n \in \mathbb{Z}^+$.

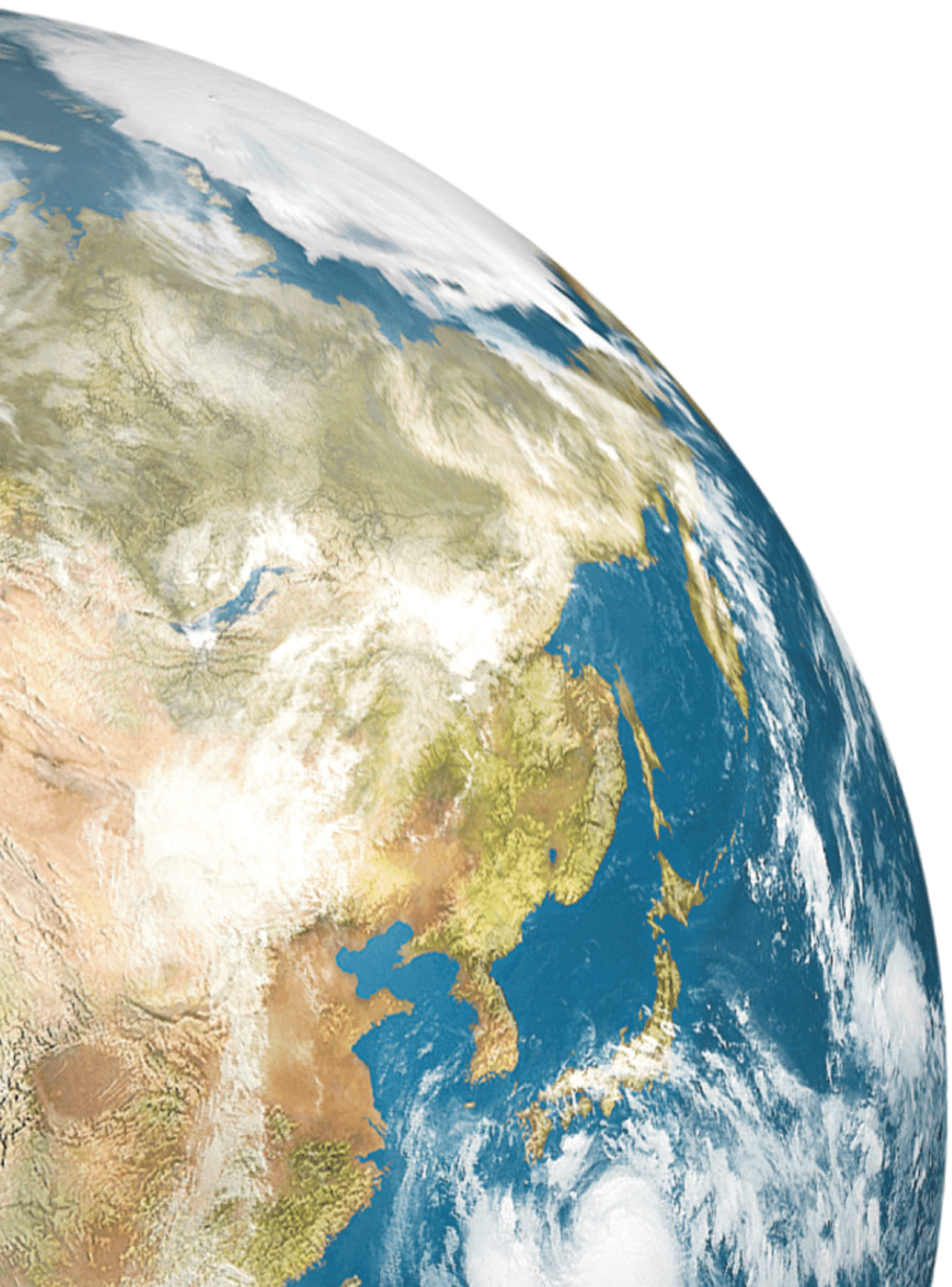
c Use the principle of mathematical induction to prove your conjecture.

Chapter 18

Properties of curves

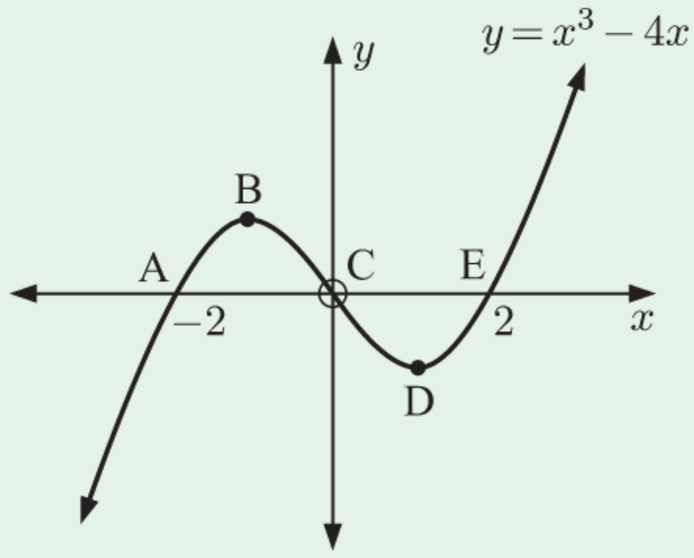
Contents:

A Tangents
B Normals
C Increasing and decreasing
D Stationary points
E Shape
F Inflection points
G Understanding functions and their derivatives
H L'Hôpital's rule



OPENING PROBLEM

The curve $y = x^3 - 4x$ is shown alongside.



Things to think about:

a What features of this curve could you describe to someone?

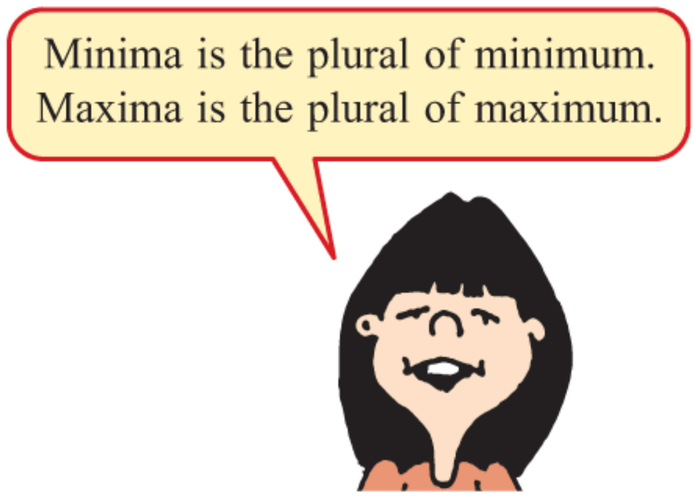
b How can we use calculus to help identify important features?

c How would you describe the *shape* of a curve? What rules would you use to identify a curve's shape?

In the previous Chapter we saw how to differentiate many types of functions.

In this Chapter we will use derivatives to find:

- tangents and normals to curves
- turning points, which are local minima and maxima
- inflection points where the curve changes shape.

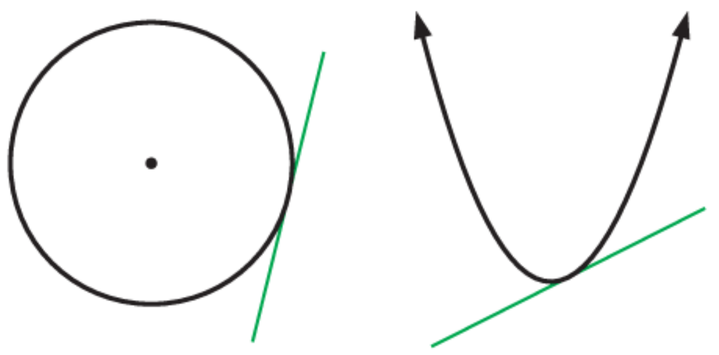


A TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

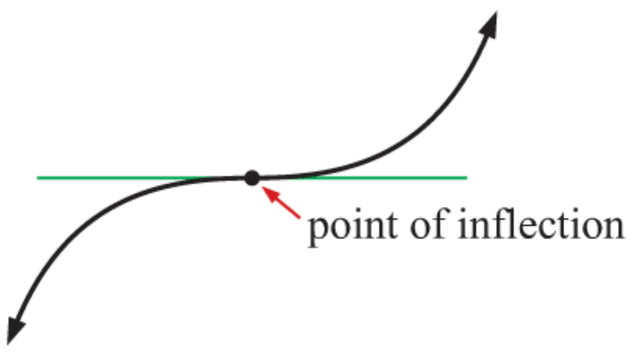
In cases we have seen already, the tangent *touches* the curve.

For example, consider the tangents to the circle and parabola shown.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency. If this happens, we call it a **point of inflection**.



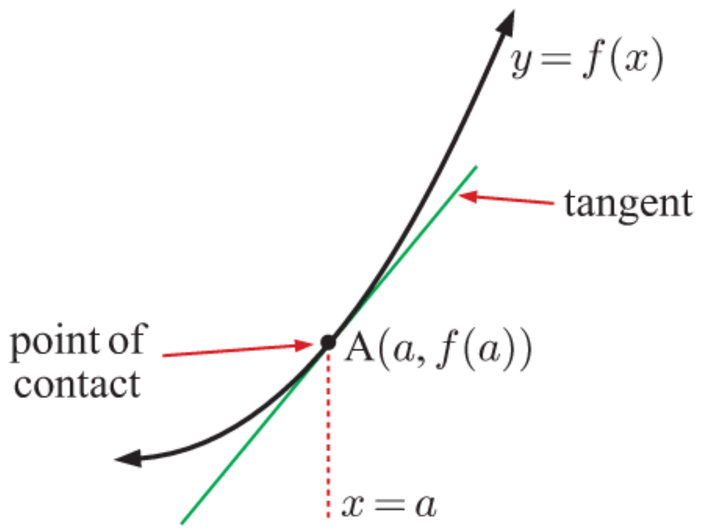
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a)$.

The equation of the tangent is

$$y - f(a) = f'(a)(x - a)$$

or $$\boldsymbol{y = f'(a)(x - a) + f(a).}$$



Example 1 **Self Tutor**

Find the equation of the tangent to $f(x) = \sqrt{x^2+5}$ at the point $(2, 3)$.

$$f(x) = \sqrt{x^2+5} = (x^2+5)^{\frac{1}{2}}$$

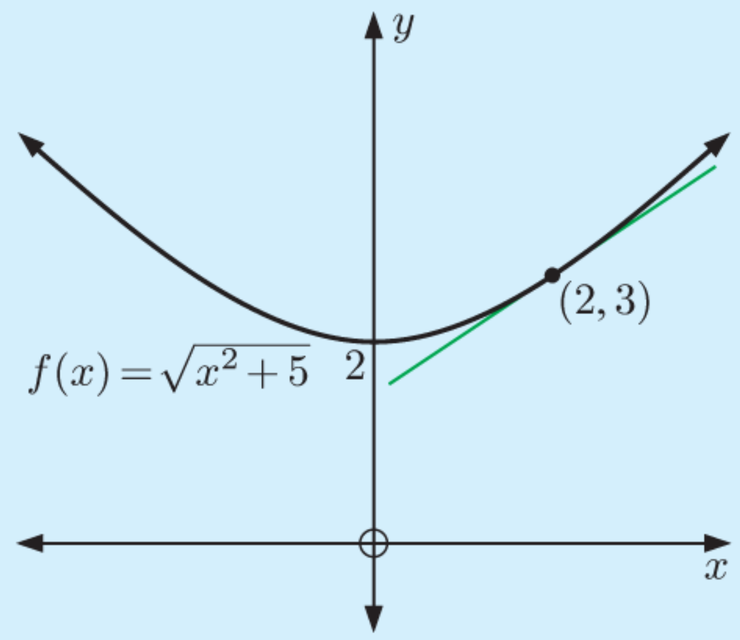
$$\therefore \quad f'(x) = \tfrac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) \qquad \{\text{chain rule}\}$$

$$= \frac{x}{\sqrt{x^2+5}}$$

$$\therefore \quad f'(2) = \frac{2}{\sqrt{2^2+5}} = \frac{2}{3}$$

So, the tangent has equation $y = \frac{2}{3}(x-2)+3$

$$\therefore \quad y = \tfrac{2}{3}x + \tfrac{5}{3}$$



Example 2 **Self Tutor**

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\dfrac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0,

so $3x^2 - 12 = 0$

$$\therefore \quad 3(x^2-4) = 0$$

$$\therefore \quad 3(x+2)(x-2) = 0$$

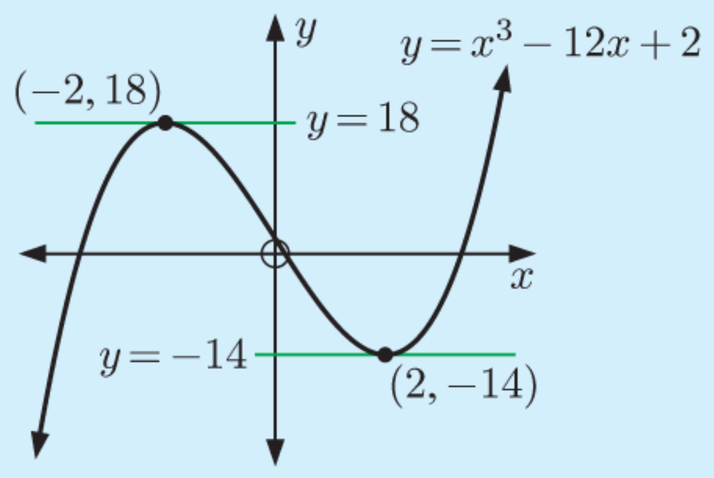
$$\therefore \quad x = -2 \text{ or } 2$$

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.

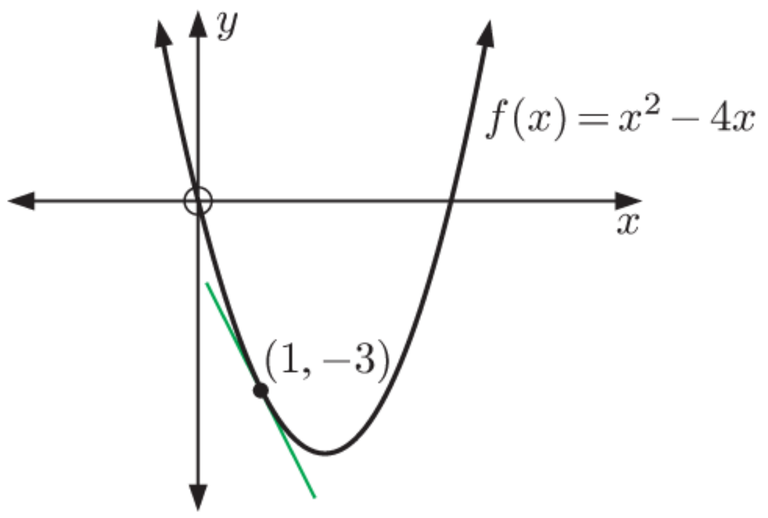


EXERCISE 18A

1 The graph of $f(x) = x^2 - 4x$ is shown alongside.

a Find $f'(x)$.

b Hence find the equation of the illustrated tangent.



2 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$ **b** $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$ **d** $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$ **f** $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$

g $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$ **h** $y = \dfrac{1}{\sqrt{3-2x}}$ at $x = -3$.

GRAPHING PACKAGE

Check your answers using technology.

3 Find the equations of any horizontal tangents to:

a $y = 2x^3 + 3x^2 - 12x + 1$ **b** $y = -x^3 + 3x^2 + 9x - 4$ **c** $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

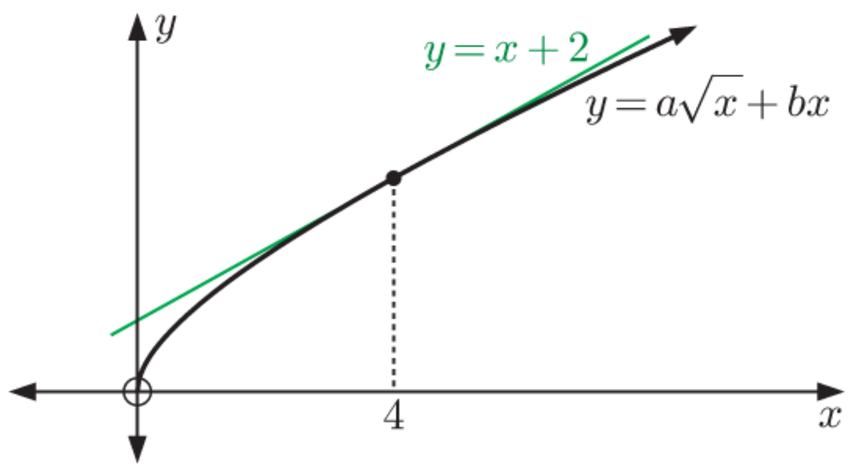
4 The tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

a Find k. **b** Hence find the equation of this tangent.

5 Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

6 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

7 Find the values of a and b.



8 Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

9 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$. Show that there is a unique horizontal tangent to the curve which touches the curve *twice*.

10 Consider the curve $y = a\sqrt{1 - bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

11 The tangent to $f(x) = (2x - 1)^4$ at $x = k$ has gradient 8.

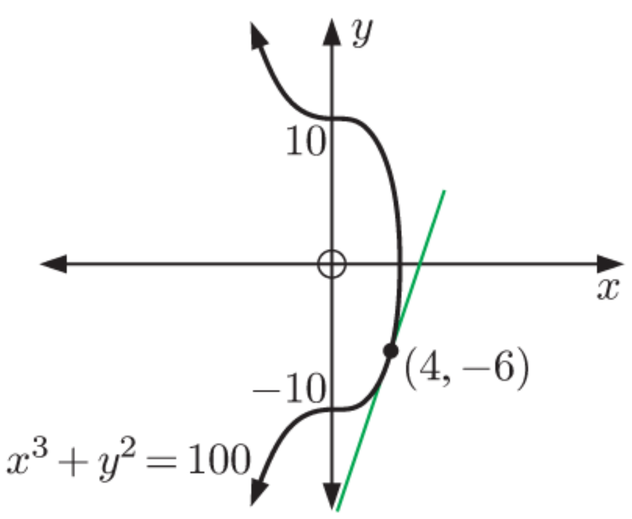
a Find k. **b** Find the equation of this tangent.

c Hence find the x-intercept of this tangent.

12 The graph of $x^3 + y^2 = 100$ is shown alongside.

a Find $\dfrac{dy}{dx}$.

b Find the equation of the illustrated tangent.



13 **a** Find the tangents to the unit circle $x^2 + y^2 = 1$ at the points where $x = \frac{1}{2}$.

b At which point on the x-axis do these tangents intersect?

Example 3 **Self Tutor**

Show that the equation of the tangent to $y = \ln x$ at $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore\ x = e^{-1} = \frac{1}{e}$

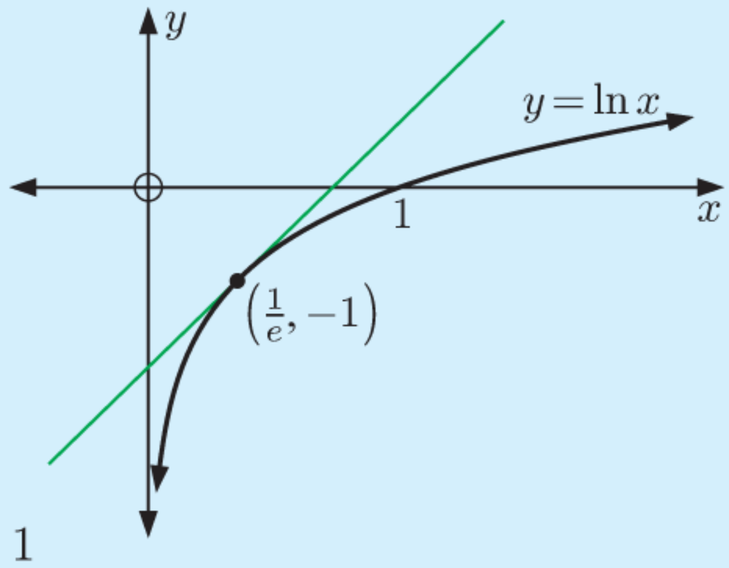
$\therefore$ the point of contact is $\left(\frac{1}{e}, -1\right)$.

Now $f(x) = \ln x$ has derivative $f'(x) = \frac{1}{x}$

$\therefore$ the tangent at $\left(\frac{1}{e}, -1\right)$ has gradient $\frac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $y = e\left(x - \frac{1}{e}\right) - 1$

which is $y = ex - 2$



14 Find the equation of the tangent to:

a $f(x) = e^{-x}$ at $x = 2$

b $y = \ln(2 - x)$ at $x = -1$

c $y = (x + 2)e^x$ at $x = 1$

d $y = \ln\sqrt{x}$ at $y = -1$

e $y = e^{3x-5}$ at $y = e$.

15 Consider $f(x) = \ln(x(x - 2))$.

a State the domain of $f(x)$.

b Find $f'(x)$.

c Find the equation of the tangent to $y = f(x)$ at the point where $x = 3$.

16 **a** Find the y-intercept of the tangent to $f(x) = x \ln x$ at the point where:

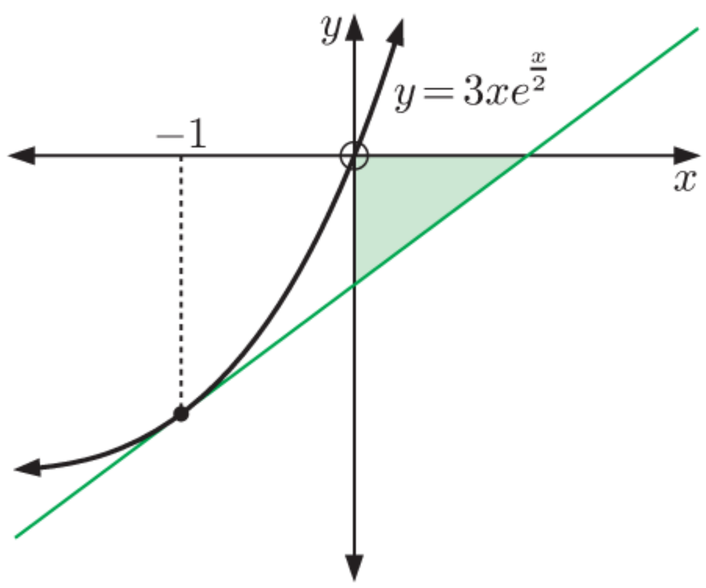
i $x = 1$ **ii** $x = 2$ **iii** $x = 3$.

b Make a conjecture about the y-intercept of the tangent to $f(x) = x \ln x$ at the point where $x = a$, $a > 0$.

c Prove your conjecture algebraically.

17 Find the axes intercepts of the tangent to $y = x^2 e^x$ at $x = 1$.

18 Find the exact area of the shaded triangle.



Example 4 — Self Tutor

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}$, $y = \tan \frac{\pi}{4} = 1$

$\therefore$ the point of contact is $(\frac{\pi}{4}, 1)$.

Now $f(x) = \tan x$ has derivative $f'(x) = \sec^2 x$

$\therefore$ the tangent at $(\frac{\pi}{4}, 1)$ has gradient $\dfrac{1}{\cos^2(\frac{\pi}{4})} = \dfrac{1}{\frac{1}{2}} = 2$

$\therefore$ the tangent has equation $y = 2(x - \frac{\pi}{4}) + 1$

which is $y = 2x + (1 - \frac{\pi}{2})$

19 Find the equation of the tangent to:

a $y = \sin x$ at the origin

b $y = \tan x$ at the origin

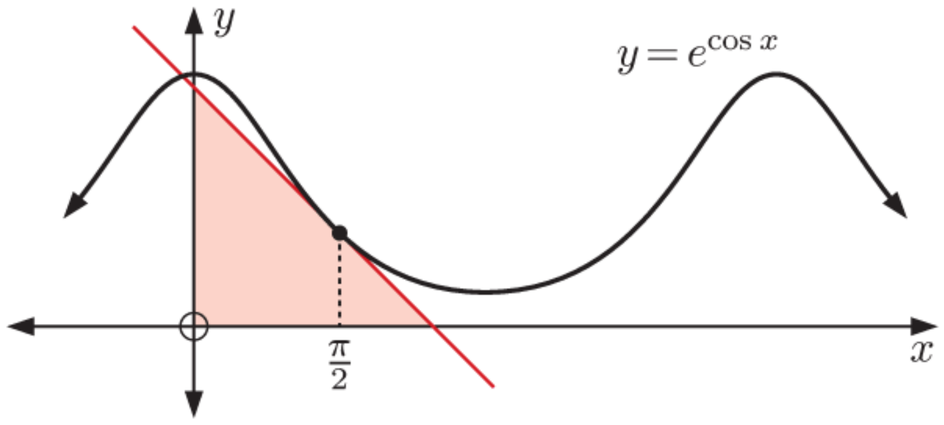
c $y = \cos x$ at $x = \frac{\pi}{6}$

d $y = \dfrac{1}{\sin 2x}$ at $x = \frac{\pi}{4}$

e $y = \cos 2x + 3 \sin x$ at $x = \frac{\pi}{2}$.

20 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

21 The graph of $y = e^{\cos x}$ is shown alongside. Find the area of the shaded triangle.



22 Find the equation of the tangent to:

a $y = \sec x$ at $x = \frac{\pi}{4}$

b $y = \cot \frac{x}{2}$ at $x = \frac{\pi}{3}$

c $y = \arctan x$ at $x = 1$

d $y = x \arccos \frac{x}{2}$ at $x = -1$

Example 5 — Self Tutor

Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore f'(x) = 3x^2 + 1$ and $\therefore f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.

The curve meets the tangent again when $x^3 + x + 2 = 4x$

$\therefore x^3 - 3x + 2 = 0$

$\therefore (x - 1)^2(x + 2) = 0$

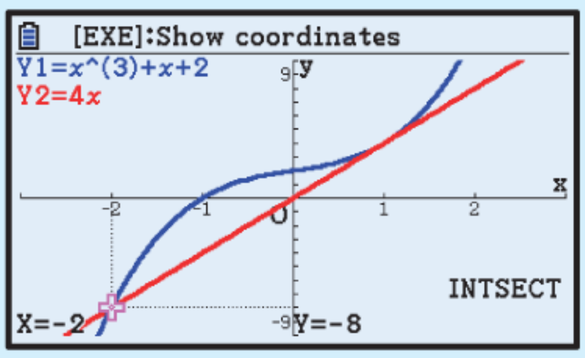


$(x - 1)^2$ must be a factor of $x^3 - 3x + 2 = 0$ since we are using the *tangent* at $x = 1$.

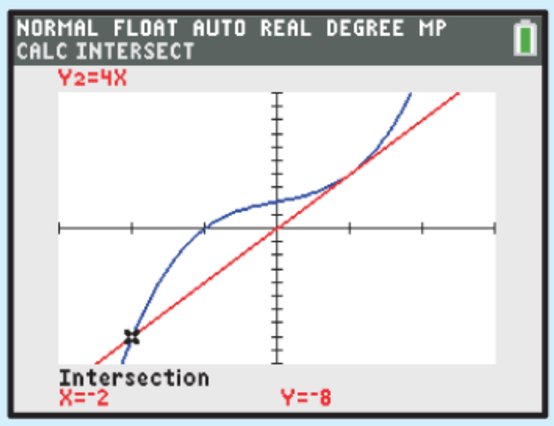
When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.

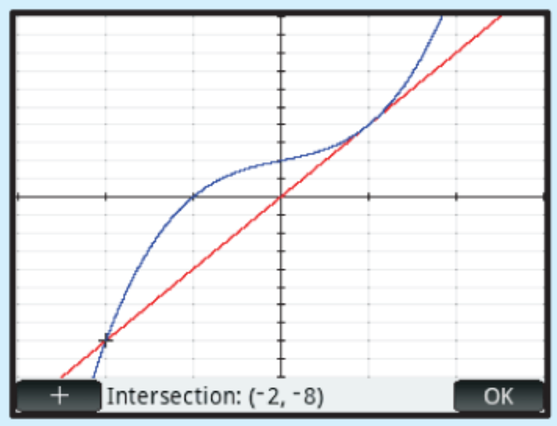
Casio fx-CG50



TI-84 Plus CE



HP Prime



23 Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

24 Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

25 Find where the tangent to the curve $y = \frac{1}{x} - \frac{1}{x^2}$ at the point where $x = 1$, meets the curve again.

26 Let $P(x) = x^3 - 3x^2 - x + 3$.

a Show that $x = 1$ is a zero of $P(x)$, and find the three real zeros.

b Sketch the graph of $y = P(x)$.

c Find the equation of the tangent to $y = P(x)$ at the point where $x = 2$.

d Find where the tangent in **c** crosses the curve again.

e Suppose a cubic has zeros a, b, and c with $a < b < c$. Prove that the tangent to the cubic at $x = \frac{a+b}{2}$ meets the cubic again at $x = c$.

Example 6 **Self Tutor**

Find the equations of the tangents to $y = -x^2 + x + 2$ which pass through $(1, 3)$.

Let $(a, -a^2 + a + 2)$ be a general point on the curve.

Now $\frac{dy}{dx} = -2x + 1$

$\therefore$ the gradient of the tangent when $x = a$ is $-2a + 1$

$\therefore$ the equation of the tangent at $(a, -a^2 + a + 2)$ is $y = (-2a + 1)(x - a) + (-a^2 + a + 2)$

which is $y = (1 - 2a)x + a^2 + 2$

The tangents which pass through $(1, 3)$ must satisfy $(1 - 2a)(1) + a^2 + 2 = 3$

$\therefore \; a^2 - 2a = 0$

$\therefore \; a(a - 2) = 0$

$\therefore \; a = 0$ or 2

$\therefore$ two tangents pass through the external point $(1, 3)$.

If $a = 0$, the tangent has equation $y = x + 2$ with point of contact $(0, 2)$.

If $a = 2$, the tangent has equation $y = -3x + 6$ with point of contact $(2, 0)$.

27 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$.

b Hence find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

28 **a** Find the equation of the tangent to $y = x^2 + 4x$ at the point where $x = a$.

b Hence find the equations of the tangents to $y = x^2 + 4x$ which pass through the external point $(1, -4)$. State the coordinates of the points of contact.

29 Find the equations of the tangents to $y = x^2 - 3x + 1$ which pass through $(1, -10)$.

30 **a** Find the equation of the tangent to $y = e^x$ at the point where $x = a$.

b Hence find the equation of the tangent to $y = e^x$ which passes through the origin.

31 Consider the function $y = 2x^2$.

a Find the equations of the tangents to the function from the external point $(1, -6)$.

b Find the points of contact for the tangents.

c Show that no tangents to the function pass through the point $(1, 4)$.

d Draw a graph of $y = 2x^2$ showing the information above.

32 Consider $f(x) = \dfrac{8}{x^2}$.

a Sketch the graph of the function.

b Find the equation of the tangent at the point where $x = a$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d Find the area of triangle OAB and discuss the area of the triangle as $a \to \infty$.

33 The graphs of $y = \sqrt{x + a}$ and $y = \sqrt{2x - x^2}$ have the same gradient at their point of intersection. Find a and the point of intersection.

34 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

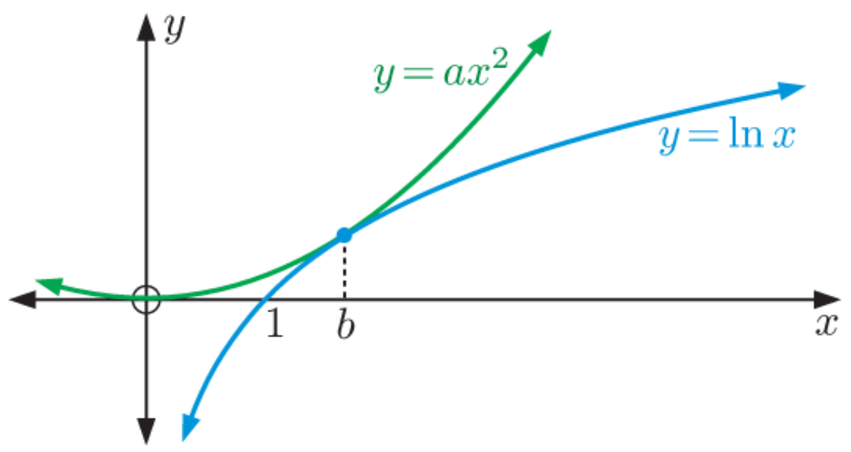
35 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.

a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.





If two curves *touch* then they share a common tangent at that point.

36 Let $p(x) = ax^2$, $a \neq 0$.

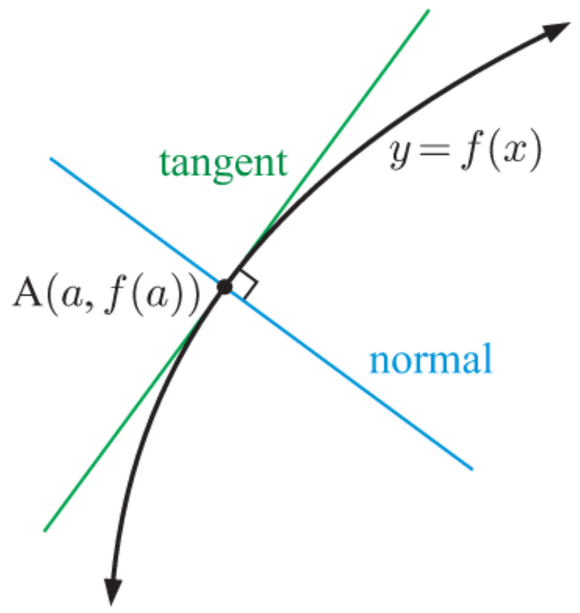
a Find the equations of the tangents to the curve at $x = s$ and $x = t$.

b Prove that the two tangent lines intersect at $x = \dfrac{s + t}{2}$.

c Prove that if the tangent lines are perpendicular then they intersect at $y = -\dfrac{1}{4a}$.

B NORMALS

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.



The gradients of perpendicular lines are negative reciprocals of each other, so:

The gradient of the normal to the curve at $x = a$ is $-\dfrac{1}{f'(a)}$.

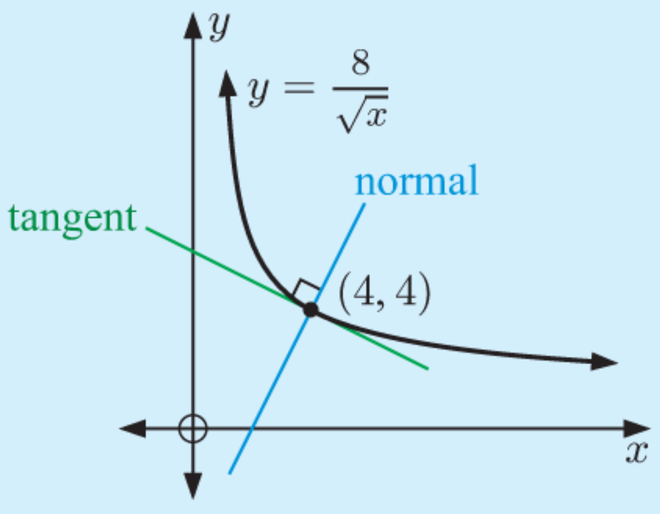
The equation of the normal to the curve at $x = a$ is

$$y = -\frac{1}{f'(a)}(x - a) + f(a).$$

Example 7 — Self Tutor

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \dfrac{8}{\sqrt{4}} = \dfrac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $\dfrac{dy}{dx} = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $\frac{2}{1}$.

$\therefore$ the equation of the normal is

$$2x - 1y = 2(4) - 1(4)$$

or $2x - y = 4$

EXERCISE 18B

1 Find the equation of the normal to:

a $y = x^2$ at $(4, 16)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$

e $f(x) = \dfrac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$

f $f(x) = \dfrac{x^2}{1 - x}$ at $(2, -4)$

g $y = \sqrt{x}(1 - x)^2$ at $x = 4$

h $f(x) = \dfrac{x^2 - 1}{2x + 3}$ at $x = -1$.

2 Suppose $f(x) = x^2 - \dfrac{8}{x}$.

a Find the equation of the tangent to $y = f(x)$ at $x = -2$.

b Find the equation of the normal to $y = f(x)$ at $x = 3$.

3 Consider $f : x \mapsto \dfrac{x}{\sqrt{2-x}}$.

a State the domain of f.

b Show that $f'(x) = \dfrac{4-x}{2(2-x)^{\frac{3}{2}}}$.

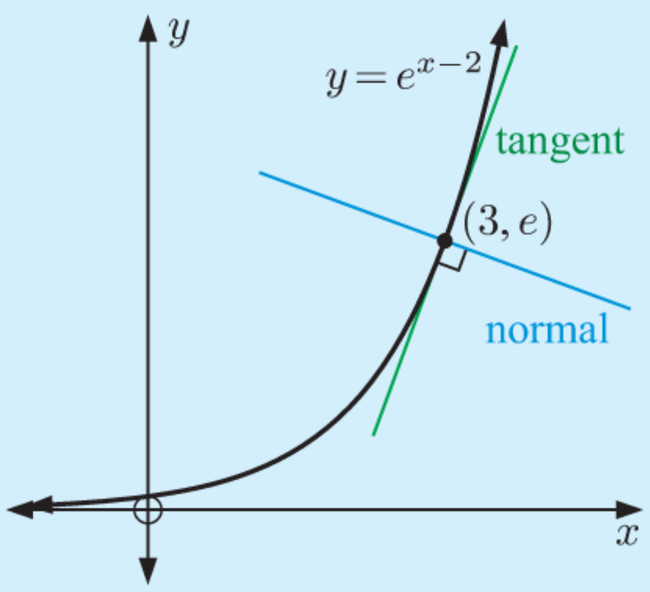
c Find the equation of the tangent to $y = f(x)$ at the point where $x = 1$.

d Find the equation of the normal to $y = f(x)$ at the point where $f(x) = -1$.

Example 8 **Self Tutor**

Find the equation of the normal to $y = e^{x-2}$ at the point where $x = 3$.

When $x = 3$, $y = e^{3-2} = e$. So, the point of contact is $(3, e)$.



Now as $y = e^{x-2}$, $\dfrac{dy}{dx} = e^{x-2}$

$\therefore$ when $x = 3$, $\dfrac{dy}{dx} = e^{3-2} = e$

$\therefore$ the normal at $(3, e)$ has gradient $-\dfrac{1}{e}$.

$\therefore$ the equation of the normal is

$$y = -\frac{1}{e}(x-3) + e$$

$\therefore$ $x + ey = 3 + e^2$

4 Find the equation of the normal to:

a $y = e^{-x}$ at $x = 0$

b $y = \ln x$ at $x = e$

c $y = e^{2x-1}$ at $x = 1$

d $y = \sin x$ at $x = \frac{\pi}{3}$

e $y = 4^x$ at $x = 0$

f $y = \tan x$ at the origin

g $y = \cos\left(2x - \frac{\pi}{3}\right)$ at $x = \frac{\pi}{2}$

h $y = \sin 3x - \cos x$ at $x = \frac{\pi}{4}$

i $y = \log_3(x^2 + 2)$ at $x = 2$

j $y = \operatorname{cosec} x$ at $x = \frac{\pi}{6}$

k $y = \sqrt{\sec \frac{x}{3}}$ at $x = \pi$.

5 Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

6 Find the points where the normal to $y = x^3 - 2x^2 + 1$ at $x = 1$, meets the curve again.

7 Find the equation of the normal to $f(x) = \cos x$ which passes through the origin.

8 Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.

9 Suppose $x = (y+1)e^y$.

a Find $\dfrac{dx}{dy}$ in terms of y.

b Find the equation of the normal to the curve at the point where $y = 2$.

10 Suppose P is $(4, \frac{19}{4})$ and Q is $(-2, -\frac{13}{4})$. The line (PQ) is a normal to $y = \dfrac{b}{(x+1)^2}$. Find b.

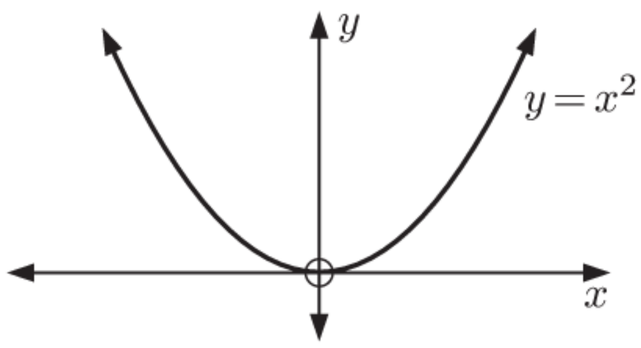
C INCREASING AND DECREASING

When we draw a graph of a function, we may notice that the function is **increasing** or **decreasing** over particular intervals.

Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

- $f(x)$ is **increasing** on $S \Leftrightarrow f(a) \leqslant f(b)$ for all $a, b \in S$ such that $a < b$.
- $f(x)$ is **decreasing** on $S \Leftrightarrow f(a) \geqslant f(b)$ for all $a, b \in S$ such that $a < b$.

For example, $y = x^2$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.



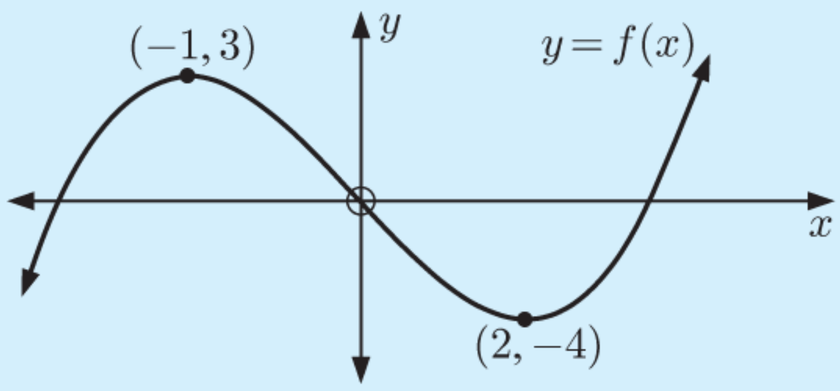
Important: In this example, people often get confused about the point $x = 0$. They wonder how the curve can be both increasing and decreasing at the same point. The answer is that the notion of increasing and decreasing are associated with *intervals*, not particular values for x. We see that $y = x^2$ is decreasing *on the interval* $x \leqslant 0$ and increasing *on the interval* $x \geqslant 0$.

Example 9 Self Tutor

Find intervals where $f(x)$ is:

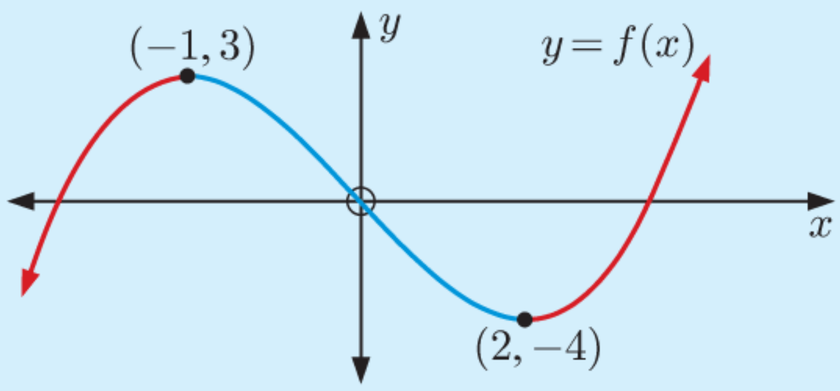
a increasing

b decreasing.



a $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 2$.

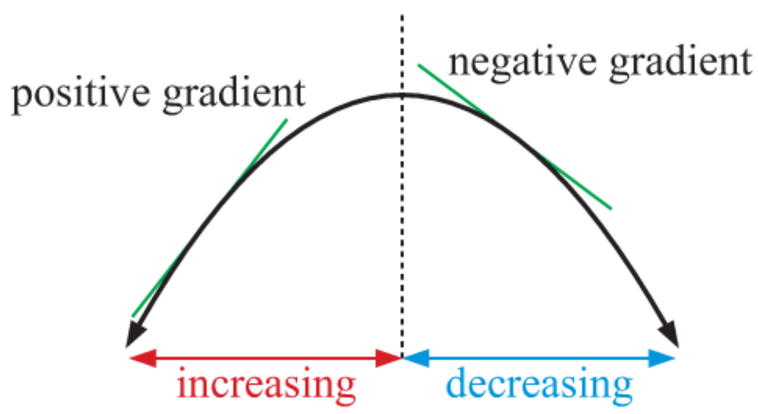
b $f(x)$ is decreasing for $-1 \leqslant x \leqslant 2$.



We can determine intervals where a curve $y = f(x)$ is increasing or decreasing by considering a **sign diagram** of the derivative function $f'(x)$.

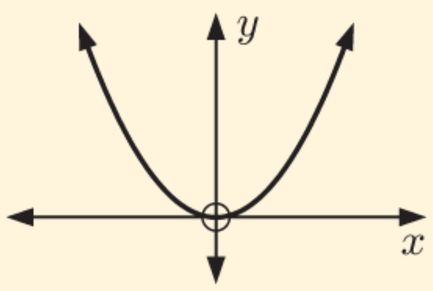
For most functions that we deal with in this course:

- $f(x)$ is **increasing** on $S \Leftrightarrow f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on $S \Leftrightarrow f'(x) \leqslant 0$ for all x in S.



Sign diagrams for the derivative are extremely useful for determining intervals where a function is increasing or decreasing. Consider the following examples:

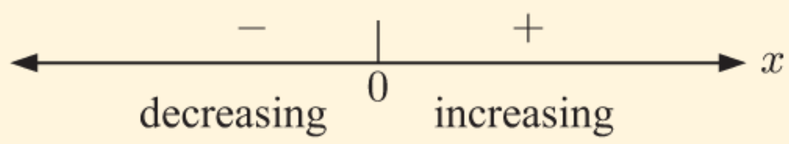
$f(x) = x^2$



DEMO

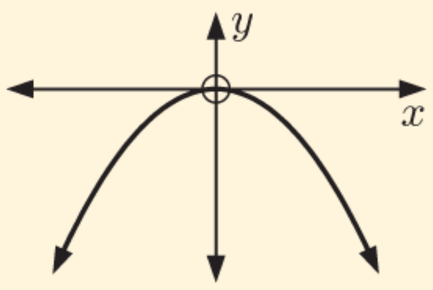


$f'(x) = 2x$ which has sign diagram



$\therefore \quad f(x) = x^2$ is $\begin{cases} \text{decreasing for } x \leqslant 0 \\ \text{increasing for } x \geqslant 0. \end{cases}$

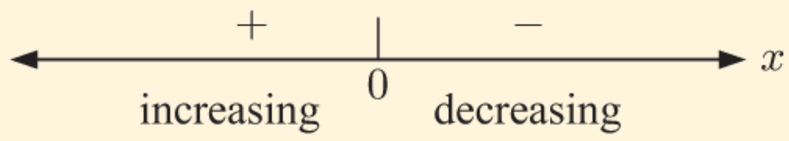
$f(x) = -x^2$



DEMO

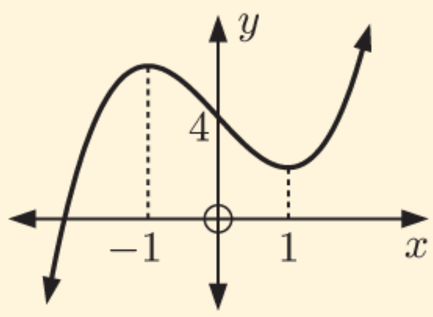


$f'(x) = -2x$ which has sign diagram



$\therefore \quad f(x) = -x^2$ is $\begin{cases} \text{increasing for } x \leqslant 0 \\ \text{decreasing for } x \geqslant 0. \end{cases}$

$f(x) = x^3 - 3x + 4$

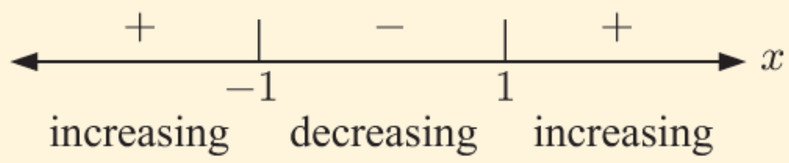


DEMO



$f'(x) = 3x^2 - 3$
$= 3(x^2 - 1)$
$= 3(x+1)(x-1)$

which has sign diagram



$\therefore \quad f(x) = x^3 - 3x + 4$ is $\begin{cases} \text{increasing for } x \leqslant -1 \text{ and for } x \geqslant 1 \\ \text{decreasing for } -1 \leqslant x \leqslant 1. \end{cases}$

Example 10

Self Tutor

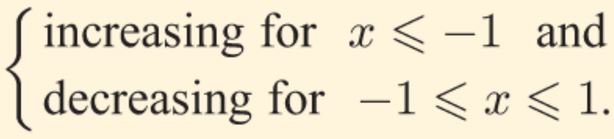
Find the intervals where the following functions are increasing or decreasing:

a $f(x) = -x^3 + 3x^2 + 5$ **b** $f(x) = 3x^4 - 8x^3 + 2$

a $f(x) = -x^3 + 3x^2 + 5$

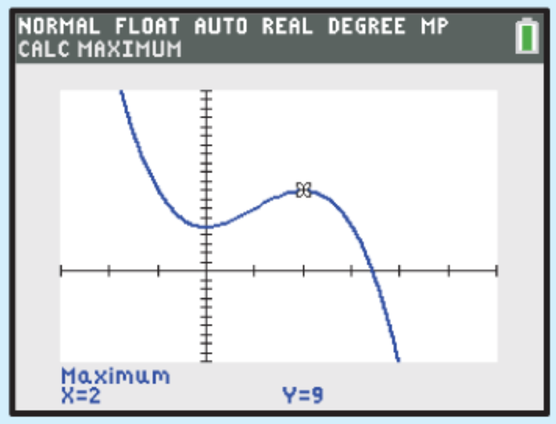
$\therefore \quad f'(x) = -3x^2 + 6x$
$= -3x(x-2)$

which has sign diagram:



So, $f(x)$ is decreasing for $x \leqslant 0$ and for $x \geqslant 2$, and increasing for $0 \leqslant x \leqslant 2$.

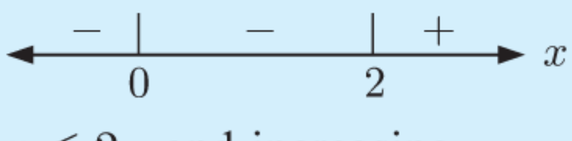
TI-84 Plus CE



b $f(x) = 3x^4 - 8x^3 + 2$

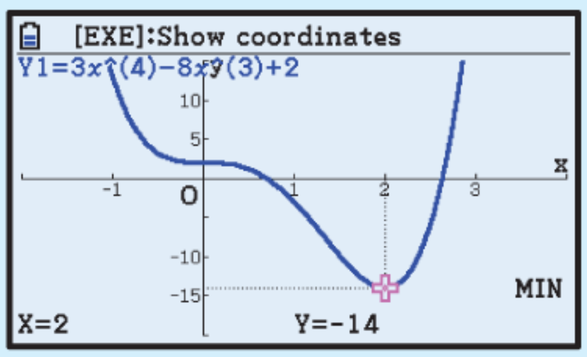
$\therefore \quad f'(x) = 12x^3 - 24x^2$
$= 12x^2(x-2)$

which has sign diagram:



So, $f(x)$ is decreasing for $x \leqslant 2$, and increasing for $x \geqslant 2$.

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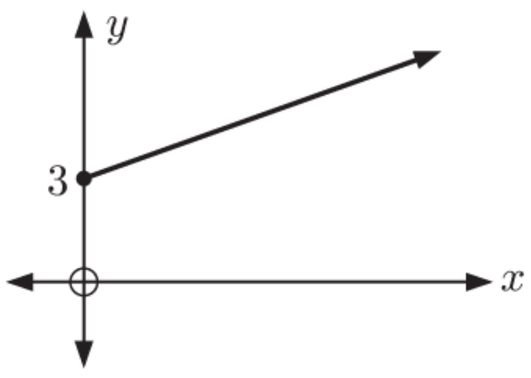
Remember that $f(x)$ must be defined for all x on an interval before we can classify the function as increasing or decreasing on that interval. We need to take care with vertical asymptotes and other values for x where the function is not defined.

EXERCISE 18C

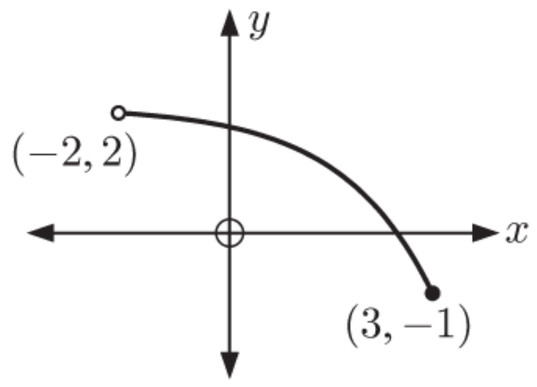
1 Write down the intervals where the graphs are:

i increasing **ii** decreasing.

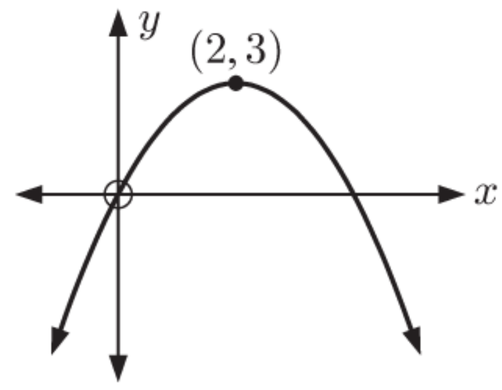
a



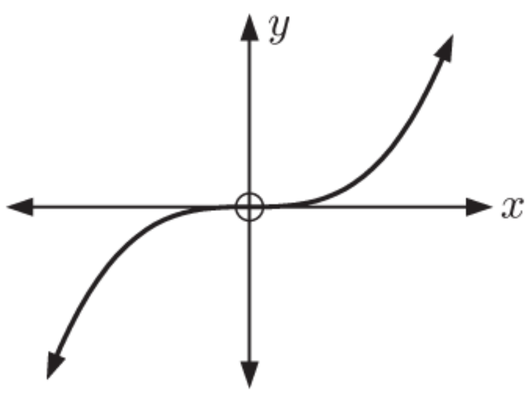
b



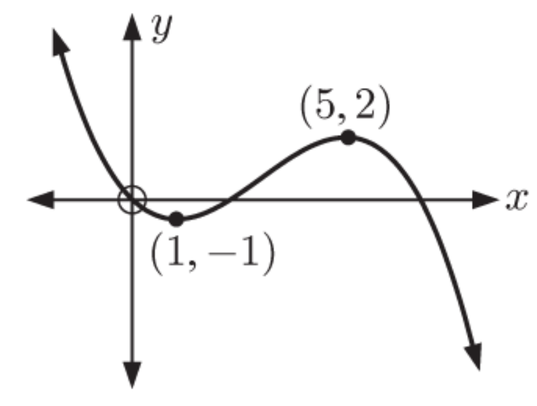
c



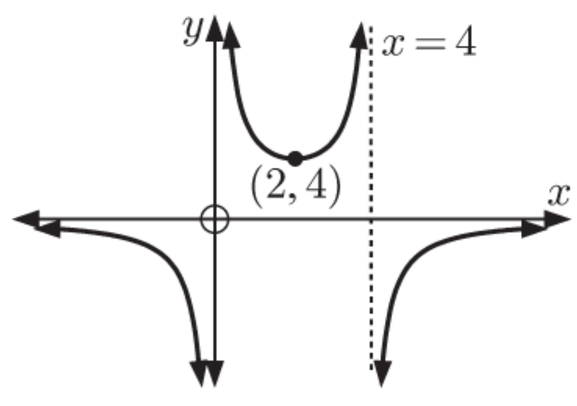
d



e



f

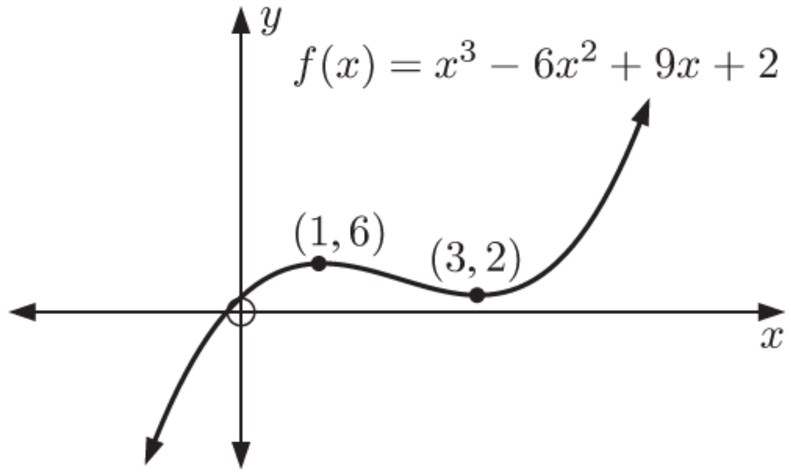


2 The graph of $f(x) = x^3 - 6x^2 + 9x + 2$ is shown alongside.

a Use the graph to write down the intervals where the function is:

i increasing **ii** decreasing.

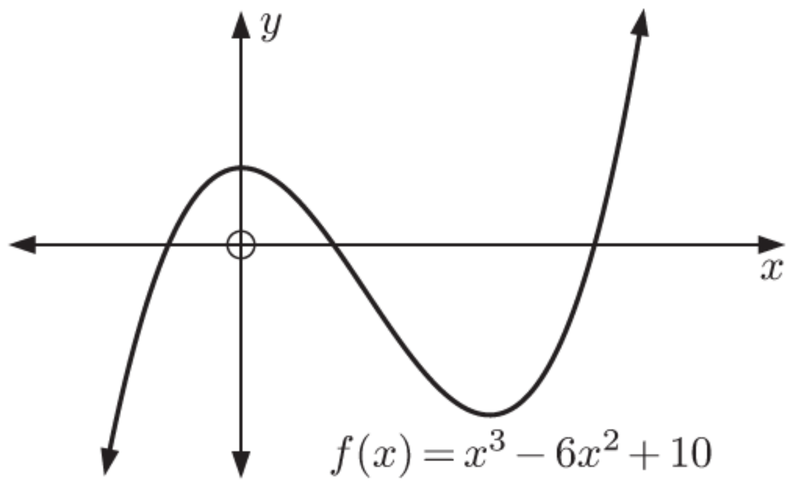
b Check your answer by finding $f'(x)$ and constructing its sign diagram.



3 The graph of $f(x) = x^3 - 6x^2 + 10$ is shown alongside.

a Find $f'(x)$, and draw its sign diagram.

b Find the intervals where $f(x)$ is increasing or decreasing.



4 Find the intervals where $f(x)$ is increasing or decreasing:

a $f(x) = x^2$ **b** $f(x) = -x^3$

c $f(x) = \sqrt{x}$ **d** $f(x) = 2x^2 + 3x - 4$

e $f(x) = \dfrac{1}{x}$ **f** $f(x) = x - 2\sqrt{x}$

g $f(x) = \dfrac{2}{\sqrt{x}}$ **h** $f(x) = x^3 - 6x^2$

i $f(x) = -2x^3 + 4x$ **j** $f(x) = -4x^3 + 15x^2 + 18x + 3$

k $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$ **l** $f(x) = x^3 - 6x^2 + 3x - 1$

5 Consider the function $f(x) = x^3 - 3x^2 + 5x + 2$.

a Find $f'(x)$.

b Show that $f'(x) > 0$ for all x, and explain the significance of this result.

c Use technology to sketch $y = f(x)$, and check your answer to **b**.

Example 11 **Self Tutor**

Consider $f(x) = \dfrac{2x-3}{x^2+2x-3}$.

a Show that $f'(x) = \dfrac{-2x(x-3)}{(x-1)^2(x+3)^2}$, and draw its sign diagram.

b Hence find the intervals where $y = f(x)$ is increasing or decreasing.

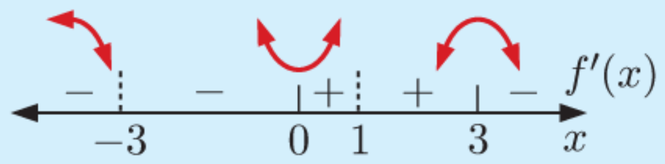
a $f(x) = \dfrac{2x-3}{x^2+2x-3}$

$\therefore \quad f'(x) = \dfrac{2(x^2+2x-3) - (2x-3)(2x+2)}{(x^2+2x-3)^2}$ {quotient rule}

$= \dfrac{2x^2+4x-6-(4x^2-2x-6)}{((x-1)(x+3))^2}$

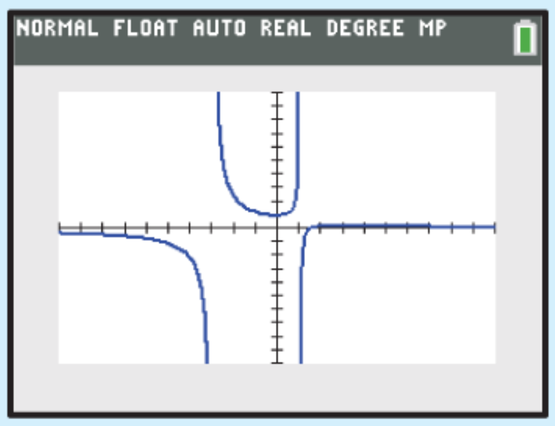
$= \dfrac{-2x^2+6x}{(x-1)^2(x+3)^2}$

$= \dfrac{-2x(x-3)}{(x-1)^2(x+3)^2}$ which has sign diagram:



b $f(x)$ is increasing for $0 \leqslant x < 1$
and for $1 < x \leqslant 3$.

$f(x)$ is decreasing for $x < -3$
and for $-3 < x \leqslant 0$
and for $x \geqslant 3$.



6 Let $f(x) = x + \dfrac{9}{x}$.

a Show that $f'(x) = \dfrac{(x+3)(x-3)}{x^2}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

7 Consider $f(x) = \dfrac{4x}{x^2+1}$.

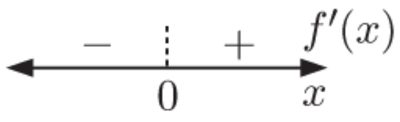
a Show that $f'(x) = \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

8 Find the intervals where $f(x)$ is increasing or decreasing:

a $f(x) = \dfrac{2-x}{x+1}$ **b** $f(x) = \dfrac{4x}{(x-1)^2}$ **c** $f(x) = \dfrac{-x^2+4x-7}{x-1}$

d $f(x) = \dfrac{x^3}{x^2-1}$ **e** $f(x) = (3x^2+1)^4$ **f** $f(x) = x^2 + \dfrac{4}{x-1}$

9 To find the intervals where $f(x) = \ln x$ is increasing or decreasing, Kenneth states that $f'(x) = \dfrac{1}{x}$, which has sign diagram 

. He therefore concludes that $f(x) = \ln x$ is increasing for $x > 0$, and decreasing for $x < 0$. Explain the mistake Kenneth has made.

10 Find the intervals where $f(x)$ is increasing or decreasing:

a $f(x) = e^x$ **b** $f(x) = \ln(x + 2)$ **c** $f(x) = 3 + e^{-x}$

d $f(x) = xe^x$ **e** $f(x) = x^3 \ln x$ **f** $f(x) = e^{-x^2}$

g $f(x) = \ln(x^2 + 4)$ **h** $f(x) = \dfrac{e^{-x}}{x}$ **i** $f(x) = \sqrt{x} \ln x$

11 Discuss intervals where the following functions are increasing or decreasing:

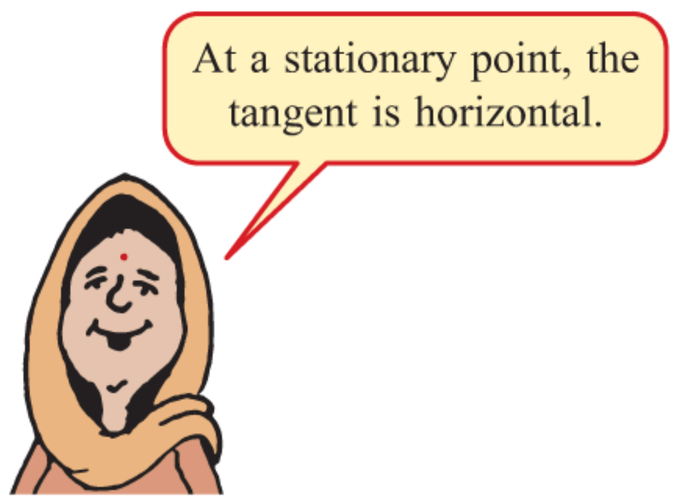
a $y = \arccos x$ **b** $y = \arcsin x$ **c** $y = \arctan x$ **d** $y = \cos x$

e $y = \sin x$ **f** $y = \tan x$ **g** $y = x - 2\sin x$ **h** $y = x + \cos x$

D STATIONARY POINTS

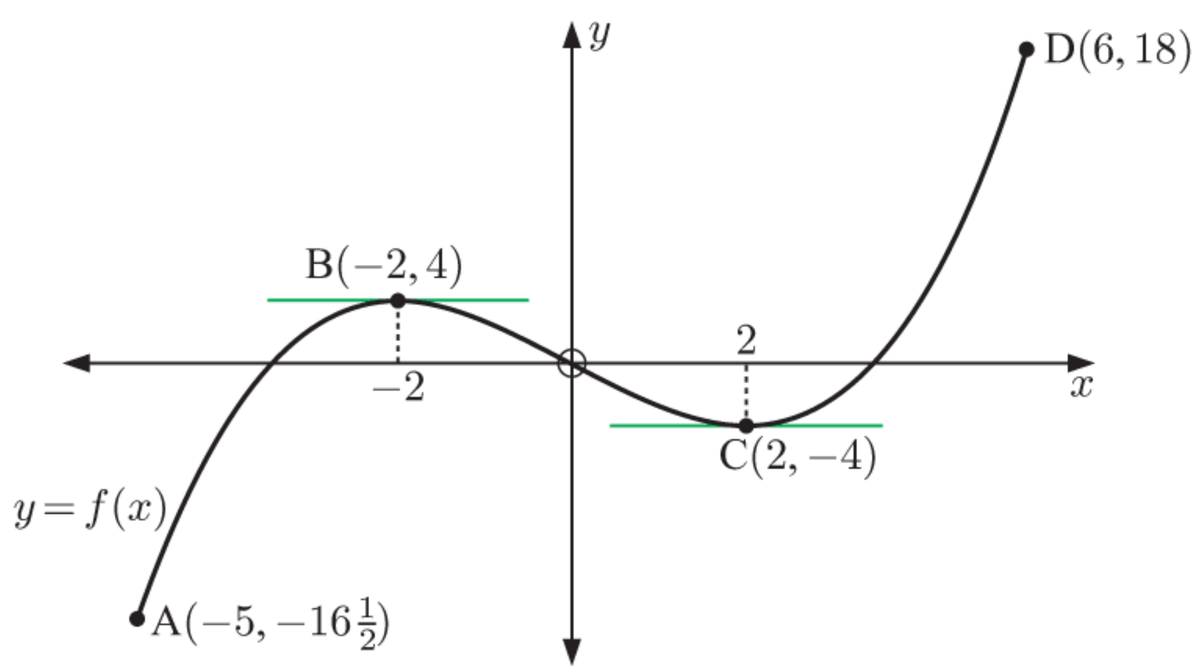
A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a **local maximum** or **local minimum**, or else a **stationary inflection**.



TURNING POINTS (MAXIMA AND MINIMA)

The graph shown has the restricted domain $-5 \leqslant x \leqslant 6$.



A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.

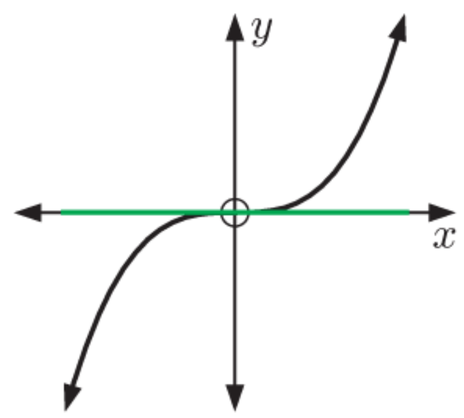
For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

For example, $f(x) = x^3$ has $f'(x) = 3x^2$, so $f'(x) = 0$ when $x = 0$.



The tangent to the curve crosses over the curve at $O(0, 0)$. This tangent is horizontal, but $O(0, 0)$ is neither a local maximum nor a local minimum.

Rather, this point is called a **stationary inflection** (or **inflexion**) as the curve changes its curvature or shape.

SIGN DIAGRAMS

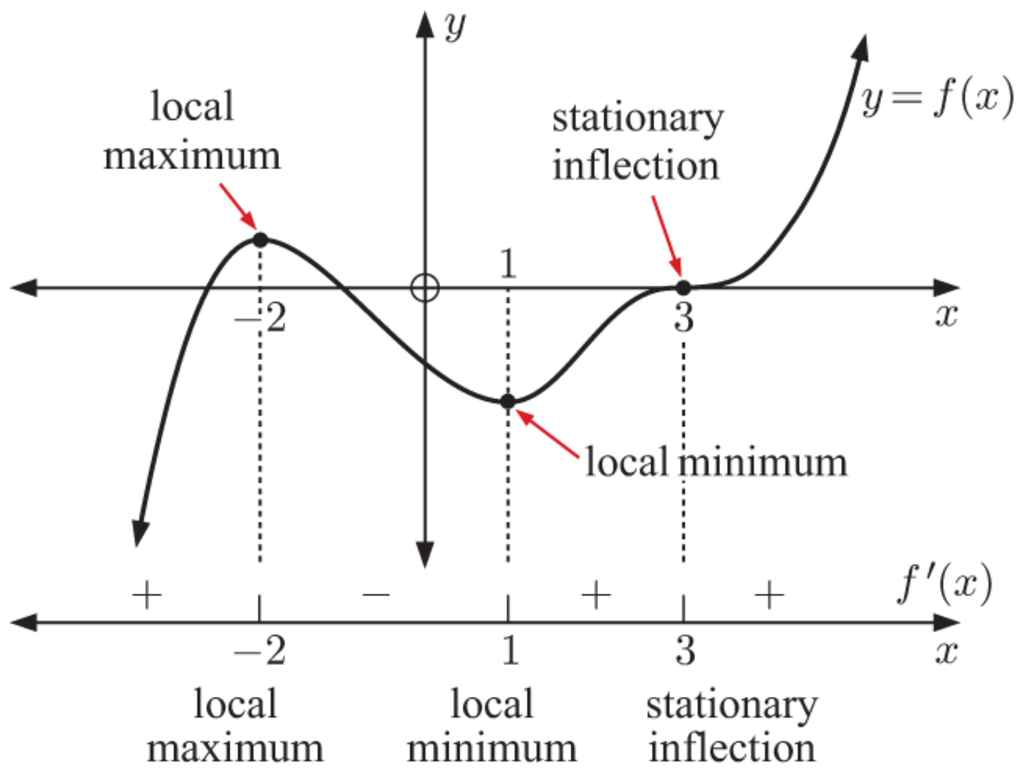
In calculus we commonly use sign diagrams of the derivative function $f'(x)$ to determine the nature of a stationary point.

Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative on that interval.





We observe the following properties:

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection		

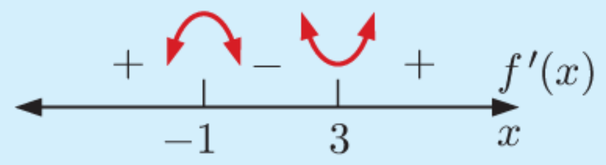
To find the stationary points of a function $f(x)$, we find the values of x for which $f'(x) = 0$. The sign diagram of $f'(x)$ tells us whether each stationary point is a local maximum, local minimum, or stationary inflection.

Example 12 **Self Tutor**

Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 5 \\ \therefore\ f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x - 3)(x + 1) \end{aligned}$$

The sign diagram of $f'(x)$ is

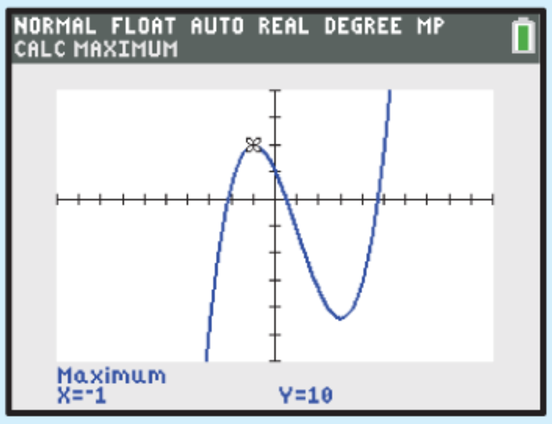


So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) + 5 \\ &= 10 \end{aligned}$$

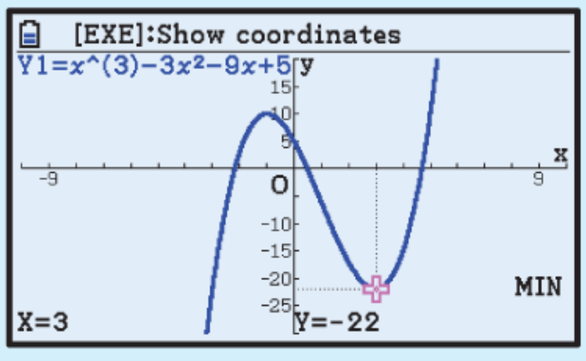
$$\begin{aligned} f(3) &= 3^3 - 3 \times 3^2 - 9 \times 3 + 5 \\ &= -22 \end{aligned}$$

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There is a local maximum at $(-1,\ 10)$.

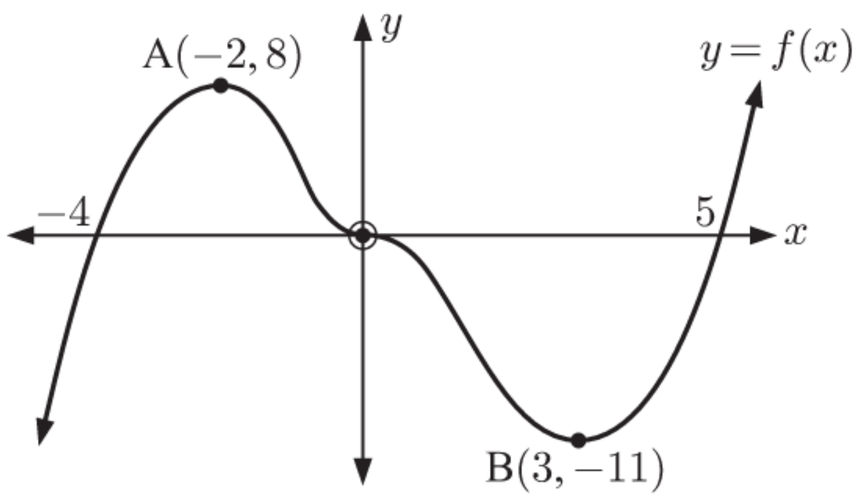
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There is a local minimum at $(3,\ -22)$.

EXERCISE 18D

1 The tangents at points A, O, and B are horizontal.



a Classify points A, O, and B.

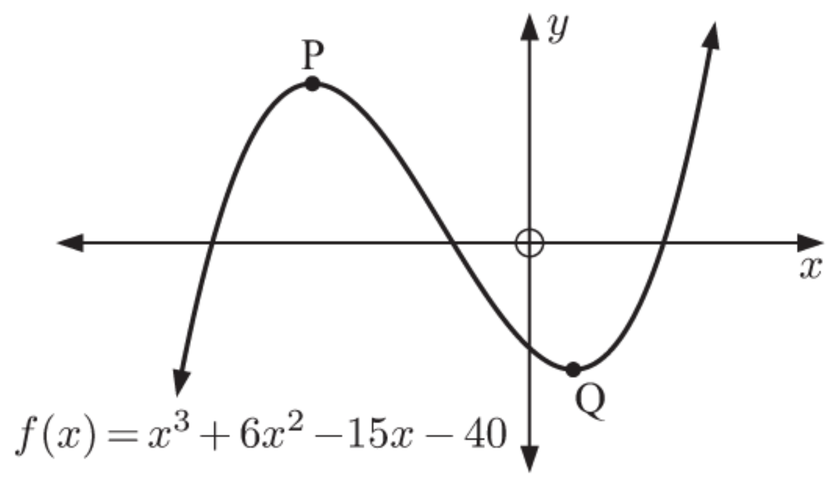
b Draw a sign diagram for the gradient function $f'(x)$ for all x.

c State intervals where $y = f(x)$ is:

i increasing **ii** decreasing.

d Draw a sign diagram for $f(x)$ for all x.

2



The graph of $f(x) = x^3 + 6x^2 - 15x - 40$ is shown alongside. P and Q are stationary points.

a Classify points P and Q.

b Find $f'(x)$.

c Find the coordinates of P and Q.

3 Consider the function $f(x) = \frac{1}{3}x^3 - 9x + 4$.

a Find $f'(x)$, and draw its sign diagram.

b Find intervals where the function is increasing and decreasing.

c Find and classify any stationary points.

d Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

e Sketch the graph of $y = f(x)$, showing the features you have found.

4 Consider the function $g(x) = -2x^3 + 6x^2 + 18x - 7$.

a Find $g'(x)$, and draw its sign diagram.

b Find intervals where the function is increasing and decreasing.

c Find and classify any stationary points.

d Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

e Sketch the graph of $y = g(x)$, showing the features you have found.

5 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

a $y = x^3 + 1$ **b** $y = x^3 - 3x + 2$ **c** $y = x^4 - 2x^2$

d $y = \sqrt{x} + 2$ **e** $y = x - \sqrt{x}$ **f** $y = x^3 - 6x^2 + 12x + 1$

g $y = 1 - x\sqrt{x}$ **h** $y = x^4 - 2x^2 - 8$ **i** $y = x^4 - 6x^2 + 8x - 3$

6 **a** At what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point?

b Under what conditions is the stationary point a local maximum or a local minimum?

7 For each function, find and classify any stationary points, and hence sketch its graph.

a $f(x) = \sqrt{x^2 - 4x + 5}$ **b** $f(x) = \dfrac{x^2}{x-1}$ **c** $f(x) = \dfrac{x}{x^2+1}$

Example 13 **Self Tutor**

Find the exact position and nature of the stationary point of $y = (x-2)e^{-x}$.

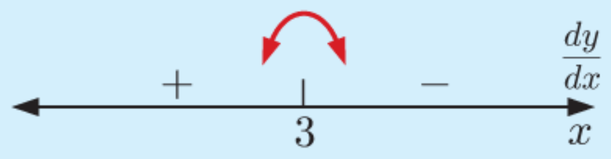
$\dfrac{dy}{dx} = (1)e^{-x} + (x-2)e^{-x}(-1)$ {product rule}

$= e^{-x}(1 - (x-2))$

$= \dfrac{3-x}{e^x}$ where e^x is positive for all x.

So, $\dfrac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\dfrac{dy}{dx}$ is



$\therefore$ at $x = 3$ we have a local maximum.

When $x = 3$, $y = (1)e^{-3} = \dfrac{1}{e^3}$ $\therefore$ the local maximum is at $\left(3, \dfrac{1}{e^3}\right)$.

8 Find the position and nature of the stationary point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \dfrac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

9 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$.

a Find a. **b** Find the coordinates of the local maximum.

10 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b.

b Find the position and nature of all stationary points.

11 $y = \dfrac{e^{ax}}{bx}$ has a stationary point at $\left(\frac{1}{3}, \frac{e}{2}\right)$. Find a and b, and the nature of the stationary point.

12 Consider $f(x) = x \ln x$.

a For what values of x is $f(x)$ defined?

b Show that the minimum value of $f(x)$ is $-\frac{1}{e}$.

13 For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

a $f(x) = \sin x$ **b** $f(x) = \cos 2x$ **c** $f(x) = \sin^2 x$

d $f(x) = e^{\sin x}$ **e** $f(x) = \cos x - \sin x$ **f** $f(x) = \sin 2x + 2\cos x$

14 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

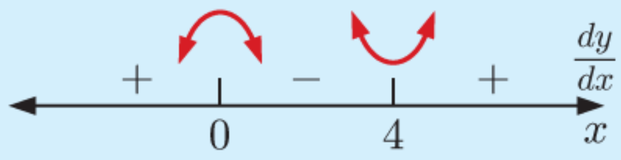
Example 14 **Self Tutor**

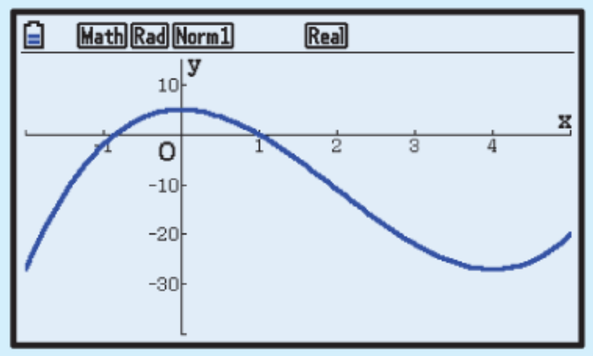
Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

Now $\dfrac{dy}{dx} = 3x^2 - 12x$

$= 3x(x - 4)$

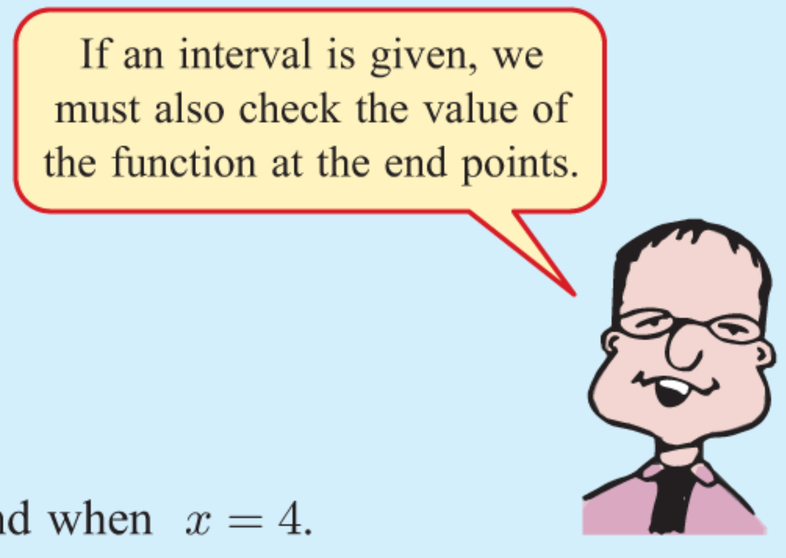
The sign diagram of $\dfrac{dy}{dx}$ is





$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

Critical value (x)	y
-2 (end point)	-27
0 (local maximum)	5
4 (local minimum)	-27
5 (end point)	-20



The greatest of these values is 5 when $x = 0$.

The least of these values is -27 when $x = -2$ and when $x = 4$.

15 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

c $x^2 + \dfrac{16}{x}$ for $1 \leqslant x \leqslant 4$ **d** $x - 4\sqrt{x}$ for $0 \leqslant x \leqslant 5$.

16 Show that $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

17 Consider $f(x) = \sin x \cos 2x$ for $0 \leqslant x \leqslant \pi$.

a Find $f'(x)$ in terms of $\cos x$ only.

b Show that $f'(x) = 0$ when $\cos x = 0$ or $\pm\sqrt{\frac{5}{6}}$.

c Hence find the position and nature of the turning points of $y = f(x)$.

d Graph $y = f(x)$, showing the features you have found.

18 $f(t) = ate^{bt^2}$ has a maximum value of 1 when $t = 2$. Find constants a and b.

19 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$.

20 Consider the function $f(x) = x - \ln x$.

a Show that $y = f(x)$ has a local minimum and that this is the only turning point.

b Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

21 Consider the function $f(x) = \sec x$.

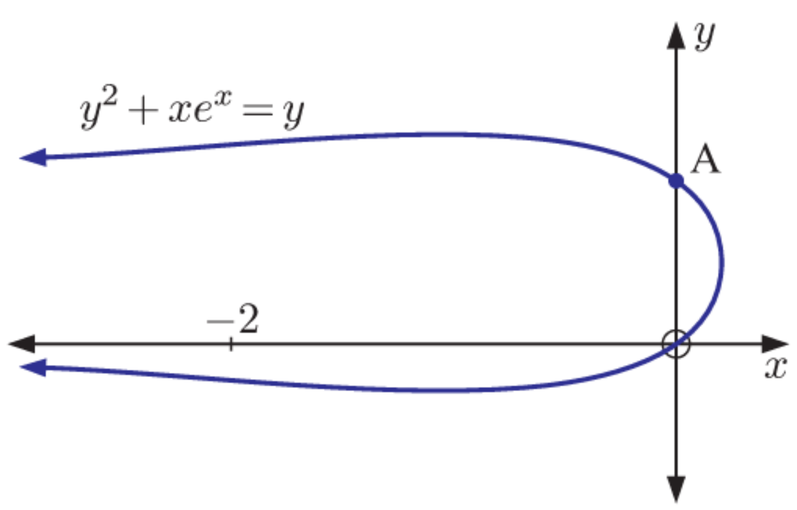
a For what values of x is $f(x)$ undefined on the interval $0 \leqslant x \leqslant 2\pi$?

b Find the position and nature of any stationary points on the interval $0 \leqslant x \leqslant 2\pi$.

c Prove that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$ where $f(x)$ is defined. Explain the geometrical significance of this result.

d Sketch the graph of $y = \sec x$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{2}$. Clearly mark its stationary points.

22



Consider the relation $y^2 + xe^x = y$.

a Find $\dfrac{dy}{dx}$.

b Find the equation of the normal to the relation at A.

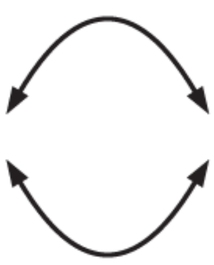
c Find the exact coordinates of the stationary points of the curve.

E SHAPE

We have seen that the first derivative $f'(x)$ gives the gradient of the curve $y = f(x)$ for any value of x.

The second derivative $f''(x)$ tells us the rate of change of the gradient $f'(x)$. It therefore gives us information about the **shape** or **curvature** of the curve $y = f(x)$.

When a curve, or part of a curve, has shape:

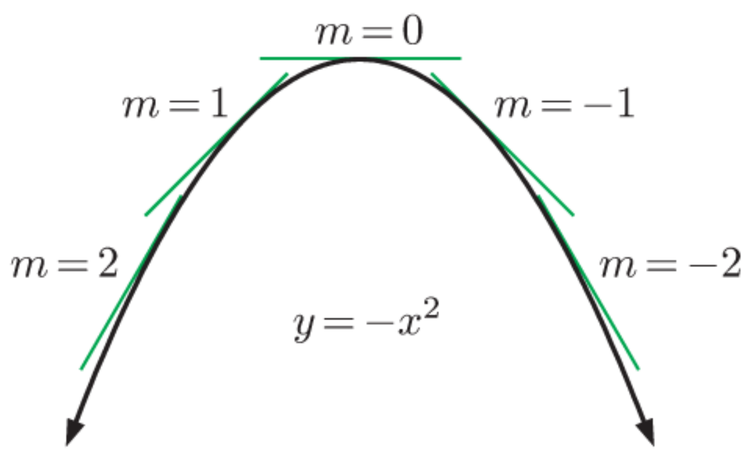


we say that the curve is **concave down**

we say that the curve is **concave up**.

For example:

- the curve $f(x) = -x^2$ is concave down

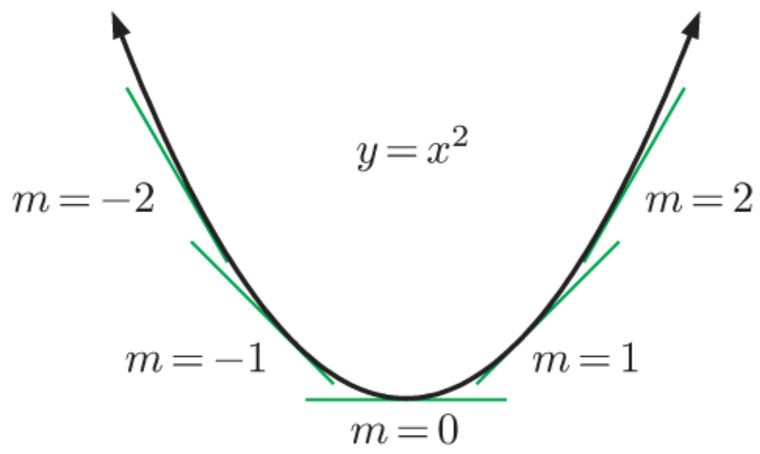


Wherever we are on the curve, as x increases, the gradient of the tangent decreases.

$\therefore$ $f'(x)$ is decreasing

$\therefore$ $f''(x) < 0$.

- the curve $f(x) = x^2$ is concave up.



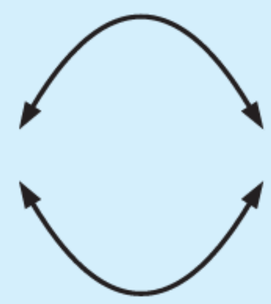
Wherever we are on the curve, as x increases, the gradient of the tangent increases.

$\therefore$ $f'(x)$ is increasing

$\therefore$ $f''(x) > 0$.

We conclude that:

- A curve is **concave down** on an interval $S \Leftrightarrow f''(x) \leqslant 0$ for all $x \in S$.
- A curve is **concave up** on an interval $S \Leftrightarrow f''(x) \geqslant 0$ for all $x \in S$.



Example 15 **Self Tutor**

Find intervals where the curve is concave up or concave down:

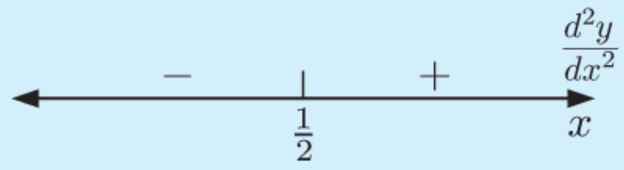
a $y = 2x^3 - 3x^2 + 4x - 6$ **b** $f(x) = \dfrac{x^2 - 5x + 4}{x - 3}$

a $y = 2x^3 - 3x^2 + 4x - 6$

$\therefore \dfrac{dy}{dx} = 6x^2 - 6x + 4$

$\therefore \dfrac{d^2y}{dx^2} = 12x - 6$

$= 6(2x - 1)$ which has sign diagram:



The curve is concave up for $x \geqslant \frac{1}{2}$ and concave down for $x \leqslant \frac{1}{2}$.

b $f(x) = \dfrac{x^2 - 5x + 4}{x - 3}$

$\therefore f'(x) = \dfrac{(2x - 5)(x - 3) - (x^2 - 5x + 4)(1)}{(x - 3)^2}$

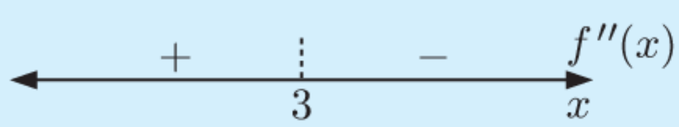
$= \dfrac{2x^2 - 6x - 5x + 15 - x^2 + 5x - 4}{(x - 3)^2}$

$= \dfrac{x^2 - 6x + 11}{(x - 3)^2}$

$\therefore f''(x) = \dfrac{(2x - 6)(x - 3)^{\cancel{2}} - (x^2 - 6x + 11) \times 2\cancel{(x - 3)}}{(x - 3)^{\cancel{4}3}}$

$= \dfrac{2x^2 - 6x - 6x + 18 - 2x^2 + 12x - 22}{(x - 3)^3}$

$= -\dfrac{4}{(x - 3)^3}$ which has sign diagram:

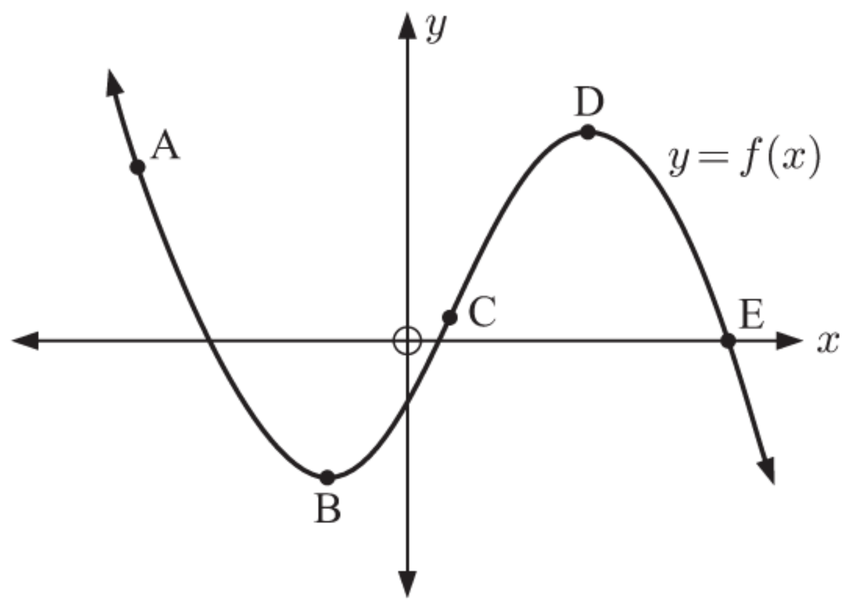


The curve is concave up for $x < 3$ and concave down for $x > 3$.

EXERCISE 18E

1 **a** Complete the table by indicating whether each value is zero, positive, or negative:

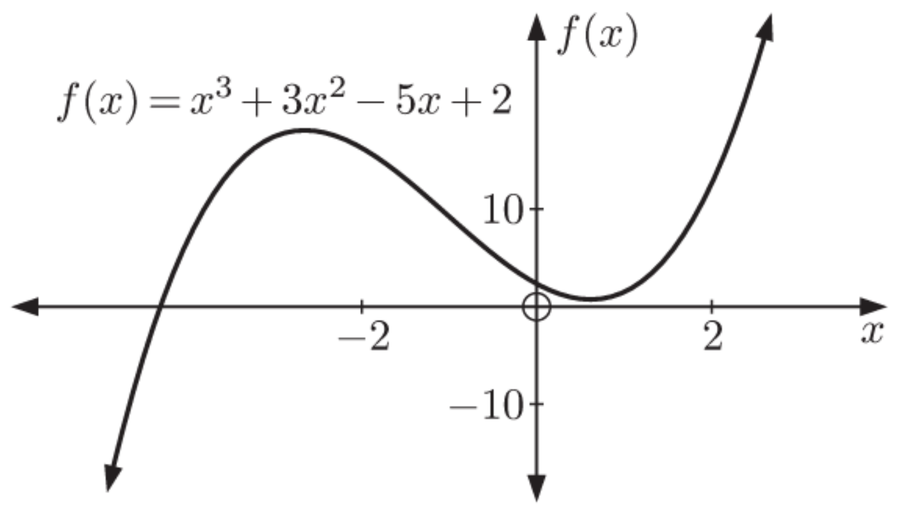
Point	$f(x)$	$f'(x)$	$f''(x)$
A	+		
B			
C			
D		0	
E			



b Describe the turning points of $y = f(x)$.

c At which point does the shape of $y = f(x)$ change?

2 The graph of $f(x) = x^3 + 3x^2 - 5x + 2$ is shown alongside.



a Find $f'(x)$ and $f''(x)$.

b Draw the sign diagram of $f''(x)$.

c State the interval on which the function is:

i concave up **ii** concave down.

3 Determine the shape of each quadratic function:

a $y = 2x^2 - 3x + 4$ **b** $y = -2(x-3)(x+1)$

c $y = -4 - x^2 + 6x$ **d** $y = (5-x)(1-2x)$

4 For each of the following functions, determine the interval(s) on which the function is:

i increasing **ii** decreasing **iii** concave up **iv** concave down.

a $f(x) = x^2 + 1$ **b** $f(x) = -x^3$ **c** $f(x) = e^x$

d $f(x) = \sqrt{x} - 2$ **e** $f(x) = -\dfrac{1}{\sqrt{x}}$ **f** $f(x) = x^4 - 12x^2$

5 Consider $f(x) = \ln(2x-1) - 3$.

a Find the x-intercept.

b Can $f(0)$ be found? What is the significance of this result?

c Find the domain of f.

d Find the gradient of the tangent to the curve at $x = 1$.

e Find $f''(x)$, and hence explain why $f(x)$ is concave down for all x in the domain of f.

f Graph the function, showing the features you have found.

6 Consider $f(x) = \ln x$.

a For what values of x is $f(x)$ defined?

b Draw the sign diagrams of $f'(x)$ and $f''(x)$, and give a geometrical interpretation of each.

c Find the equation of the normal to $y = f(x)$ at the point where $y = 1$.

7 Find intervals where the curve is concave up or concave down:

a $f(x) = -x^3 + 3x - 2$ **b** $y = -\dfrac{4}{x^2}$ **c** $f(x) = \dfrac{1}{x} + \ln x$

d $f(x) = \dfrac{3-x}{x+2}$ **e** $y = \dfrac{x^2+x-3}{x+1}$ **f** $f(x) = \dfrac{\ln x}{x-2}$

8 Find intervals where the curve $f(x) = \dfrac{x}{x^2+1}$ is:

a increasing **b** decreasing **c** concave up **d** concave down.

9 Consider the function $f(x) = \dfrac{e^x}{x}$.

a Does the graph of $y = f(x)$ have any x or y-intercepts?

b Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

c Find and classify any stationary points of $y = f(x)$.

d Find the intervals where $f(x)$ is:

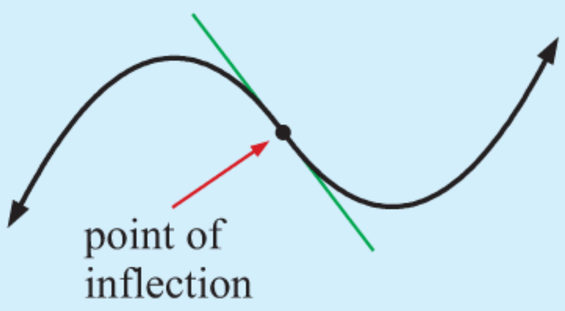
i concave up **ii** concave down.

e Sketch the graph of $y = f(x)$, showing all important features.

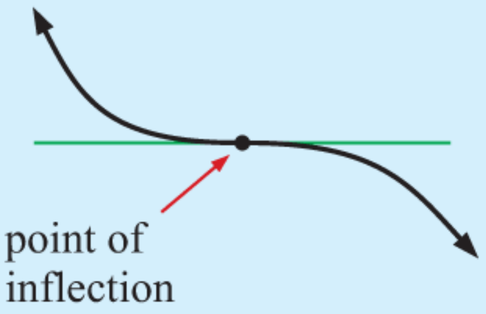
f Find the equation of the tangent to $f(x) = \dfrac{e^x}{x}$ at the point where $x = -1$.

F INFLECTION POINTS

A **point of inflection** is a point at which the tangent to the curve crosses the curve.
At a point of inflection, $f''(x) = 0$.



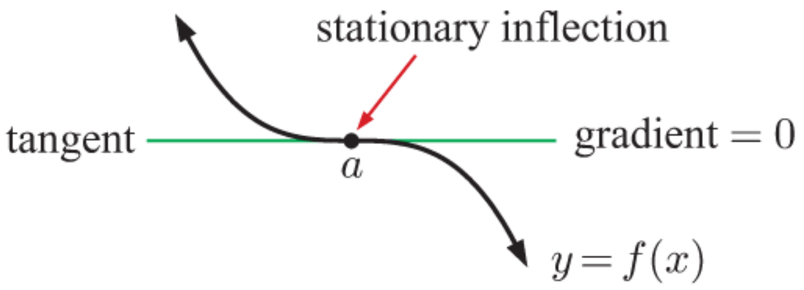
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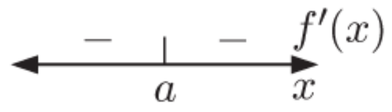
DEMO



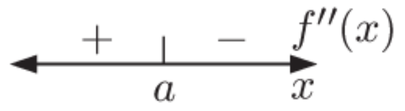
If the tangent at a point of inflection is horizontal, then this is a **stationary inflection point**.



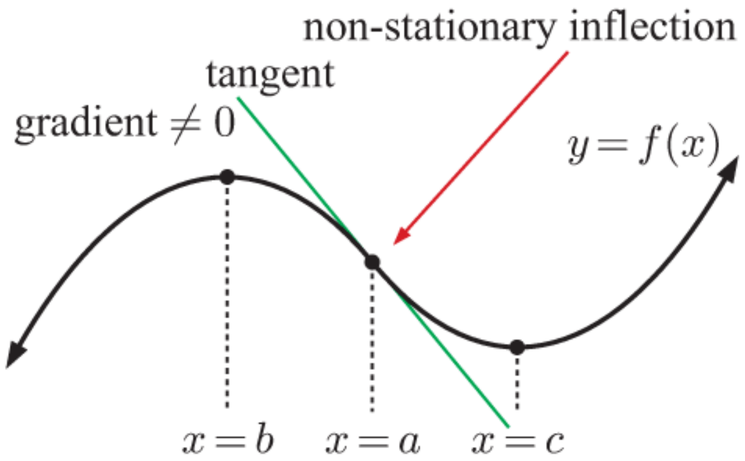
$f'(x)$ has sign diagram



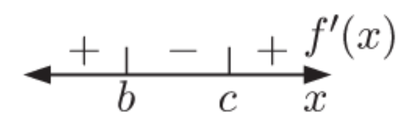
$f''(x)$ has sign diagram



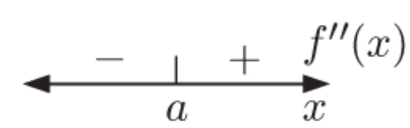
If the tangent at a point of inflection is *not* horizontal, then this is a **non-stationary inflection point**.



$f'(x)$ has sign diagram



$f''(x)$ has sign diagram



The tangent at the point of inflection, also called the **inflecting tangent**, crosses the curve at that point.

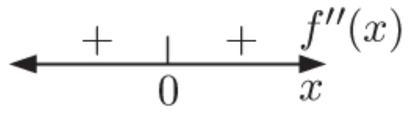
There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes at $x = a$.

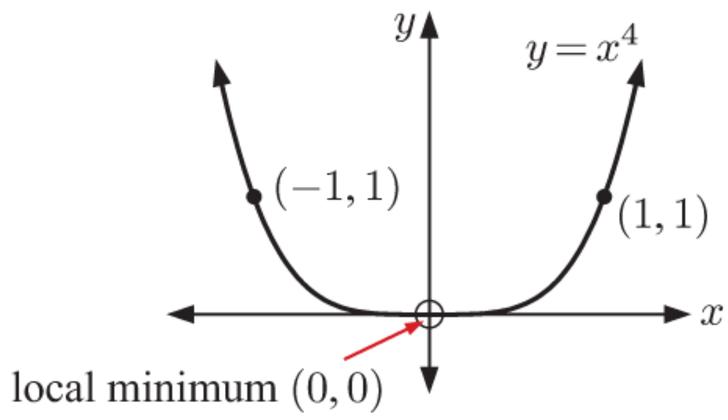
The point of inflection is a:

- **stationary inflection** if $f'(a) = 0$
- **non-stationary inflection** if $f'(a) \neq 0$.

Notice that if $f(x) = x^4$ then $f'(x) = 4x^3$
and $f''(x) = 12x^2$.

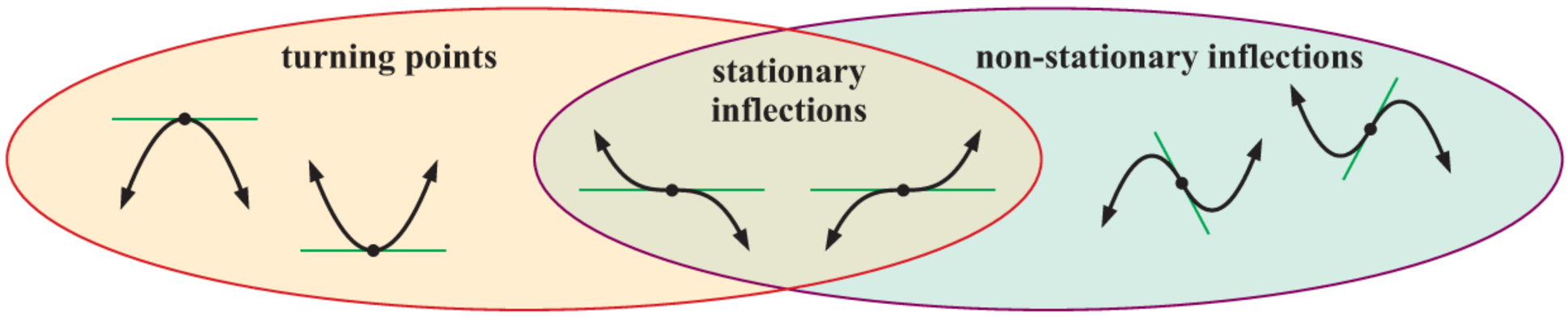
$f''(x)$ has sign diagram





Although $f''(0) = 0$ we do not have a point of inflection at $(0, 0)$ because the sign of $f''(x)$ does not change at $x = 0$.

SUMMARY



stationary points	**inflections**
$f'(x) = 0$	$f''(x) = 0$ and change of sign

Click on the demo icon to examine some common functions for turning points, points of inflection, and intervals where the function is increasing, decreasing, and concave up or down.



Example 16 **Self Tutor**

Consider $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$.

a Find and classify all points where $f'(x) = 0$.

b Find and classify all points of inflection.

c Find intervals where the function is increasing or decreasing.

d Find intervals where the function is concave up or down.

e Sketch the function showing the features you have found.

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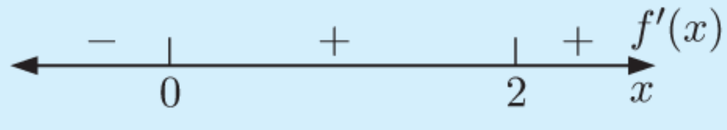
a $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$

$\therefore f'(x) = 12x^3 - 48x^2 + 48x$

$= 12x(x^2 - 4x + 4)$

$= 12x(x - 2)^2$

$\therefore$ $f'(x)$ has sign diagram:

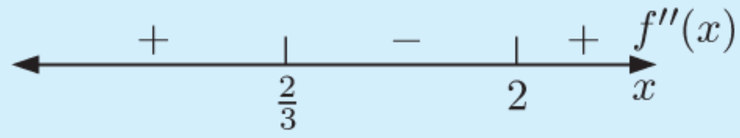


Now $f(0) = -9$ and $f(2) = 7$

$\therefore$ $(0, -9)$ is a local minimum and $(2, 7)$ is a stationary inflection.

b $f''(x) = 36x^2 - 96x + 48$
$= 12(3x^2 - 8x + 4)$
$= 12(x - 2)(3x - 2)$

$\therefore$ $f''(x)$ has sign diagram:



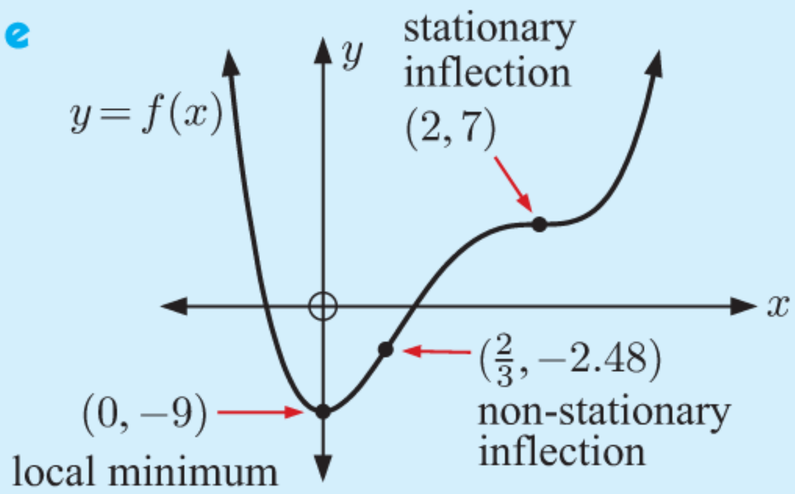
Now $f(\frac{2}{3}) \approx -2.48$

$\therefore$ $(2, 7)$ is a stationary inflection and $(\frac{2}{3}, -2.48)$ is a non-stationary inflection.

c $f(x)$ is decreasing for $x \leqslant 0$
$f(x)$ is increasing for $x \geqslant 0$.

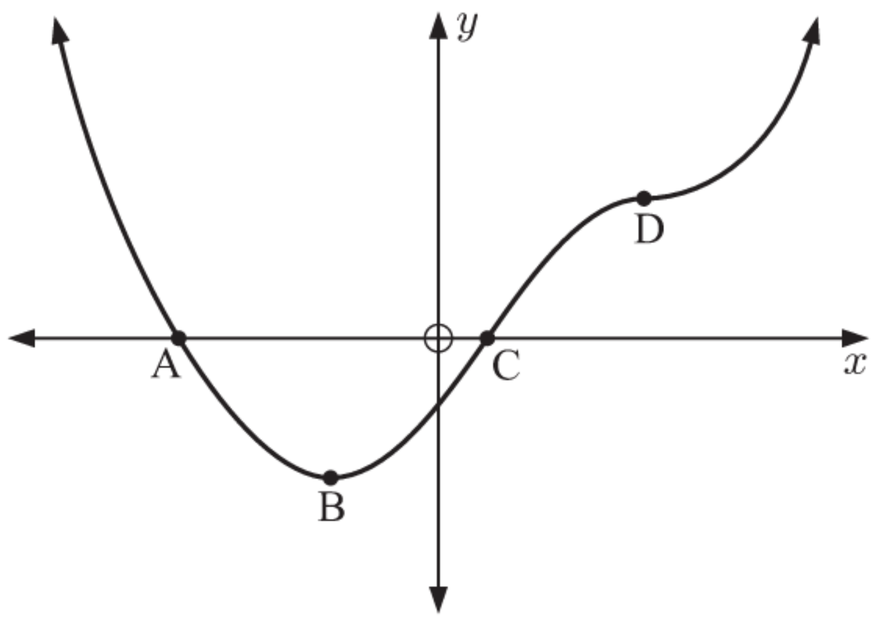
d $f(x)$ is concave up for $x \leqslant \frac{2}{3}$ and $x \geqslant 2$
$f(x)$ is concave down for $\frac{2}{3} \leqslant x \leqslant 2$.

e



EXERCISE 18F

1 In the diagram alongside, each labelled point corresponds to a zero of $f(x)$, $f'(x)$, or $f''(x)$.



a Complete the table by indicating whether each value is zero, positive, or negative:

Point	$f(x)$	$f'(x)$	$f''(x)$
A			
B			
C			
D			

b Describe the turning point of $y = f(x)$.

c Describe the inflection points of $y = f(x)$.

2 Find and classify all points of inflection of:

a $y = 2 - x^3$

b $y = x^3 - 6x^2 + 9x + 1$

c $y = -3x^4 - 8x^3 + 2$

d $y = 3 - \dfrac{1}{\sqrt{x}}$

e $y = x^3 + 6x^2 + 12x + 5$

f $y = x^2 + 8\sqrt{x}$

g $y = x^4 - 6x^2 + 10$

h $y = 2x^2 - \dfrac{6}{x^2}$

i $y = \dfrac{1}{\ln x}$

3 For each of the following functions:

i Find and classify all turning points.
ii Find and classify all points of inflection.
iii Find intervals where the function is increasing or decreasing.
iv Find intervals where the function is concave up or down.
v Sketch the function showing the features you have found.

a $f(x) = x^2 - 5x + 4$

b $f(x) = x^3 + 4x^2$

c $f(x) = \sqrt{x}$

d $f(x) = x^3 - 3x^2 - 24x + 1$

e $f(x) = 3x^4 + 4x^3 - 2$

f $f(x) = (x - 1)^4$

g $f(x) = x^4 - 4x^2 + 3$

h $f(x) = 3 - \dfrac{4}{\sqrt{x}}$

Example 17 **Self Tutor**

Consider the function $y = 2 - e^{-x}$.

a Find the x-intercept. **b** Find the y-intercept.

c Show algebraically that the function is increasing for all x.

d Show algebraically that the function is concave down for all x.

e Explain why $y = 2$ is a horizontal asymptote.

f Sketch $y = 2 - e^{-x}$, showing the features you have found.

a When $y = 0$, $e^{-x} = 2$

$\therefore\ -x = \ln 2$

$\therefore\ x = -\ln 2$

$\therefore$ the x-intercept is $-\ln 2 \approx -0.693$

b When $x = 0$, $y = 2 - e^0 = 1$

$\therefore$ the y-intercept is 1.

c $\dfrac{dy}{dx} = 0 - e^{-x}(-1) = e^{-x} = \dfrac{1}{e^x}$

Now $e^x > 0$ for all x,

so $\dfrac{dy}{dx} > 0$ for all x.

$\therefore$ the function is increasing for all x.

d $\dfrac{d^2y}{dx^2} = e^{-x}(-1)$

$= -\dfrac{1}{e^x}$ which is < 0 for all x.

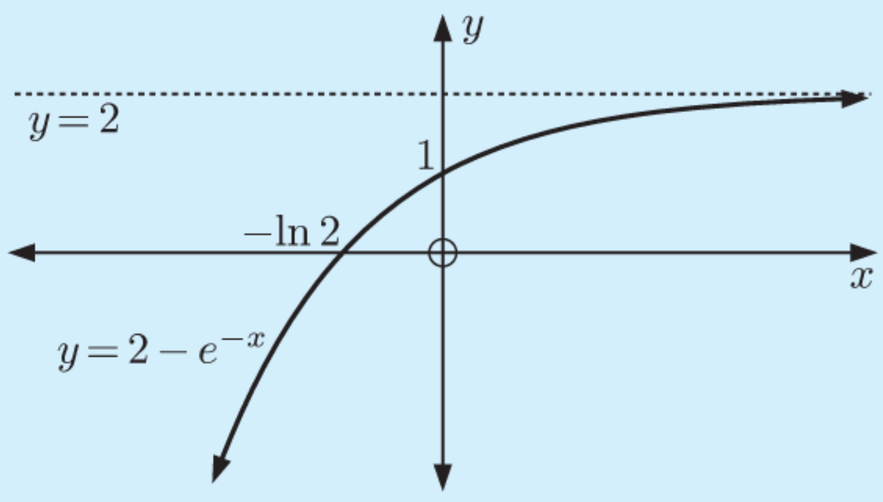
$\therefore$ the function is concave down for all x.

e As $x \to \infty$, $e^{-x} \to 0$

$\therefore\ y \to 2$

Hence the horizontal asymptote is $y = 2$.

f



4 Consider the function $f(x) = e^{2x} - 3$.

a Find the x and y-intercepts.

b Show algebraically that the function is increasing for all x.

c Find $f''(x)$, and hence explain why $f(x)$ is concave up for all x.

d Explain why $y = -3$ is a horizontal asymptote.

e Sketch $y = e^{2x} - 3$, showing the features you have found.

5 Suppose $f(x) = e^x - 3$ and $g(x) = 3 - 5e^{-x}$.

a Find the x and y-intercepts of both functions.

b Discuss $f(x)$ and $g(x)$ as $x \to \infty$ and as $x \to -\infty$.

c Draw the sign diagrams of $f'(x)$, $f''(x)$, $g'(x)$, and $g''(x)$ and give a geometrical interpretation of each.

d Find algebraically the point(s) of intersection of the functions.

e Sketch the graphs of both functions on the same set of axes. Show all important features on your graph.

6 Consider the function $y = e^x - 3e^{-x}$.

a Determine the x and y-intercepts.

b Prove that the function is increasing for all x.

c Show that $\dfrac{d^2y}{dx^2} = y$. What can be deduced about the concavity of the function above and below the x-axis?

d Use technology to help graph $y = e^x - 3e^{-x}$. Show the features you have found.

7 Consider $f(x) = \ln(3x^2)$.

a For what values of x is $f(x)$ defined?

b Draw the sign diagrams of $f'(x)$ and $f''(x)$, and give a geometrical interpretation of each.

c Sketch the graph of $f(x) = \ln(3x^2)$.

8 A function commonly used in statistics is the *standard normal curve* $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

a Find the turning points of the function, and find the intervals where the function is increasing and decreasing.

b Find all points of inflection.

c Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

d Sketch the graph of $y = f(x)$, showing all important features.

9 Consider the function $f(x) = \cos x$.

a Show that $f''(x) = -f(x)$. What does this tell us about the location of the inflection points?

b Find and classify the inflection points of $f(x)$ on $0 \leqslant x \leqslant 2\pi$.

c Find the intervals on $0 \leqslant x \leqslant 2\pi$ where $f(x)$ is:

i increasing **ii** decreasing **iii** concave up **iv** concave down.

d Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$, showing all important features.

10 Consider the function $y = 4^x - 2^x$.

a Find the axes intercepts.

b Discuss the graph as $x \to \infty$ and as $x \to -\infty$.

c Find the position and nature of any stationary points.

d Discuss the shape of the function.

e Sketch the function, showing the features you have found.

11 Consider the *surge function* $f(t) = Ate^{-bt}$, $t \geqslant 0$, where A and b are positive constants.

a Prove that the function has:

i a local maximum at $t = \dfrac{1}{b}$ **ii** a point of inflection at $t = \dfrac{2}{b}$.

b Sketch the function, showing the features you have found.

12 Consider the *logistic function* $f(t) = \dfrac{C}{1 + Ae^{-bt}}$, $t \geqslant 0$, where A, b, and C are positive constants.

a Find the y-intercept.

b Prove that $y = C$ is its horizontal asymptote.

c Prove that if $A > 1$, there is a point of inflection with y-coordinate $\dfrac{C}{2}$.

d Sketch the function, showing the features you have found.

13 Find the point of inflection of:

a $y = \arcsin x$ **b** $y = \arctan\left(x - \frac{1}{2}\right)$

G UNDERSTANDING FUNCTIONS AND THEIR DERIVATIVES

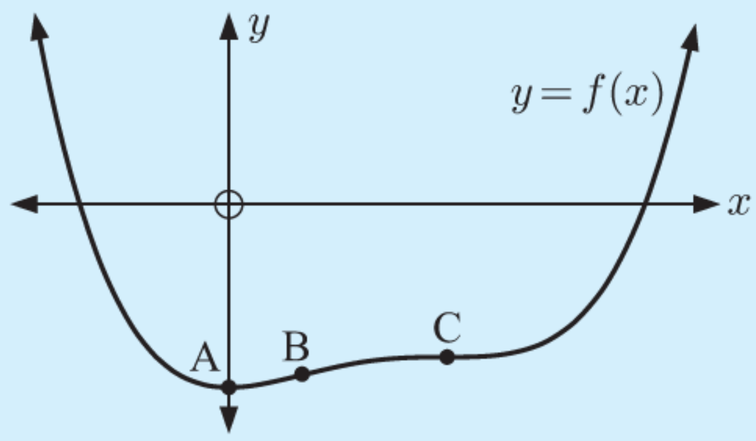
In this Chapter we have seen that, given a function $f(x)$:

- The sign of $f'(x)$ tells us intervals where $f(x)$ is increasing or decreasing.
- The zeros of $f'(x)$ indicate stationary points.
- The sign of $f''(x)$ tells us the *shape* of $f(x)$.
- The zeros of $f''(x)$ indicate points of inflection provided $f''(x)$ changes sign at that point.

We can use these properties to analyse how the graphs of $f(x)$, $f'(x)$, and $f''(x)$ are related.

Example 18 **Self Tutor**

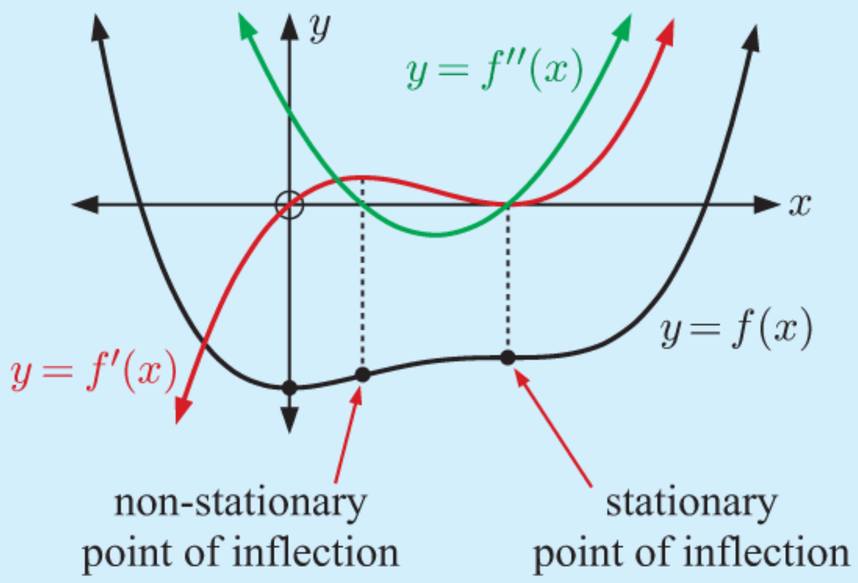
Using the graph of $y = f(x)$ alongside, sketch the graphs of $y = f'(x)$ and $y = f''(x)$.



The local minimum A corresponds to $f'(x) = 0$ and $f''(x) \neq 0$.

The non-stationary point of inflection B corresponds to $f'(x) \neq 0$ and $f''(x) = 0$.

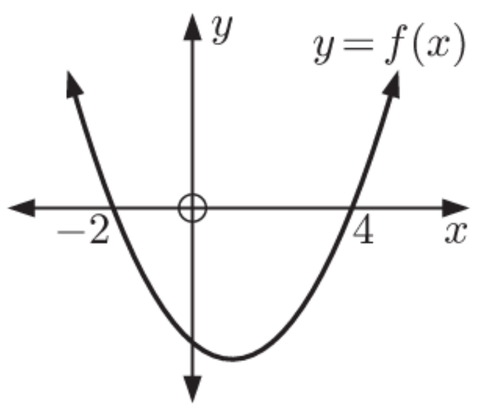
The stationary point of inflection C corresponds to $f'(x) = 0$ and $f''(x) = 0$.



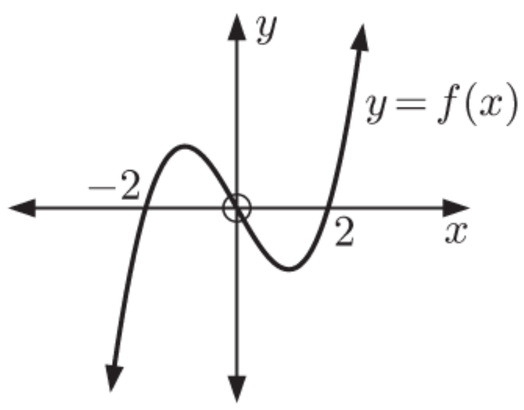
EXERCISE 18G

1 Using the graphs of $y = f(x)$ below, sketch the graphs of $y = f'(x)$ and $y = f''(x)$. Show clearly the axes intercepts and turning points.

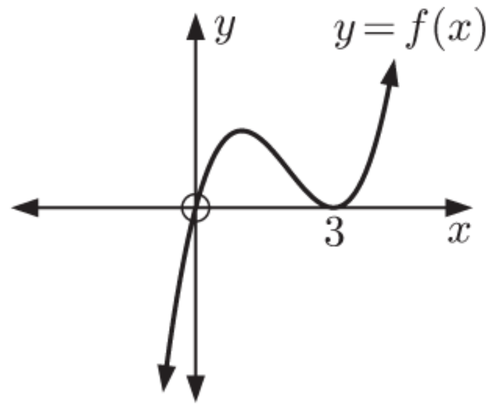
a



b



c

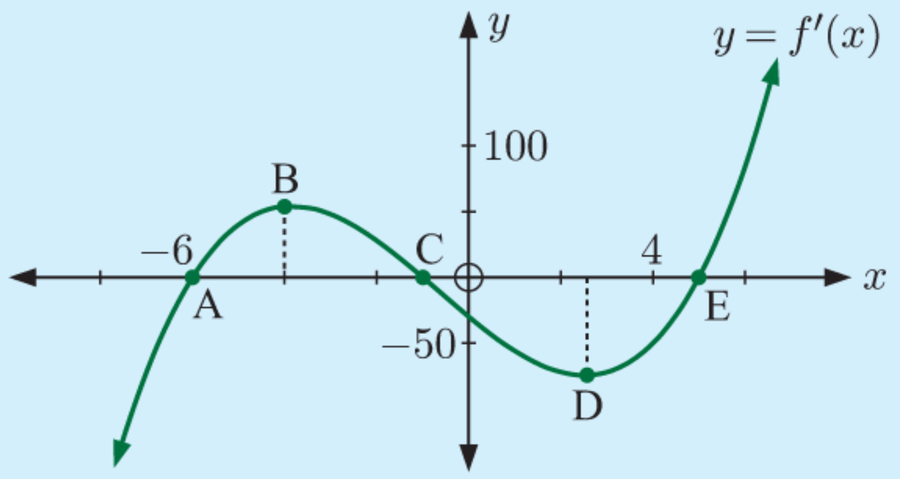


Example 19

Self Tutor

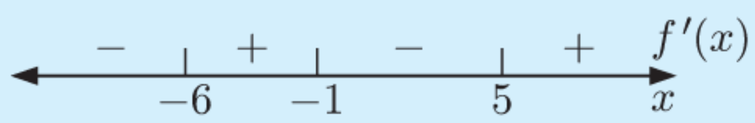
The graph alongside shows a gradient function $y = f'(x)$.

Sketch a graph which could be $y = f(x)$, showing clearly the x-values corresponding to all stationary points and points of inflection.



The stationary points of $f(x)$ are when $f'(x) = 0$. These correspond to A, C, and E.

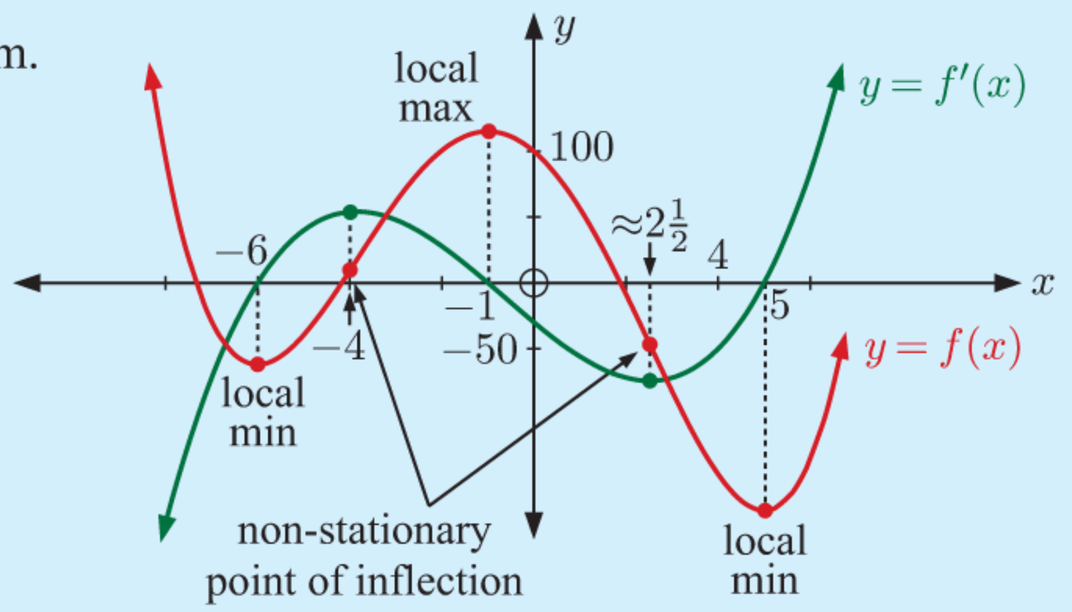
The sign diagram of $f'(x)$ is



$\therefore$ A and E correspond to local minima, and C corresponds to a local maximum.

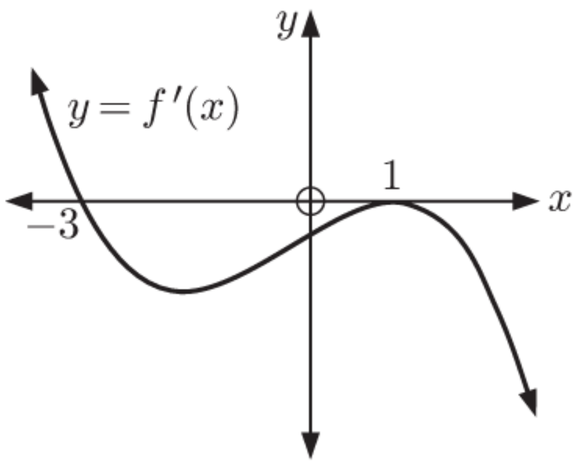
$f'(x)$ is a maximum when $x = -4$ and a minimum when $x \approx 2\frac{1}{2}$.

At these points $f''(x) = 0$ but $f'(x) \neq 0$, so they correspond to non-stationary points of inflection.

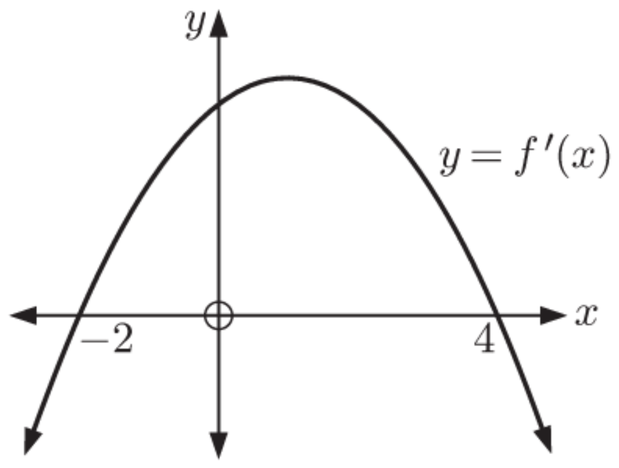


2 For each graph of $y = f'(x)$ below, sketch a graph which could be $y = f(x)$. Show clearly the location of any stationary points and points of inflection.

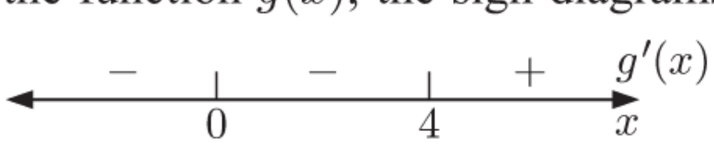
a

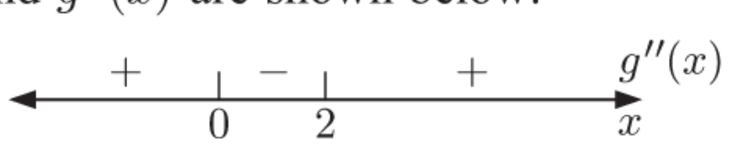


b



3 For the function $g(x)$, the sign diagrams for $g'(x)$ and $g''(x)$ are shown below:





The points A(0, 2), B(2, 0), and C(4, −2) all lie on $y = g(x)$.
Sketch $y = g(x)$, labelling the stationary points.

ACTIVITY

Click on the icon to run a card game on curve properties.



H L'HÔPITAL'S RULE

We have previously studied the behaviour of curves near asymptotes, points of discontinuity, and for large values of $|x|$, by considering limits. However, it is not always obvious whether a limit exists, and if it does, how it can be calculated.

For example:

- The curve $f(x) = \dfrac{\sin x}{x}$ has a discontinuity at $x = 0$.

  In the limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$, both $\lim\limits_{x\to 0} \sin x = 0$ and $\lim\limits_{x\to 0} x = 0$.

  We therefore need to consider something of the form $\frac{0}{0}$.

  In this particular case, we have seen previously that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

- For the curve $f(x) = x^2 e^{-x}$, consider the behaviour as $x \to \infty$.

  We know that x^2 becomes infinitely large, and that $\lim\limits_{x\to\infty} e^{-x} = 0$.

  We therefore need to consider something of the form $\infty \times 0$.

There are three **indeterminate forms** which we need to deal with:

Type	*Description*
$\frac{0}{0}$	$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$
$\frac{\infty}{\infty}$	$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$
$0 \times \infty$	$\lim\limits_{x\to a} [f(x)\,g(x)]$ where $\lim\limits_{x\to a} f(x) = 0$ and $g(x) \to \pm\infty$ as $x \to a$

To address these types of limits, we use **l'Hôpital's rule**:

Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ on an interval that contains the point $x = a$.

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, or, if as $x \to a$, $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, then

$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ provided the limit on the right exists.

Proof for the case of l'Hôpital's rule where $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, $f'(x)$ and $g'(x)$ continuous, $g'(a) \neq 0$:

Since $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ where f and g are continuous, $f(a) = g(a) = 0$

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \qquad \{\text{since } x \neq a\}$$

$$= \frac{\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \frac{g(x)-g(a)}{x-a}} \qquad \{\text{limit laws}\}$$

$$= \frac{f'(a)}{g'(a)}$$
$$= \frac{\lim\limits_{x\to a} f'(x)}{\lim\limits_{x\to a} g'(x)} \quad \{\text{since } f'(x) \text{ and } g'(x) \text{ are continuous}\}$$
$$= \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Important: One limit we should not attempt to prove using l'Hôpital's rule is the Fundamental Trigonometric Limit $\lim\limits_{x\to 0} \frac{\sin x}{x}$. This is because the derivative of $\sin x$ from first principles itself requires the use of the limit.

Example 20 **Self Tutor**

Use l'Hôpital's rule to evaluate:

a $\lim\limits_{x\to 0} \frac{2^x - 1}{x}$ **b** $\lim\limits_{x\to\infty} xe^{-x}$ **c** $\lim\limits_{x\to\infty} \frac{\ln x}{x}$

a $\lim\limits_{x\to 0} (2^x - 1) = 0$ and $\lim\limits_{x\to 0} x = 0$, so we can use l'Hôpital's rule if the resulting limit exists.

$$\therefore \lim_{x\to 0} \frac{2^x - 1}{x} = \frac{\lim\limits_{x\to 0} \frac{d}{dx}(2^x - 1)}{\lim\limits_{x\to 0} \frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\}$$
$$= \frac{\lim\limits_{x\to 0} 2^x \ln 2}{\lim\limits_{x\to 0} 1}$$
$$= \frac{\ln 2}{1}$$
$$= \ln 2$$

$\lim\limits_{x\to 0} \frac{2^x - 1}{x}$ has indeterminate form $\frac{0}{0}$.



b As $x \to \infty$, $e^{-x} \to 0$, so we can use l'Hôpital's rule.

$$\therefore \lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \quad \{\text{convert to the form } \tfrac{\infty}{\infty}\}$$
$$= \frac{\lim\limits_{x\to\infty} \frac{d}{dx}(x)}{\lim\limits_{x\to\infty} \frac{d}{dx}(e^x)} \quad \{\text{l'Hôpital's rule}\}$$
$$= \frac{\lim\limits_{x\to\infty} 1}{\lim\limits_{x\to\infty} (e^x)}$$
$$= 0 \quad \{\lim\limits_{x\to\infty} 1 = 1 \text{ and as } x \to \infty,\ e^x \to \infty\}$$

$\lim\limits_{x\to\infty} xe^{-x}$ has indeterminate form $\infty \times 0$.



c As $x \to \infty$, $\ln x \to \infty$ and $x \to \infty$, so we can use l'Hôpital's rule.

$$\therefore \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\lim\limits_{x\to\infty} \frac{d}{dx}(\ln x)}{\lim\limits_{x\to\infty} \frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\}$$
$$= \frac{\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)}{\lim\limits_{x\to\infty} 1}$$
$$= 0 \quad \{\lim\limits_{x\to\infty} \left(\frac{1}{x}\right) = 0\}$$

$\lim\limits_{x\to\infty} \frac{\ln x}{x}$ has indeterminate form $\frac{\infty}{\infty}$.



EXERCISE 18H

1 Evaluate, if possible, using l'Hôpital's rule.

a $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ **b** $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2}$ **c** $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

d $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$ **e** $\lim\limits_{x\to 0^+} x\ln x$ **f** $\lim\limits_{x\to 0} \dfrac{\arctan x}{x}$

g $\lim\limits_{x\to 0} \dfrac{x^2+x}{\sin 2x}$ **h** $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$ **i** $\lim\limits_{x\to 0} \dfrac{x+\sin x}{x-\sin x}$

j $\lim\limits_{x\to 0^+} x^2\ln x$ **k** $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{\sin x}$, where $a, b > 0$ are real constants.

2 Discuss the behaviour of $f(x) = x^2e^{-x}$ as $x \to \infty$.

3 Attempt to find $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{\tan x}{\sec x}$ using l'Hôpital's rule.

4 Does $y = \dfrac{\frac{\pi}{2} - \arccos x - x}{x^3}$ have a vertical asymptote when $x = 0$? Explain your answer.

5 Consider the function $f(x) = \dfrac{e^x-1}{x^2+x}$.

a State the domain of f. **b** Find $f'(x)$ and state its domain.

c Find the turning point of $y = f(x)$.

d Discuss the behaviour of $f(x)$: **i** as $x \to -\infty$ **ii** near $x = 0$.

e Sketch $y = f(x)$, showing the features you have found.

Example 21 Self Tutor

Find $\lim\limits_{x\to 0^+} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)}$.

$\lim\limits_{x\to 0^+} \ln(\cos 3x) = 0$ and $\lim\limits_{x\to 0^+} \ln(\cos 2x) = 0$, so we can use l'Hôpital's rule.

$$\therefore \lim_{x\to 0^+} \frac{\ln(\cos 3x)}{\ln(\cos 2x)} = \lim_{x\to 0^+} \left(\frac{\frac{-3\sin 3x}{\cos 3x}}{\frac{-2\sin 2x}{\cos 2x}} \right) \quad \{\text{l'Hôpital's rule}\}$$

$$= \lim_{x\to 0^+} \left(\frac{3\sin 3x \cos 2x}{2\sin 2x \cos 3x} \right)$$

$$= \left(\lim_{x\to 0^+} \frac{\sin 3x}{\sin 2x} \right) \times \left(\lim_{x\to 0^+} \frac{3\cos 2x}{2\cos 3x} \right)$$

$$= \left(\lim_{x\to 0^+} \frac{\sin 3x}{\sin 2x} \right) \times \frac{3}{2}$$

$$= \lim_{x\to 0^+} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 2x}{2x}} \times \frac{9}{4}$$

$$= \frac{\lim\limits_{x\to 0^+} \frac{\sin 3x}{3x}}{\lim\limits_{x\to 0^+} \frac{\sin 2x}{2x}} \times \frac{9}{4}$$

$$= \frac{9}{4} \quad \{\text{Fundamental Trigonometric Limit}\}$$

6 Evaluate, if possible:

a $\lim\limits_{x \to 0^+} \dfrac{\ln(\cos 5x)}{\ln(\cos 3x)}$

b $\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\ln(\sin 2x)}{\ln(\sin 3x)}$

Example 22 **Self Tutor**

Evaluate $\lim\limits_{x \to \frac{\pi}{2}^-} (\sec x - \tan x)$.

As $x \to \frac{\pi}{2}^-$, both $\sec x \to \infty$ and $\tan x \to \infty$.

Now $\sec x - \tan x = \dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x} = \dfrac{1 - \sin x}{\cos x}$ {converting to a quotient}

$\therefore \lim\limits_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) = \lim\limits_{x \to \frac{\pi}{2}^-} \left(\dfrac{1 - \sin x}{\cos x} \right)$

But $\lim\limits_{x \to \frac{\pi}{2}^-} (1 - \sin x) = 0$ and $\lim\limits_{x \to \frac{\pi}{2}^-} \cos x = 0$

$$\therefore \lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) = \frac{\lim\limits_{x \to \frac{\pi}{2}^-} (-\cos x)}{\lim\limits_{x \to \frac{\pi}{2}^-} (-\sin x)} \quad \{\text{l'Hôpital's rule}\}$$

$$= 0$$

7 Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$ using l'Hôpital's rule *twice*.

8 Evaluate, if possible:

a $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin 2x} \right)$

b $\lim\limits_{x \to \frac{\pi}{2}^-} (\sec^2 x - \tan x)$

Example 23 **Self Tutor**

Evaluate, if possible: $\lim\limits_{x \to \infty} \dfrac{e^x}{x^n}$ where $n \in \mathbb{Z}^+$.

For all $n \in \mathbb{Z}^+$, as $x \to \infty$, $e^x \to \infty$ and $x^n \to \infty$, so we can use l'Hôpital's rule.

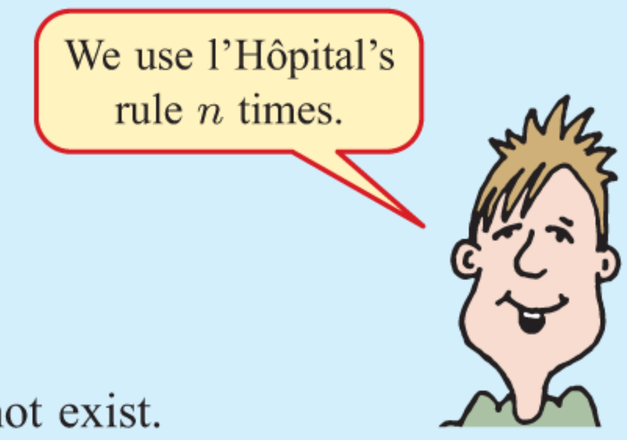
$$\therefore \lim_{x \to \infty} \frac{e^x}{x^n} = \lim_{x \to \infty} \frac{e^x}{nx^{n-1}}$$

$$= \lim_{x \to \infty} \frac{e^x}{n(n-1)x^{n-2}}$$

$$\vdots$$

$$= \lim_{x \to \infty} \frac{e^x}{n!}$$

$$= \frac{1}{n!} \lim_{x \to \infty} e^x$$



As $x \to \infty$, $e^x \to \infty$, so the given limit does not exist.

9 **a** Evaluate, if possible: $\lim\limits_{x\to\infty} \dfrac{x^k}{e^x}$, $k \in \mathbb{Z}^+$.

b Hence explain why, as $x \to \infty$, the exponential function e^x increases more rapidly than any fixed positive power of x.

10 Prove that as $x \to \infty$, $\ln x$ increases more slowly than any fixed positive power of x.

11 **a** Prove that $\lim\limits_{x\to\infty} x \ln\left(1+\dfrac{1}{x}\right) = 1$.

b By writing $\left(1+\dfrac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$ and using the fact that $f(x) = e^x$ is continuous on $\mathbb{R}$, prove that $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$.

c Prove that for $a \neq 0$, $\lim\limits_{x\to\infty} \left(1+\dfrac{a}{x}\right)^x = e^a$.

12 By writing $x^{\sin x} = e^{\sin x \ln x}$ and using the fact that $f(x) = \ln x$ is continuous on $x \in \mathbb{R}$, $x > 0$, prove that $\lim\limits_{x\to 0^+} x^{\sin x} = 1$.

13 Discuss the behaviour of $y = x^{\frac{1}{x}}$ as $x \to \infty$.

HISTORICAL NOTE

L'Hôpital's rule was discovered by the Swiss mathematician **Johann Bernoulli** (1667 - 1748) in 1694. However, Bernoulli had entered into a business arrangement with the Frenchman **Marquis Guillaume de l'Hôpital** (1661 - 1704) whereby l'Hôpital had the rights to Bernoulli's mathematical discoveries. The rule bearing l'Hôpital's name was therefore first published in l'Hôpital's calculus textbook *Analyse des Infiniment Petits* or "Analysis of the Infinitely Small" in 1696.



Johann Bernoulli

REVIEW SET 18A

1 Find the equation of the tangent to:

a $y = -2x^2$ at $x = -1$

b $y = x^3 - 5x + 2$ at $(2, 0)$

c $y = \dfrac{1-2x}{x^2}$ at $(1, -1)$

d $f(x) = e^{3x-1}$ at $x = 0$

e $f(x) = \ln(x^2)$ at $x = e$.

2 Find the equation of the normal to:

a $y = \sqrt{3x+4}$ at $(4, 4)$

b $y = 3e^{2x}$ at $x = 1$

c $y = \dfrac{x+1}{x^2-2}$ at $x = 1$

d $y = 3x^e - e^x$ at $x = e$.

3 At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find p and q.

4 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

5 The line through $A(2, 4)$ and $B(0, 8)$ is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find a.

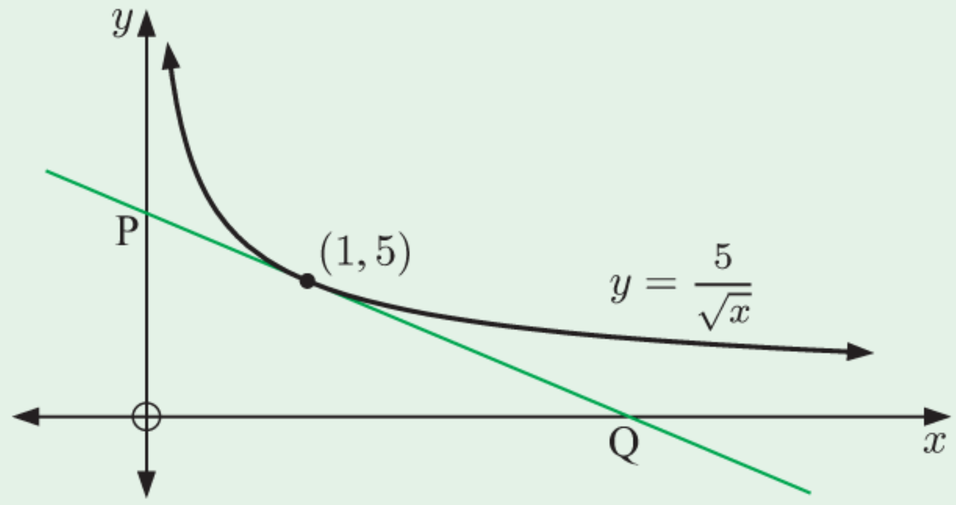
6 $y = \dfrac{x}{\sqrt{1-x}}$ has a tangent with equation $5x + by = a$ at the point where $x = -3$.
Find the values of a and b.

7 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ meets the curve again.

8 **a** Find the equation of the normal to $y = e^{2x}$ at the point where $x = a$.

b Hence find the equation of the normal to $y = e^{2x}$ which passes through the origin.

9 Find the coordinates of P and Q if (PQ) is the tangent to $y = \dfrac{5}{\sqrt{x}}$ at $(1, 5)$.



10 The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at points A and B.
Determine the area of triangle OAB.

11 The normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$ cuts the axes at B and C. Find the length of [BC].

12 Find the equation of:

a the tangent to $y = \sec x$ at the point where $x = \frac{\pi}{3}$

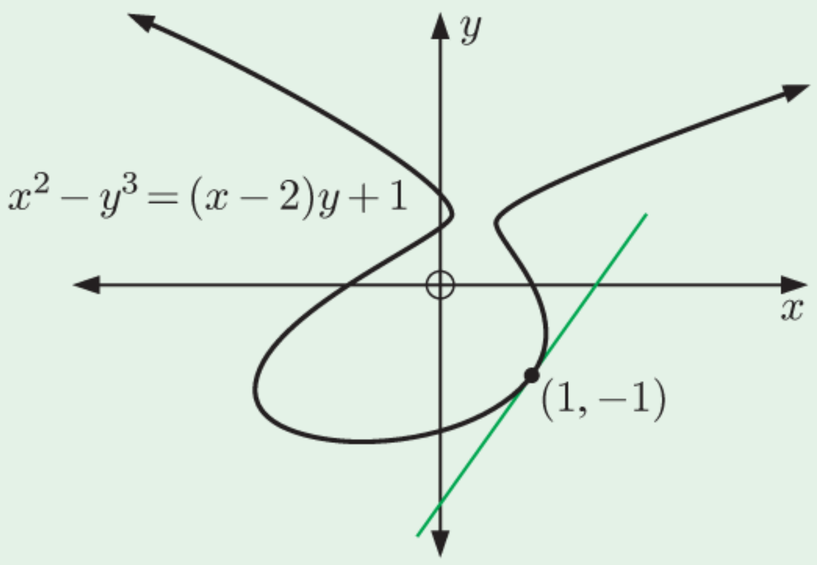
b the normal to $y = \arctan x$ at the point where $x = \sqrt{3}$.

13 Prove that every normal to the unit circle $x^2 + y^2 = 1$ passes through the origin.

14 The graph of $x^2 - y^3 = (x-2)y + 1$ is shown alongside.

a Find $\dfrac{dy}{dx}$.

b Find the equation of the illustrated tangent.



15 Find intervals where $f(x) = -x^3 - 6x^2 + 36x - 17$ is:

a increasing **b** decreasing.

16 Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 7$.

a Find and classify all stationary points.

b Find intervals where the function is increasing and decreasing.

c Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

d Sketch the graph of $y = f(x)$ showing the features you have found.

17 Consider the function $f(x) = \dfrac{3x-2}{x+3}$.

a State the domain of $f(x)$.

b Find the axes intercepts.

c Find $f'(x)$ and draw its sign diagram.

d Does $f(x)$ have any stationary points?

18 Find the greatest and least values of $x + \dfrac{32}{x^2}$ for $2 \leqslant x \leqslant 10$.

19 Consider $f(x) = xe^{1-2x}$.

a Show that $f'(x) = e^{1-2x}(1-2x)$.

b Find values of x for which:

i $f(x) > 0$ **ii** $f'(x) > 0$

c Find the stationary point of $y = f(x)$, and determine its nature.

20 Find intervals where $f(x)$ is increasing or decreasing:

a $f(x) = x^3 - 6x$ **b** $f(x) = e^x(x-2)$ **c** $f(x) = 2x - \sin x$

21 Find and classify the stationary points of:

a $f(x) = -x^3 + 2x^2 - x + 3$ **b** $f(x) = \dfrac{x^2}{x+3}$

22 For each of the following, determine the position and nature of the stationary points on the interval $-\pi \leqslant x \leqslant \pi$, then show them on a graph of the function.

a $y = \sin\frac{x}{2}$ **b** $y = \cos^2 x$ **c** $y = \cos 2x - 2\sin x$

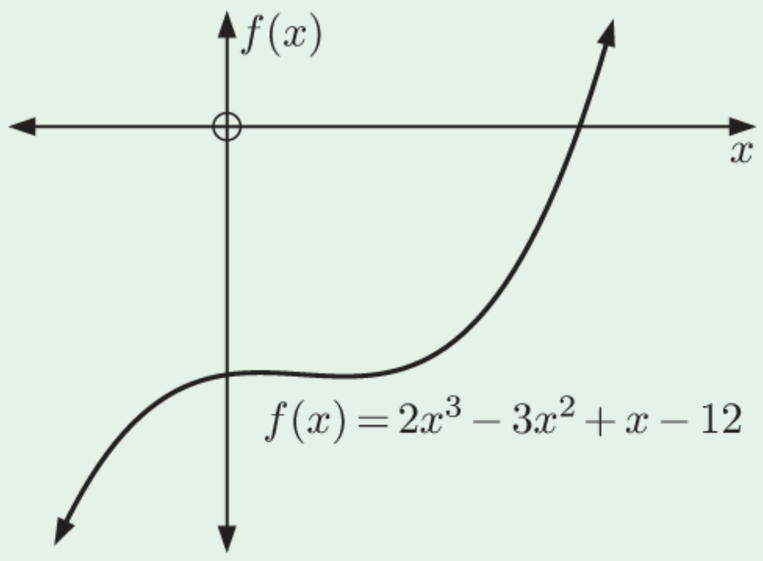
23 The graph of $f(x) = 2x^3 - 3x^2 + x - 12$ is shown alongside.

a Find $f'(x)$ and $f''(x)$.

b Draw the sign diagram of $f''(x)$.

c State the interval on which the function is concave down.

d Find the point at which the shape of $f(x)$ changes.



24 Find intervals where the curve is concave up or concave down:

a $y = x^3 - 4x^2 + 11$ **b** $y = -\dfrac{x+1}{x^2}$ **c** $y = \dfrac{x+2}{x(x+4)}$

25 Consider the function $f(x) = x + \ln x$.

a Find the values of x for which $f(x)$ is defined.

b Draw the sign diagrams of $f'(x)$ and $f''(x)$, and give a geometrical interpretation of each.

c Sketch the graph of $y = f(x)$.

d Find the equation of the normal to the curve at the point where $x = 1$.

26 Consider the function $f(x) = e^{x\sqrt{3}} \sin x$.

a Find $f'(x)$.

b Find x on $0 \leqslant x \leqslant 2\pi$ such that $f'(x) = 0$.

c Draw the sign diagram for $f'(x)$ on $0 \leqslant x \leqslant 2\pi$.

d Determine the intervals on $0 \leqslant x \leqslant 2\pi$ for which $f(x)$ is:

i increasing **ii** decreasing.

27 Consider the function $f(x) = \ln(x^2 + 5)$.

a Find and classify any turning points.

b Find and classify any points of inflection.

c Find intervals where the function is increasing or decreasing.

d Find intervals where the function is concave up or down.

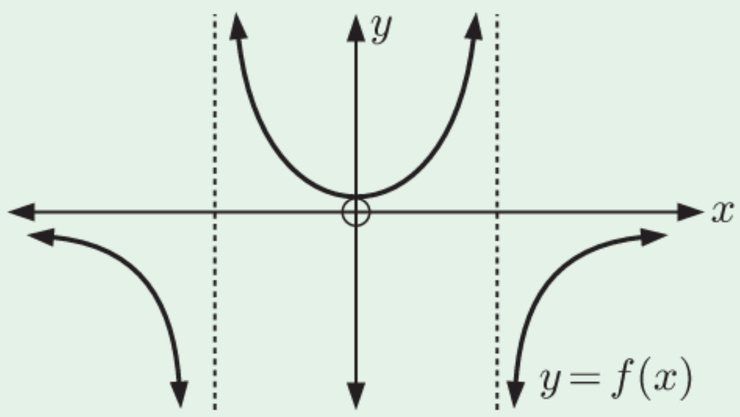
e Sketch the function, showing the features you have found.

28 Consider the function $f(x) = e^{\sin^2 x}$, $0 \leqslant x \leqslant \pi$.

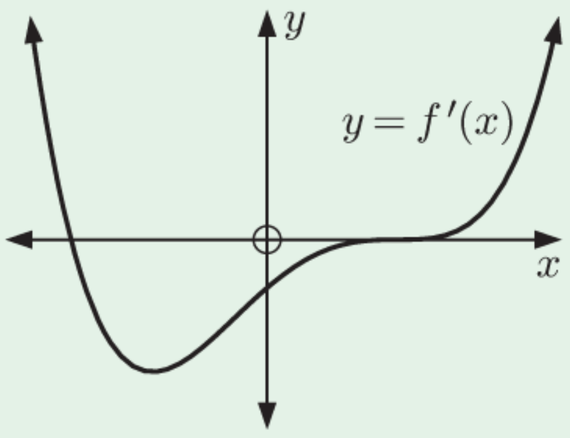
a Find the exact value(s) of x at which $f(x)$ has a maximum turning point.

b Find $f''(x)$. **c** Find any points of inflection in the given domain.

29 The graph of $y = f(x)$ is given.
On the same set of axes, sketch the graph of $y = f'(x)$.



30



The graph of $y = f'(x)$ is drawn.
On the same set of axes, clearly draw a possible graph of $y = f(x)$. Show all turning points and points of inflection.

31 Evaluate, if possible, using l'Hôpital's rule:

a $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$ **b** $\lim\limits_{x \to 0} \dfrac{\arcsin x}{x}$ **c** $\lim\limits_{x \to 0} \dfrac{\sin x}{2^x - 1}$

32 Consider the function $f(x) = xe^{-x^2}$.

a Find $f'(x)$ and $f''(x)$.

b Find and classify the turning points of $y = f(x)$.

c Use l'Hôpital's rule to consider the behaviour of $f(x)$ as $x \to \pm\infty$.

d Find the inflection point of $y = f(x)$.

e Sketch $y = f(x)$, showing the features you have found.

33 Use l'Hôpital's rule several times to evaluate $\lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{x - \sin x}$.

34 **a** Attempt to evaluate $\lim_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ by directly applying l'Hôpital's rule. Describe what happens.

b Multiply the numerator and denominator of $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ by e^x, then evaluate the limit as $x \to \infty$.

35 Consider the general quadratic $p(x) = ax^2 + bx + c$.

a Perform a "shift-expand-shift" on $p(x)$ using these steps:

(1) Substitute $x = z + r$ into p.
(2) Expand the brackets.
(3) Substitute $z = x - r$ into the result, thus writing $p(x)$ in the form $p(x) = A(x-r)^2 + B(x-r) + C$.

b Prove that for any number r and any quadratic, that
$p(x) = p(r) + p'(r)(x-r) + \dfrac{p''(r)}{2}(x-r)^2$.

c Explain how this form can be used to find the tangent to $p(x)$ at $x = r$ without using calculus.

REVIEW SET 18B

1 Find the equation of the tangent to:

a $f(x) = x^4 - 2x^2 + 7x - 3$ at $(2, 19)$

b $f(x) = \dfrac{1}{\sqrt{x+7}}$ at $(9, \frac{1}{4})$

c $f(x) = 3\sin 2x$ at $x = \frac{\pi}{6}$

d $f(x) = \dfrac{e^x}{2-x}$ at $x = 0$

e $y = \ln(x^2 + 3)$ at $x = 0$.

2 Find the equation of the normal to:

a $y = \dfrac{1}{x^2} - \dfrac{2}{x}$ at $x = 1$

b $y = x\sin x$ at the origin

c $y = e^{-x^2}$ at $x = 1$.

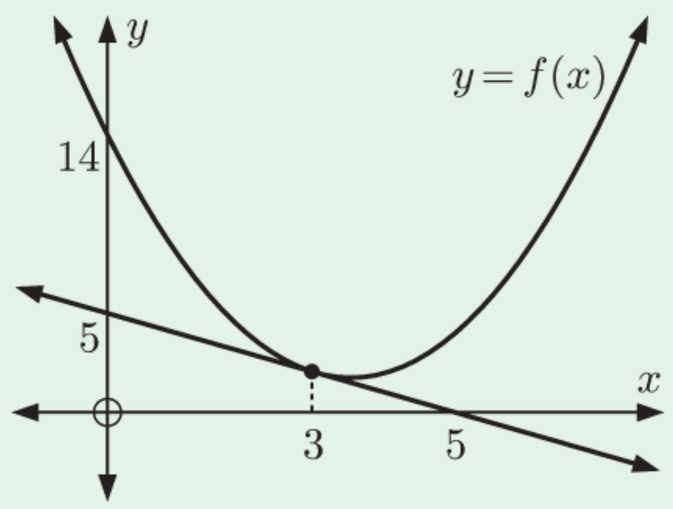
3 Show that the equation of the tangent to $y = x\tan x$ at $x = \frac{\pi}{4}$ is $(2+\pi)x - 2y = \frac{\pi^2}{4}$.

4 The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

5 $y = f(x)$ is the parabola shown.

a Find $f(3)$ and $f'(3)$.

b Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.



6 Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

7 The tangent to $y = x^3 + ax^2 - 4x + 3$ at $x = 1$ is parallel to the line $y = 3x$.

a Find a. **b** Find the equation of the tangent at $x = 1$.

c Where does the tangent cut the curve again?

8 Find the point where the normal to $y = x^2 - 4x + 2$ at $x = 3$, meets the curve again.

9 Find where the tangent to $y = \ln(x^4 + 3)$ at $x = 1$ cuts the y-axis.

10 Find the equation of the tangent to:

a $y = \operatorname{cosec} x$ at the point where $x = \frac{\pi}{3}$ **b** $y = \cos\frac{x}{2}$ at the point where $x = \frac{\pi}{2}$.

11 Find the equation of the normal to:

a $y = \sec 2x$ at $x = \frac{\pi}{3}$ **b** $y = \operatorname{cosec}\frac{x}{2}$ at $x = \frac{\pi}{2}$.

12 Show that the curves with equations $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have a common tangent at their point of intersection. Find the equation of this common tangent.

13 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ has equation $2x - y = 1$. Find a and b.

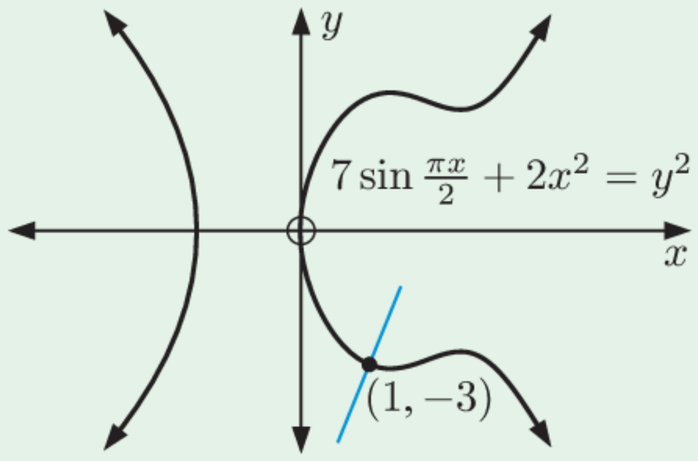
14 **a** Find the equations of the tangents to the circle $x^2 + y^2 = 4$ at the points where $y = 1$.

b Find the point of intersection of these tangents.

15 The graph of $7\sin\frac{\pi x}{2} + 2x^2 = y^2$ is shown alongside.

a Find $\dfrac{dy}{dx}$.

b Find the equation of the illustrated normal.



16 Find intervals where $f(x) = x^4 - 4x^3 - 8x^2 + 5$ is:

a increasing **b** decreasing.

17 $f(x) = x^3 - 3x^2 + ax + 50$ has a stationary point at $x = 3$.

a Find a. **b** Find the position and nature of all stationary points.

18 Consider the function $f(x) = x^3 - 4x^2 + 4x$.

a Find all axes intercepts.

b Find and classify all stationary points.

c Find intervals where the function is increasing and decreasing.

d Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

e Sketch the graph of $y = f(x)$ showing the features you have found.

19 Suppose $f(x) = \dfrac{x+1}{x^2 - 2x - 8}$.

a Show that $f'(x) = -\dfrac{x^2 + 2x + 6}{(x^2 - 2x - 8)^2}$ and draw its sign diagram.

b Hence show that $f(x)$ is never increasing.

20 Consider the function $f(x) = e^x - x$.

a Find and classify any stationary points of $y = f(x)$.

b Discuss what happens to $f(x)$ as $x \to \infty$.

c Find $f''(x)$ and draw its sign diagram. Give a geometrical interpretation for the sign of $f''(x)$.

d Sketch the graph of $y = f(x)$.

e Deduce that $e^x \geqslant x + 1$ for all x.

21 Find and describe the stationary point of $y = \dfrac{x + a}{e^x}$, where a is a constant.

22 $f(x) = \dfrac{\ln(ax)}{bx}$ has a stationary point at $\left(\dfrac{e}{2}, \dfrac{2}{3e}\right)$. Find a and b.

23 Determine the interval(s) on which the function $f(x) = -\frac{1}{2}x^4 + x^3 + 6x^2 - 3x + 2$ is:

a increasing **b** decreasing **c** concave up **d** concave down.

24 Find and classify the inflection points of:

a $y = x^4 - 3x^3 + 9$ **b** $y = -x^4 + x^3 + 9x^2 + 1$

25 Consider $f(x) = \sqrt{\cos x}$ for $0 \leqslant x \leqslant 2\pi$.

a For what values of x in this interval is $f(x)$ defined?

b Find $f'(x)$ and hence find intervals where $f(x)$ is increasing or decreasing.

c Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$.

26 Consider the function $f(x) = x^4 - 4x^3 + 7$.

a Find $f'(x)$ and $f''(x)$, and draw their sign diagrams.

b Find and classify any turning points.

c Find and classify any points of inflection.

d Find intervals where the function is:

i increasing **ii** decreasing **iii** concave up **iv** concave down.

e Sketch the function, showing all important features.

27 For the function $f(x) = \cos^2 x$, $0 \leqslant x \leqslant 2\pi$:

a Find and classify all turning points.

b Find and classify all points of inflection.

c Sketch the function, showing the features you have found.

28 Consider the function $f(x) = \dfrac{e^x}{x - 1}$.

a Find the y-intercept of the function.

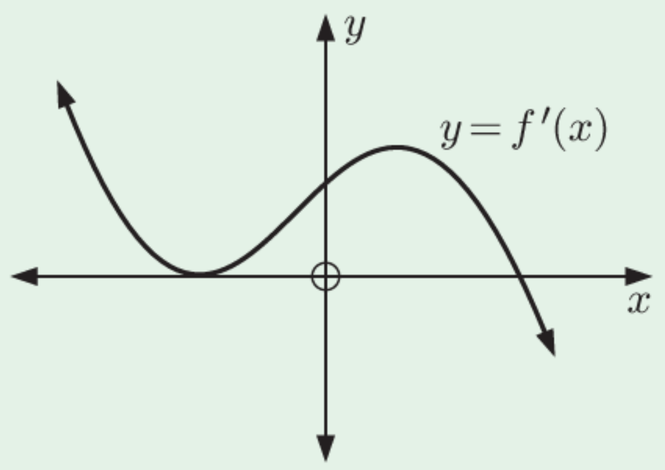
b For what values of x is $f(x)$ defined?

c Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.

d Sketch the graph of $y = f(x)$.

e Find the equation of the tangent at the point where $x = 2$.

29 Given the graph of $y = f'(x)$ drawn alongside, sketch a possible curve for $y = f(x)$. Show clearly any turning points and points of inflection.



30 For the function $f(x)$, $f'(x) > 0$ and $f''(x) < 0$ for all $x \in \mathbb{R}$, $f(2) = 1$, and $f'(2) = 2$.

a Find the equation of the tangent to $f(x)$ where $x = 2$.

b On the same set of axes, sketch $y = f(x)$ and the tangent to the curve where $x = 2$.

c Explain why $f(x)$ has exactly one zero.

d Estimate an interval in which the zero of $f(x)$ lies.

31 Evaluate, if possible, using l'Hôpital's rule:

a $\lim\limits_{x \to 0} \dfrac{x^2}{x + e^{-x} - 1}$ **b** $\lim\limits_{x \to \pi^-} \dfrac{\ln(\sin 3x)}{\ln(\sin 5x)}$

32 Use l'Hôpital's rule to prove that $\lim\limits_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x} \right) = \dfrac{1}{2}$.

33 Explain why it would not be valid to use l'Hôpital's rule to prove the derivative formula for powers of x, $\lim\limits_{h \to 0} \dfrac{(x+h)^n - x^n}{h} = n\,x^{n-1}$.

34 Evaluate, if possible, using l'Hôpital's rule:

a $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$ **b** $\lim\limits_{x \to \infty} \dfrac{e^{2x} + x^2}{e^x + 4x}$

35 Consider the parabola $f(x) = \dfrac{1}{4b}x^2$ where $b > 0$.

a Show that the tangent to $y = f(x)$ at $\mathrm{P}(a,\, f(a))$ meets the x-axis at the point $\mathrm{P}'\left(\dfrac{a}{2},\, 0\right)$.

b **i** Find the equation of the line perpendicular to this tangent line, and which passes through P'.

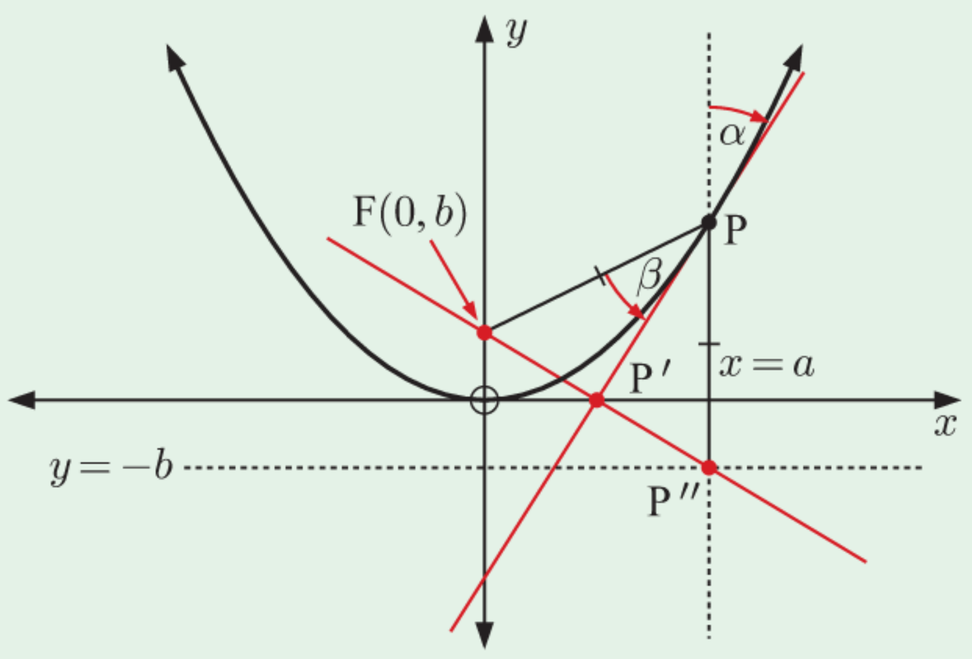
ii Show that this line has y-intercept $\mathrm{F}(0,\, b)$.

iii Show that the distance FP equals the distance from P to the line $y = -b$.

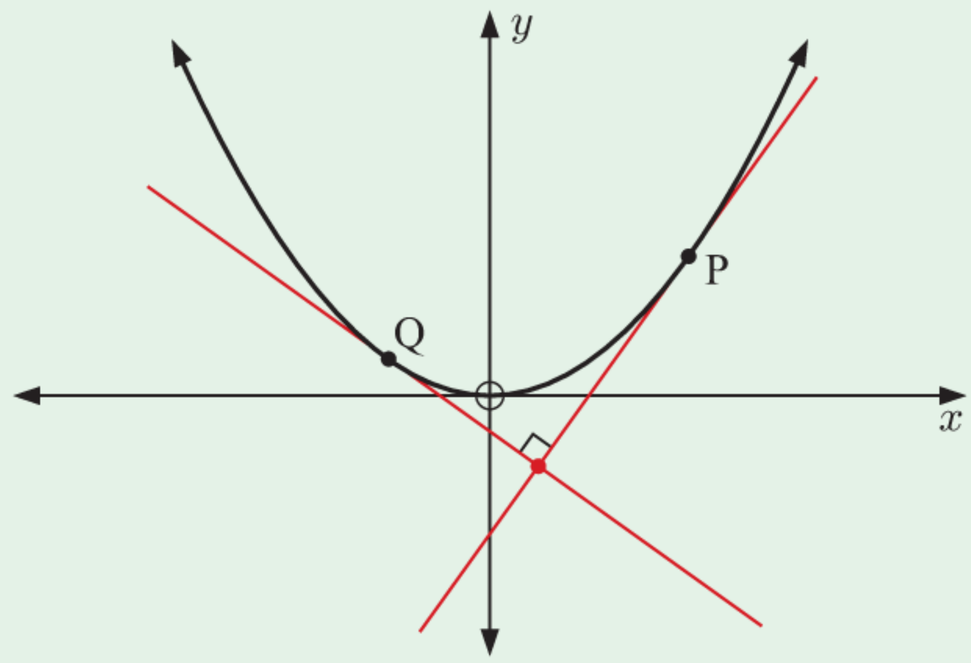
c The point $\mathrm{F}(0,\, b)$ is *invariant* since it is independent of our choice of a. F is called the *focus* of the parabola. The line $y = -b$ is called the *directrix*.

i Prove the *reflective* property of the parabola, that any vertical ray will be reflected off the parabola into the focus.

Hint: Show that $\alpha = \beta$.



ii Suppose $a \neq 0$ and that $Q(c,\ f(c))$ is another point on the parabola such that the tangents from P and Q are perpendicular. Show that the intersection of the tangents occurs on the directrix.

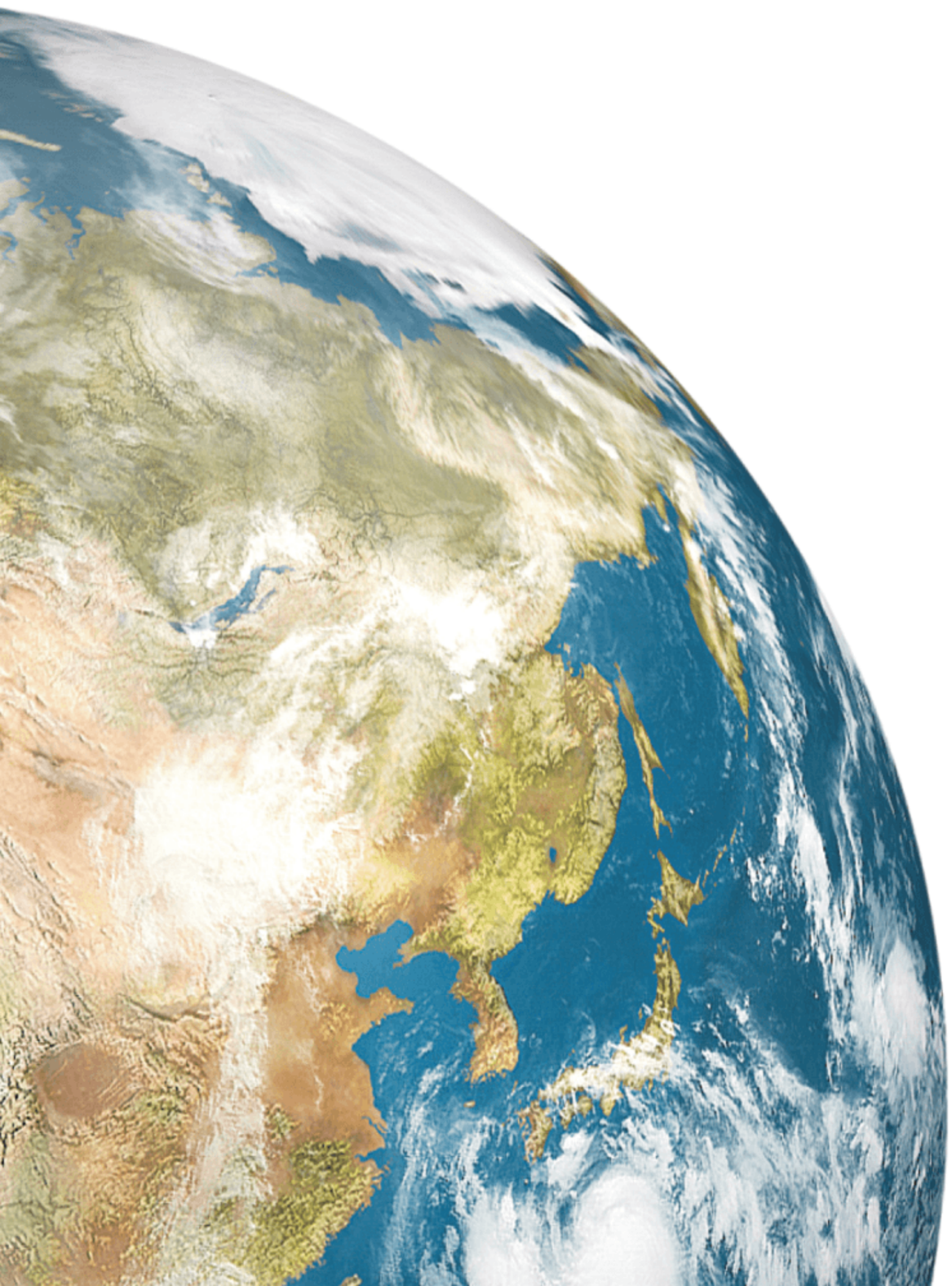


Chapter 19

Applications of differentiation

Contents:

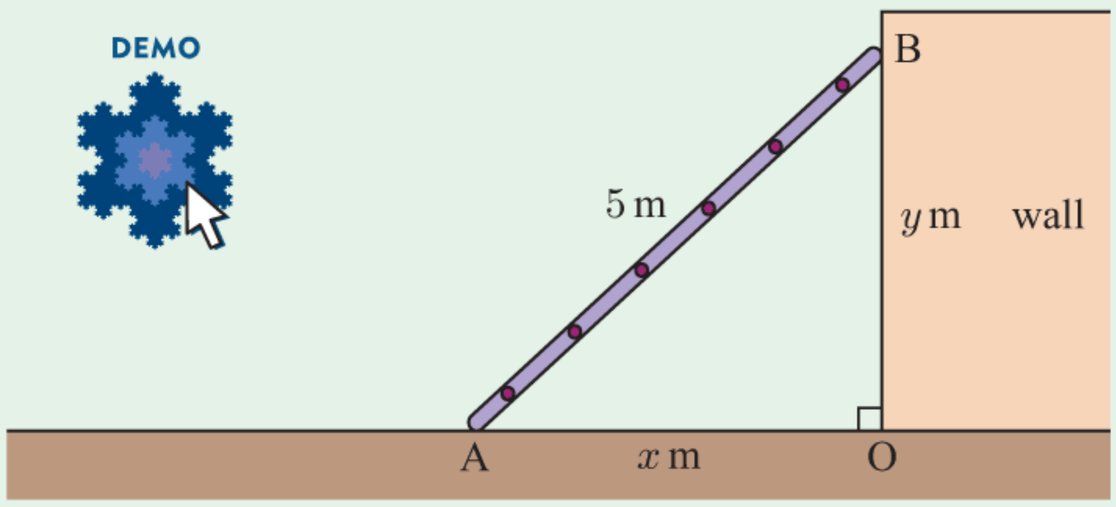
A Rates of change
B Optimisation
C Related rates



OPENING PROBLEM

A 5 m ladder rests against a vertical wall at point B. Its feet are at point A on horizontal ground.

Suppose the ladder slips and slides down the wall. As this happens, the value of x increases.



Things to think about:

a Why *must* the value of y change when the value of x changes?

b As x increases will y increase or decrease?

c What is the relationship between x and y?

d How can we establish a relationship between the *rate of change* in x and the *rate of change* in y?

We have already seen that if $y = f(x)$ then $f'(x)$ or $\frac{dy}{dx}$ gives the gradient of the tangent to $y = f(x)$ for any value of x.

In this Chapter we consider some real-world applications of differential calculus, using derivatives to tell us how one variable changes relative to another.

A RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:

- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

$\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x.**

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\frac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or $\text{m}\,\text{s}^{-1}$.
- $\frac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or $\text{L}\,\text{s}^{-1}$.



Example 1 **Self Tutor**

According to a psychologist, the ability of a child to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a child is:

i 9 years old **ii** 16 years old.

b Show that $\frac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Show that $\frac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$

$\therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$

i When $t = 9$, $\frac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old child.

ii When $t = 16$, $\frac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old child.

b Since $\sqrt{t}$ is never negative, $\frac{1}{6\sqrt{t}}$ is never negative

$\therefore \frac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

$\therefore \frac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$

$\therefore \frac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE

EXERCISE 19A

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

a What is the current annual profit?

b Find $\frac{dP}{dt}$ and state its units.

c Find $\frac{dP}{dt}$ when $t = 8$. Explain what this value means.

2 In a hot, dry summer, water is evaporating from a desert oasis. The volume of water remaining after t days is $V = 2(50 - t)^2$ m^3. Find:

a the average rate at which the water evaporates in the first 5 days

b the instantaneous rate at which the water is evaporating at $t = 5$ days.

3 The quantity of a chemical in human skin which is responsible for its "elasticity" is given by $Q(t) = 100 - 10\sqrt{t}$ where t is the age of a person in years.

a Find $Q(t)$ when:

i $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

b At what rate is the quantity of the chemical changing when the person is aged:

i 25 years **ii** 50 years?

c Show that the quantity of the chemical is decreasing for all $t > 0$.

4 The height of *pinus sylvestris* is given by $H = 35 - \dfrac{172.5}{t+5}$ metres, where t is the number of years after the tree was planted from an established seedling.

a How high was the tree when it was planted?

b Find the height of the tree after:

i 4 years **ii** 8 years **iii** 12 years.

c Find the rate at which the tree was growing after 0, 5, and 10 years.

d Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$. Explain the significance of this result.

Example 2 **Self Tutor**

The cost in dollars of producing x items in a factory each day is given by $C(x) = 9500 + 12x + 8x^{0.8}$.

a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

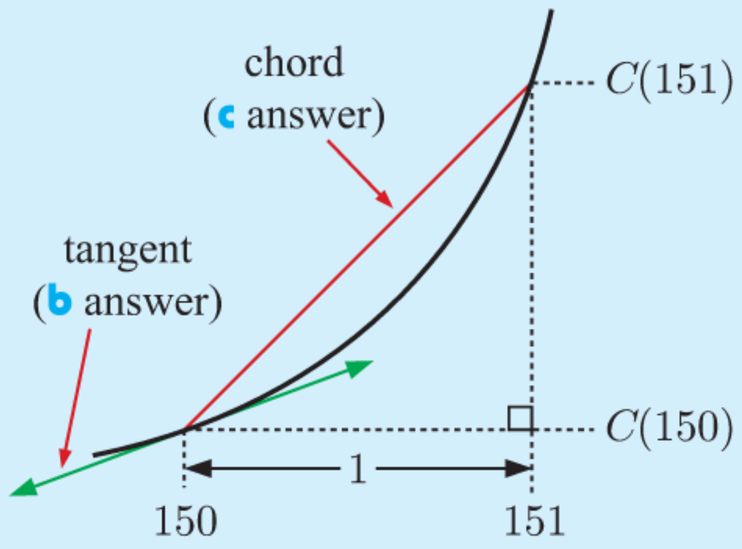
c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is $C'(x) = 12 + 6.4x^{-0.2}$ dollars per item.

b $C'(150) \approx \$14.35$

This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.

It gives an estimate of the cost of making the 151st item each day.



c $C(151) - C(150) \approx \$11\,754.87 - \$11\,740.52$
$\approx \$14.35$

This is the actual cost of making the 151st item each day, so the answer in **b** gives a very good estimate.

5 The cost for making x pairs of denim jeans per day is $C(x) = 7800 + 6x + 12x^{0.7}$ dollars.

a Find the marginal cost function $C'(x)$. **b** Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. Discuss your answer.

6 The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.



a Find the total cost of the journey if the average speed is:

i $50\ \text{km h}^{-1}$ **ii** $100\ \text{km h}^{-1}$.

b Find the rate of change in the cost of running the train for the average speed:

i $30\ \text{km h}^{-1}$ **ii** $90\ \text{km h}^{-1}$.

c At what speed will the cost be a minimum?

7 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

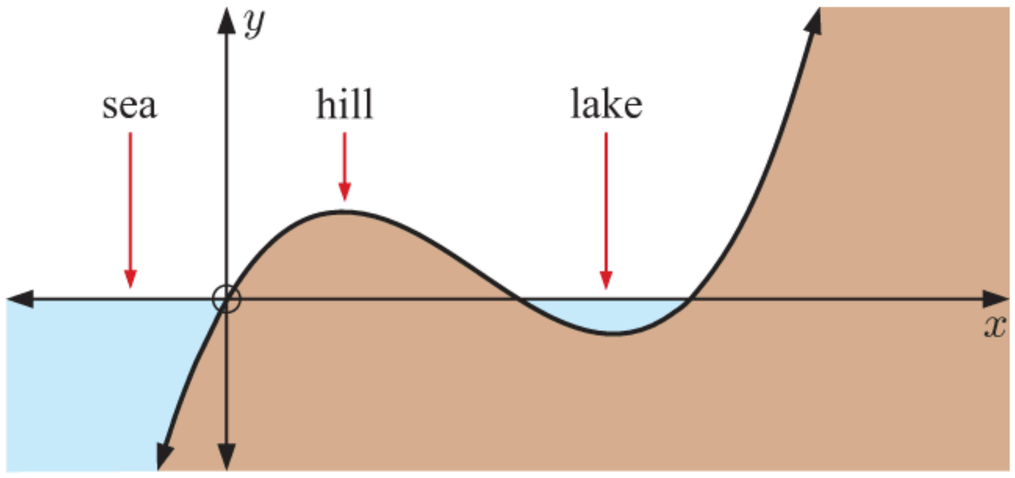
a Find $\frac{dV}{dt}$, and draw the graph of $\frac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Show that $\frac{d^2V}{dt^2}$ is always constant and positive. Interpret this result.

8 Alongside is a land and sea profile where the x-axis is sea level.

The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.



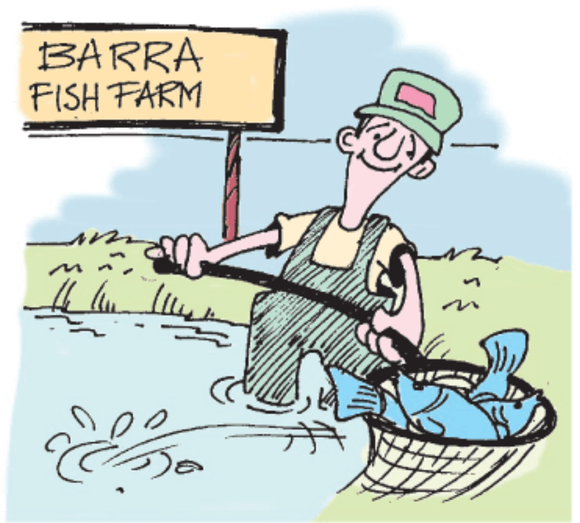
a Find where the lake is located relative to the shore line of the sea.

b Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake, and the depth at this point.

9 A fish farm grows and harvests barramundi in a large pond. The population of fish after t years is given by the function $P(t)$. The rate of change in the population is given by

$\frac{dP}{dt} = aP\left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ where a, b, and c are known constants. a is the birth rate of the barramundi, b is the carrying capacity of the pond, and c is the percentage of barramundi that are harvested each year.



a Explain why the fish population is stable when $\frac{dP}{dt} = 0$.

b If the birth rate is 6%, the maximum carrying capacity is 24 000, and 5% are harvested each year, find the stable population.

c If the harvest rate is reduced to 4%, what will the stable population increase to?

d Given an initial population P_0, explain why the population tends to the stable population.

10 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

a Find k given that the weight is 10 grams after 50 hours.

b Find the weight of radioactive substance present:

i initially **ii** after 24 hours **iii** after 1 week.

c How long will it take for the weight to reach 1 gram?

d Find the rate of radioactive decay after:

i 100 hours **ii** 1000 hours.

e Show that $\frac{dW}{dt} = bW$ for some constant b.

11 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ °C where k is a positive constant and t is the time in minutes.

a Find k if the temperature of the liquid is 20°C after 15 minutes.

b What was the temperature of the liquid when it was first placed in the refrigerator?

c Show that $\frac{dT}{dt} = c(T - 5)$ for some constant c.

d At what rate is the temperature changing:

i initially **ii** after 10 minutes **iii** after 20 minutes?

12 The height of a shrub t years after it was planted is given by $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$.

a How high was the shrub when it was planted?

b How long will it take for the shrub to reach a height of 1 m?

c At what rate is the shrub's height changing:

i 3 years after being planted **ii** 10 years after being planted?

13 In the conversion of sugar solution to alcohol, the amount of alcohol produced t hours after the reaction commenced is given by $A = s(1 - e^{-kt})$ litres, where s is the original sugar concentration (%), $t \geqslant 0$.

a Find A when $t = 0$.

b Suppose $s = 10$, and $A = 5$ after 3 hours.

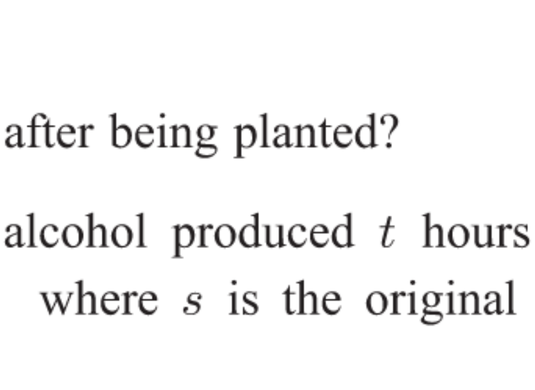
i Find k. **ii** Find the speed of the reaction after 5 hours.

Example 3 **Self Tutor**

Cathy is using a compass to draw a circle. The arm with the needle is 8 cm long, and the arm with the pencil is 9 cm in total. The angle between the arms is θ.

a Write the radius of the circle to be drawn in terms of θ.

b Hence find the rate of change in r with respect to θ when $\theta = 60°$.



a Using the cosine rule, $r^2 = 9^2 + 8^2 - 2 \times 9 \times 8 \times \cos\theta$

$\therefore\ r = \sqrt{145 - 144\cos\theta}$ {since $r > 0$}

b $r = (145 - 144\cos\theta)^{\frac{1}{2}}$

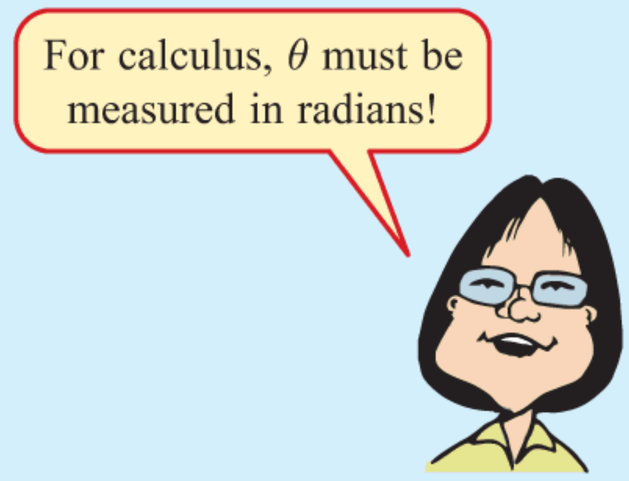
$\therefore \quad \dfrac{dr}{d\theta} = \frac{1}{2}(145 - 144\cos\theta)^{-\frac{1}{2}}(144\sin\theta)$

$= \dfrac{72\sin\theta}{\sqrt{145 - 144\cos\theta}}$

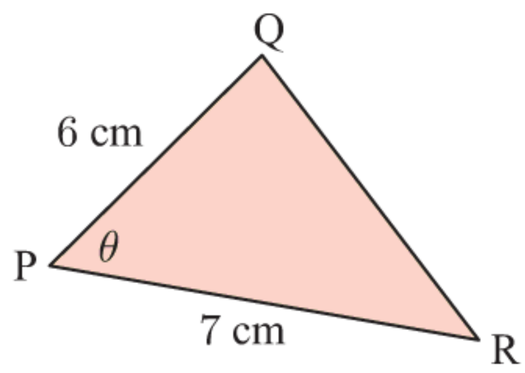
When $\theta = \frac{\pi}{3}$, $\dfrac{dr}{d\theta} = \dfrac{72\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{145 - 144\left(\frac{1}{2}\right)}}$

≈ 7.30 cm per radian

≈ 0.127 cm per degree



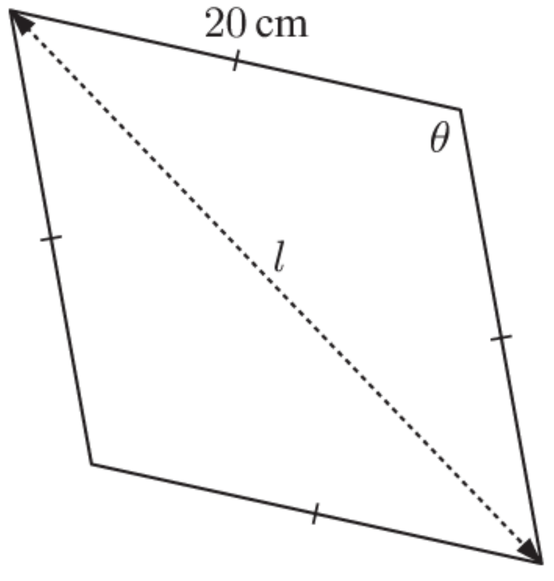
14 Find exactly the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45°$.

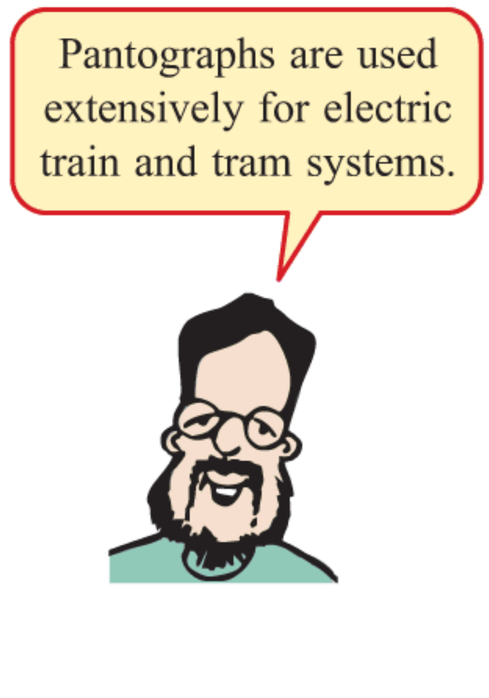


15



A set of retractable stairs is used to gain access to an attic. The frame uses a pantograph mechanism which is a set of rhombuses with variable angles to control the retraction. Each rhombus has side length 20 cm, and the angle where the arms meet is θ, as shown.





a Find a formula for the length l between the pivots, in terms of θ.

b Hence find the rate of change in l at the time when $\theta = 120°$.

16 The voltage in a circuit is given by $V(t) = 340\sin(100\pi t)$ volts where t is the time in seconds.

a Find the voltage in the circuit:

i initially **ii** after 0.125 seconds.

b At what rate is the voltage changing:

i when $t = 0.01$ **ii** when $V(t)$ is a maximum?

17 The number of bees in a hive after t months is modelled by $B(t) = \dfrac{3000}{1 + 0.5e^{-1.73t}}$.

a Find the initial bee population.

b Find the percentage increase in the population after 1 month.

c Is there a limit to the population size? If so, what is it?

d Find $B'(t)$, and use it to explain why the population is increasing over time.

e Find the rate at which the population is increasing after 6 months.

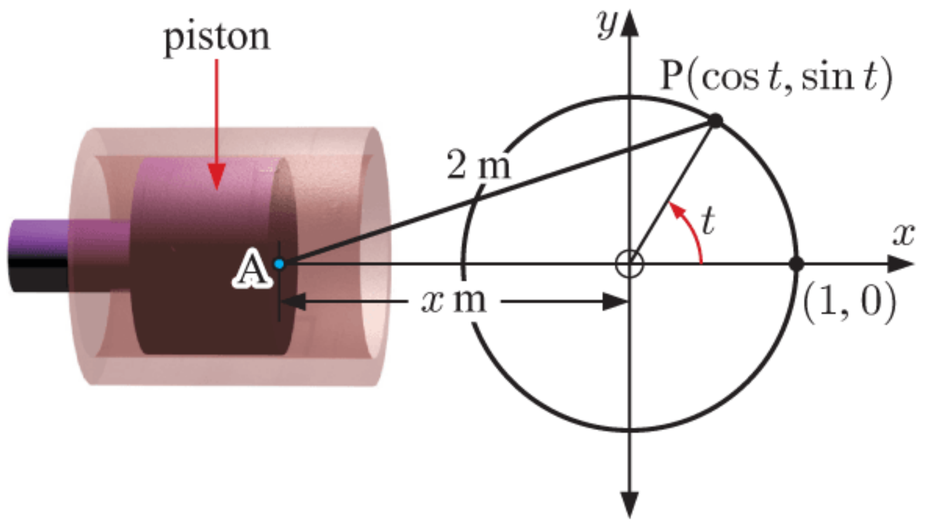
f Sketch the graph of $B(t)$.

18 A piston is operated by rod [AP] attached to a flywheel of radius 1 m. AP = 2 m. P has coordinates $(\cos t, \sin t)$ and point A is $(-x, 0)$.

a Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.

b Find the rate at which x is changing at the instant when:

i $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \frac{2\pi}{3}$



B OPTIMISATION

Optimisation is the process of finding the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimal solution**.

We can find optimal solutions in several ways:

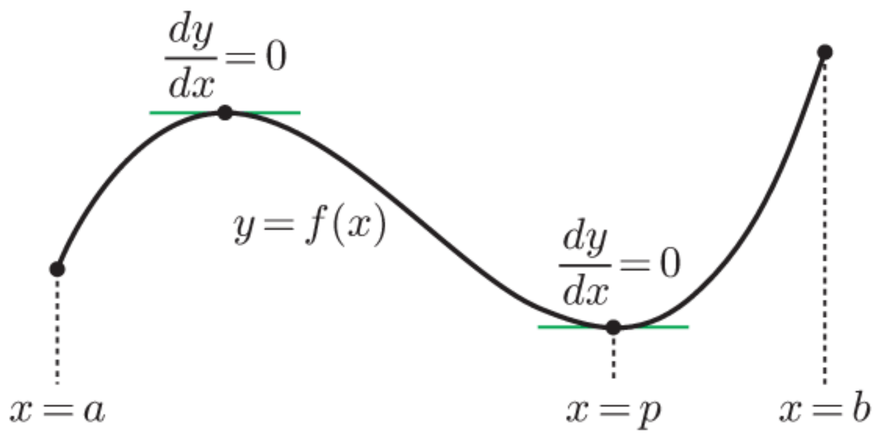
- using technology to graph the function and search for the maximum or minimum value
- using analytical methods such as the formula $x = -\dfrac{b}{2a}$ for the vertex of a parabola
- using differential calculus to locate the turning points of a function.

These last two methods are useful especially when exact solutions are required.

You should always be aware that:

> The maximum or minimum value does not always occur when the first derivative is zero.
>
> It is essential to also examine the values of the function at the end point(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the end point $x = b$.

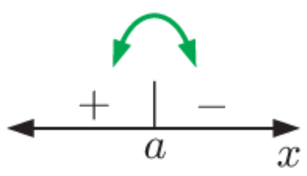
The minimum value of y occurs at the local minimum $x = p$.

TESTING FOR LOCAL MAXIMA AND MINIMA

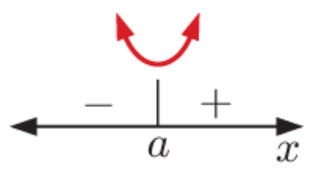
If we find a value $x = a$ such that $f'(a) = 0$, there are several tests we can use to see whether we have a local maximum or a local minimum at this point.

SIGN DIAGRAM TEST

If, near to $x = a$, the sign diagram is:

- 

we have a **local maximum**

- 

we have a **local minimum**.

SECOND DERIVATIVE TEST

- If $f''(a) < 0$ we have  shape, which indicates a **local maximum**.
- If $f''(a) > 0$ we have  shape, which indicates a **local minimum**.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

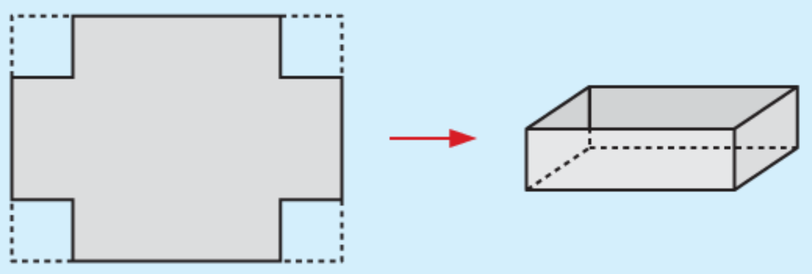
Step 5: Identify the optimal solution, also considering end points where appropriate.

Step 6: Write your answer in a sentence, making sure you specifically answer the question.

Example 4 **Self Tutor**

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?

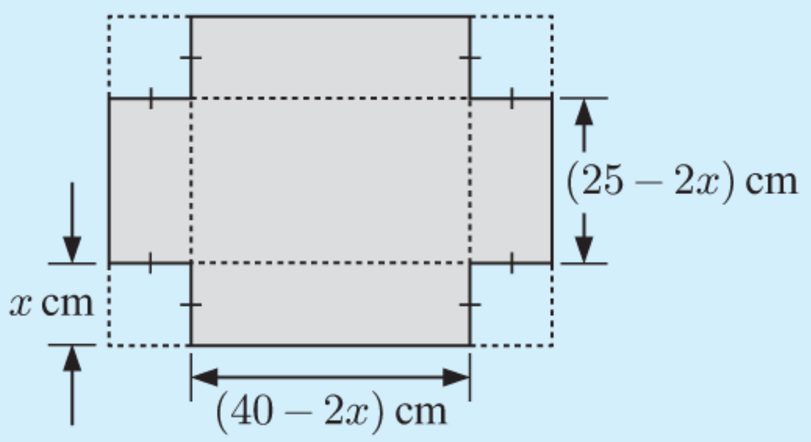


Step 1: Let x cm be the side lengths of the squares that are cut out.

Step 2: Volume = length × width × depth

$$= (40 - 2x)(25 - 2x)x$$
$$= (1000 - 80x - 50x + 4x^2)x$$
$$= 1000x - 130x^2 + 4x^3 \text{ cm}^3$$

Since the side lengths must be positive,
$x > 0$ and $25 - 2x > 0$.
$\therefore \quad 0 < x < 12.5$



Step 3: $\dfrac{dV}{dx} = 12x^2 - 260x + 1000$

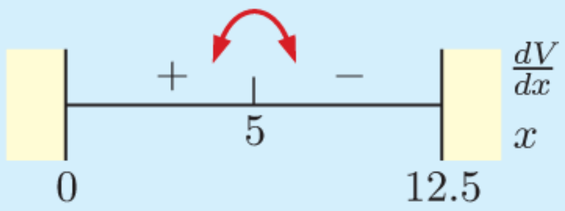
$= 4(3x^2 - 65x + 250)$

$= 4(3x - 50)(x - 5)$



On the domain $0 < x < 12.5$, $\dfrac{dV}{dx} = 0$ only when $x = 5$.

Step 4: $\dfrac{dV}{dx}$ has sign diagram:



Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

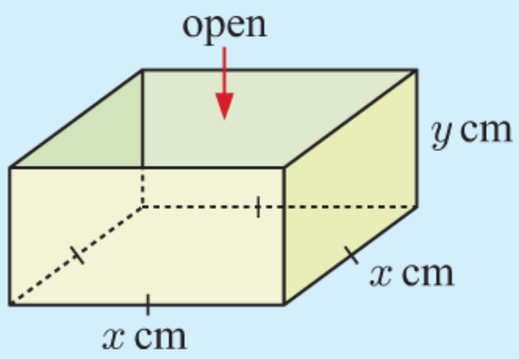
Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

Example 5

Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.

Step 1:



Let the base lengths be x cm and the depth be y cm.

The volume $V =$ length $\times$ width $\times$ depth

$\therefore \ V = x^2 y$

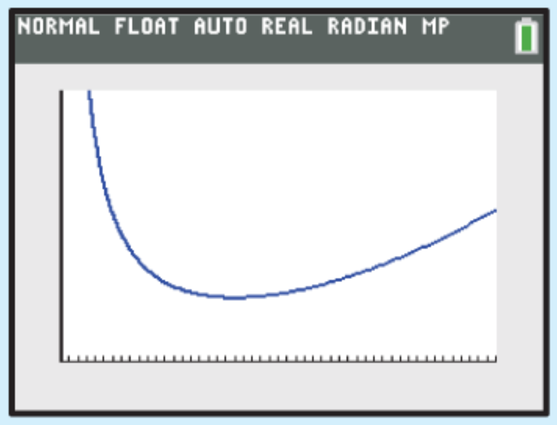
$\therefore \ 4000 = x^2 y \quad \ldots. (1) \quad \{1 \text{ litre} \equiv 1000 \text{ cm}^3\}$

Step 2: The total surface area $A =$ area of base $+ 4$(area of one side)

$= x^2 + 4xy$

$= x^2 + 4x\left(\dfrac{4000}{x^2}\right)$ {using (1)}

$\therefore \ A(x) = x^2 + 16\,000x^{-1}$ where $x > 0$



Step 3: $\therefore \ A'(x) = 2x - 16\,000x^{-2}$

$\therefore \ A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

$\therefore \ 2x^3 = 16\,000$

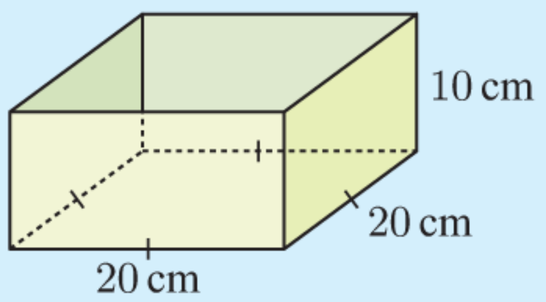
$\therefore \ x = \sqrt[3]{8000} = 20$

Step 4: $A''(x) = 2 + 32\,000x^{-3}$

$\therefore \ A''(20) = 2 + \dfrac{32\,000}{20^3} = 6$

Since $A''(20) > 0$, there is a local minimum at $x = 20$.

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.



Step 6: The most economical shape has a square base 20 cm $\times$ 20 cm, and height 10 cm.

Use **calculus techniques** to answer the following problems.

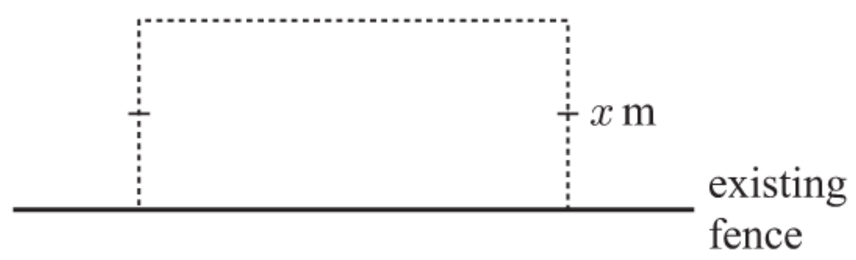
In cases where finding the zeros of the derivatives is difficult you may use the **graphing package** to help you.



EXERCISE 19B

1 When a manufacturer makes x items per day, the profit function is $P(x) = -0.022x^2 + 11x - 720$ pounds. Find the production level that will maximise profits.

2 60 metres of fencing is used to build a rectangular enclosure along an existing fence. Suppose the sides adjacent to the existing fence are x m long.



a Show that the area A of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b Find the dimensions which maximise the area of the enclosure.

3 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.



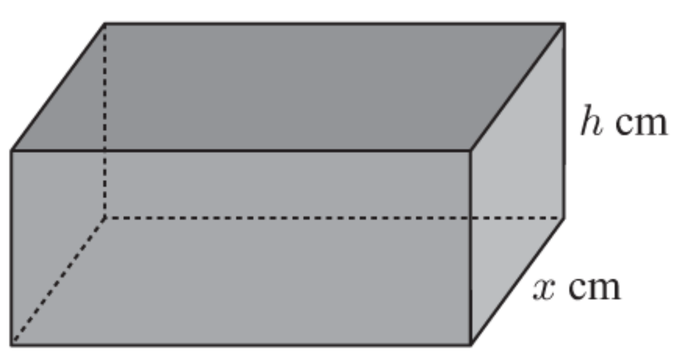
a If the sides adjacent to the existing fence have length x m, show that the required length of wire netting to be purchased is $L = 2x + \frac{100}{x}$.

b Find the minimum value of L and the corresponding value of x when this occurs.

c Sketch the optimal situation, showing all dimensions.

4 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ pounds, and for this production level each blanket may be sold for $\left(23 - \frac{1}{2}x\right)$ pounds.
How many blankets should be produced per day to maximise the total profit?

5 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of a box has dimensions in the ratio $2 : 1$.



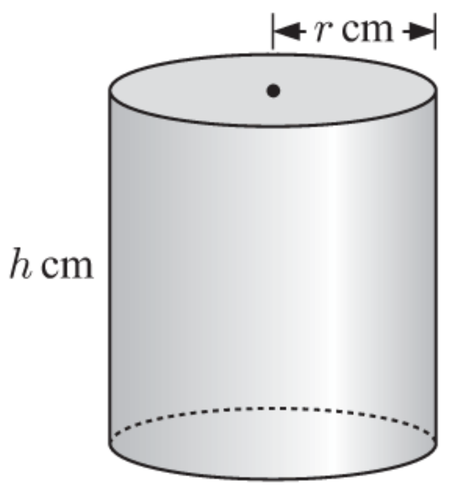
a Show that $x^2h = 100$.

b Show that the inner surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2.

c Find the minimum inner surface area of the box and the corresponding value of x.

d Sketch the optimal box shape, showing all dimensions.

6 Brenda is designing a cylindrical tin can for a canned fruit company. The cans must have capacity 1 litre, and they must use as little metal as possible.

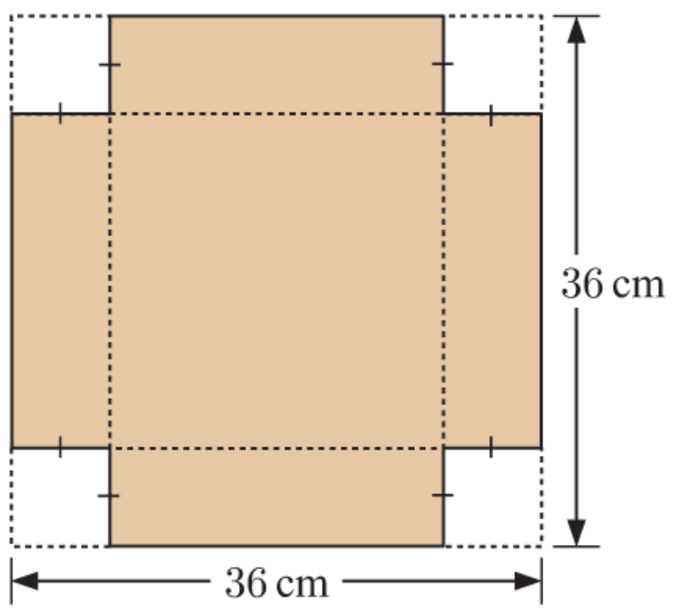


a Explain why the height h is given by $h = \frac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by $A = 2\pi r^2 + \frac{2000}{r}$ cm^2.

c Find the dimensions of the can which make A as small as possible.

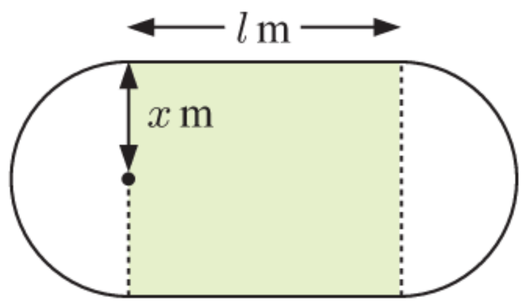
7 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.



a Show that the volume of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

8 An athletics track has two "straights" of length l m, and two semi-circular ends of radius x m. The perimeter of the track is 400 m.



a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?

9 A 60 cm length of wire is bent into a rectangle with length x cm and width y cm.

a Write an expression for y in terms of x.

b Write an expression for the area $A(x)$ of the rectangle enclosed by the wire.

c Find $A'(x)$.

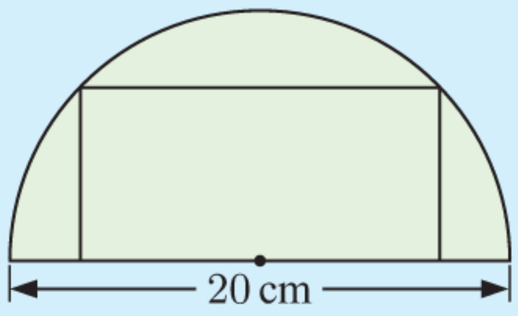
d Hence find the value of x which maximises the area, and the dimensions of the rectangle in this case.

Example 6

Self Tutor

Infinitely many rectangles can be inscribed in a semi-circle of diameter 20 cm.

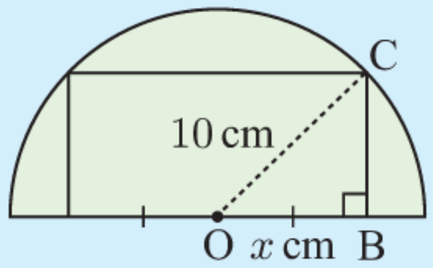
Find the shape of the largest rectangle which can be inscribed.



Step 1: Let OB $= x$ cm, $0 < x < 10$

In $\triangle$OBC, $BC^2 + x^2 = 10^2$ {Pythagoras}

$\therefore$ $BC = \sqrt{100 - x^2}$ {as $BC > 0$}



Step 2: The rectangle has area $A =$ length $\times$ width

$\therefore$ $A = 2x\sqrt{100 - x^2}$

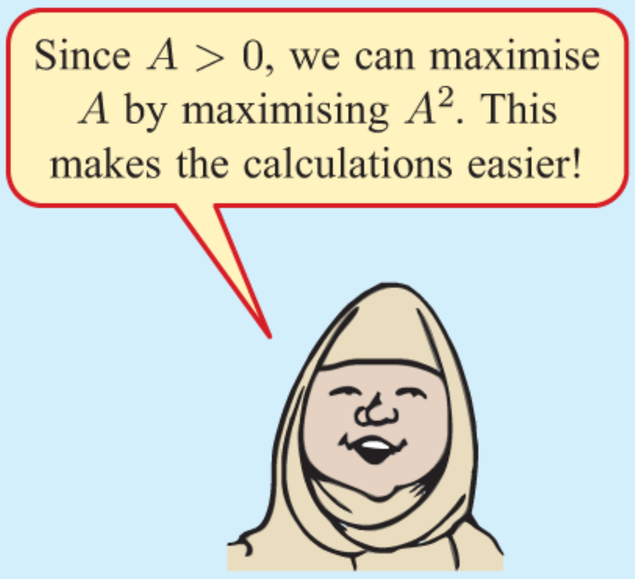
$\therefore$ $A^2 = 4x^2(100 - x^2)$

$= 400x^2 - 4x^4$

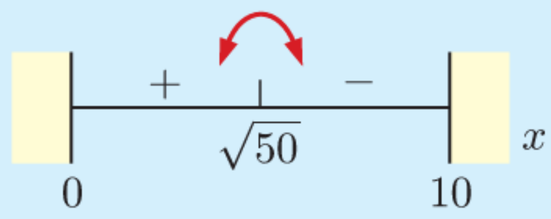
Step 3: $\frac{d}{dx}(A^2) = 800x - 16x^3$

$= 16x(50 - x^2)$

So, $\frac{d}{dx}(A^2) = 0$ when $x = 0$ or $\pm\sqrt{50}$.

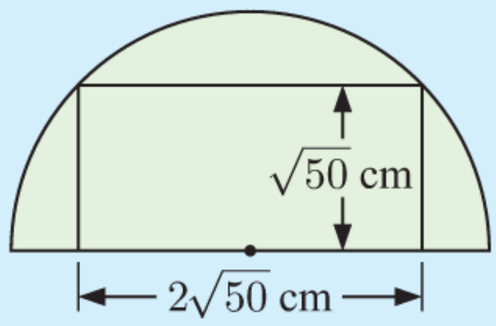


Step 4: $\frac{d}{dx}(A^2)$ has sign diagram:



Step 5: The area is maximised when $x = \sqrt{50}$ and
$\text{BC} = \sqrt{100 - 50} = \sqrt{50}$ cm

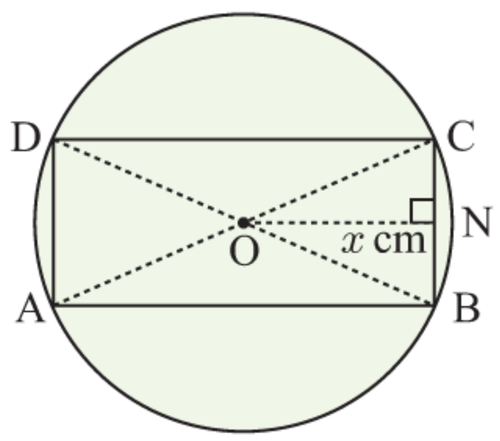
Step 6: The largest rectangle which can be inscribed is $2\sqrt{50}$ cm long and $\sqrt{50}$ cm wide.



10 Infinitely many rectangles can be inscribed in a circle of diameter 10 cm. In the diagram alongside, suppose $\text{ON} = x$ cm.

a Find the area of ABCD in terms of x only.

b Find the dimensions of ABCD which maximises its area.

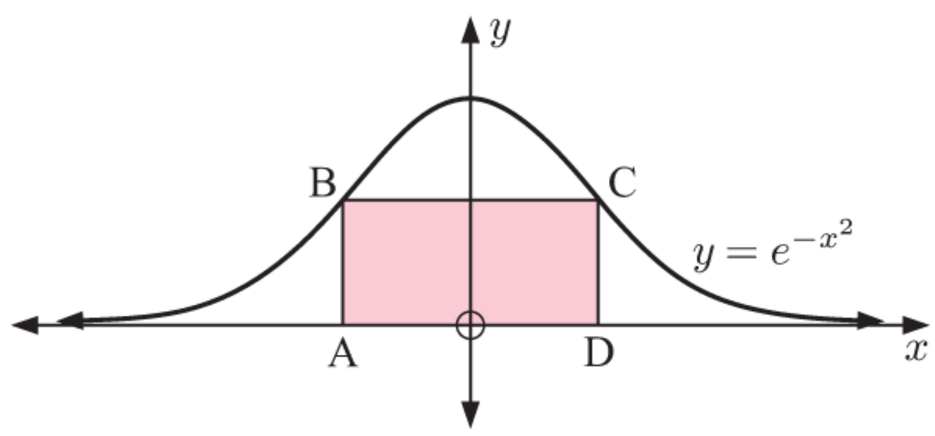


11 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by $C(x) = 4\ln x + \left(\frac{30-x}{10}\right)^2$ pounds with a minimum production capacity of 10 kettles per day.

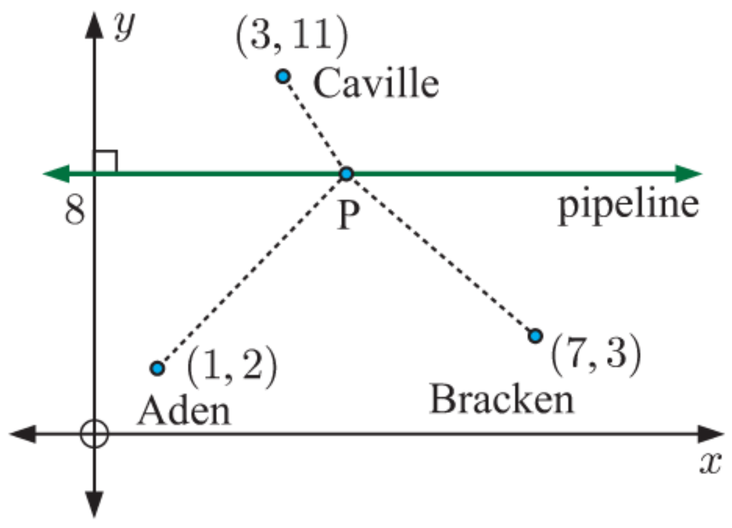
How many kettles should be manufactured to keep the cost per kettle to a minimum?

12 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.

Determine the coordinates of C such that rectangle ABCD has maximum area.



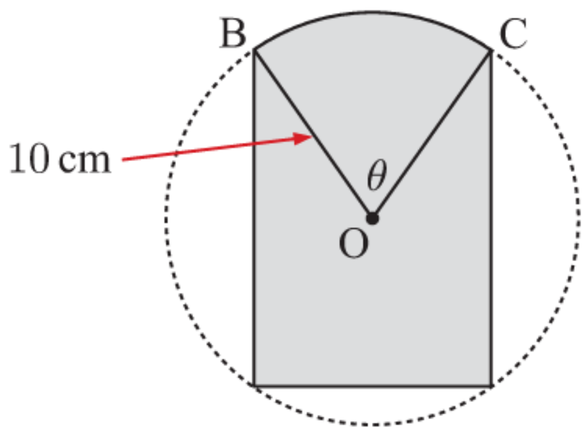
13



Three towns and their grid references are marked on the diagram. A pumping station is to be located at P on the pipeline, to pump water to the three towns. The grid units are kilometres.

Where should P be located so that the total length of the pipelines to the three towns is minimised? What is the shortest length of pipe required? Give your answers to 4 significant figures.

14



A circular piece of tin-plate with radius 10 cm has 3 segments removed as illustrated. The angle θ is measured in radians.

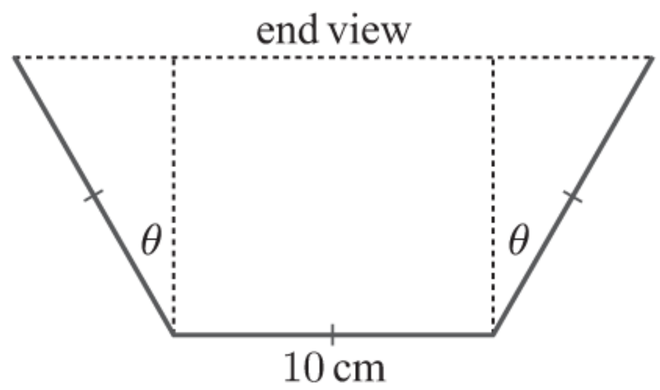
a Show that the remaining area is given by $A = 50(\theta + 3\sin\theta)$ cm^2.

b Find θ such that the area A is a maximum, and find the area A in this case.

15 When a new anaesthetic is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.

a Find $E'(t)$. **b** At what time is the anaesthetic most effective?

16 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

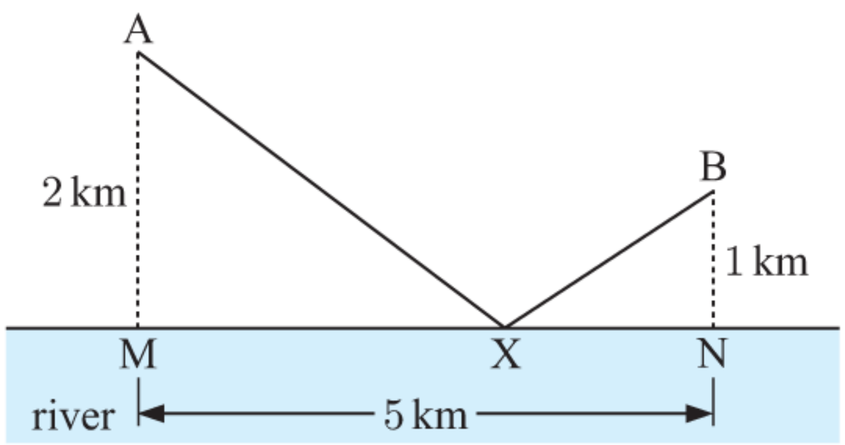


a Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1 + \sin\theta)$ cm^2.

b Show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

c For what value of θ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of θ.

17



A pumphouse is to be placed at some point X along a river.
Two pipelines will then connect the pumphouse to homesteads A and B.
How far should point X be from M so that the total length of pipeline is minimised?

18 A small population of wasps is observed. After t weeks the population is modelled by

$P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.

Find when the wasp population is growing fastest.

Hint: You need to maximise $P'(t)$.

19 At 1:00 pm ship A leaves port P. It sails in the direction 30° east of north at 12 km h^{-1}. At the same time, ship B is 100 km due east of P, and is sailing at 8 km h^{-1} towards P.

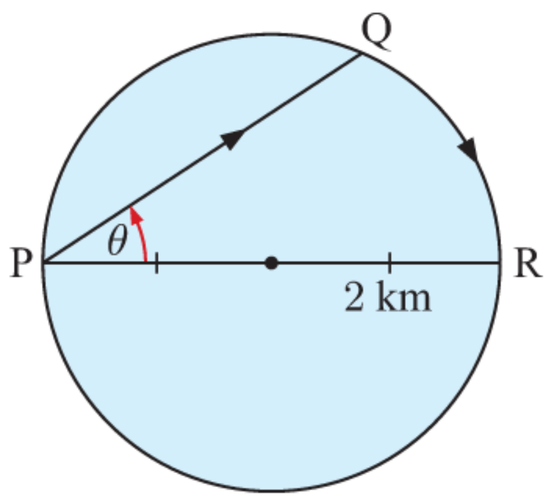
a Show that the distance D between the two ships is given by $D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ km, where t is the number of hours after 1:00 pm.

b Find the minimum value of D^2 for all $t \geqslant 0$.

c At what time, to the nearest minute, are the ships closest?

20 A cone has radius r cm and slant height s cm.
Find the ratio of $s : r$ which maximises the volume of a cone.

21 Hieu can row a boat at 3 km h^{-1}, and can walk at 6 km h^{-1}. He is currently at point P on the shore of a lake 2 km in radius. He will row to point Q, then walk around the shore to point R which is opposite P.



a Show that PQ $= 4\cos\theta$ km.

b Show that the time taken for Hieu's journey is given by $T = \frac{4}{3}\cos\theta + \dfrac{2\theta}{3}$ hours where $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

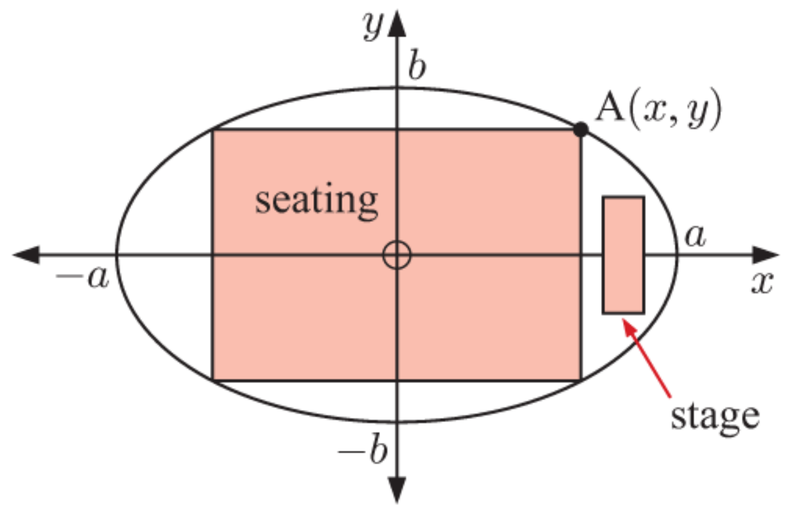
c Find θ such that $\dfrac{dT}{d\theta} = 0$ on $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

d Draw a sign diagram for $\frac{dT}{d\theta}$.

e What route should Hieu take to travel from P to R in:

i the longest time **ii** the shortest time?

22 A company constructs rectangular seating arrangements for pop concerts on sports grounds. The oval shown has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the lengths of the semi-major and semi-minor axes.



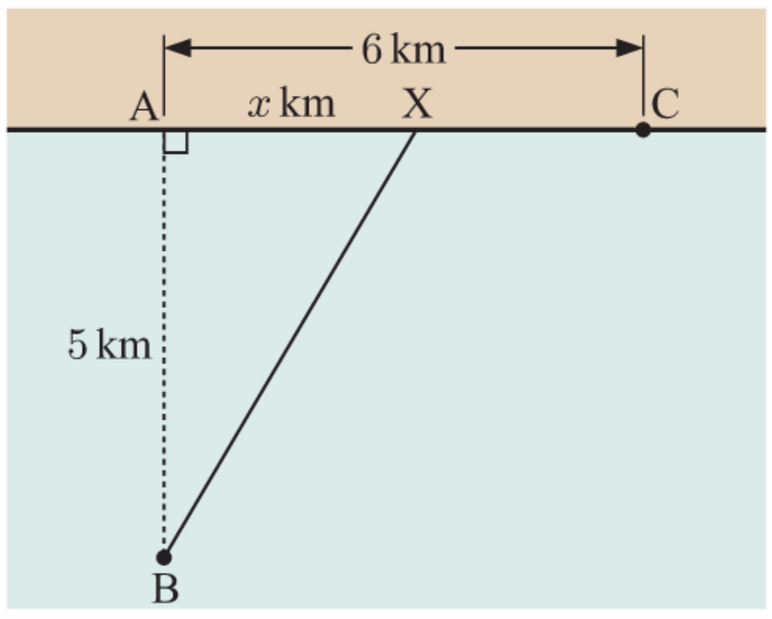
a Show that $y = \frac{b}{a}\sqrt{a^2 - x^2}$ for A as shown.

b Show that the seating area is given by $A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}$.

c Prove that the seating area is maximised when $x = \frac{a}{\sqrt{2}}$.

d Given that the area of the ellipse is πab, what percentage of the ground is occupied by the seats in the optimal case?

23 B is a boat 5 km out at sea from A. [AC] is a straight sandy beach, 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that AX $= x$ km.

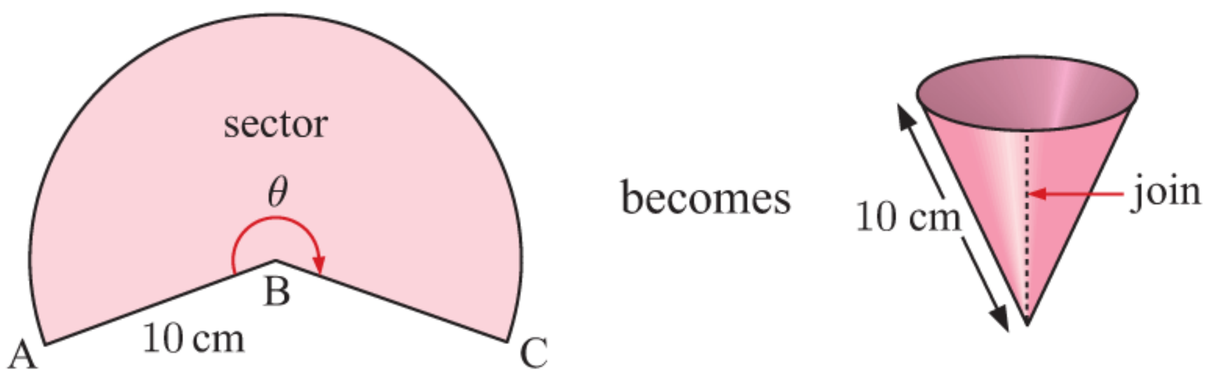


a Explain why $0 \leqslant x \leqslant 6$.

b Show that the *total time* Peter takes to row to X and then run along the beach to C, is given by $T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17}$ hours, $0 \leqslant x \leqslant 6$.

c Find x such that $\frac{dT}{dx} = 0$. Explain the significance of this value.

24 A sector of radius 10 cm and angle θ is bent to form a conical cup as shown.



when edges [AB] and [CB] are joined with tape.

Suppose the resulting cone has base radius r cm and height h cm.

a Show that $r = \frac{5\theta}{\pi}$ and $h = \sqrt{100 - \left(\frac{5\theta}{\pi}\right)^2}$.

b Find the cone's capacity V in terms of θ only.

c Use technology to sketch the graph of $V(\theta)$.

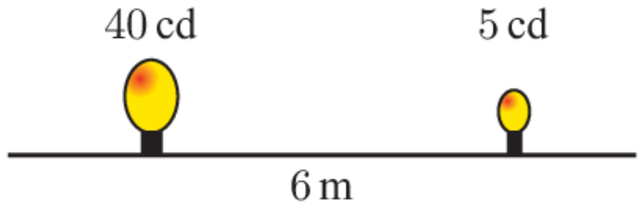
d Find the angle θ which maximises V.

25 A mosquito flying with position $M(x, y, z)$ is repelled by scent emitted from the origin O. At time t seconds, the coordinates of the mosquito are given by $x(t) = 3 - t^2$, $y(t) = 2 + \sqrt{t}$, and $z(t) = 2 - \sqrt{t}$, where all distance units are metres.

a Show that if the mosquito is D m from the origin at time t, then $D^2 = t^4 - 6t^2 + 2t + 17$.

b *Hence* find the shortest distance between the mosquito and the repellent.

26 Two lamps with luminous intensities 40 candelas and 5 candelas are placed 6 m apart. The intensity of illumination from each lamp is directly proportional to the power of the lamp, and inversely proportional to the square of the distance from the lamp. Find the darkest point on the line segment joining the two lamps.

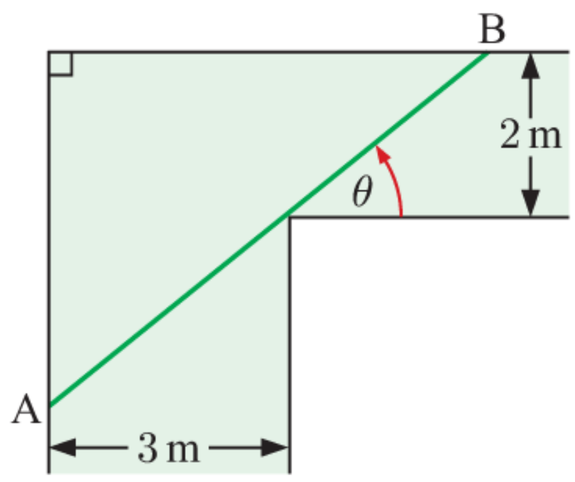


27 Two corridors meet at right angles and are 2 m and 3 m wide respectively. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.

a Show that the length AB is given by $L = 3\sec\theta + 2\operatorname{cosec}\theta$.

b Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \arctan\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$.

c Find L when $\theta = \arctan\left(\sqrt[3]{\frac{2}{3}}\right)$, and comment on the significance of this value.



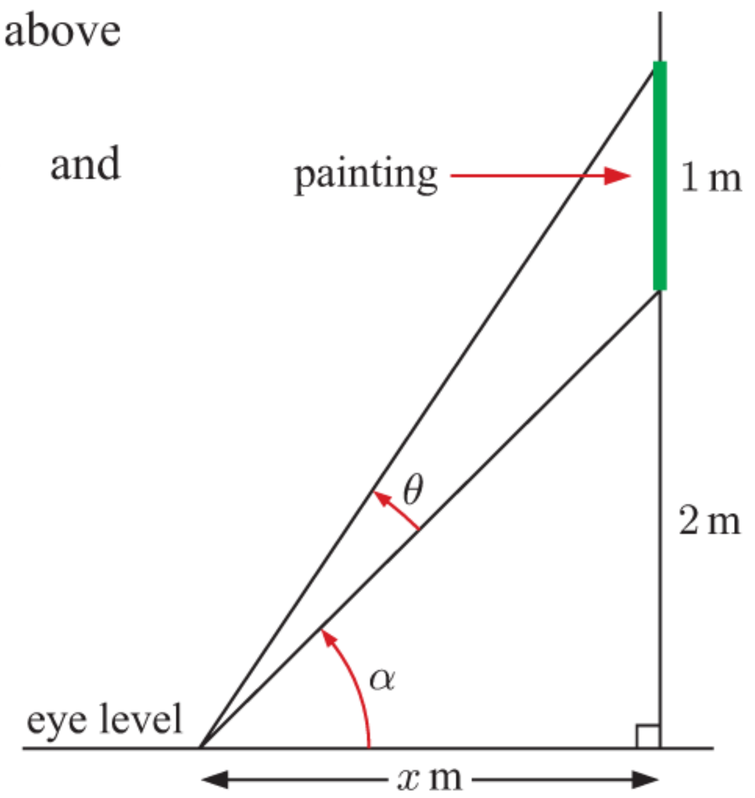
28 Sonia approaches a painting which has its bottom edge 2 m above eye level and its top edge 3 m above eye level.

a Given α and θ as shown in the diagram, find $\tan\alpha$ and $\tan(\alpha + \theta)$.

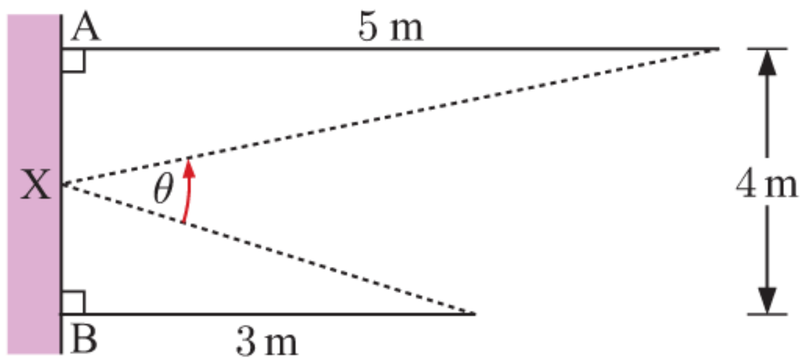
b Find θ in terms of x only.

c Show that $\dfrac{d\theta}{dx} = \dfrac{2}{x^2+4} - \dfrac{3}{x^2+9}$.

d Find the distance Sonia should stand from the wall in order to maximise her viewing angle of the painting.



29



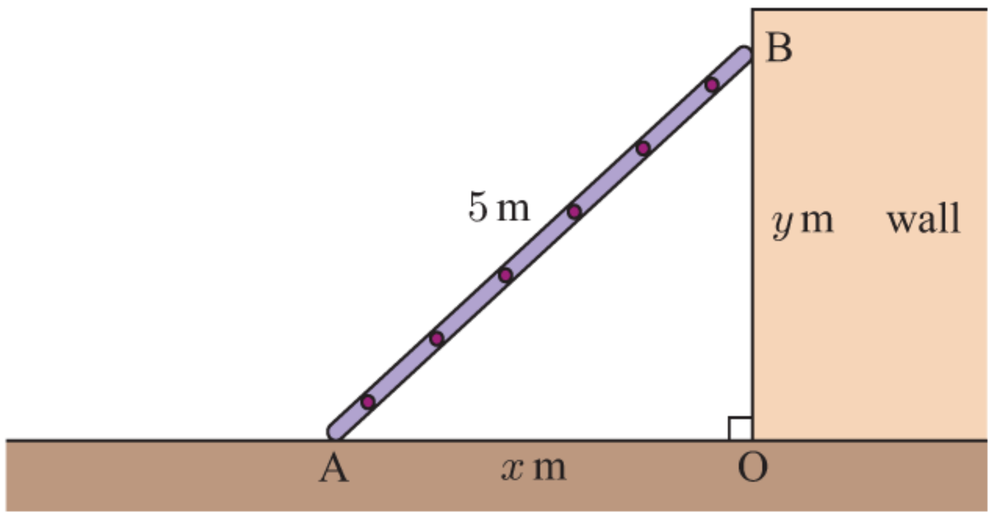
How far should X be from A for angle θ to be maximised?

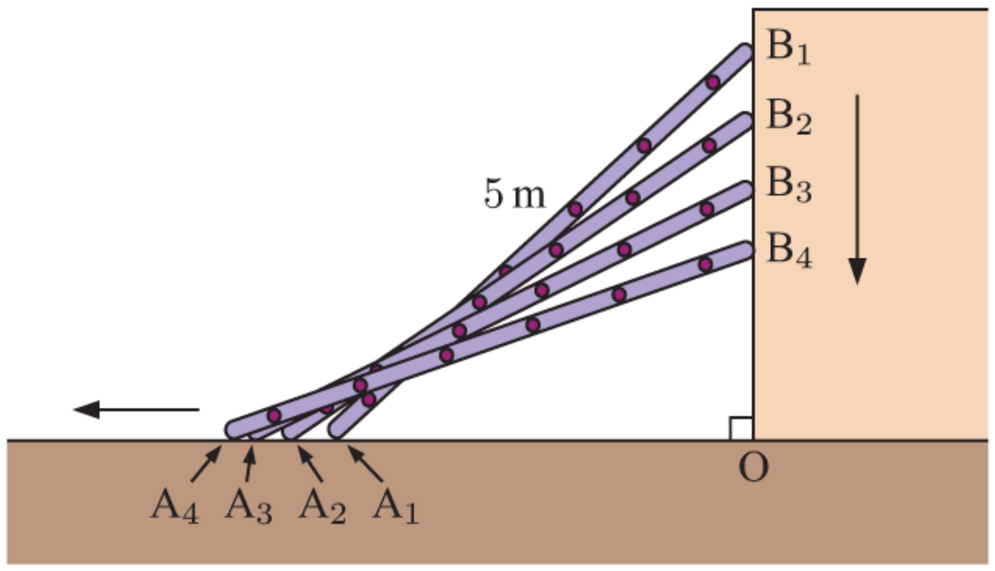
C RELATED RATES

In the **Opening Problem** on page **518**, we observed a ladder slide down a wall.

Because the ladder has a fixed length, when the distance x m from the wall to the foot of the ladder increases, the height y m that the ladder reaches up the wall must decrease.

The variables $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are **related rates**.





The method for solving related rates problems is:

Step 1: Draw a large, clear **diagram** of the situation.

Step 2: Write down the information, label the diagram, and make sure you distinguish between the **variables** and the **constants**.

Step 3: Write an **equation** connecting the variables. You may need to use:

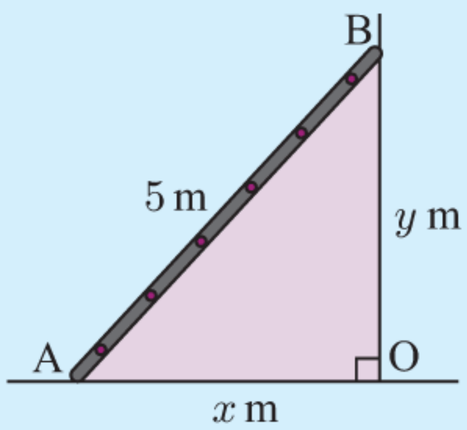
- Pythagoras' theorem
- similar triangles where corresponding sides are in proportion
- right angled triangle trigonometry
- sine and cosine rules.

Step 4: **Differentiate** the equation with respect to time t.

Step 5: Substitute the values for the **particular case** corresponding to some instant in time, and solve to find the required unknown.

Example 7 **Self Tutor**

A 5 m long ladder rests against a vertical wall with its feet on horizontal ground. The feet on the ground slip, and at the instant when they are 3 m from the wall, they are moving at 10 m s^{-1}. At what speed is the other end of the ladder moving at this instant?



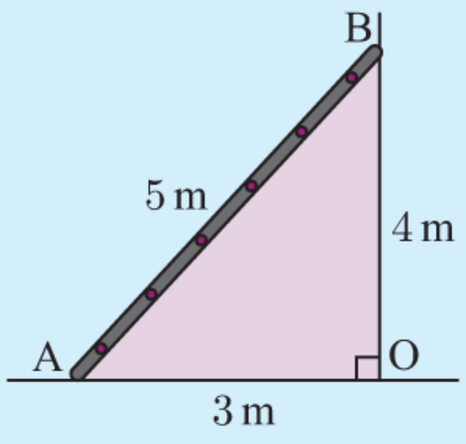
Let OA $= x$ m and OB $= y$ m

$\therefore \quad x^2 + y^2 = 5^2$ {Pythagoras}

Differentiating with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$



Particular case:

At the instant when $x = 3$, we find $y = 4$ and $\frac{dx}{dt} = 10 \text{ m s}^{-1}$.

$\therefore \quad 3(10) + 4\frac{dy}{dt} = 0$

$\therefore \quad \frac{dy}{dt} = -\frac{30}{4} = -7.5 \text{ m s}^{-1}$

The other end of the ladder is moving down the wall at 7.5 m s^{-1} at this instant.

Warning:

You **must not** substitute values for the particular case too early. Otherwise you will incorrectly treat variables as constants. The differentiated equation in fully generalised form must be established first.

EXERCISE 19C

1 a and b are variables related by the equation $ab^3 = 40$.

a Differentiate this equation with respect to t.

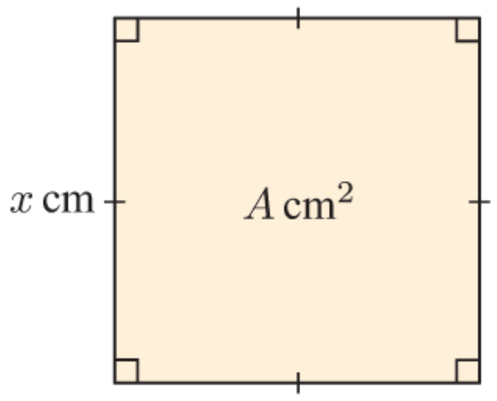
b At the instant when $a = 5$, b is increasing at 1 unit per second. What is happening to a at this instant?

2 A square has side length x cm and area A cm^2. The side length of the square is increasing at 2 cm per second.

a State the relationship between A and x.

b Differentiate the relationship in **a** with respect to t.

c At what rate is the area increasing when the side length is 6 cm?



3 The length of a rectangle is decreasing at 1 cm per minute. However, the area of the rectangle remains constant at 100 cm^2.

Let x cm be the length of the rectangle, and y cm be the width of the rectangle.

a State the relationship between x and y.

b Differentiate the relationship in **a** with respect to t.

c At what rate is the width of the rectangle increasing at the instant when:

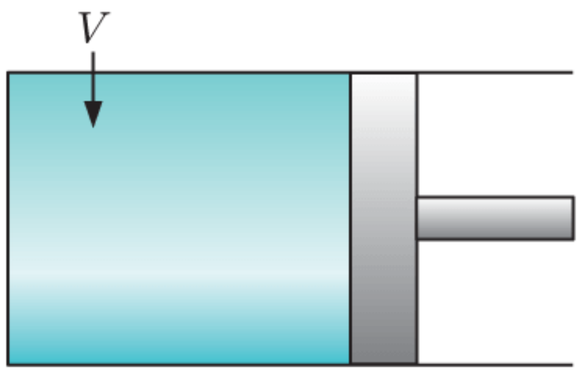
i the length is 20 cm **ii** the rectangle is a square?

4 A stone is thrown into a lake, and a circular ripple moves out at a constant speed of 1 m s^{-1}. Find the rate at which the circle's area is increasing at the instant when:

a $t = 2$ seconds **b** $t = 4$ seconds.

5 Air is pumped into a spherical weather balloon at a constant rate of 6π m^3 per minute. Find the rate of change in the balloon's radius at the instant when the radius is 2 m.

6



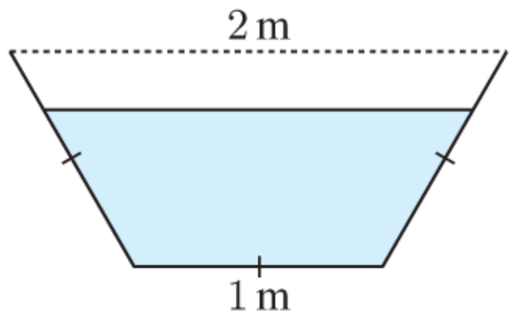
For a given mass of gas in a piston, $pV^{1.5} = 400$ where p is the pressure in N m^{-2}, and V is the volume in m^3.

Suppose the pressure increases at a constant rate of 3 N m^{-2} per minute. Find the rate at which the volume is changing at the instant when the pressure is 50 N m^{-2}.

7 Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius is treble its height. After 1 minute, the height of the heap is 20 cm. Find the rate at which the height is rising at this instant.

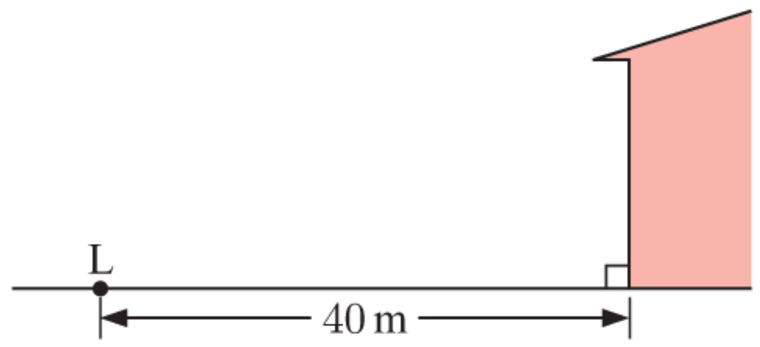
8 Two jet aeroplanes fly on parallel courses which are 12 km apart. Their air speeds are 200 m s^{-1} and 250 m s^{-1}. How fast is the distance between them changing at the instant when the slower jet is 5 km ahead of the faster one?

9 A trough of length 6 m has a uniform cross-section which is a trapezium with side lengths shown. Water leaks from the bottom of the trough at a constant rate of 0.1 m^3 per minute.
Find the rate at which the water level is falling at the instant when the water is 20 cm deep.



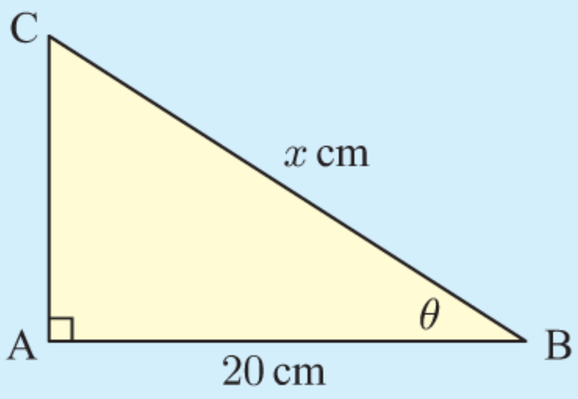
10 A ground-level floodlight located 40 m from the foot of a building shines in the direction of the building.
A 2 m tall person walks directly from the floodlight towards the building at 1 m s^{-1}. How fast is the person's shadow on the building shortening at the instant when the person is:

a 20 m **b** 10 m from the building?



Example 8 **Self Tutor**

Triangle ABC is right angled at A, and AB = 20 cm. $A\hat{B}C$ increases at a constant rate of 1° per minute. At what rate is BC changing at the instant when $A\hat{B}C$ measures 30°?



Let $A\hat{B}C = \theta$ and BC = x cm

Now $\cos\theta = \dfrac{20}{x} = 20x^{-1}$

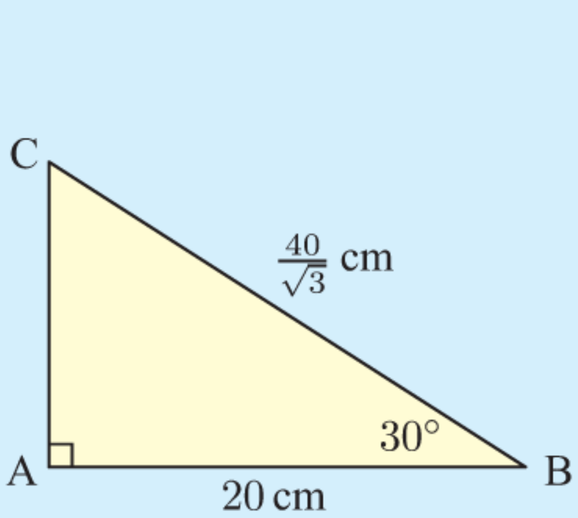
$\therefore \quad -\sin\theta \dfrac{d\theta}{dt} = -20x^{-2} \dfrac{dx}{dt}$

Particular case:

When $\theta = 30°$, $\cos 30° = \dfrac{20}{x}$

$\therefore \quad \dfrac{\sqrt{3}}{2} = \dfrac{20}{x}$

$\therefore \quad x = \dfrac{40}{\sqrt{3}}$



Also, $\dfrac{d\theta}{dt} = 1°$ per minute

$= \frac{\pi}{180}$ radians per minute

Thus $-\sin 30° \times \frac{\pi}{180} = -20 \times \dfrac{3}{1600} \times \dfrac{dx}{dt}$

$\therefore \quad -\frac{1}{2} \times \frac{\pi}{180} = -\frac{3}{80} \dfrac{dx}{dt}$

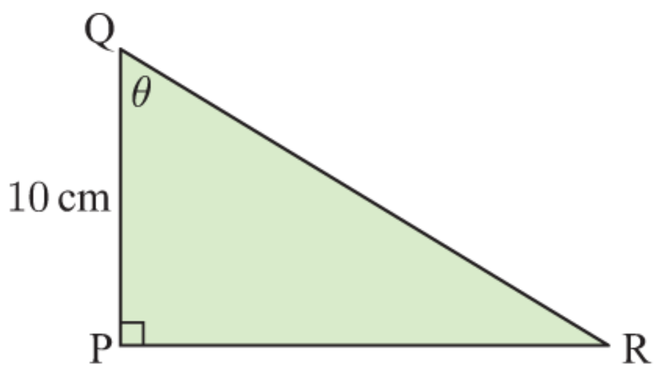
$\therefore \quad \dfrac{dx}{dt} = \frac{\pi}{360} \times \frac{80}{3}$ cm per minute

≈ 0.233 cm per minute

$\therefore$ BC is increasing at approximately 0.233 cm per minute.



11 Consider a right angled triangle PQR in which QP $= 10$ cm and $\text{P}\widehat{\text{Q}}\text{R} = \theta$. $\text{P}\widehat{\text{Q}}\text{R}$ is increasing at $2°$ per minute.



a As $\text{P}\widehat{\text{Q}}\text{R}$ increases, would you expect QR to increase or decrease?

b At what rate is QR changing when $\text{P}\widehat{\text{Q}}\text{R}$ measures $60°$?

12 A right angled triangle ABC has a fixed hypotenuse [AC] of length 10 cm. Side [AB] increases in length at 0.1 cm per second. At what rate is $\text{C}\widehat{\text{A}}\text{B}$ decreasing at the instant when the triangle is isosceles?

13 An aeroplane passes directly overhead, then flies horizontally away from an observer with altitude of 5000 m and air speed 200 m s^{-1}. At what rate is its angle of elevation to the observer changing at the instant when the angle of elevation is:

a $60°$ **b** $30°$?

14 Rectangle PQRS has [PQ] of fixed length 20 cm, and [QR] increases in length at a constant rate of 2 cm s^{-1}. At what rate is the acute angle between the diagonals of the rectangle changing at the instant when [QR] is 15 cm long?

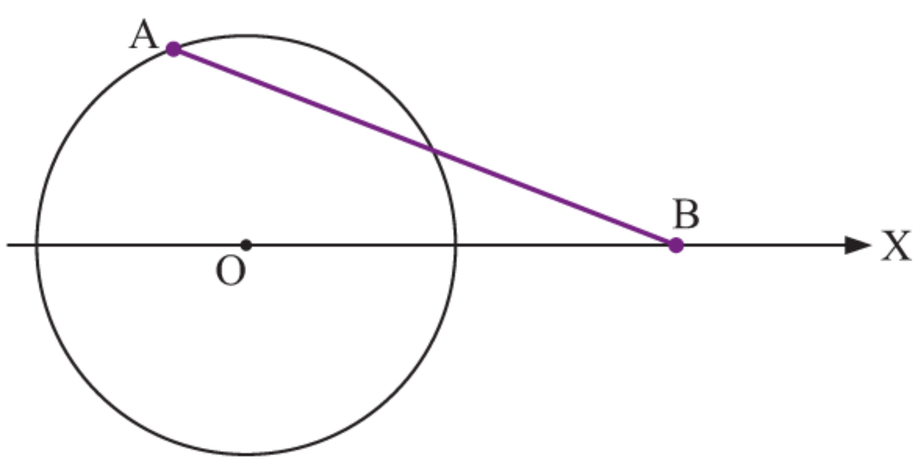
15 Triangle PQR is right angled at Q, and [PQ] is 6 cm long. [QR] increases in length at 2 cm per minute. Find the rate of change in $\text{Q}\widehat{\text{P}}\text{R}$ at the instant when [QR] is 8 cm long.

16 Two cyclists A and B leave X simultaneously at $120°$ to one another, with constant speeds of 12 m s^{-1} and 16 m s^{-1} respectively. Find the rate at which the distance between them is changing after 2 minutes.

17 [AB] is a fixed diameter of a circle of radius 5 cm. Point P moves around the circle at a constant rate of 1 revolution in 10 seconds. Find the rate at which the distance AP is changing at the instant when:

a AP $= 5$ cm and increasing **b** P is at B.

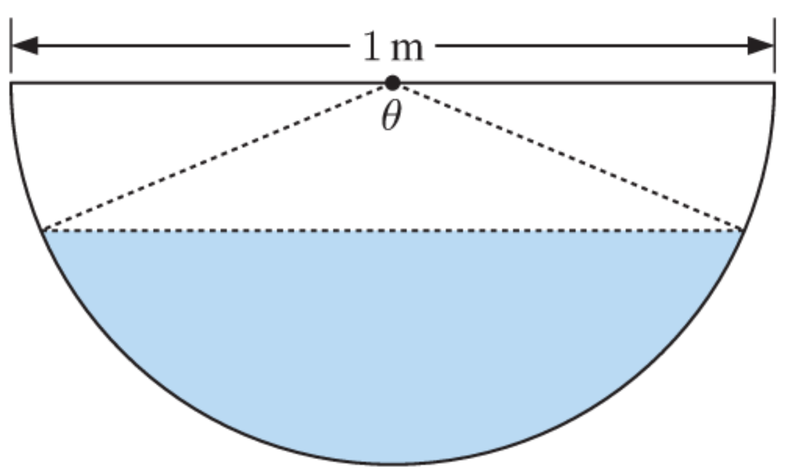
18



Shaft [AB] is 30 cm long and is attached to a flywheel at A. B is confined to motion along [OX]. The radius of the wheel is 15 cm, and the wheel rotates clockwise at 100 revolutions per second. Find the rate of change in $\text{A}\widehat{\text{B}}\text{O}$ when $\text{A}\widehat{\text{O}}\text{X}$ is:

a $120°$ **b** $180°$

19 A farmer has a water trough of length 8 m which has a semi-circular cross-section of diameter 1 m. Water is pumped into the trough at a constant rate of 0.1 m^3 per minute.



a Show that the volume of water in the trough is given by $V = \theta - \sin\theta$, where θ is the angle illustrated (in radians).

b Find the rate at which the water level is rising at the instant when the water is 25 cm deep.

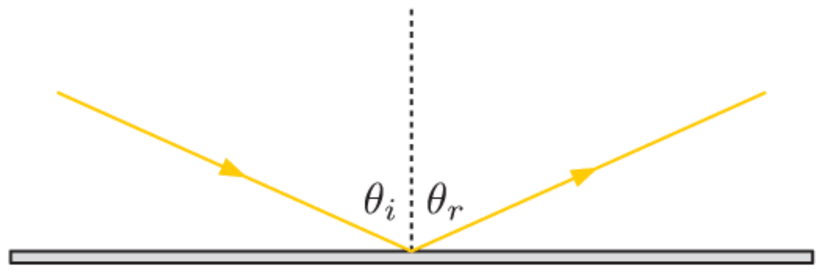
Hint: First find $\frac{d\theta}{dt}$ and then find $\frac{dh}{dt}$ at the given instant.

20 In about 1650, **Pierre de Fermat** (1607 - 1665) developed the *principle of least time* to explain the behaviour of light:

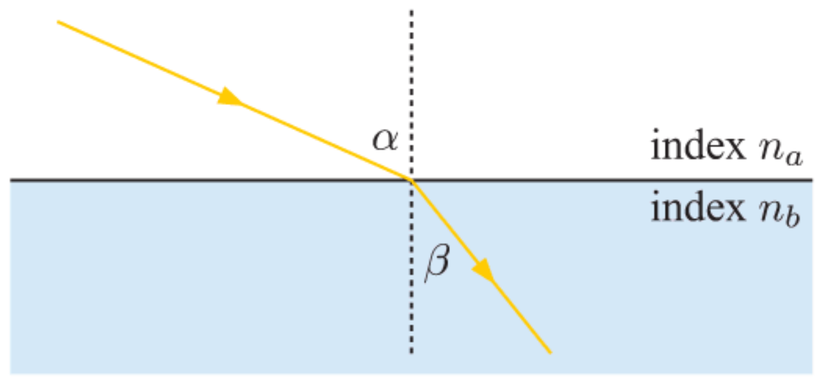
> Out of all possible paths that it might take to get from one point to another, light takes the path which requires the shortest time.

Use Fermat's principle of least time to prove the following laws of optics:

a The *law of reflection* states that when light hits a mirror, the angle of incidence θ_i is equal to the angle of reflection θ_r.



b Snell's *law of refraction* states that when light travels from a substance with index of refraction n_a to another with index of refraction n_b, the angle of incidence α and the angle of refraction β are related by $n_a \sin\alpha = n_b \sin\beta$.



Hint: The speed of light in a substance is given by $v = \frac{c}{n}$ where v is the speed of light in the substance, c is the speed of light in a vacuum, and n is the index of refraction of the substance.

THEORY OF KNOWLEDGE

Snell's law was first discovered in 984 AD by the Persian scientist **Ibn Sahl**, who was studying the shape of lenses. However, it is named after **Willebrord Snellius**, who rediscovered it during the Renaissance. The law was published by **René Descartes** in his *Discourse on the Method* published in 1637.

Both Snell's law and the law of reflection are optimisation problems governed by Fermat's principle of least time.



Willebrord Snellius

1 Is optimisation a mathematical principle?

2 Is mathematics an intrinsic or natural part of other subjects?

ACTIVITY CUBIC SPLINES

A *spline* is a function which uses several polynomials to draw a smooth curve through a set of data points. The spline can be used in modelling to interpolate values between the data.

In this online Activity, you will use a set of cubic functions to construct a smooth curve through a set of data points.



REVIEW SET 19A

1 The height of a tree t years after it was planted is given by $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$.

a How tall was the tree when it was planted?

b How long will it take for the tree to reach:

i 150 cm **ii** 300 cm?

c At what rate is the tree's height increasing after:

i 2 years **ii** 20 years?



2 The value of a car t years after its purchase is given by $V = 20\,000e^{-0.4t}$ pounds. Calculate:

a the purchase price of the car

b the rate at which the value of the car is decreasing 10 years after it was purchased.

3 The cost per hour of running a freight train is given by $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

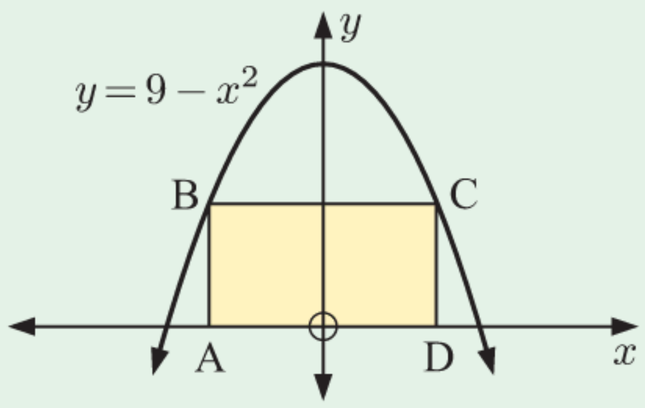
a Find the cost of running the train for 5 hours at 64 km h^{-1}.

b Find the rate of change in the hourly cost of running the train at speeds of:

i 75 km h^{-1} **ii** 90 km h^{-1}.

c At what speed will the cost per hour be a minimum?

4



Rectangle ABCD is inscribed within the parabola $y = 9 - x^2$ and the x-axis, as shown.

a If OD $= x$, show that the rectangle ABCD has area function $A(x) = 18x - 2x^3$.

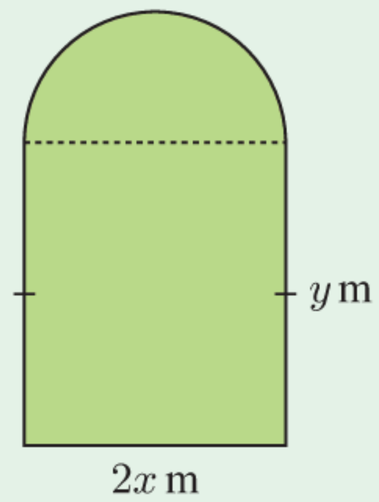
b Find the coordinates of C when rectangle ABCD has maximum area.

5 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

a Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.

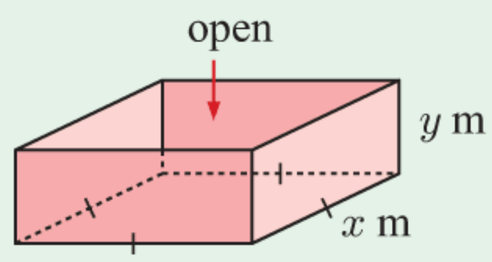
b Find the area of the lawn A in terms of x only.

c Find the dimensions of the lawn of maximum area.



6 A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 kL. The steel costs £2 per square metre.

a If the base measures x m by x m and the height is y m, find y in terms of x.



b Hence show that the total cost of the steel is $C(x) = 2x^2 + \frac{8}{x}$ pounds.

c Find the dimensions of the steel box which would cost the least to make.

7 On the Indonesian coast, the depth of water t hours after midnight is given by $D = 9.3 + 6.8\cos(0.507t)$ metres.



a What is the derivative function $\frac{dD}{dt}$ and what does it tell us?

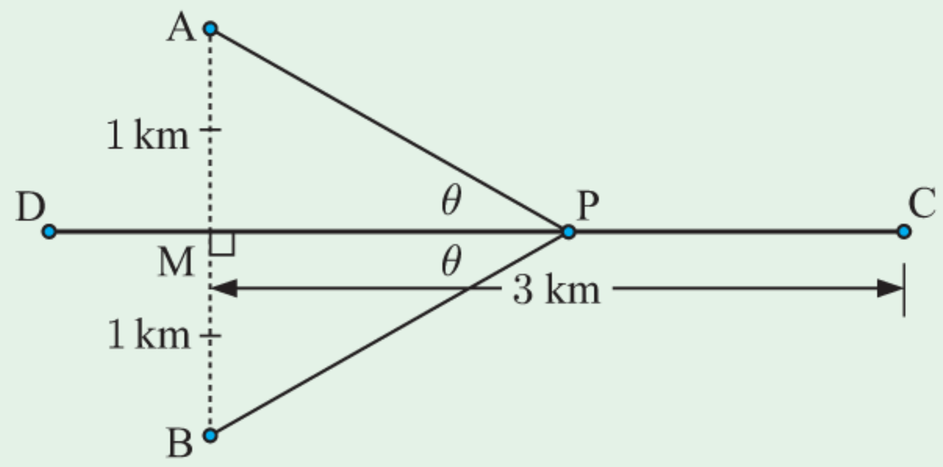
b What is the depth of water at 8 am?

c Is the tide rising or falling at 8 am? Explain your answer.

d At what time(s) is the tide highest on this day? What is the maximum depth of water?

8 A and B are two houses directly opposite one another and 1 km from a straight road [DC]. MC is 3 km and C is a house at the roadside.

A power unit is to be located on [DC] at P such that PA + PB + PC is minimised. This ensures that the cost of trenching and cable will be as small as possible.



a What cable length would be required if the power unit P is placed at:

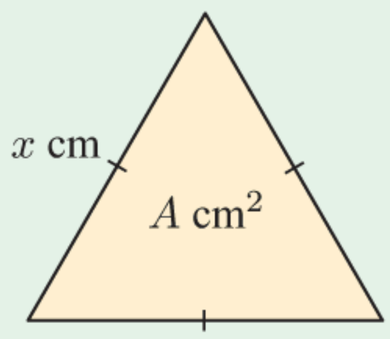
i M **ii** C?

b Show that if $\theta = A\widehat{P}M = B\widehat{P}M$, then the length of cable is given by

$$L(\theta) = 3 + \frac{2 - \cos\theta}{\sin\theta} \text{ km}.$$

c Show that $\frac{dL}{d\theta} = \frac{1 - 2\cos\theta}{\sin^2\theta}$, and hence show that the minimum length of cable required is $(3 + \sqrt{3})$ km.

9

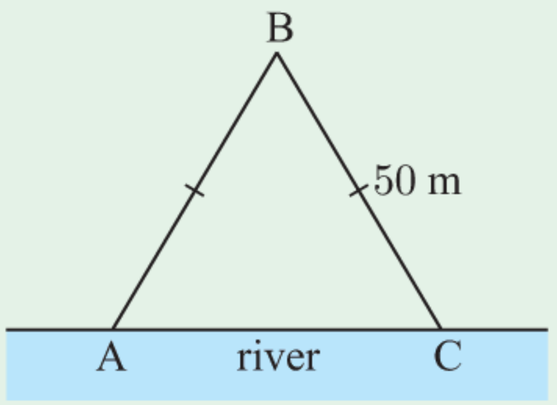


An equilateral triangle has side length x cm and area A cm^2. The side length of the triangle is increasing at 3 cm per minute.

a State the relationship between A and x.

b At what rate is the area increasing when the side length is 15 cm?

10 A triangular pen is enclosed by two fences [AB] and [BC], each of length 50 m, with the river as the third side.

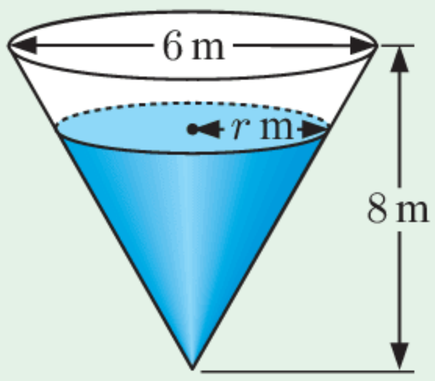


a If AC $= 2x$ m, show that the area of triangle ABC is $A = x\sqrt{2500 - x^2}$ m^2.

b Find $\frac{d}{dx}(A^2)$.

c Hence find x such that the area is maximised.

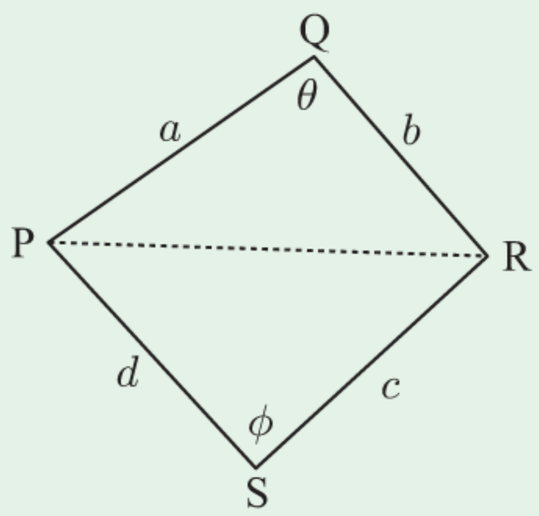
11 Water exits a conical tank at a constant rate of 0.2 m^3 per minute. The surface of the water has radius r m.



a Find $V(r)$, the volume of the water remaining.

b Find the rate at which the surface radius is changing at the instant when the water is 5 m deep.

12



Four straight sticks of fixed lengths a, b, c, and d are hinged together at P, Q, R, and S.

a Use the cosine rule to find an equation which connects a, b, c, d, $\cos\theta$, and $\cos\phi$.

b Show that $\dfrac{d\theta}{d\phi} = \dfrac{cd\sin\phi}{ab\sin\theta}$.

c Hence show that the area of quadrilateral PQRS is maximised when PQRS is a cyclic quadrilateral.

REVIEW SET 19B

1 The cost in euros of producing x items in a factory each day is given by $C(x) = 850 + 3.3x^{0.85} + 2.8x^{0.5}$.

a Find the marginal cost function $C'(x)$.

b Find $C'(1000)$ and explain what this result estimates.

c Find $C(1001) - C(1000)$. Discuss your answer.

2 The size of a population at time t years is given by $P(t) = 60\,000\left(1 + 2e^{-\frac{t}{4}}\right)^{-1}$, $t \geqslant 0$.

a Find the initial population. **b** Find $P'(t)$.

c Show that $P'(t) > 0$ for all $t \geqslant 0$. Explain what this means.

d Find $P''(t)$.

e Find the maximum growth rate of the population, and the exact time when this occurs.

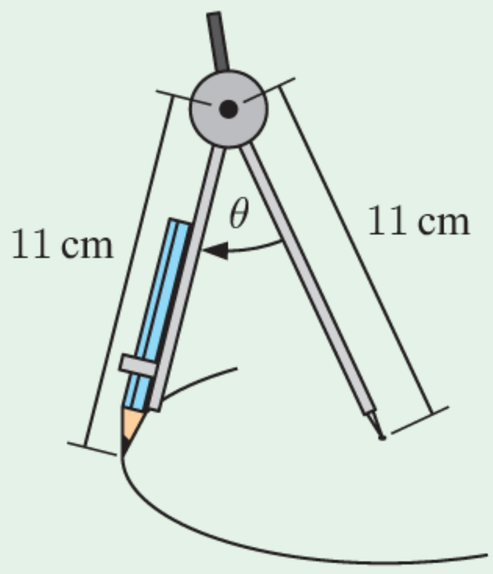
f Discuss $P(t)$ as $t \to \infty$.

g Sketch the population function, showing the information you have found.

3 Mark has set his compass so that both arms have length 11 cm. The angle between the arms is θ.

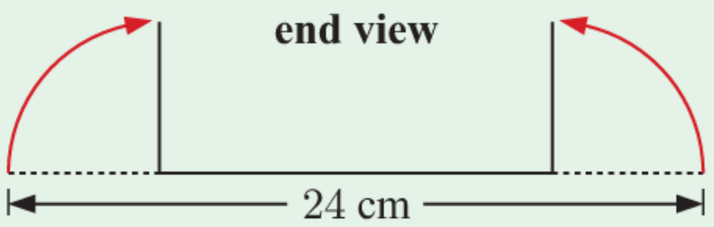
a Write the area A of the circle Mark will draw in terms of θ.

b Hence find the rate of change in A with respect to θ when $\theta = \frac{\pi}{4}$.

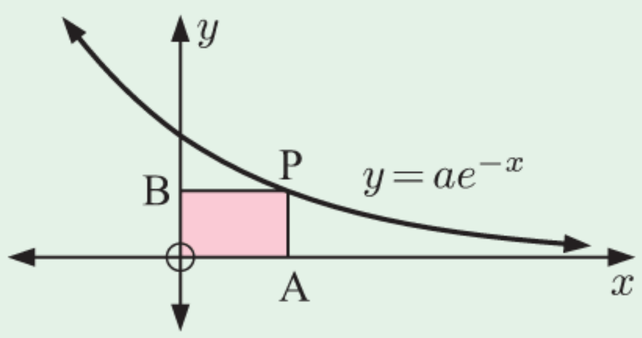


4 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.

Where must the bends be made in order to maximise the capacity of the gutter?



5

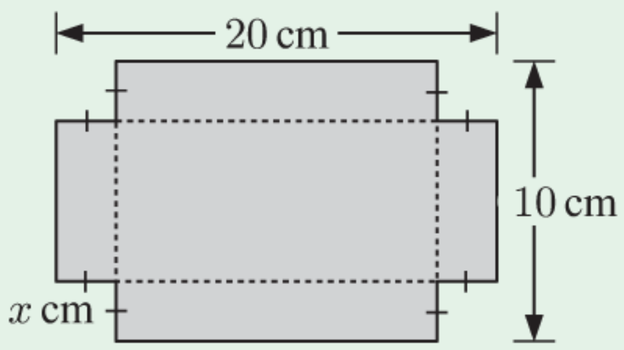


The graph of $y = ae^{-x}$ for $a > 0$ is shown.

P is a moving point on the graph, and A and B lie on the axes as shown so that OAPB is a rectangle.

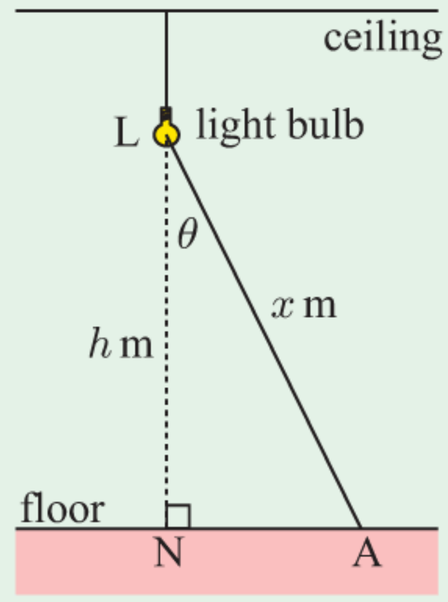
Find the x-coordinate of P, in terms of a, such that the rectangle OAPB has minimum perimeter.

6



A rectangular sheet of tin-plate is 20 cm by 10 cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.

7 A light bulb hangs from the ceiling at height h metres above the floor, directly above point N. At any point A on the floor which is x metres from the light bulb, the illumination I is given by $I = \dfrac{\sqrt{8}\cos\theta}{x^2}$ units.

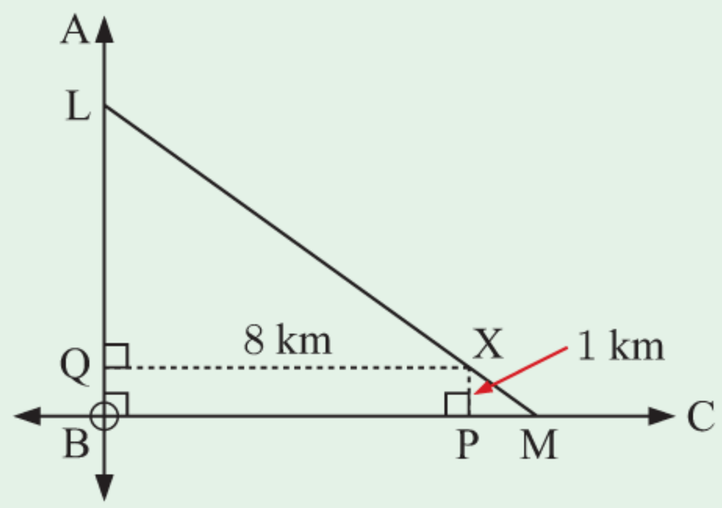


a If NA $= 1$ metre, show that at A, $I = \sqrt{8}\cos\theta\sin^2\theta$.

b The light bulb may be lifted or lowered to change the intensity at A. Assuming NA $= 1$ metre, find the height the bulb should be above the floor to provide the greatest illumination at A.

8 Two roads AB and BC meet at right angles. A straight pipeline [LM] is to be laid between the two roads so that it passes through the point X shown.

Let PM $= x$ km.



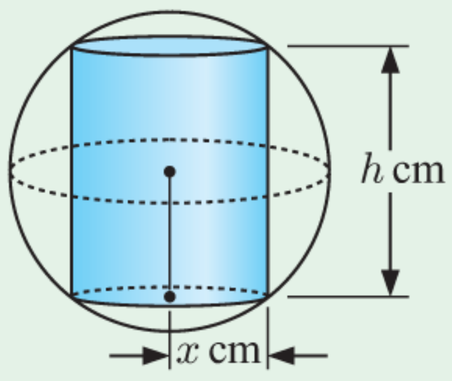
a Write LQ in terms of x.

b Hence show that the length of the pipeline is given by $L = \sqrt{x^2+1}\left(1+\dfrac{8}{x}\right)$ km.

c Find $\dfrac{d}{dx}(L^2)$.

d Hence find the shortest possible length for the pipeline.

9 A ball is made up of a blue plastic cylinder fitting exactly in a transparent rubber sphere with diameter 10 cm.

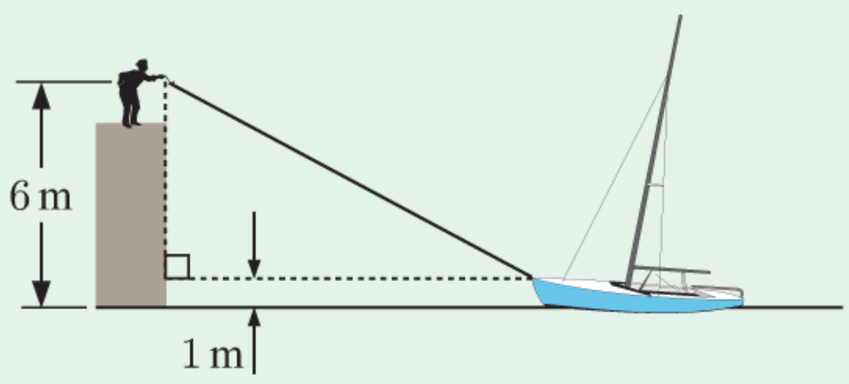


a If the cylinder has radius x cm, show that the cylinder's volume is given by $V(x) = \pi x^2\sqrt{100-4x^2}$ cm^3.

b Hence find the dimensions of the cylinder of largest volume.

10 Two runners run in different directions, $60°$ apart. Alex runs at 5 m s^{-1} and Barry runs at 4 m s^{-1}. Barry passes through X 3 seconds after Alex passes through X. At what rate is the distance between them increasing at the instant when Alex is 20 metres past X?

11



A man on a jetty pulls a boat towards him by hauling a rope at the rate 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty?

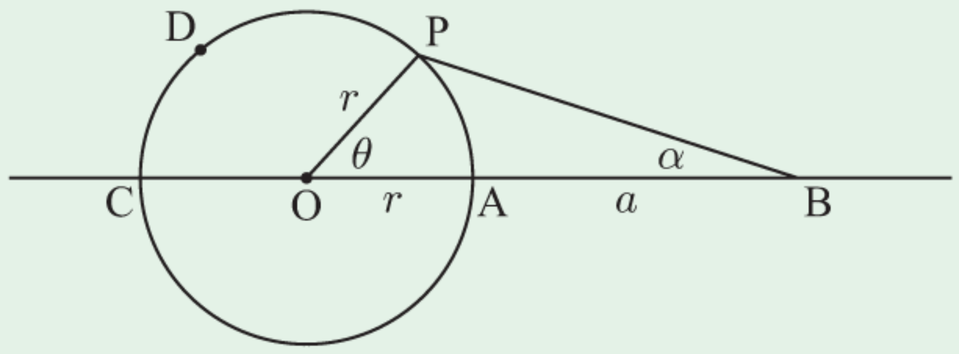
12 The point $(3\cos\theta,\ 2\sin\theta)$ lies on a curve with $0 \leqslant \theta \leqslant 2\pi$.

a Find the equation of the curve in Cartesian form.

b Find $\dfrac{dy}{dx}$ in terms of θ.

c A tangent to the curve meets the x-axis at A and the y-axis at B.
Find the smallest possible area of triangle OAB and the value(s) of θ when it occurs.

13 Consider a circle with centre O and radius r. A, B, and C are fixed points. An ant starts at B and moves at constant speed v in a straight line to point P. The ant then moves along the arc from P to C, via D, at constant speed w where $w > v$.



a Show that the total time for the journey is given by

$$T = \frac{\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}}{v} + \frac{r(\pi - \theta)}{w}.$$

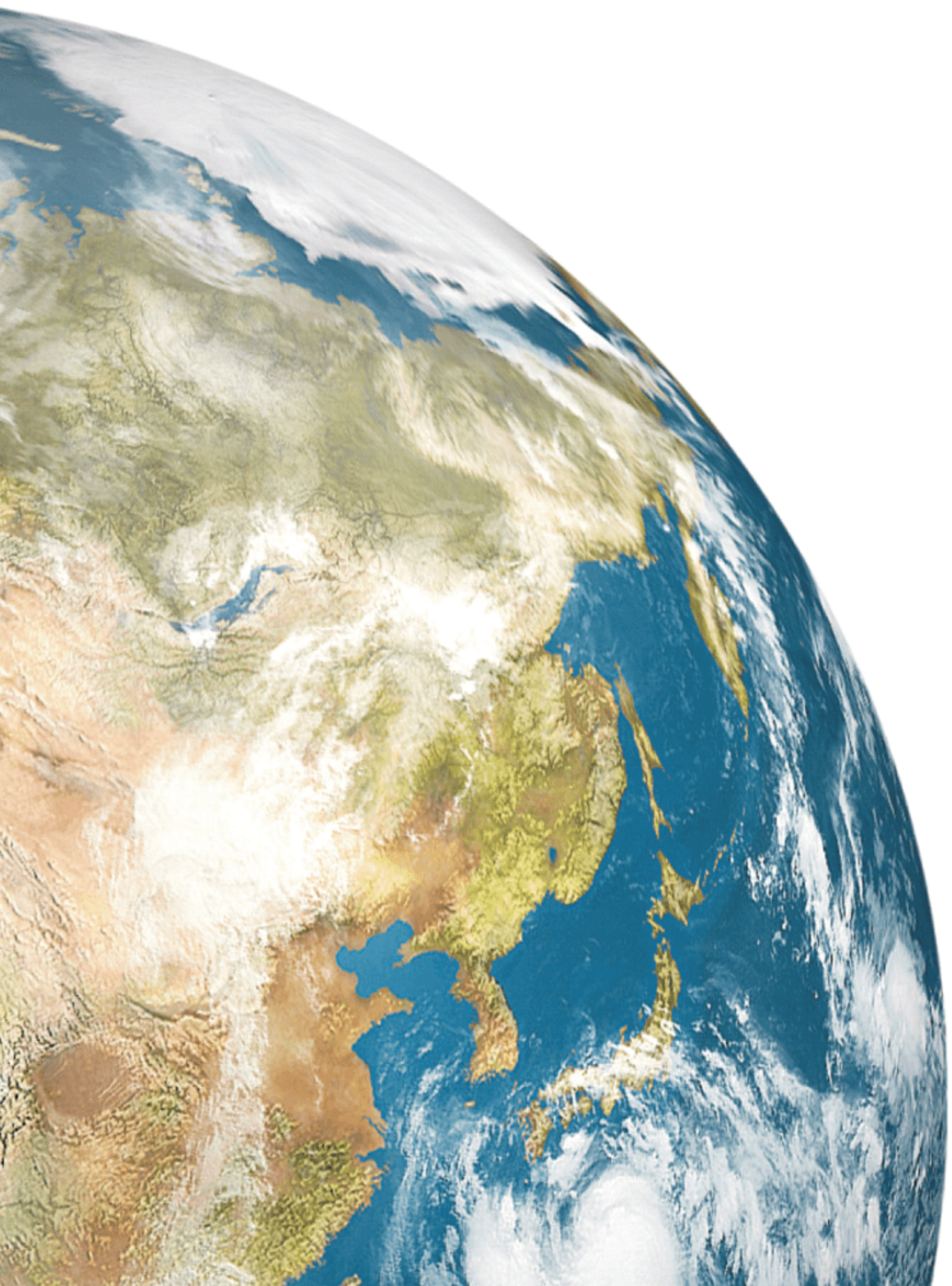
b Hence show that T is minimised when $\alpha = \arcsin\left(\dfrac{rv}{(a+r)w}\right)$.

Chapter 20

Introduction to integration

Contents:

A Approximating the area under a curve
B The Riemann integral
C Antidifferentiation
D The Fundamental Theorem of Calculus

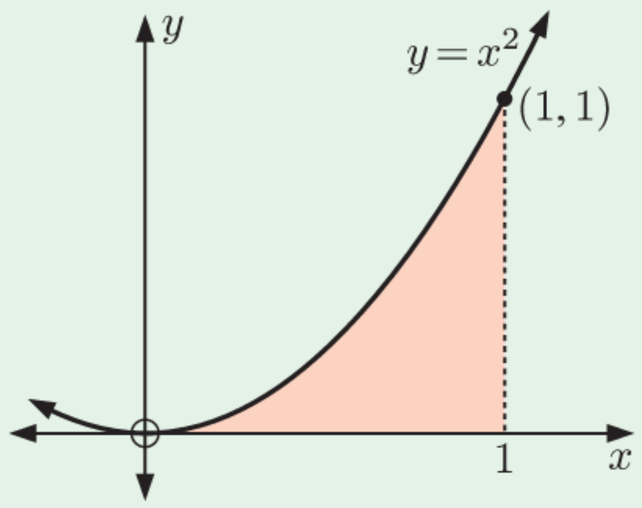


OPENING PROBLEM

Another of **Archimedes'** achievements was devising a method for calculating the area under a curve.

In an article containing 24 propositions, he provided essential theory for what, over 1800 years later, would be developed into **integral calculus**.

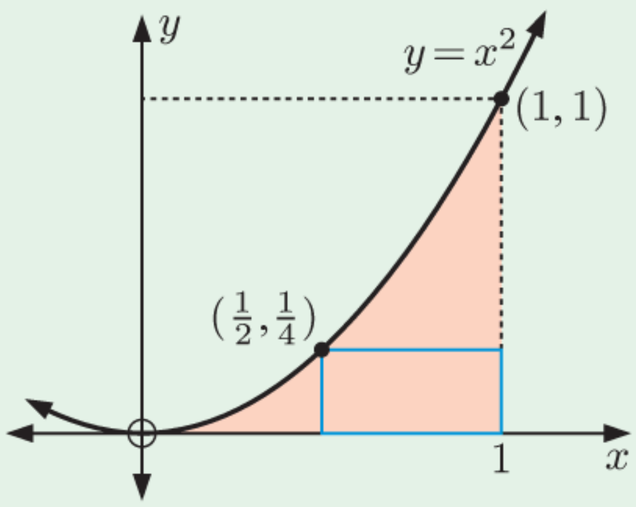
In the process, Archimedes found the exact area A between the curve $y = x^2$ and the x-axis, on the interval $0 \leqslant x \leqslant 1$.



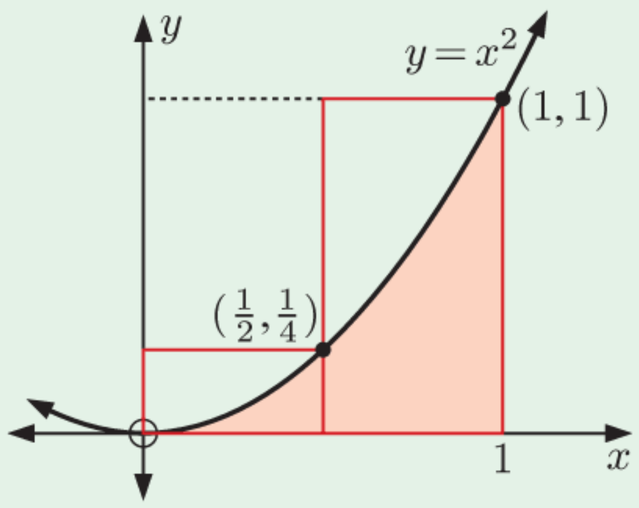
Things to think about:

a Can you use the:

i blue rectangle to explain why $A > \frac{1}{8}$



ii red rectangles to explain why $A < \frac{5}{8}$?



b How can we obtain a better estimate for A?

c What function has x^2 as its derivative?

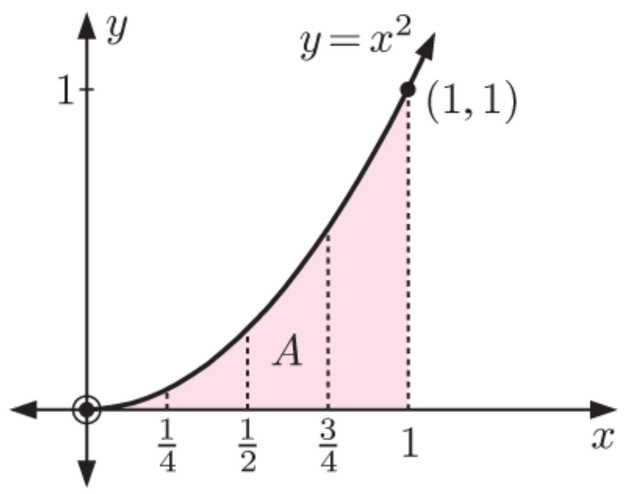
In this Chapter we consider **integral calculus**. This involves **antidifferentiation**, which is the reverse process of differentiation.

A APPROXIMATING THE AREA UNDER A CURVE

Consider the function $f(x) = x^2$ in the **Opening Problem**.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical line $x = 1$.

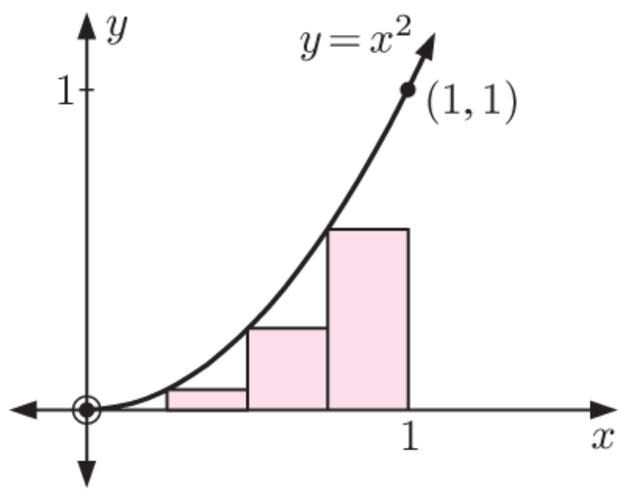
Suppose we divide the interval $0 \leqslant x \leqslant 1$ into 4 strips of width $\frac{1}{4}$ unit as shown. We obtain 4 subintervals of equal width.



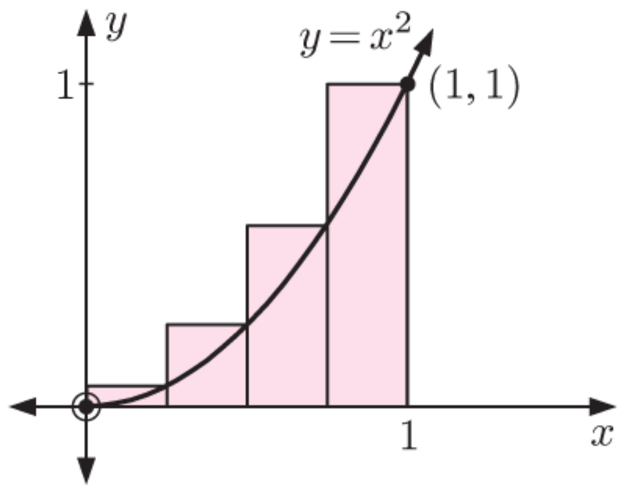
The diagram alongside shows **lower rectangles**, which are rectangles with height equal to the *lower* value of the function at the endpoints of the subinterval.

The total area of the lower rectangles is

$$\begin{aligned} A_L &= \tfrac{1}{4} \times f(0) + \tfrac{1}{4} \times f(\tfrac{1}{4}) + \tfrac{1}{4} \times f(\tfrac{1}{2}) + \tfrac{1}{4} \times f(\tfrac{3}{4}) \\ &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &= 0.218\,75 \end{aligned}$$



The next diagram shows **upper rectangles**, which are rectangles with height equal to the *upper* value of the function at the endpoints of the subinterval.



The total area of the upper rectangles is

$$\begin{aligned} A_U &= \tfrac{1}{4} \times f(\tfrac{1}{4}) + \tfrac{1}{4} \times f(\tfrac{1}{2}) + \tfrac{1}{4} \times f(\tfrac{3}{4}) + \tfrac{1}{4} \times f(1) \\ &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2 \\ &= 0.468\,75 \end{aligned}$$

Clearly, for increasing functions such as $f(x) = x^2$, $A_L < A < A_U$, so the area A lies between $0.218\,75$ units^2 and $0.468\,75$ units^2.

If the interval $0 \leqslant x \leqslant 1$ was divided into 8 subintervals instead, each of width $\frac{1}{8}$, then

$$\begin{aligned} A_L &= \tfrac{1}{8}\left[f(0) + f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8})\right] \\ &= \tfrac{1}{8}\left[0 + \tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64}\right] \\ &\approx 0.273\,44 \\ A_U &= \tfrac{1}{8}\left[f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8}) + f(1)\right] \\ &= \tfrac{1}{8}\left[\tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64} + 1\right] \\ &\approx 0.398\,44 \end{aligned}$$

From this refinement we conclude that the area A lies between $0.273\,44$ units^2 and $0.398\,44$ units^2.

As we create more subintervals, the estimates A_L and A_U will become more and more accurate. In fact, as the subinterval width is reduced further and further, both A_L and A_U will **converge** to A.

Now suppose there are n subintervals between $x = 0$ and $x = 1$, each of width $\frac{1}{n}$.

You can use the **area finder** software or your **graphics calculator** to help calculate A_L and A_U for large values of n.

AREA FINDER

GRAPHICS CALCULATOR INSTRUCTIONS

The table alongside summarises the results you should obtain for $n = 4$, 8, 16, 50, 200, 1000, and 10 000.

From the table, it appears that both A_L and A_U are *converging* to $\frac{1}{3}$ as n increases.

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
8	0.273 44	0.398 44	0.335 50
16	0.302 73	0.365 23	0.333 98
50	0.323 40	0.343 40	0.333 40
200	0.330 84	0.335 84	0.333 38
1000	0.332 83	0.333 83	0.333 33
10 000	0.333 28	0.333 38	0.333 33

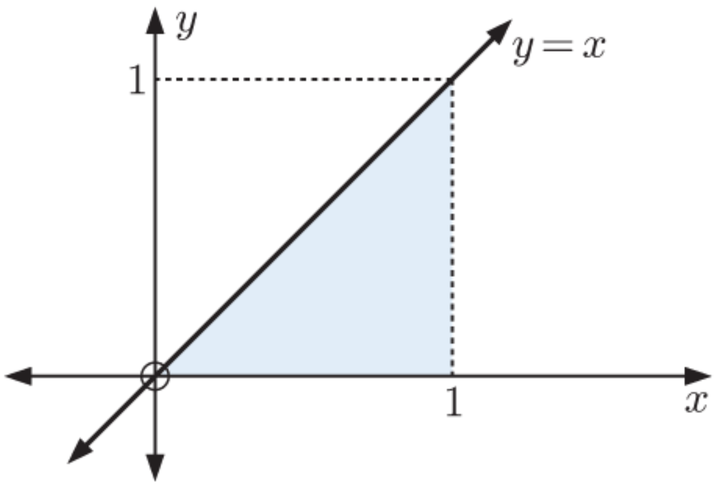
EXERCISE 20A

1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

a Divide the interval into 5 subintervals of equal width, then estimate the area using:

i lower rectangles **ii** upper rectangles.

b Calculate the actual area and compare it with your answers in **a**.



2 Consider the area between $y = \frac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 subintervals of equal width, then estimate the area using:

a lower rectangles **b** upper rectangles.

3 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100, and 500. Give your answers to 4 decimal places.

As n gets larger, both A_L and A_U converge to the same rational number. What is it?



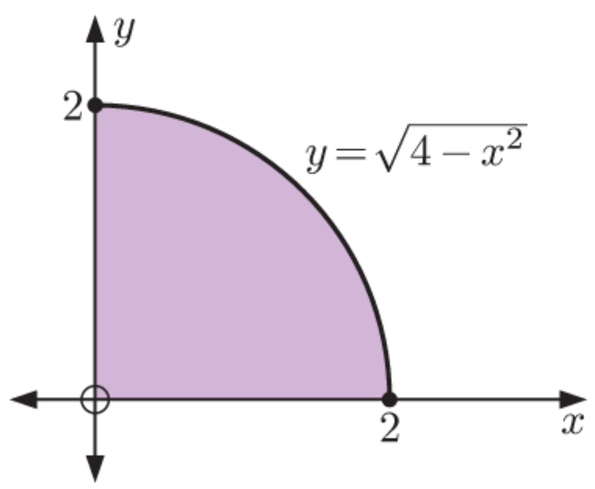
4 **a** Use lower and upper rectangle sums to estimate the area between each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$. Use $n = 5$, 10, 50, 100, 500, 1000, and 10 000. Give your answer to 5 decimal places in each case.

i $y = x^3$ **ii** $y = x$ **iii** $y = x^{\frac{1}{2}}$ **iv** $y = x^{\frac{1}{3}}$

b For each case in **a**, write down the value to which A_L and A_U converge.

c Using your answer to **b**, predict the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$.

5 Consider the quarter circle with centre $(0, 0)$ and radius 2 units illustrated.



Its area is $\frac{1}{4}$(full circle with radius 2 units) $= \frac{1}{4} \times \pi \times 2^2$
$= \pi$ units^2

a Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000, and 10 000. Hence find rational bounds for π.

b Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

For what value of n is your estimate for π better than that of Archimedes?

INVESTIGATION 1 THE AREA UNDER $f(x) = x^2$

For the curve $f(x) = x^2$, we already have the tools necessary to calculate the area A between $y = f(x)$ and the x-axis on the interval $0 \leqslant x \leqslant 1$ exactly.

Suppose we divide the interval $0 \leqslant x \leqslant 1$ into n subintervals, each of width $\frac{1}{n}$.

What to do:

1 **a** Explain why the total area of lower rectangles can be written as $A_L = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$.

b Use the sum of series formulae $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to show that $A_L = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$.

c What value does A_L approach as $n \to \infty$?

2 **a** Explain why the total area of upper rectangles can be written as $A_U = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)$.

b Hence show that $A_U = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$.

c What value does A_U approach as $n \to \infty$?

3 Use the results from **1** and **2** to explain why $A = \frac{1}{3}$ units^2.

B THE RIEMANN INTEGRAL

We have seen that for the special case of a quadratic, we can use series formulae to evaluate the area under the curve exactly.

However, for most functions we do not have such formulae. We therefore need a more general method for finding the area under a curve.

HISTORICAL NOTE

Italian Mathematician **Bonaventura Cavalieri** (1598 - 1647) became Professor of Mathematics at Bologna in 1629. He published tables for many trigonometric and logarithmic functions. However, his best known contribution to mathematics was the invention of **indivisibles**.

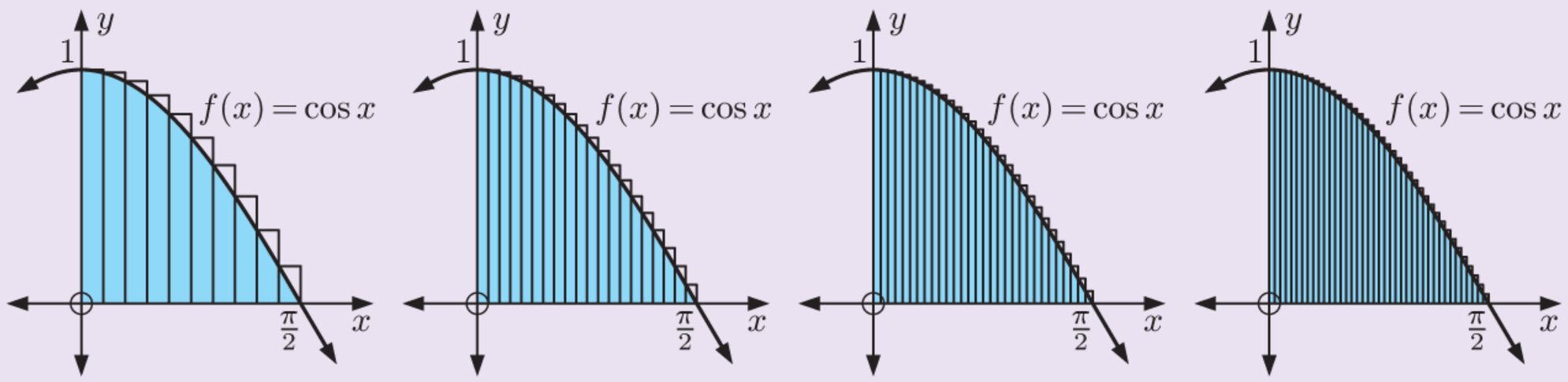


Bonaventura Cavalieri

In his Method of Indivisibles, Cavalieri considered that a moving point could be used to sketch a curve. The curve could therefore be considered as the set of an infinite number of points, each with no length.

In a similar way, the "indivisibles" that made up a surface were an infinite number of lines. Almost every introduction to integral calculus starts with the division of an area into a number of rectangular strips with finite width.

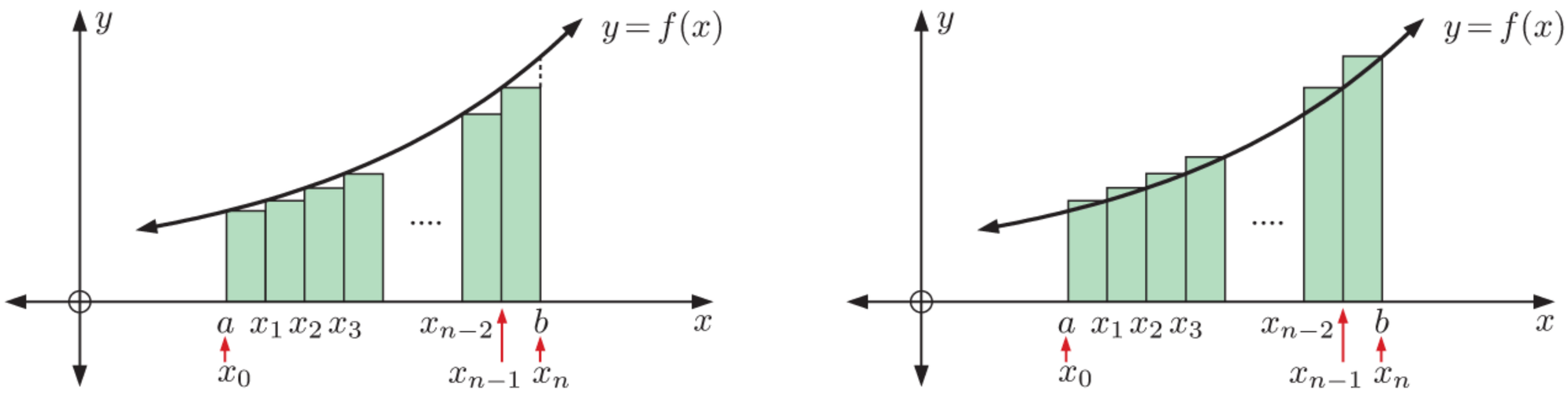
Cavalieri's important step was to make the strips narrower and narrower until they were infinitely thin lines. This reduces the "jagged" steps of the strips until they exactly define the curved boundary of the area.



It was not until limits were formally introduced that Cavalieri's Method of Indivisibles progressed into the foundation for Integral Calculus.

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subintervals, each of width $w = \dfrac{b-a}{n}$.



Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + ... + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + ... + w\,f(x_{n-1}) + w\,f(x_n) = w\sum_{i=1}^{n} f(x_i)$$

$$\therefore \quad A_U - A_L = w\,(f(x_n) - f(x_0))$$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

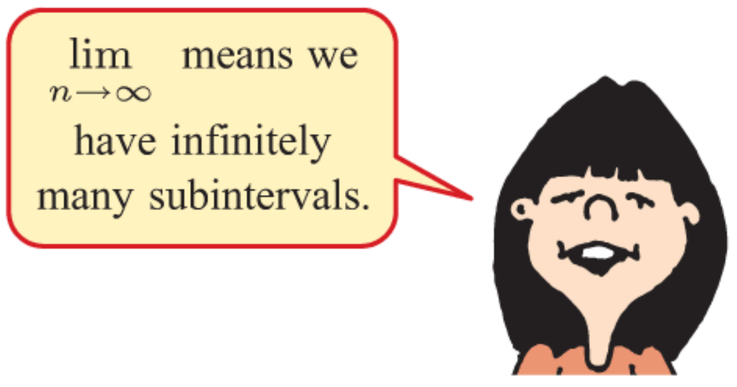
Following Cavalieri's suggestion, we allow there to be infinitely many subintervals, so $n \to \infty$.

In this case $\lim_{n\to\infty} (A_U - A_L) = 0$ {since $\lim_{n\to\infty} \frac{1}{n} = 0$}

$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U$ {provided both limits exist}

$\therefore$ since $A_L < A < A_U$ for all values of n, it follows that

$$\lim_{n\to\infty} A_L = A = \lim_{n\to\infty} A_U$$



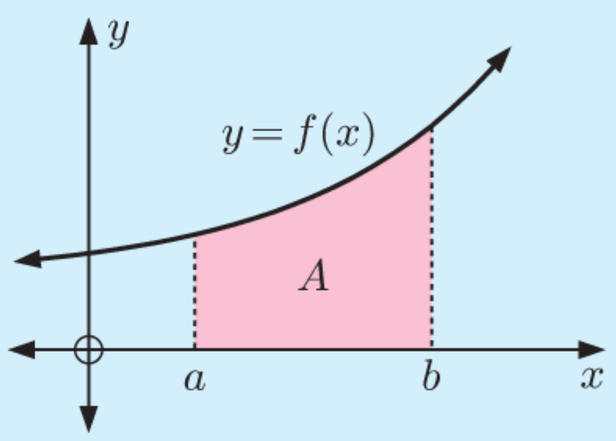
We can obtain a result like this for every increasing and decreasing interval of a positive function provided the function is *continuous*. This means that the function must have a defined value $f(k)$ for all $a \leqslant k \leqslant b$, and that $\lim_{x\to k} f(x) = f(k)$ for all $a \leqslant k \leqslant b$.

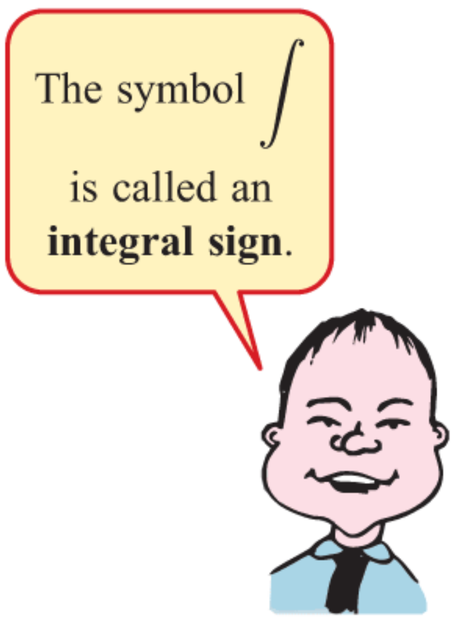
If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\int_a^b f(x)\,dx$ is defined as the shaded area A.

This is known as the **Riemann integral**.

We would say "the integral of $f(x)$ from a to b with respect to x".



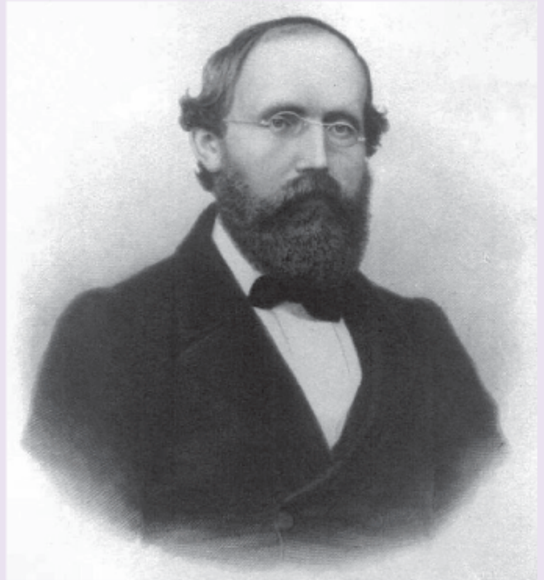


HISTORICAL NOTE

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas of the thin rectangular strips are put together into one whole area.

The theory of integration was developed independently by **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**.

It was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866), whose name is given to the integral which calculates the area under a curve.



Bernhard Riemann

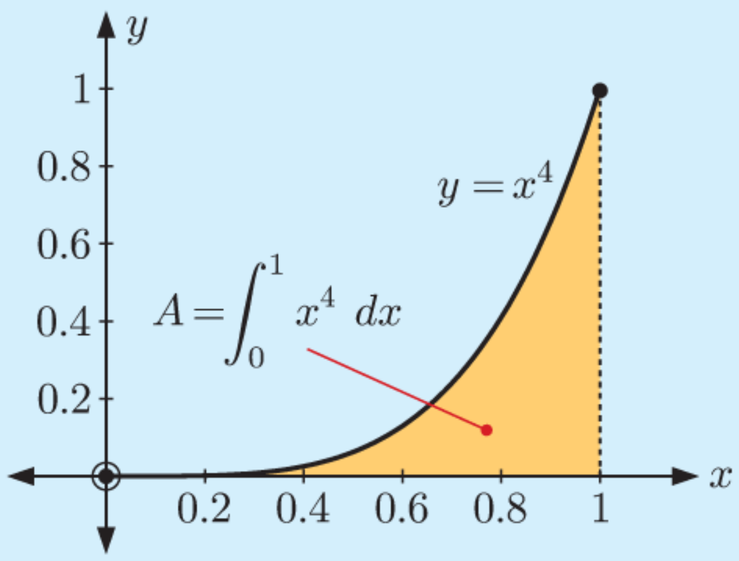
Example 1 **Self Tutor**

a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4 \, dx$.

b Use technology to calculate the lower and upper rectangle sums for n equal subintervals where $n = 5$, 10, 50, 100, and 500.

c Hence evaluate $\int_0^1 x^4 \, dx$ to 2 significant figures.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4 \, dx < A_U$, $\int_0^1 x^4 \, dx \approx 0.20$

EXERCISE 20B

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 \sqrt{x} \, dx$.

b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

c Hence evaluate $\int_0^1 \sqrt{x} \, dx$ to 2 significant figures.

AREA FINDER

2 Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

a Write expressions for the lower and upper rectangle sums using n subintervals where $n \in \mathbb{N}$.

b Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

GRAPHING PACKAGE

c Hence estimate $\int_0^2 \sqrt{1 + x^3} \, dx$.

3 The integral $\int_{-3}^{3} e^{-\frac{x^2}{2}} \, dx$ is of considerable interest to statisticians.

a Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

b Calculate the lower and upper rectangle sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.

c Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to find lower and upper rectangle sums for $-3 \leqslant x \leqslant 0$ for $n = 2250$.

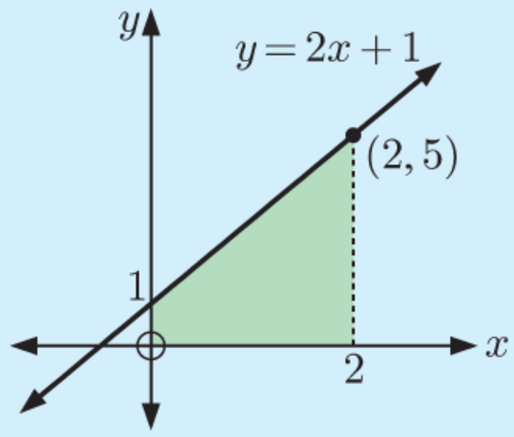
d Hence estimate $\int_{-3}^{3} e^{-\frac{x^2}{2}} \, dx$. Compare your answer with $\sqrt{2\pi}$.

Example 2 **Self Tutor**

Use graphical evidence and known area facts to find:

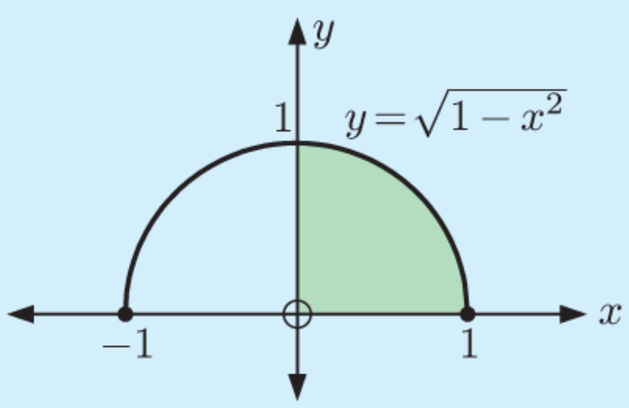
a $\int_0^2 (2x+1)\,dx$ **b** $\int_0^1 \sqrt{1-x^2}\,dx$

a



$$\int_0^2 (2x+1)\,dx = \text{shaded area}$$
$$= \left(\frac{1+5}{2}\right) \times 2$$
$$= 6$$

b If $y = \sqrt{1-x^2}$ then $y^2 = 1-x^2$ and so $x^2+y^2 = 1$. This is the equation of the unit circle, and $y = \sqrt{1-x^2}$ is the upper half.

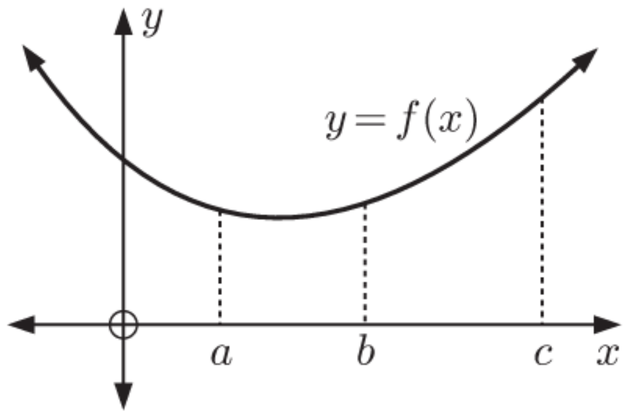


$$\int_0^1 \sqrt{1-x^2}\,dx = \text{shaded area}$$
$$= \tfrac{1}{4} \times \pi \times 1^2$$
$$= \tfrac{\pi}{4}$$

4 Use graphical evidence and known area facts to find:

a $\int_1^3 (1+4x)\,dx$ **b** $\int_{-1}^2 (2-x)\,dx$ **c** $\int_{-2}^2 \sqrt{4-x^2}\,dx$

5 **a** Use the diagram alongside to show that for any positive function $f(x)$:



i $\int_a^a f(x)\,dx = 0$

ii $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

b For a positive function $f(x)$, $\int_2^5 f(x) = 10$, and $\int_5^9 f(x) = 12$. Find:

i $\int_5^5 f(x)\,dx$ **ii** $\int_2^9 f(x)\,dx$

ACTIVITY THE TRAPEZOIDAL RULE

In this online Activity we see a more accurate method to numerically approximate the area under a curve.



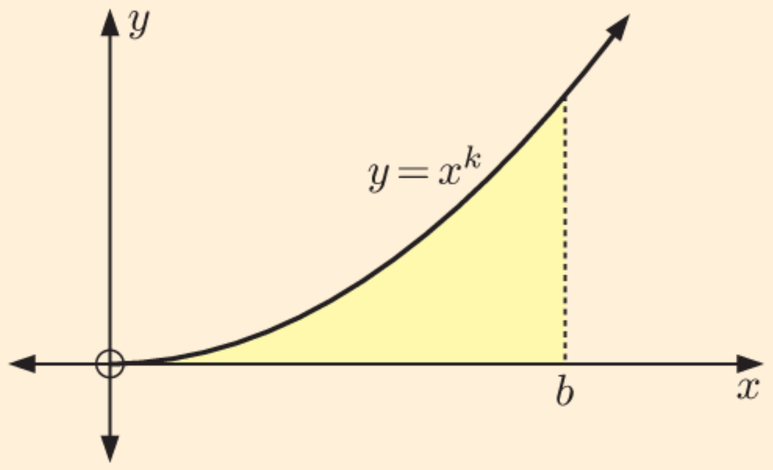
INVESTIGATION 2

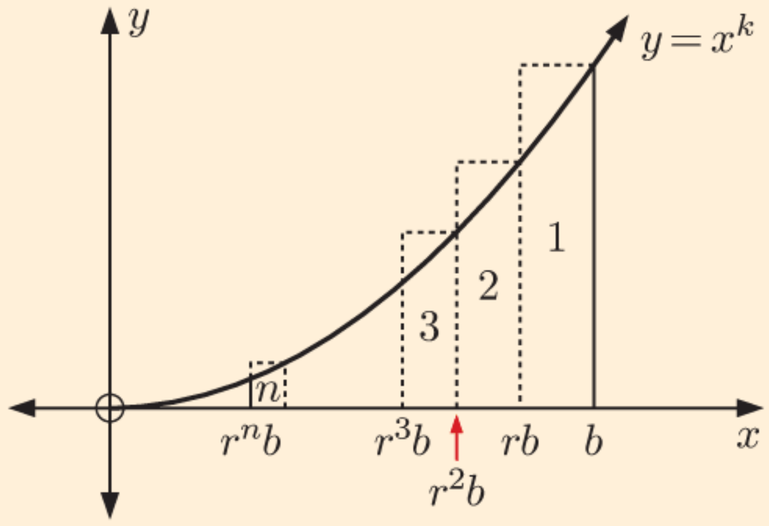
$\int_0^b x^k \, dx$

In the 17th century, the French mathematician **Pierre de Fermat** devised a formula for $\int_0^b x^k \, dx$, where $b > 0$.

In this Activity we shall only consider $k \in \mathbb{N}$.

However, the result is correct for all real $k \neq -1$.





For a given $k \in \mathbb{N}$ and $b > 0$, we select a number r such that $0 < r < 1$, and generate a sequence of points with x-coordinates rb, r^2b, r^3b, ..., r^nb. These form the corner points of n upper rectangles of *unequal* width between $x = r^nb$ and $x = b$.

Suppose we number the rectangles from right to left as shown.

What to do:

1 Show that the ith rectangle has area $b^{k+1}(1-r)(r^{k+1})^{i-1}$.

2 Hence show that the total area of the rectangles is $b^{k+1}(1-r)\dfrac{1-(r^{k+1})^n}{1-r^{k+1}}$.

3 Suppose we let $n \to \infty$.

a Explain why, in the limit as $n \to \infty$, the rectangles include the entire area under the curve from $x = 0$ to $x = b$.

b Show that, in the limit as $n \to \infty$, the total area of the rectangles is $\dfrac{b^{k+1}}{1 + r + r^2 + ... + r^k}$.

4 Now suppose we let $r \to 1$.

a Explain why, in the limit as $r \to 1$, the rectangles now form the area under the curve between $x = 0$ and $x = b$.

b Hence show that $\int_0^b x^k \, dx = \dfrac{b^{k+1}}{k+1}$.

5 Use this result to find $\int_0^1 x^4 \, dx$. Use **Example 1** to check your answer.

C ANTIDIFFERENTIATION

In many problems in calculus, we know the rate of change of one variable with respect to another, but we do not have a formula which directly relates the variables. In other words, we know $\dfrac{dy}{dx}$, but we need to know y in terms of x.

The process of finding y from $\frac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.



Consider $\frac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\frac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\frac{dy}{dx} = x^2$.

This is the correct result. However, for *all* of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\frac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant, which will give $\frac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of x^2. It is the simplest function which, when differentiated, gives x^2.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

Example 3 Self Tutor

Find the antiderivative of:

a x^3 b e^{2x} c $\frac{1}{\sqrt{x}}$

a

$\frac{d}{dx}(x^4) = 4x^3$

$\therefore \frac{d}{dx}(\frac{1}{4}x^4) = x^3$

$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b

$\frac{d}{dx}(e^{2x}) = e^{2x} \times 2 = 2e^{2x}$

$\therefore \frac{d}{dx}(\frac{1}{2}e^{2x}) = \frac{1}{2} \times 2e^{2x} = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c

$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Now $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})\,x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 20C

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2}

v x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{3}}$ **viii** $x^{\frac{2}{3}}$

b Predict a general rule for the antiderivative of x^n, for $n \neq -1$.

2 **a** Find the antiderivative of:

i e^{3x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b Predict a general rule for the antiderivative of e^{kx} where k is a constant.

3 **a** Find the antiderivative of:

i 2^x **ii** 2^{3x} **iii** 2^{-x} **iv** $2^{\frac{x}{2}}$

b Predict a general rule for the antiderivative of 2^{kx} where k is a constant.

4 Find the antiderivative of:

a $6x^2 + 4x$ by first differentiating $x^3 + x^2$ **b** $\sqrt{x}$ by first differentiating $x\sqrt{x}$

c $\dfrac{1}{x\sqrt{x}}$ by first differentiating $\dfrac{1}{\sqrt{x}}$ **d** $\sin x$ by first differentiating $\cos x$

e $(4x+1)^3$ by first differentiating $(4x+1)^4$ **f** $\sec^2\left(\frac{x}{3}\right)$ by first differentiating $\tan\frac{x}{3}$

g $\cos(3x-1)$ by first differentiating $\sin(3x-1)$.

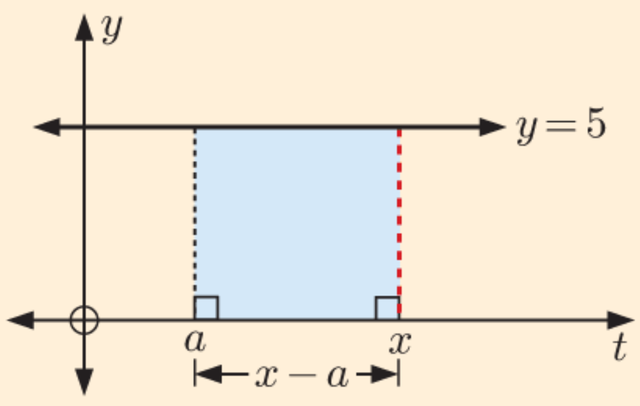
D THE FUNDAMENTAL THEOREM OF CALCULUS

We can now use the Riemann integral to explain the link between differential calculus and the definite integral or limit of an area sum we saw in **Section B**. This link is called the **Fundamental Theorem of Calculus**.

INVESTIGATION 3 THE AREA FUNCTION

Consider the constant function $f(t) = 5$.

We wish to find an **area function** which will give the area under the function between $t = a$ and some other value of t which we will call x.



The area function is $A(x) = \int_a^x 5 \, dt$

$= \text{shaded area}$

$= (x-a)5$

$= 5x - 5a$

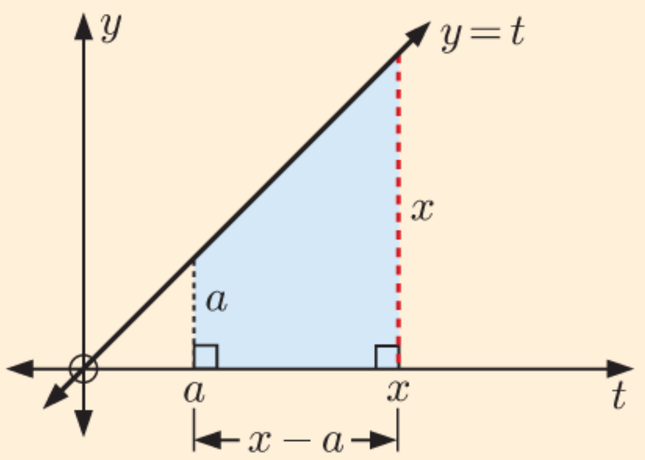
Since $f(t) = 5$ has the antiderivative $F(t) = 5t$, we can write $A(x)$ in the form $F(x) - F(a)$.

What to do:

1 What is the derivative $F'(t)$ of the function $F(t) = 5t$? How is this related to the function $f(t)$?

2 Consider the simplest linear function $f(t) = t$.
The corresponding area function is

$$A(x) = \int_a^x t \, dt$$
$$= \text{shaded area}$$
$$= \left(\frac{x+a}{2}\right)(x-a)$$



a Write $A(x)$ in the form $F(x) - F(a)$.

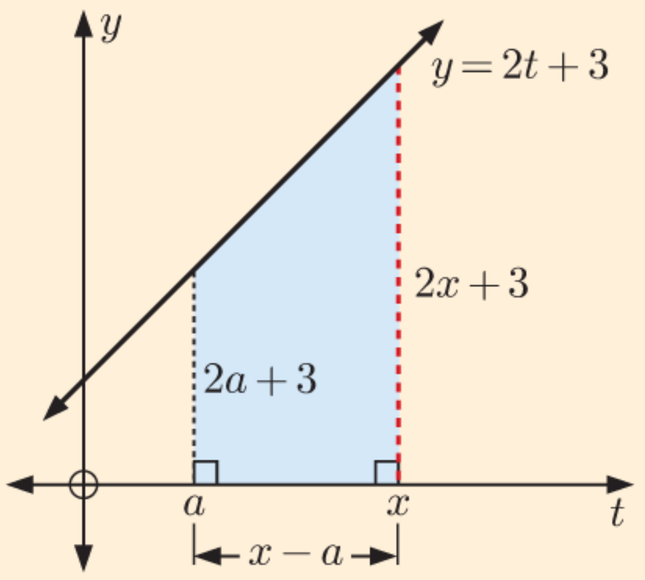
b What is the derivative $F'(t)$? How is this related to the function $f(t)$?

3 Consider $f(t) = 2t + 3$. The corresponding area function is

$$A(x) = \int_a^x (2t+3) \, dt$$
$$= \text{shaded area}$$
$$= \left(\frac{2x+3+2a+3}{2}\right)(x-a)$$

a Write $A(x)$ in the form $F(x) - F(a)$.

b What is the derivative $F'(t)$?
How is this related to the function $f(t)$?



4 Repeat the procedure in **2** and **3** to find area functions for:

a $f(t) = \frac{1}{2}t + 3$ **b** $f(t) = 5 - 2t$

Do your results fit with your earlier observations?

5 If $f(t) = 3t^2 + 4t + 5$, predict what $F(t)$ will be without performing the algebraic procedure.

From the **Investigation** you should have found that, for $f(t) \geqslant 0$,

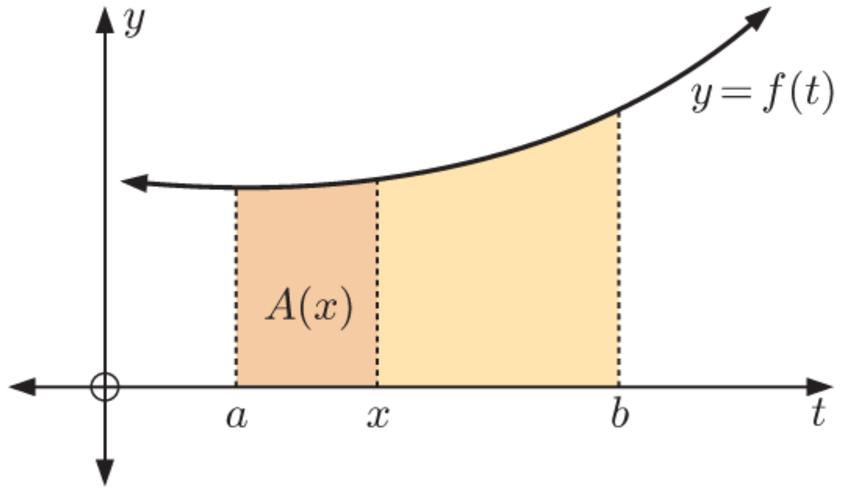
$$\int_a^x f(t) \, dt = F(x) - F(a) \quad \text{where} \quad F'(t) = f(t).$$

$F(t)$ is the **antiderivative** of $f(t)$.

The following argument shows why this is true for all functions $f(t) \geqslant 0$.

Consider a function $y = f(t)$ which has antiderivative $F(t)$ and an area function $A(x) = \int_a^x f(t) \, dt$ which is the area from $t = a$ to $t = x$.

$A(x)$ is an increasing function since $f(x) \geqslant 0$, and
$A(a) = 0$ .... (1)



Consider the narrow strip between $t = x$ and $t = x+h$. The area of this strip is $A(x+h) - A(x)$, but we also know it must lie between a lower and upper rectangle on the interval $x \leqslant t \leqslant x+h$ of width h.

$$\begin{matrix}\text{area of lower}\\ \text{rectangle}\end{matrix} \leqslant A(x+h) - A(x) \leqslant \begin{matrix}\text{area of upper}\\ \text{rectangle}\end{matrix}$$

If $f(t)$ is increasing on this interval then

$$hf(x) \leqslant A(x+h) - A(x) \leqslant hf(x+h)$$

$$\therefore \quad f(x) \leqslant \frac{A(x+h) - A(x)}{h} \leqslant f(x+h)$$

Equivalently, if $f(t)$ is decreasing on this interval then

$$f(x+h) \leqslant \frac{A(x+h) - A(x)}{h} \leqslant f(x)$$

Taking the limit as $h \to 0$ gives

$$f(x) \leqslant A'(x) \leqslant f(x)$$
$$\therefore \quad A'(x) = f(x)$$

So, the area function $A(x)$ must only differ from the antiderivative of $f(x)$ by a constant.

$$\therefore \quad A(x) = F(x) + c$$

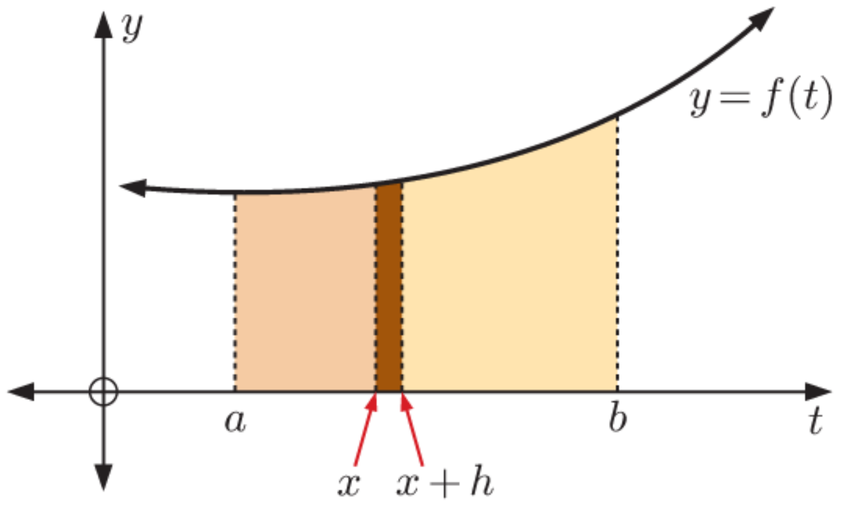
Letting $x = a$, $\quad A(a) = F(a) + c$

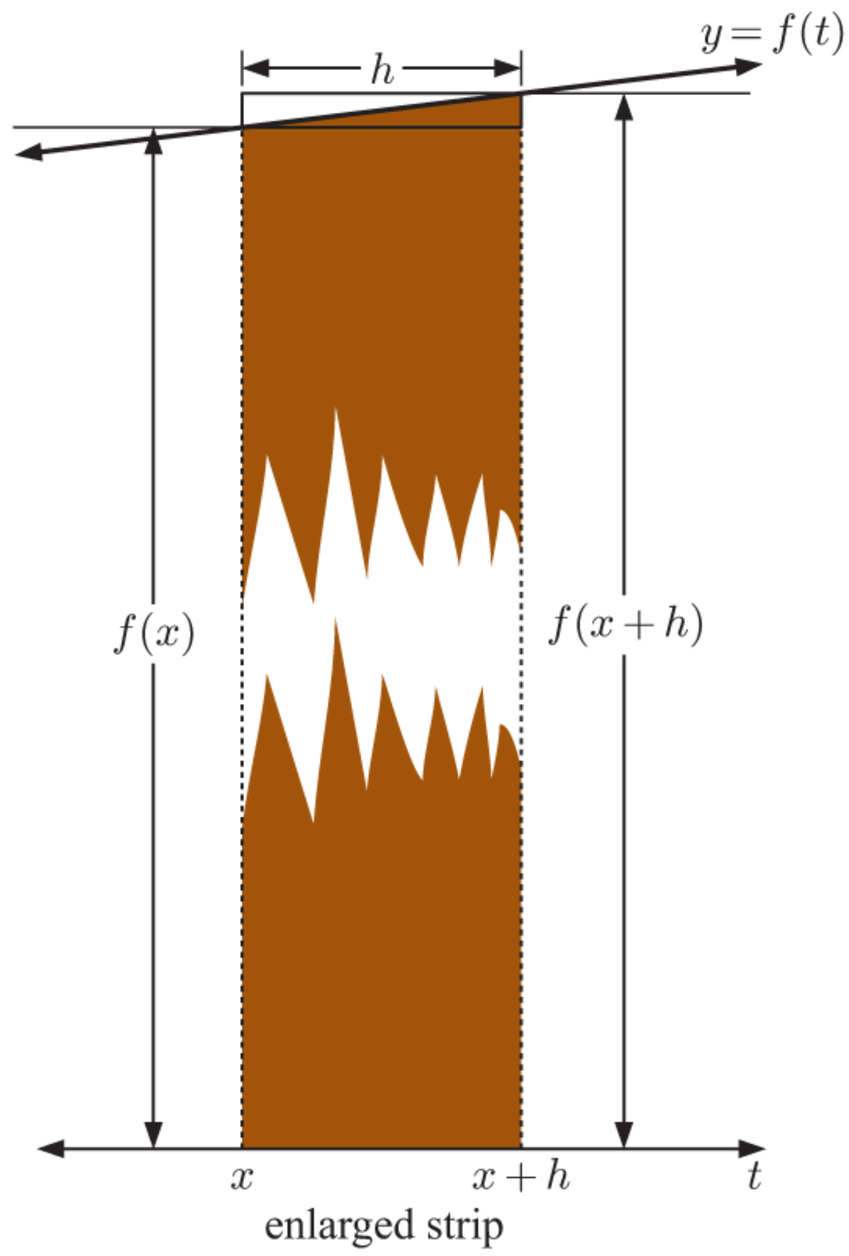
But from (1), $\quad A(a) = 0 \quad \therefore \quad c = -F(a)$

$$\therefore \quad A(x) = F(x) - F(a)$$

and so $\quad \displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$

Letting $x = b$, $\quad \displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$





This result is in fact true for *all* continuous functions $f(t)$, even if they are negative.

However, in situations where a function is negative, the area between the curve and the x-axis is counted as negative. We therefore refer to $A(x)$ as a **signed area function**.

THE FUNDAMENTAL THEOREM OF CALCULUS

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

In general, $\displaystyle\int_a^b f(x)\,dx$ is called a **definite integral**.

PROPERTIES OF DEFINITE INTEGRALS

The following properties of definite integrals can all be deduced from the Fundamental Theorem of Calculus:

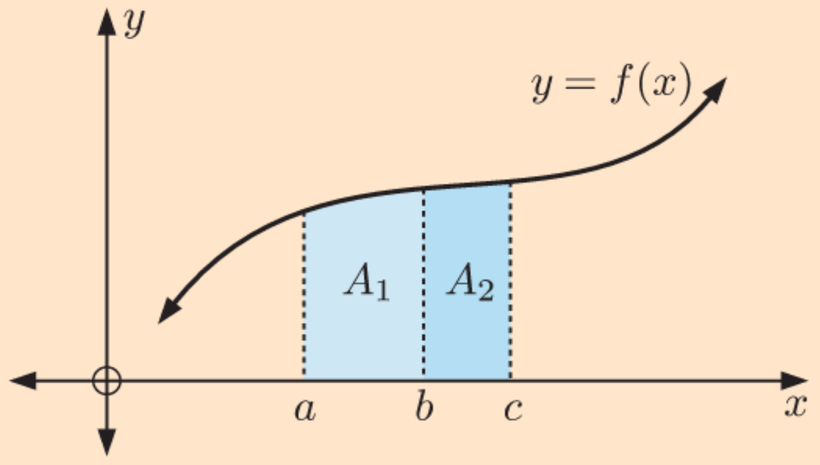
- $\int_a^a f(x)\,dx = 0$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b k\,dx = k(b-a) \quad \{k \text{ is a constant}\}$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

Example proof:

$$\begin{aligned}\int_a^b f(x)\,dx + \int_b^c f(x)\,dx &= F(b) - F(a) + F(c) - F(b)\\ &= F(c) - F(a)\\ &= \int_a^c f(x)\,dx\end{aligned}$$

In particular, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\begin{aligned}\int_a^b f(x)\,dx + \int_b^c f(x)\,dx &= A_1 + A_2\\ &= \int_a^c f(x)\,dx\end{aligned}$$



The Fundamental Theorem of Calculus allows us to calculate areas under curves that we could previously only estimate.

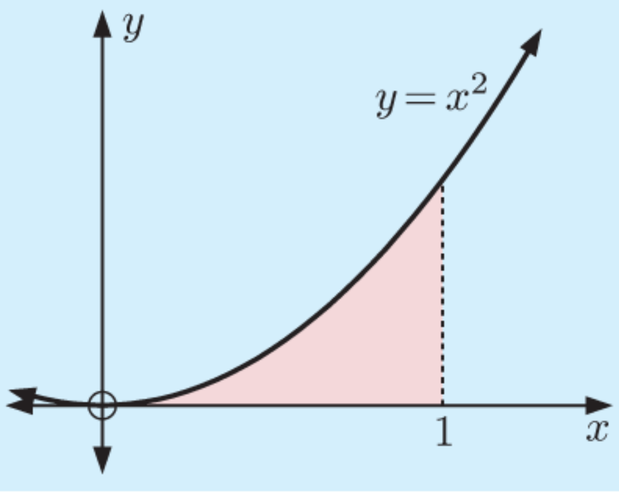
Example 4 **Self Tutor**

Use the Fundamental Theorem of Calculus to find the area between:

a the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

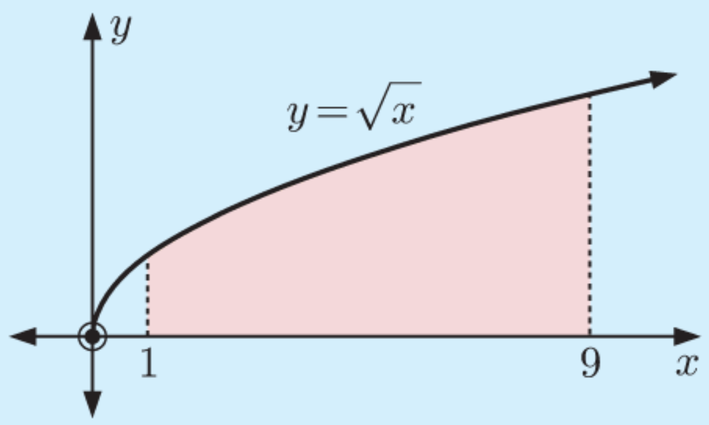
a



$f(x) = x^2$ has antiderivative $F(x) = \dfrac{x^3}{3}$

$$\begin{aligned}\therefore \text{ shaded area} &= \int_0^1 x^2\,dx\\ &= F(1) - F(0)\\ &= \tfrac{1}{3} - 0\\ &= \tfrac{1}{3} \text{ units}^2\end{aligned}$$

b



$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$\therefore$ shaded area $= \displaystyle\int_1^9 x^{\frac{1}{2}}\,dx$

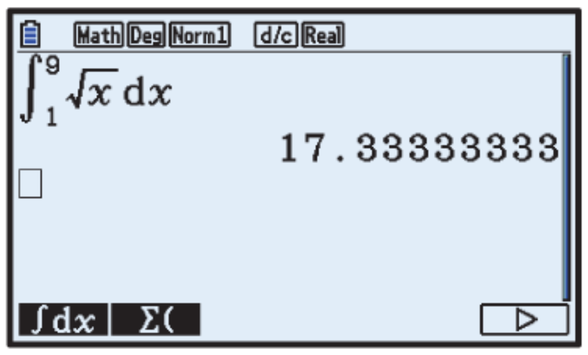
$= F(9) - F(1)$

$= \frac{2}{3} \times 27 - \frac{2}{3} \times 1$

$= 17\frac{1}{3}$ units^2

Instructions for evaluating definite integrals on your calculator can be found by clicking on the icon.



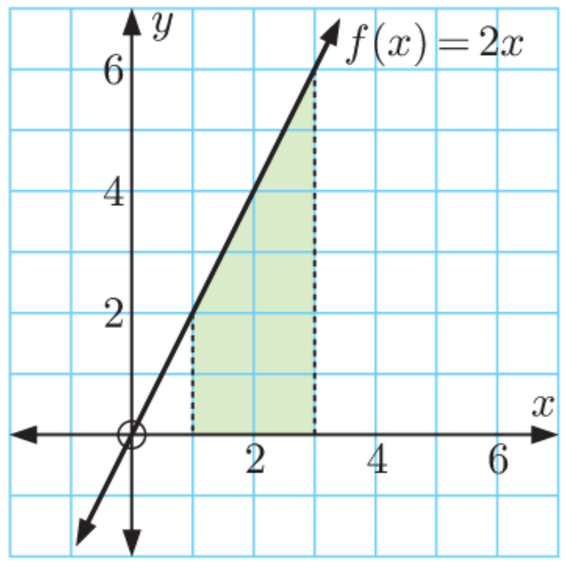


EXERCISE 20D

1 **a** Differentiate x^2, and hence find the antiderivative of $2x$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $f(x) = 2x$ from $x = 1$ to $x = 3$.

c Use graphical methods to check your answer.



2 **a** Find the antiderivative of $\sqrt{x}$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

c Compare your answer to **Exercise 20B** question **1**.

3 **a** Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^3$ from:

i $x = 0$ to $x = 2$ **ii** $x = 2$ to $x = 3$ **iii** $x = 0$ to $x = 3$.

b Comment on your answers in **a**.

4 Use the Fundamental Theorem of Calculus to find the area between the x-axis and:

a $y = x^3$ from $x = 1$ to $x = 2$

b $y = x^2$ from $x = 1$ to $x = 3$

c $y = \sqrt{x}$ from $x = 1$ to $x = 2$

d $y = e^x$ from $x = 0$ to $x = 1.5$

e $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$.



5 Use the Fundamental Theorem of Calculus to show that:

a $\int_a^a f(x)\,dx = 0$ and explain the result graphically

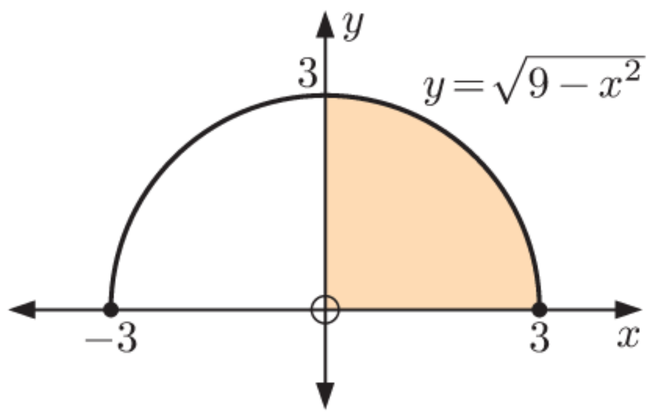
b $\int_a^b k\,dx = k(b-a)$ where k is a constant

c $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

d $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant

e $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

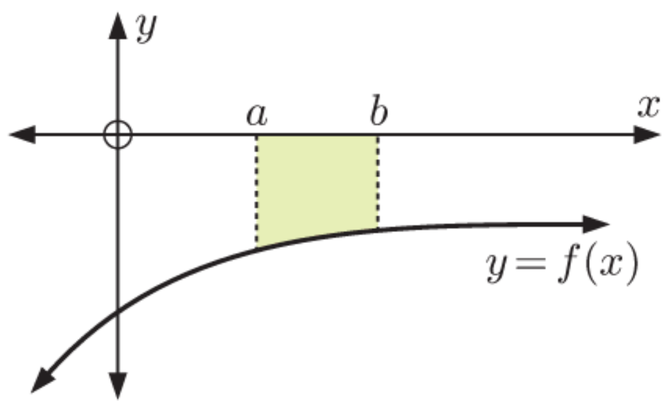
6 Use technology to find the area between the x-axis and $y = \sqrt{9-x^2}$ from $x=0$ to $x=3$.
Check your answer by direct calculation of the area.



7 **a** Use the Fundamental Theorem of Calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$

b Hence show that if $f(x) \leqslant 0$ for all x on $a \leqslant x \leqslant b$ then the shaded area $= -\int_a^b f(x)\,dx$ units^2.



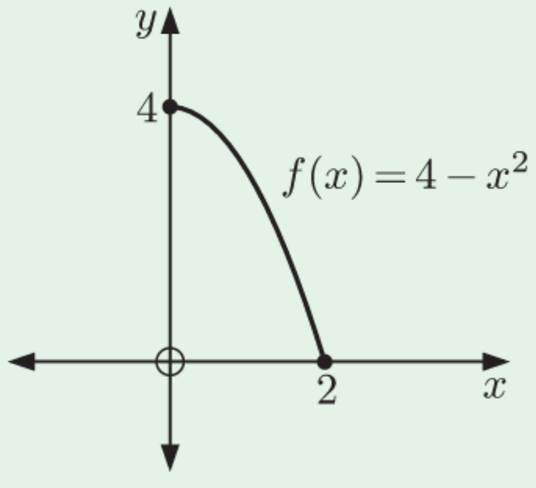
c Calculate the following integrals, and give graphical interpretations of your answers:

i $\int_0^1 (-x^2)\,dx$ **ii** $\int_0^1 (x^2-x)\,dx$ **iii** $\int_{-2}^0 3x\,dx$

d Use graphical evidence and known area facts to find $\int_0^2 \left(-\sqrt{4-x^2}\right)dx$.

REVIEW SET 20A

1



a Use *four* lower and upper rectangles to find rational numbers A and B such that:

$$A < \int_0^2 (4-x^2)\,dx < B.$$

b Hence estimate $\int_0^2 (4-x^2)\,dx$.

2 Find the antiderivative of:

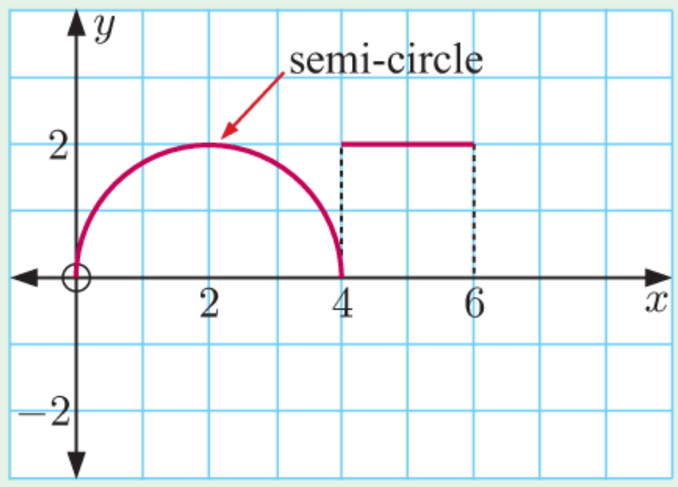
a x^4 **b** $\dfrac{1}{2x^2}$ **c** $e^{-\frac{1}{2}x}$ **d** $\cos x$ **e** 3^x

3 **a** Sketch the region between $y = \sin x$ and the x-axis for $0 \leqslant x \leqslant \frac{\pi}{2}$.

b Divide the interval into 3 equal parts and display the 3 upper and 3 lower rectangles. Hence find boundaries for the value of $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

4 The graph of $y = f(x)$ is illustrated.
Evaluate the following using area interpretation:

a $\int_0^4 f(x) \, dx$ **b** $\int_4^6 f(x) \, dx$



5 **a** Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^2$ from:

i $x = 0$ to $x = 1$ **ii** $x = 1$ to $x = 2$ **iii** $x = 0$ to $x = 2$.

b Comment on your answers in **a**.

REVIEW SET 20B

1 **a** Sketch the region between the curve $y = \dfrac{4}{1 + x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$.
Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Use the area finder software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.

AREA FINDER



c Give your best estimate for $\int_0^1 \dfrac{4}{1 + x^2} \, dx$ and compare this answer with π.

2 Use graphical evidence and known area facts to find:

a $\int_2^4 (2x - 1) \, dx$ **b** $\int_{-1}^1 \sqrt{1 - x^2} \, dx$

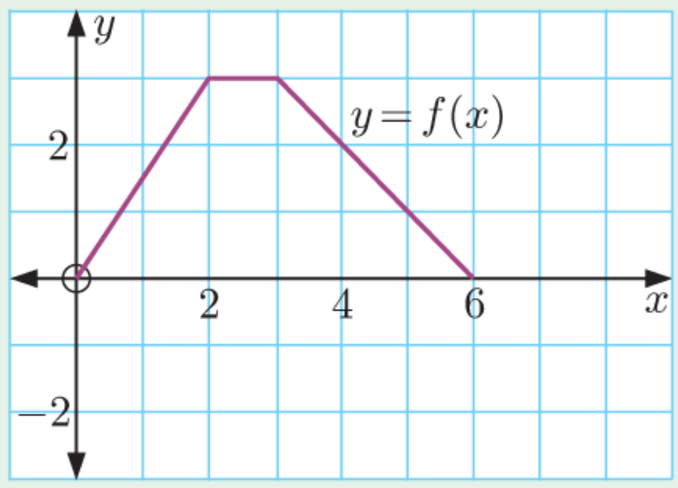
3 Find the antiderivative of:

a $3x^2 - 2$ by first differentiating $x^3 - 2x$ **b** $\sqrt[3]{x}$ by first differentiating $x^{\frac{4}{3}}$

c $\dfrac{1}{\sqrt{1 - x^2}}$ by first differentiating $\arccos x$.

4 The graph of $y = f(x)$ is illustrated.
Evaluate the following using area interpretation:

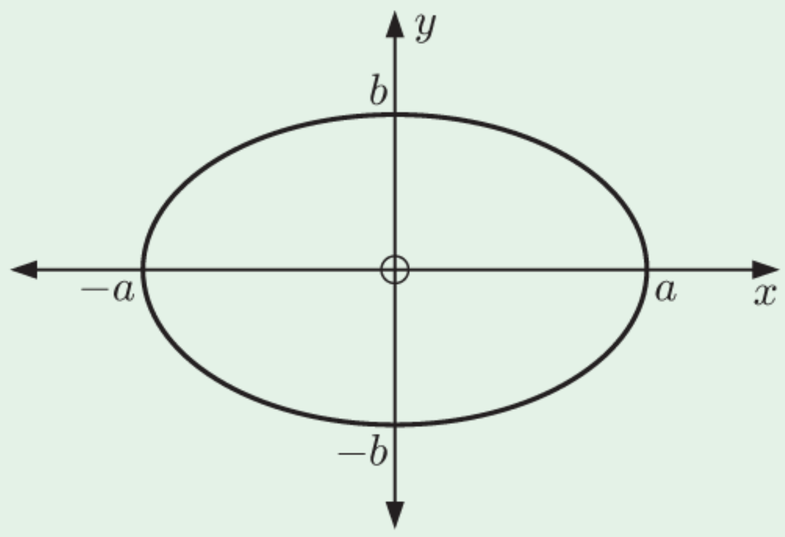
a $\int_0^2 f(x) \, dx$ **b** $\int_2^6 f(x) \, dx$



5 Use the Fundamental Theorem of Calculus to find the area between the x-axis and:

a $y = 4x$ from $x = 0$ to $x = 3$ **b** $y = \sqrt{x}$ from $x = 0$ to $x = 9$.

6 The ellipse alongside is defined by the relation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.



a Show that the top half of the ellipse is defined by the function $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

b Explain why the total area of the ellipse is given by $\frac{b}{a}\int_{-a}^{a} 2\sqrt{a^2 - x^2}\,dx$ units^2.

c Find the derivative of $a^2 \arcsin\frac{x}{a} + x\sqrt{a^2 - x^2}$.

d *Hence* show that the area of the ellipse is given by πab units^2.

Chapter 21

Techniques for integration

Contents:

A Discovering integrals
B Rules for integration
C Particular values
D Integrating $f(ax+b)$
E Partial fractions
F Integration by substitution
G Integration by parts



OPENING PROBLEM

The Fundamental Theorem of Calculus developed by Newton and Leibniz identifies the link between differential calculus and a definite integral.

If we are to use the Fundamental Theorem of Calculus to calculate areas, we need to be able to identify an area function from its derivative.

Things to think about:

a Can you identify the function which has derivative:

i $3x^2$ **ii** $x^3 - x + 1$ **iii** $\cos 2x$?

b What can the rules of differentiation teach us about the reverse process of integration?

In our previous study of the Fundamental Theorem of Calculus, we showed that the **antiderivative** of x^2 is $\frac{1}{3}x^3$, and that any function of the form $\frac{1}{3}x^3 + c$ where c is a constant, has derivative x^2.

We say that the **indefinite integral** or **integral** of x^2 is $\frac{1}{3}x^3 + c$, and write $\int x^2\,dx = \frac{1}{3}x^3 + c$.

We read this as "the integral of x^2 with respect to x is $\frac{1}{3}x^3 + c$, where c is a constant".

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

The constant c is called the **constant of integration**.

This process of finding an indefinite integral is called **indefinite integration**.

A DISCOVERING INTEGRALS

Just as we did in antidifferentiation, we can sometimes discover integrals by differentiation.

The following rules will prove useful:

- Any constant within the integral may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

Proof: Consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$

$$\therefore \int k\,f(x)\,dx = k\,F(x) + c$$

$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 1 **Self Tutor**

Find the derivative of x^4+2x^3, and hence find $\int (2x^3+3x^2)\,dx$.

$$\frac{d}{dx}(x^4+2x^3) = 4x^3+6x^2$$

$$\therefore \int (4x^3+6x^2)\,dx = x^4+2x^3+c$$

$$\therefore \int 2(2x^3+3x^2)\,dx = x^4+2x^3+c$$

$$\therefore 2\int (2x^3+3x^2)\,dx = x^4+2x^3+c$$

$$\therefore \int (2x^3+3x^2)\,dx = \tfrac{1}{2}x^4+x^3+c$$

c represents an arbitrary constant, so is simply any value $c \in \mathbb{R}$. Instead of writing $\frac{c}{2}$, we can therefore still write just c.



EXERCISE 21A

1 Find the derivative of kx where k is a constant. Hence find $\int k\,dx$, $k \in \mathbb{R}$.

2 **a** Find the derivative of x^7. Hence find $\int x^6\,dx$.

b Find the derivative of $x^{\frac{3}{2}}$. Hence find $\int \sqrt{x}\,dx$.

c Find the derivative of $x^{-\frac{1}{2}}$. Hence find $\int x^{-\frac{3}{2}}\,dx$.

d Find the derivative of x^{n+1}, $n \neq -1$. Hence find $\int x^n\,dx$, $n \neq -1$.

We can check that an integral is correct by differentiating the answer. It should give us the **integrand**, the function we originally integrated.



3 **a** Find the derivative of e^{4x}. Hence find $\int e^{4x}\,dx$.

b Find the derivative of $e^{-\frac{x}{2}}$. Hence find $\int e^{-\frac{x}{2}}\,dx$.

c Find the derivative of e^{kx}, $k \neq 0$. Hence find $\int e^{kx}\,dx$, $k \neq 0$.

4 **a** Find the derivative of $\sin x$. Hence find $\int \cos x\,dx$.

b Find the derivative of $\cos x$. Hence find $\int (-\sin x)\,dx$.

c Find the derivative of $\tan x$. Hence find $\int \sec^2 x\,dx$.

5 **a** Find the derivative of x^3+x^2. Hence find $\int (3x^2+2x)\,dx$.

b Find the derivative of $3x^4-2x^2$. Hence find $\int (3x^3-x)\,dx$.

6 Suppose $F(x)$ and $G(x)$ have the derivative functions $f(x)$ and $g(x)$ respectively.

a Find the derivative of $F(x) + G(x)$.

b Hence show that $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

7 **a** Find the derivative of 3^x. Hence find $\int 3^x\,dx$.

b Find the derivative of a^x where $a > 0,\ a \neq 1$. Hence find $\int a^x\,dx,\ a > 0,\ a \neq 1$.

8 **a** Find the derivative of $\operatorname{cosec} x$. Hence find $\int (-\operatorname{cosec} x \cot x)\,dx$.

b Find the derivative of $\sec x$. Hence find $\int \sec x \tan x\,dx$.

c Find the derivative of $\cot x$. Hence find $\int (-\operatorname{cosec}^2 x)\,dx$.

9 **a** Find the derivative of $\arcsin x$. Hence find $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

b Find the derivative of $\arccos x$. Hence find $\int \left(-\frac{1}{\sqrt{1-x^2}}\right) dx$.

c Find the derivative of $\arctan x$. Hence find $\int \frac{1}{1+x^2}\,dx$.

10 **a** Find the derivative of $\ln x,\ x > 0$.

b Find the derivative of $\ln(-x),\ x < 0$.

c Hence explain why $\int \frac{1}{x}\,dx = \ln|x| + c,\ x \neq 0$.

11 **a** Find the derivative of $\log_a x,\ x > 0$.

b Find the derivative of $\log_a(-x),\ x < 0$.

c Hence explain why $\int \frac{1}{x \ln a}\,dx = \log_a|x| + c,\ x \neq 0$.

Example 2 **Self Tutor**

Find the derivative of $\cos\left(2x + \frac{\pi}{3}\right)$, and hence find $\int \sin\left(2x + \frac{\pi}{3}\right) dx$.

$$\frac{d}{dx}\left(\cos\left(2x + \tfrac{\pi}{3}\right)\right) = -\sin\left(2x + \tfrac{\pi}{3}\right)(2)$$
$$= -2\sin\left(2x + \tfrac{\pi}{3}\right)$$

$$\therefore \int \left(-2\sin\left(2x + \tfrac{\pi}{3}\right)\right) dx = \cos\left(2x + \tfrac{\pi}{3}\right) + c$$

$$\therefore -2\int \sin\left(2x + \tfrac{\pi}{3}\right) dx = \cos\left(2x + \tfrac{\pi}{3}\right) + c$$

$$\therefore \int \sin\left(2x + \tfrac{\pi}{3}\right) dx = -\tfrac{1}{2}\cos\left(2x + \tfrac{\pi}{3}\right) + c$$

12 **a** Find the derivative of $\sin 3x$. Hence find $\int \cos 3x \, dx$.

b Find the derivative of $\cos\left(\frac{\pi}{3} - x\right)$. Hence find $\int \sin\left(\frac{\pi}{3} - x\right) dx$.

c Find the derivative of $\sin(1 - 5x)$. Hence find $\int \cos(1 - 5x) \, dx$.

d Find the derivative of e^{3x+1}. Hence find $\int e^{3x+1} \, dx$.

e Find the derivative of $\sqrt{5x - 1}$. Hence find $\int \frac{1}{\sqrt{5x - 1}} \, dx$.

f Find the derivative of $(2x + 1)^4$. Hence find $\int (2x + 1)^3 \, dx$.

g Find the derivative of $(x^2 - x)^3$. Hence find $\int (2x - 1)(x^2 - x)^2 \, dx$.

h Find the derivative of 2^{3x-5}. Hence find $\int 2^{3x-5} \, dx$.

i Find the derivative of $\arctan \frac{x}{2}$. Hence find $\int \frac{-3}{x^2 + 4} \, dx$.

B RULES FOR INTEGRATION

In **Chapter 17** we developed a set of rules to help us differentiate functions more efficiently. These rules or combinations of them can be used to differentiate all of the functions we consider in this course.

However, the task of finding **antiderivatives** is not so easy. Many functions simply do not have antiderivatives which can be expressed easily using standard functions.

HISTORICAL NOTE

Robert Henry Risch (1939 -) is an American mathematician. He studied at the University of California, Berkeley.

In his doctorate studies in 1968, Risch devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

After completing his doctorate, he worked at the IBM Thomas Watson Research Centre.

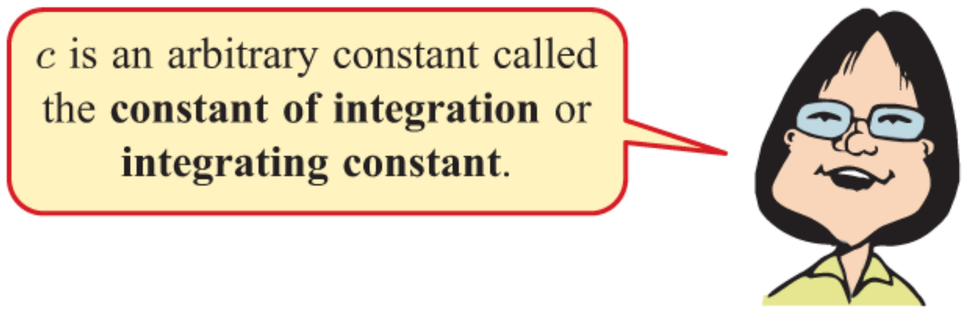
We *can* construct some rules which will allow us to integrate most of the function types we consider in this course.

INTEGRATING BASIC FUNCTION TYPES

In the previous Exercise, you should have found the following integrals:

Function	*Integral*
k, a constant	$kx + c$
$x^n, \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
e^x	$e^x + c$
$a^x \ln a, \quad a > 0, \quad a \neq 1$	$a^x + c$
$\dfrac{1}{x}$	$\ln \lvert x \rvert + c, \quad x \neq 0$
$\dfrac{1}{x \ln a}, \quad a > 0, \quad a \neq 1$	$\log_a \lvert x \rvert + c$

Function	*Integral*
$\cos x$	$\sin x + c$
$-\sin x$	$\cos x + c$
$\sec^2 x$	$\tan x + c$
$-\operatorname{cosec} x \cot x$	$\operatorname{cosec} x + c$
$\sec x \tan x$	$\sec x + c$
$-\operatorname{cosec}^2 x$	$\cot x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\dfrac{-1}{\sqrt{1-x^2}}$	$\arccos x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$



GENERAL RULES FOR INTEGRATION

These rules can be combined with the other general rules we proved in the previous Section:

- $\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k$ is a constant
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

There are no power or quotient rules for integration, so we often have to perform the multiplication or division first and then integrate.

Example 3 — Self Tutor

Find:

a $\int (-2x^3 + 5x - 2)\,dx$ **b** $\int \left(5e^x + \dfrac{2}{\sqrt{x}}\right) dx$ **c** $\int (2\sin x - \cos x)\,dx$

a
$$\int (-2x^3 + 5x - 2)\,dx$$
$$= \frac{-2x^4}{4} + \frac{5x^2}{2} - 2x + c$$
$$= -\tfrac{1}{2}x^4 + \tfrac{5}{2}x^2 - 2x + c$$

b
$$\int \left(5e^x + \frac{2}{\sqrt{x}}\right) dx$$
$$= \int (5e^x + 2x^{-\frac{1}{2}})\,dx$$
$$= 5e^x + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 5e^x + 4\sqrt{x} + c$$

c
$$\int (2\sin x - \cos x)\,dx$$
$$= 2(-\cos x) - \sin x + c$$
$$= -2\cos x - \sin x + c$$

Example 4 **Self Tutor**

Integrate with respect to x:

a $-\frac{2}{x}+3e^x$ **b** $3\sec^2 x - x^2$ **c** $2\sec x\tan x - \frac{6}{1+x^2}$

a $\int \left(-\frac{2}{x}+3e^x\right)dx$

$= -2\ln|x| + 3e^x + c$

b $\int \left(3\sec^2 x - x^2\right)dx$

$= 3\tan x - \frac{1}{3}x^3 + c$

c $\int \left(2\sec x\tan x - \frac{6}{1+x^2}\right)dx$

$= 2\sec x - 6\arctan x + c$

EXERCISE 21B

1 Find:

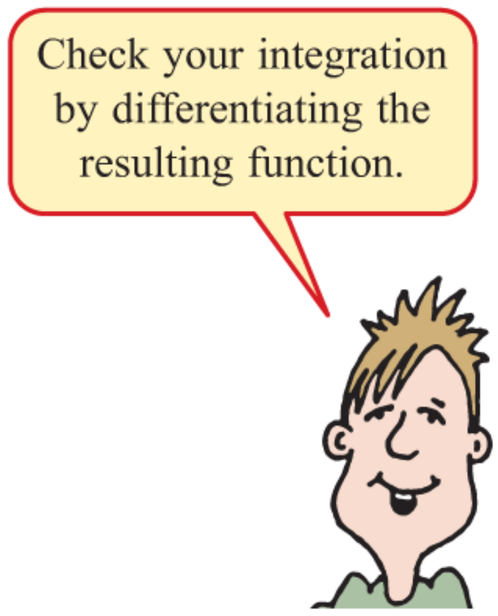
a $\int (x^2+3x-2)\,dx$ **b** $\int (2x^2-3x+1)\,dx$

c $\int (-x^3+4x^2-3)\,dx$ **d** $\int \left(\frac{1}{2}x + x^2 + x^3\right)dx$

e $\int (x^4 - x^2 - x + 2)\,dx$ **f** $\int \left(4x^2 + \frac{1}{x}\right)dx$

g $\int \left(2\sqrt{x} - \frac{3}{\sqrt{x}}\right)dx$ **h** $\int \left(\frac{1}{3x} - \frac{2}{x^2}\right)dx$

i $\int (x\sqrt{x} - 9)\,dx$ **j** $\int (3x^{-\frac{3}{2}} + x^{\frac{1}{4}})\,dx$



2 Find:

a $\int (2e^x - 3x)\,dx$ **b** $\int \left(\frac{4}{x} + x^2 - e^x\right)dx$ **c** $\int \left(5e^x + \frac{1}{2}x^2\right)dx$

d $\int (\sqrt{x} + e^x)\,dx$ **e** $\int \left(3e^x - \frac{1}{x\sqrt{x}}\right)dx$ **f** $\int \left(\frac{1}{2x} + x^2 - \frac{1}{2}e^x\right)dx$

3 Integrate with respect to x:

a $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $\sin x - 2\cos x + e^x$

d $x^2\sqrt{x} - 10\sin x$ **e** $\frac{x(x-1)}{3} + \sec^2 x$ **f** $-\sin x + 2\sqrt{x}$

4 Find y if:

a $\frac{dy}{dx} = 6$ **b** $\frac{dy}{dx} = 4x^2$ **c** $\frac{dy}{dx} = \frac{1}{x^2}$

d $\frac{dy}{dx} = \frac{2}{\sqrt[3]{x}}$ **e** $\frac{dy}{dx} = 2x^3 - 4$ **f** $\frac{dy}{dx} = 4x^3 + 3x^2$

g $\frac{dy}{dx} = 2 - \frac{1}{x}$ **h** $\frac{dy}{dx} = \sin x + 2\cos x$ **i** $\frac{dy}{dx} = 2e^x - 5 + x$

5 In the rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$, explain why we excluded the value $n = -1$.

6 Find:

a $\int 4^x \, dx$ **b** $\int \left(\frac{3}{x} - \frac{1}{x \ln 2}\right) dx$ **c** $\int (5^x - 2 \times 7^x) \, dx$

d $\int \operatorname{cosec} x \cot x \, dx$ **e** $\int (\cos x + \operatorname{cosec}^2 x) \, dx$ **f** $\int \left(\frac{4}{x^2\sqrt{x}} - \frac{\sec x \tan x}{9}\right) dx$

7 Find:

a $\int (2e^t - 4 \sin t) \, dt$ **b** $\int \left(3 \cos t - \frac{1}{t}\right) dt$ **c** $\int (5 \sin t - \sqrt{t}) \, dt$

d $\int (\sec^2 x + 2 \sin x) \, dx$ **e** $\int (\theta - \sin \theta) \, d\theta$ **f** $\int \left(\frac{2}{\theta} - \sec^2 \theta\right) d\theta$

g $\int (2 \operatorname{cosec}^2 \theta + 3 \operatorname{cosec} \theta \cot \theta) \, d\theta$ **h** $\int 4 \sec \theta \tan \theta \, d\theta$ **i** $\int \tan^2 \theta \, d\theta$

8 Find:

a $\int \frac{4}{\sqrt{1 - x^2}} \, dx$ **b** $\int -\frac{5}{\sqrt{1 - x^2}} \, dx$ **c** $\int \frac{7}{1 + x^2} \, dx$

d $\int \left(\frac{3}{x} + \frac{5}{\sqrt{1 - x^2}}\right) dx$ **e** $\int \left(-\frac{1}{\sqrt{1 - x^2}} + 3x\right) dx$ **f** $\int \left(6 - \frac{2}{1 + x^2}\right) dx$

9 To find $\int -\frac{3}{\sqrt{1 - x^2}} \, dx$, Teegan's answer was $\int -\frac{3}{\sqrt{1 - x^2}} \, dx = 3 \arccos x + c$ and Jill's answer was $\int -\frac{3}{\sqrt{1 - x^2}} \, dx = -3 \int \frac{1}{\sqrt{1 - x^2}} \, dx = -3 \arcsin x + c$.

Which of them is correct? Explain your answer.

Example 5 Self Tutor

Find:

a $\int \left(x - \frac{1}{x}\right)^3 dx$ **b** $\int \frac{3\sqrt{x} + x^2}{x} \, dx$

a
$$\int \left(x - \frac{1}{x}\right)^3 dx$$
$$= \int \left(x^3 + 3x^2\left(-\frac{1}{x}\right) + 3x\left(-\frac{1}{x}\right)^2 + \left(-\frac{1}{x}\right)^3\right) dx$$
{binomial theorem}
$$= \int (x^3 - 3x + 3x^{-1} - x^{-3}) \, dx$$
$$= \frac{x^4}{4} - \frac{3x^2}{2} + 3 \ln|x| - \frac{x^{-2}}{(-2)} + c$$
$$= \tfrac{1}{4}x^4 - \tfrac{3}{2}x^2 + 3 \ln|x| + \frac{1}{2x^2} + c$$

b
$$\int \frac{3\sqrt{x} + x^2}{x} \, dx$$
$$= \int (3x^{-\frac{1}{2}} + x) \, dx$$
$$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^2}{2} + c$$
$$= 6\sqrt{x} + \tfrac{1}{2}x^2 + c$$

10 Find:

a $\int (2x+1)^2\,dx$ **b** $\int \left(x+\frac{1}{x}\right)^2 dx$ **c** $\int \frac{1-4x}{x\sqrt{x}}\,dx$

d $\int \frac{2x-1}{\sqrt{x}}\,dx$ **e** $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$ **f** $\int \frac{1-x^2}{x}\,dx$

g $\int \left(\frac{2}{x}+1\right)^2 dx$ **h** $\int (x+1)^3\,dx$ **i** $\int (x-1)^4\,dx$

j $\int \frac{x^2-4}{x\sqrt{x}}\,dx$ **k** $\int \frac{x^2-4x+10}{x}\,dx$ **l** $\int \frac{3x^3-2x^2+5}{x^2}\,dx$

11 Find $f(x)$ if:

a $f'(x)=(1-2x)^2$ **b** $f'(x)=\sqrt{x}-\frac{2}{\sqrt{x}}$ **c** $f'(x)=\frac{x^2-5}{x^2}$

d $f'(x)=\frac{2x^2+5}{x^2+1}$ **e** $f'(x)=\operatorname{cosec} x\left(\frac{\cot x}{2}-\frac{\operatorname{cosec} x}{3}\right)$

12 Find:

a $\int \left(x^2-\frac{1}{x}\right)^2 dx$ **b** $\int \frac{x^2-4x+2}{\sqrt{x}}\,dx$ **c** $\int \sqrt{x}(3x-1)^2\,dx$

Example 6 **Self Tutor**

Find $\frac{d}{dx}(x\sin x)$ and hence determine $\int x\cos x\,dx$.

$\frac{d}{dx}(x\sin x)=(1)\sin x+(x)\cos x$ {product rule}

$=\sin x+x\cos x$

$\therefore \int (\sin x+x\cos x)\,dx=x\sin x+c$

$\therefore \int \sin x\,dx+\int x\cos x\,dx=x\sin x+c$

$\therefore -\cos x+\int x\cos x\,dx=x\sin x+c$

$\therefore \int x\cos x\,dx=x\sin x+\cos x+c$

13 Find $\frac{d}{dx}(x\cos x)$ and hence find $\int x\sin x\,dx$.

14 Find $\frac{d}{dx}(xe^x)$ and hence find $\int xe^x\,dx$.

15 By considering $\frac{d}{dx}(x\ln x)$, find $\int \ln x\,dx$.

16 **a** Find $\frac{d}{dx}(2x\sin x)$ and $\frac{d}{dx}(-x^2\cos x)$. **b** Hence find $\int x^2\sin x\,dx$.

C PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 7 Self Tutor

Find $f(x)$ given that:

a $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$

b $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

a
$$f'(x) = x^3 - 2x^2 + 3$$
$$\therefore\ f(x) = \int (x^3 - 2x^2 + 3)\,dx$$
$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$
But $f(0) = 2$, so $c = 2$
Thus $f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + 2$.

b
$$f'(x) = 2\sin x - \sqrt{x}$$
$$\therefore\ f(x) = \int (2\sin x - x^{\frac{1}{2}})\,dx$$
$$\therefore\ f(x) = 2(-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$
But $f(0) = 4$, so $-2\cos 0 - 0 + c = 4$
$\therefore\ c = 6$
Thus $f(x) = -2\cos x - \frac{2}{3}x^{\frac{3}{2}} + 6$.

EXERCISE 21C

1 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = e^x + \frac{1}{\sqrt{x}}$ and $f(1) = 1$

d $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

e $f'(x) = \sqrt{x} - 2$ and $f(4) = 0$

f $f'(x) = \frac{1}{x}$ and $f(e) = 2$.

2 A curve has gradient function $\frac{dy}{dx} = x - 2x^2$ and passes through $(2, 4)$. Find the equation of the curve.

3 A curve has gradient function $\frac{dy}{dx} = 1 - e^x$ and passes through $(3, e^3)$. Find the equation of the curve.

4 Find $f(x)$ given that:

a $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

b $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

c $f'(x) = \sqrt{x} - 2\sec^2 x$ and $f(\pi) = 0$

d $f'(x) = e^x + 3\cos x$ and $f(\pi) = 0$

e $f'(x) = 2\sec x \tan x$ and $f\left(\frac{\pi}{3}\right) = 7$

f $f'(x) = \frac{10}{\pi\sqrt{1 - x^2}} - \frac{3}{\pi(x^2 + 1)}$ and $f(1) = 4$.

5 A curve has gradient function $f'(x) = ax + 1$ where a is a constant. Find $f(x)$ given that $f(0) = 3$ and $f(3) = -3$.

6 A curve has gradient function $f'(x) = ax^2 + bx$ where a, b are constants. Find $f(x)$ given that $f(-1) = -2$, $f(0) = 1$, and $f(1) = 4$.

Example 8 **Self Tutor**

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$, and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

$$\therefore\ f'(x) = \int (12x^2 - 4)\,dx$$

$$= \frac{12x^3}{3} - 4x + c$$

$$= 4x^3 - 4x + c$$

But $f'(0) = -1$, so $c = -1$

$$\therefore\ f'(x) = 4x^3 - 4x - 1$$

$$\therefore\ f(x) = \int (4x^3 - 4x - 1)\,dx$$

$$= \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$$

$$= x^4 - 2x^2 - x + d$$

But $f(1) = 4$, so $1 - 2 - 1 + d = 4$ and hence $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

7 Find $f(x)$ given that:

a $f''(x) = 2x + 1$, $f'(1) = 3$, and $f(2) = 7$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}}$, $f'(1) = 12$, and $f(0) = 5$

c $f''(x) = \cos x$, $f'\left(\frac{\pi}{2}\right) = 0$, and $f(0) = 3$

d $f''(x) = 3^x$, $f'(0) = 0$, and $f(1) = -\dfrac{1}{\ln 3}$

e $f''(x) = 2x$ and the points $(1, 0)$ and $(0, 5)$ lie on the curve $y = f(x)$.

8 Suppose $f''(x) = \sin x - \cos x$, $f\left(\frac{\pi}{2}\right) = -1$, and $f(\pi) = 2$. Find $f(x)$.

D INTEGRATING $f(ax + b)$

In this Section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$.

$$\therefore\ \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \quad \text{for } a \neq 0.$$

Likewise if $n \neq -1$, $\dfrac{d}{dx}((ax+b)^{n+1}) = (n+1)(ax+b)^n \times a$
$= a(n+1)(ax+b)^n$

$$\therefore \quad \boxed{\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for} \quad n \neq -1, \quad a \neq 0.}$$

We can also observe that $\dfrac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$.

So, $$\boxed{\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c \quad \text{for} \quad a \neq 0.}$$

Likewise, $$\boxed{\int (-\sin(ax+b)) \, dx = \frac{1}{a}\cos(ax+b) + c \quad \text{for} \quad a \neq 0.}$$

and $$\boxed{\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + c \quad \text{for} \quad a \neq 0.}$$

Finally, $\dfrac{d}{dx}\left(\dfrac{1}{a}\ln(ax+b)\right) = \dfrac{1}{a}\left(\dfrac{a}{ax+b}\right) = \dfrac{1}{ax+b}$ for $ax+b > 0, \quad a \neq 0$

$$\therefore \int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln(ax+b) + c \quad \text{for} \quad ax+b > 0, \quad a \neq 0$$

We can similarly show that $\displaystyle\int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln(-(ax+b)) + c$ for $ax+b < 0, \quad a \neq 0$

$$\therefore \quad \boxed{\int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln|ax+b| + c \quad \text{for} \quad a \neq 0.}$$

For a, b constants with $a \neq 0$, we have:

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c$
$(ax+b)^n, \quad n \neq -1$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert + c$

Function	*Integral*
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$-\sin(ax+b)$	$\frac{1}{a}\cos(ax+b) + c$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b) + c$

Example 9 — Self Tutor

Find: **a** $\displaystyle\int (2x+3)^4 \, dx$ **b** $\displaystyle\int \frac{1}{\sqrt{1-3x}} \, dx$

a $\displaystyle\int (2x+3)^4 \, dx = \tfrac{1}{2} \times \frac{(2x+3)^5}{5} + c$
$= \frac{1}{10}(2x+3)^5 + c$

b $\displaystyle\int \frac{1}{\sqrt{1-3x}} \, dx = \int (1-3x)^{-\frac{1}{2}} \, dx$
$= \frac{1}{-3} \times \frac{(1-3x)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= -\frac{2}{3}\sqrt{1-3x} + c$

EXERCISE 21D

1 Find:

a $\int (2x+5)^3 \, dx$ **b** $\int (4x-3)^7 \, dx$ **c** $\int 3(1-x)^4 \, dx$

d $\int \frac{1}{(3-2x)^2} \, dx$ **e** $\int \frac{4}{(2x-1)^4} \, dx$ **f** $\int \sqrt{3x-4} \, dx$

g $\int \frac{10}{\sqrt{1-5x}} \, dx$ **h** $\int \frac{4}{\sqrt{3-4x}} \, dx$ **i** $\int \frac{5}{(3x-2)^3} \, dx$

2 Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x-7}$ and that $f(8) = 11$.

3 The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.

Find the point on the graph of $y = f(x)$ with x-coordinate -8.

4 Find:

a $\int 3(2x-1)^2 \, dx$ **b** $\int (4x-5)^2 \, dx$ **c** $\int (1-3x)^3 \, dx$

d $\int (2-5x)^2 \, dx$ **e** $\int 4\sqrt{5-x} \, dx$ **f** $\int (7x+1)^4 \, dx$

5 Find an expression for y given that $\frac{dy}{dx} = x - \frac{5}{(1-x)^2}$ and that the curve passes through $(2, 0)$.

Example 10 — Self Tutor

Integrate with respect to x:

a $2e^{2x} - e^{-3x}$ **b** $2\sin 3x + \cos(4x+\pi)$

a
$$\int (2e^{2x} - e^{-3x}) \, dx$$
$$= 2(\tfrac{1}{2})e^{2x} - (\tfrac{1}{-3})e^{-3x} + c$$
$$= e^{2x} + \tfrac{1}{3}e^{-3x} + c$$

b
$$\int (2\sin 3x + \cos(4x+\pi)) \, dx$$
$$= 2 \times -\tfrac{1}{3}\cos 3x + \tfrac{1}{4}\sin(4x+\pi) + c$$
$$= -\tfrac{2}{3}\cos 3x + \tfrac{1}{4}\sin(4x+\pi) + c$$

6 Integrate with respect to x:

a $\sin 3x$ **b** $2\cos(-4x) + 1$ **c** $\sec^2 2x$

d $3\cos\frac{x}{2}$ **e** $3\sin 2x - e^{-x}$ **f** $e^{2x} - 2\sec^2\left(\frac{x}{2}\right)$

g $2\sin\left(2x + \frac{\pi}{6}\right)$ **h** $-3\cos\left(\frac{\pi}{4} - x\right)$ **i** $4\sec^2\left(\frac{\pi}{3} - 2x\right)$

j $\cos 2x + \sin 2x$ **k** $2\sin 3x + 5\cos 4x$ **l** $\frac{1}{2}\cos 8x - 3\sin x$

7 Find:

a $\int (2e^x + 5e^{2x}) \, dx$ **b** $\int (3e^{5x-2}) \, dx$ **c** $\int (e^{7-3x}) \, dx$

d $\int (e^x + e^{-x})^2 \, dx$ **e** $\int (e^{-x} + 2)^2 \, dx$ **f** $\int \frac{(e^{2x}-5)^2}{e^x} \, dx$

8 Find an expression for y given that $\frac{dy}{dx} = (1 - e^x)^2$, and that the graph has y-intercept 4.

9 **a** Show that $\int k^{ax+b}\,dx = \frac{k^{ax+b}}{a \ln k} + c, \quad k > 0, \quad k \neq 1, \quad a \neq 0.$

b Hence find:

i $\int 3^{2x-1}\,dx$ **ii** $\int 5^{-x}\,dx$ **iii** $\int (2^{5x} - 7^{1-2x})\,dx$

10 Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

11 Suppose $f'(x) = p \sin \frac{x}{2}$, $f(0) = 1$, and $f(2\pi) = 0$. Find p and hence $f(x)$.

12 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

13 A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

14 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$, and hence determine $\int (\sin x + \cos x)^2\,dx$.

15 **a** Rearrange the double angle formulae $\cos 2x = 1 - 2\sin^2 x$ and $\cos 2x = 2\cos^2 x - 1$ to write expressions for $\sin^2 x$ and $\cos^2 x$.

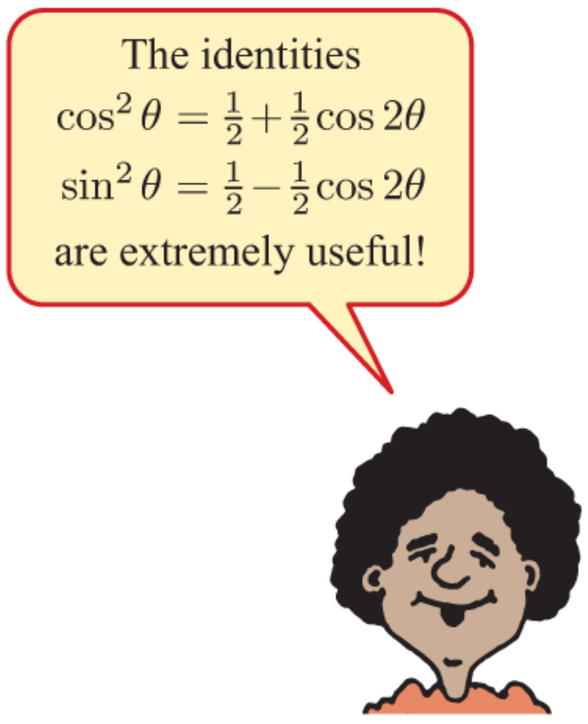
b Hence find:

i $\int \sin^2 x\,dx$ **ii** $\int \cos^2 x\,dx$

Example 11 **Self Tutor**

Integrate $(2 - \sin x)^2$ with respect to x.

$$\int (2 - \sin x)^2\,dx$$
$$= \int (4 - 4\sin x + \sin^2 x)\,dx$$
$$= \int \left(4 - 4\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)dx$$
$$= \int \left(\tfrac{9}{2} - 4\sin x - \tfrac{1}{2}\cos 2x\right)dx$$
$$= \tfrac{9}{2}x + 4\cos x - \tfrac{1}{2} \times \tfrac{1}{2}\sin 2x + c$$
$$= \tfrac{9}{2}x + 4\cos x - \tfrac{1}{4}\sin 2x + c$$



16 Integrate with respect to x:

a $\cos^2 x + 2$ **b** $\sin^2 x + 4x$ **c** $1 + \cos^2 2x$ **d** $3 - \sin^2 3x$

e $\frac{1}{2}\cos^2 4x$ **f** $(1 + \cos x)^2$ **g** $\sin x(2\sin x - 1)$ **h** $(1 - 3\sin x)^2$

17 **a** Use the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ to show that $\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$.

b Hence find $\int \cos^4 x \, dx$.

Example 12 **Self Tutor**

Find: **a** $\int \frac{1}{5x-3}\,dx$ **b** $\int \frac{4}{1-2x}\,dx$

a $\int \frac{1}{5x-3}\,dx$
$= \frac{1}{5}\ln|5x-3| + c$

b $\int \frac{4}{1-2x}\,dx$
$= 4\left(\frac{1}{-2}\right)\ln|1-2x| + c$
$= -2\ln|1-2x| + c$

18 Find:

a $\int \frac{6}{x+4}\,dx$ **b** $\int \frac{1}{2x-1}\,dx$ **c** $\int \frac{5}{1-3x}\,dx$

d $\int \left(4 + \frac{1}{5x-2}\right)dx$ **e** $\int \left(1 - 2x + \frac{4}{x-3}\right)dx$ **f** $\int \left(\sin 2x - \frac{3}{1-2x}\right)dx$

g $\int \left(e^{-x} - \frac{4}{2x+1}\right)dx$ **h** $\int \left(\frac{1}{x+2} + \frac{2}{x-3}\right)dx$ **i** $\int \left(\frac{5}{x-6} - \frac{2}{3x-1}\right)dx$

19 To find $\int \frac{1}{4x}\,dx$, Tracy's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\ln|4x| + c$.

Nadine's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln|x| + c$.

Which of them has found the correct answer? Prove your statement.

20 Show that $\frac{3x-1}{x+2}$ may be written in the form $3 - \frac{7}{x+2}$. Hence find $\int \frac{3x-1}{x+2}\,dx$.

21 Find $\int \frac{x^3 - x^2 + 4x - 3}{x-1}\,dx$.

22 Find $f(x)$ given $f'(x) = 2x - \frac{2}{1-x}$ and $f(-1) = 3$.

23 **a** Show that $\int \frac{1}{(ax+b)\ln k}\,dx = \frac{1}{a}\log_k|ax+b| + c, \quad k > 0, \quad k \neq 1, \quad a \neq 0$.

b Hence find:

i $\int \frac{1}{(2x+5)\ln 3}\,dx$ **ii** $\int \frac{3}{(1-x)\ln 2}\,dx$

24 **a** Show that $\int (-\operatorname{cosec}(ax+b)\cot(ax+b))\,dx = \frac{1}{a}\operatorname{cosec}(ax+b) + c, \quad a \neq 0$.

b Use similar methods to write an expression for:

i $\int \sec(ax+b)\tan(ax+b)\,dx$ **ii** $\int (-\operatorname{cosec}^2(ax+b))\,dx$

c Hence find:

i $\int \text{cosec}\left(2x - \frac{\pi}{6}\right) \cot\left(2x - \frac{\pi}{6}\right) dx$ **ii** $\int 3\sec\left(\frac{\pi}{3} - \theta\right) \tan\left(\frac{\pi}{3} - \theta\right) d\theta$

iii $\int \frac{\text{cosec}^2(4x - \pi)}{5} dx$

25 **a** Show that $\int \frac{1}{\sqrt{1-(ax+b)^2}} dx = \frac{1}{a}\arcsin(ax+b) + c, \quad a \neq 0.$

b Use similar methods to write an expression for:

i $\int \frac{-1}{\sqrt{1-(ax+b)^2}} dx$ **ii** $\int \frac{1}{1+(ax+b)^2} dx$

c Hence find:

i $\int \frac{1}{\sqrt{1-(x+2)^2}} dx$ **ii** $\int \frac{-2}{\sqrt{-x^2+10x-24}} dx$ **iii** $\int \frac{1}{4x^2+12x+10} dx$

26 **a** Show that:

i $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\frac{x}{a} + c$ **ii** $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \arccos\frac{x}{a} + c$

iii $\int \frac{a}{a^2+x^2} dx = \arctan\frac{x}{a} + c$

b Hence find:

i $\int \frac{1}{\sqrt{25-x^2}} dx$ **ii** $\int -\frac{4}{\sqrt{9-x^2}} dx$ **iii** $\int \frac{18}{9+x^2} dx$

iv $\int \frac{1}{\sqrt{\frac{1}{4}-x^2}} dx$ **v** $\int -\frac{5}{\sqrt{\frac{1}{9}-x^2}} dx$ **vi** $\int \frac{1}{\frac{1}{16}+x^2} dx$

E PARTIAL FRACTIONS

In **Chapter 6** we saw how rational functions can be decomposed into the sum of partial fractions. The resulting sum can then be integrated one term at a time.

Example 13 Self Tutor

Find $\int \frac{7x-1}{2x^2-x-1} dx.$

$2x^2 - x - 1 = (2x+1)(x-1)$

Let $\frac{7x-1}{2x^2-x-1} = \frac{A}{2x+1} + \frac{B}{x-1}$

$\therefore \quad 7x - 1 = A(x-1) + B(2x+1)$

Substituting $x = 1$, $\quad 7 - 1 = 3B$

$\therefore \quad B = 2$

Substituting $x = -\frac{1}{2}$, $7\left(-\frac{1}{2}\right) - 1 = A\left(-\frac{1}{2} - 1\right)$

$$\therefore \quad -\tfrac{9}{2} = -\tfrac{3}{2}A$$

$$\therefore \quad A = 3$$

$$\therefore \quad \frac{7x-1}{2x^2-x-1} = \frac{3}{2x+1} + \frac{2}{x-1}$$

$$\therefore \quad \int \frac{7x-1}{2x^2-x-1}\,dx = \int \left(\frac{3}{2x+1} + \frac{2}{x-1}\right) dx$$

$$= \tfrac{3}{2}\ln|2x+1| + 2\ln|x-1| + c$$

EXERCISE 21E

1 **a** Show that $\frac{3}{x+2} - \frac{1}{x-2} = \frac{2x-8}{(x+2)(x-2)}$. **b** Hence find $\int \frac{2x-8}{x^2-4}\,dx$.

2 **a** Show that $\frac{1}{2x-1} - \frac{1}{2x+1} = \frac{2}{(2x-1)(2x+1)}$. **b** Hence find $\int \frac{2}{4x^2-1}\,dx$.

3 **a** Write $\frac{20}{(2x-3)(x+1)}$ as the sum of partial fractions. **b** Hence find $\int \frac{20}{2x^2-x-3}\,dx$.

4 **a** Write $\frac{x-9}{x^2-2x-3}$ as the sum of partial fractions. **b** Hence find $\int \frac{x-9}{x^2-2x-3}\,dx$.

5 Integrate with respect to x:

a $\frac{3}{(x-2)(x+1)}$ **b** $\frac{4x}{x^2+2x-3}$ **c** $\frac{1-x}{x^2+2x}$

d $\frac{2x-5}{x^2-x-2}$ **e** $\frac{2-x}{2x^2-5x-3}$ **f** $\frac{2x}{6x^2-x-15}$

6 **a** Write $\frac{6x^2+x-19}{(x+3)(x-1)^2}$ as the sum of partial fractions in the form $\frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$.

b Hence find $\int \frac{6x^2+x-19}{(x+3)(x-1)^2}\,dx$.

F INTEGRATION BY SUBSTITUTION

Consider the integral $\int (x^2+3x)^4(2x+3)\,dx$.

The function we are integrating is the product of two expressions:

- The first expression is $(x^2+3x)^4$, which has the form $f(u(x))$ where $f(u) = u^4$ and $u(x) = x^2+3x$.
- The second expression is $2x+3$, and we notice that this is $\frac{du}{dx}$.

So, we can write the integral in the form $\int f(u(x))\,\frac{du}{dx}\,dx$ where $f(u) = u^4$ and $u(x) = x^2+3x$.

Integrals in this form can be found using a method called **integration by substitution**.

Integration by substitution is the reverse process of differentiating using the **chain rule**.

Suppose $F(u)$ is the antiderivative of $f(u)$, so $\dfrac{dF}{du} = f(u)$

$$\therefore \int f(u)\,du = F(u) + c \quad \ldots. \ (1)$$

But $\dfrac{dF}{dx} = \dfrac{dF}{du}\dfrac{du}{dx}$ {chain rule}

$$= f(u)\,\frac{du}{dx}$$

$$\therefore \int f(u)\,\frac{du}{dx}\,dx = F(u) + c \quad \ldots. \ (2)$$

Comparing (1) and (2), we conclude that: $\displaystyle\int f(u)\,\frac{du}{dx}\,dx = \int f(u)\,du$

Using this method,

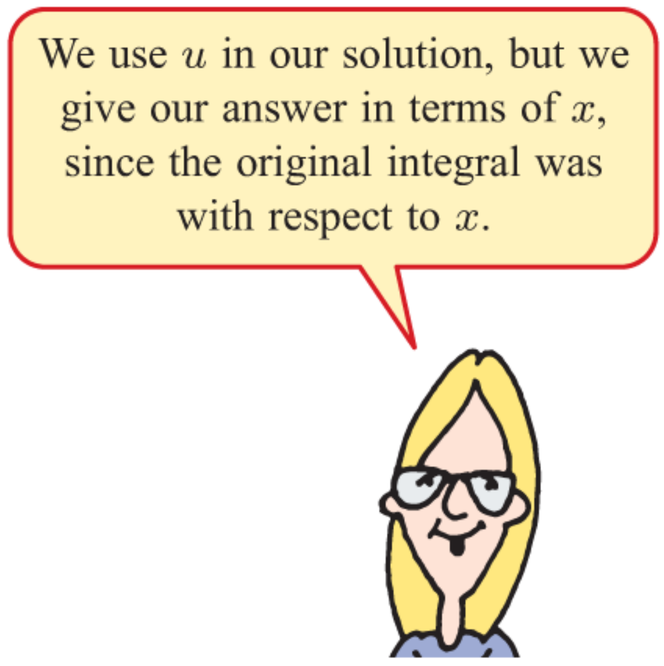
$$\int (x^2+3x)^4(2x+3)\,dx$$

$$= \int u^4\,\frac{du}{dx}\,dx \qquad \{u = x^2+3x,\ \ \frac{du}{dx} = 2x+3\}$$

$$= \int u^4\,du \qquad \{\text{replacing } \frac{du}{dx}\,dx \text{ by } du\}$$

$$= \frac{u^5}{5} + c$$

$$= \tfrac{1}{5}(x^2+3x)^5 + c$$



Example 14 — Self Tutor

Use substitution to find:

a $\displaystyle\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$ **b** $\displaystyle\int xe^{1-x^2}\,dx$

a

$$\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$$

$$= \int \sqrt{u}\,\frac{du}{dx}\,dx \qquad \{u = x^3+2x,\ \ \frac{du}{dx} = 3x^2+2\}$$

$$= \int u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \tfrac{2}{3}(x^3+2x)^{\frac{3}{2}} + c$$

b

$$\int xe^{1-x^2}\,dx$$

$$= -\tfrac{1}{2}\int (-2x)e^{1-x^2}\,dx$$

$$= -\tfrac{1}{2}\int e^u\,\frac{du}{dx}\,dx \qquad \{u = 1-x^2,\ \ \frac{du}{dx} = -2x\}$$

$$= -\tfrac{1}{2}\int e^u\,du$$

$$= -\tfrac{1}{2}e^u + c$$

$$= -\tfrac{1}{2}e^{1-x^2} + c$$

Example 15 **Self Tutor**

Integrate with respect to x:

a $\cos^3 x \sin x$ **b** $\dfrac{\cos x}{\sin x}$

a
$$\int \cos^3 x \sin x \, dx$$
$$= \int (\cos x)^3 \sin x \, dx$$
$$= \int u^3 \left(-\frac{du}{dx}\right) dx \qquad \{u = \cos x, \ \frac{du}{dx} = -\sin x\}$$
$$= -\int u^3 \, du$$
$$= -\frac{u^4}{4} + c$$
$$= -\tfrac{1}{4}\cos^4 x + c$$

b
$$\int \frac{\cos x}{\sin x} \, dx$$
$$= \int \frac{1}{u}\frac{du}{dx} \, dx \qquad \{u = \sin x, \ \frac{du}{dx} = \cos x\}$$
$$= \int \frac{1}{u} \, du$$
$$= \ln|u| + c$$
$$= \ln|\sin x| + c$$



$\int \dfrac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$ is a useful result.

CHOOSING WHAT TO SUBSTITUTE

The substitutions we make need to be chosen carefully.

For example, in **Example 15** part **b**, if we let $u = \cos x$, then $\dfrac{du}{dx} = -\sin x$ and we obtain $\displaystyle\int \frac{\cos x}{\sin x} \, dx = \int \frac{u}{-\frac{du}{dx}} \, dx$. This is not in the correct form to apply our theorem, so we have made the wrong substitution and we need to try another option.

A useful guide is that when an integral involves a composition of functions $f \circ g$, try substituting $u = g(x)$.

For example, if a function contains $\sqrt{g(x)}$, try substituting $u = g(x)$.

However, this guide is not perfect. With practice you will learn to recognise situations where other substitutions are useful.

For example, if a function contains $\sqrt{a^2 - x^2}$, try substituting $x = a\sin\theta$.

For $\sqrt{x^2 + a^2}$ or $f(x^2 + a^2)$, try substituting $x = a\tan\theta$.

For $\sqrt{x^2 - a^2}$, try substituting $x = a\sec\theta$.

EXERCISE 21F

1 Differentiate $(2x^2 - 5x)^3$, and hence find $\displaystyle\int (4x - 5)(2x^2 - 5x)^2 \, dx$.

2 Differentiate $\sin(x^2)$, and hence find $\displaystyle\int x\cos(x^2) \, dx$.

3 Differentiate $\ln(5 - 3x + x^2)$, and hence find $\displaystyle\int \frac{4x - 6}{5 - 3x + x^2} \, dx$.

4 Use the substitution given to perform each integration:

a $\int 3x^2(x^3+1)^4\,dx$ using $u = x^3+1$

b $\int x^2 e^{x^3+1}\,dx$ using $u = x^3+1$

c $\int \sin^4 x \cos x\,dx$ using $u = \sin x$

d $\int 2x\cos(x^2-3)\,dx$ using $u = x^2-3$

e $\int \dfrac{e^{\frac{x-1}{x}}}{x^2}\,dx$ using $u = \dfrac{x-1}{x}$

5 Integrate by substitution:

a $4x^3(2+x^4)^3$

b $\dfrac{2x}{\sqrt{x^2+3}}$

c $\dfrac{6x^2}{(2x^3-1)^4}$

d $\sqrt{x^3+x}\,(3x^2+1)$

e $(x^3+2x+1)^4(3x^2+2)$

f $\dfrac{x}{(1-x^2)^5}$

g $\dfrac{x+2}{(x^2+4x-3)^2}$

h $\dfrac{x^2}{(3x^3-1)^4}$

i $x^4(x+1)^4(2x+1)$

6 Find:

a $\int -2e^{1-2x}\,dx$

b $\int 2xe^{x^2}\,dx$

c $\int \dfrac{e\sqrt{x}}{\sqrt{x}}\,dx$

d $\int (2x-1)e^{x-x^2}\,dx$

7 Find:

a $\int \dfrac{2x}{x^2+1}\,dx$

b $\int \dfrac{x}{2-x^2}\,dx$

c $\int \dfrac{4x-10}{5x-x^2}\,dx$

d $\int \dfrac{1-x^2}{x^3-3x}\,dx$

e $\int \dfrac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

f $\int \dfrac{x^3}{1+x^2}\,dx$

8 Integrate with respect to x:

a $x^2(3-x^3)^2$

b $x\sqrt{1-x^2}$

c xe^{1-x^2}

d $\dfrac{(\ln x)^3}{x}$

e $\dfrac{4}{x\ln x}$

f $x^2 2^{x^3}$

g $\dfrac{e^{2x}}{\sqrt{5-e^{2x}}}$

h $\cos x\, 3^{\sin x}$

9 Integrate with respect to x:

a $\sin^7 x\cos x$

b $\cos^5 x\sin x$

c $\dfrac{\sin x}{\sqrt{\cos x}}$

d $\tan x$

e $\sqrt{\sin x}\cos x$

f $\dfrac{\cos x}{(2+\sin x)^2}$

g $\dfrac{\sin x}{\cos^3 x}$

h $\dfrac{\sin x}{1-\cos x}$

i $\dfrac{\sin x+\cos x}{\sin x-\cos x}$

j $\dfrac{\cos 2x}{\sin 2x-3}$

k $x\sin(x^2)$

l $\dfrac{\sin^3 x}{\cos^5 x}$

m $\dfrac{\arctan x}{1+x^2}$

n $\tan^3 x$

o $\cos^3 x$

p $\sin^5 x\cos^5 x$

10 Find:

a $\int \sin^5 x\,dx$

b $\int \sin^4 x\cos^3 x\,dx$

c $\int \sin^3 2x\cos 2x\,dx$

11 Find $f(x)$ if $f'(x)$ is:

a $\sin x\, e^{\cos x}$

b $\dfrac{e^{\tan x}}{\cos^2 x}$

c $\dfrac{e^{\cot x}}{\sin^2 x}$

12 Find:

a $\int \cot 3x \, dx$ **b** $\int \sec^3 x \sin x \, dx$ **c** $\int \operatorname{cosec}^4 x \cos x \, dx$

d $\int \sec^2 x \sqrt{\tan x} \, dx$ **e** $\int x \operatorname{cosec}^2(x^2) \, dx$ **f** $\int \dfrac{\sec x \tan x}{\sqrt[3]{\sec x}} \, dx$

g $\int \dfrac{\operatorname{cosec}^2 x}{\sqrt{\cot x}} \, dx$ **h** $\int \operatorname{cosec}^3(2x) \cot 2x \, dx$ **i** $\int \sec^5 x \tan x \, dx$

13 Find:

a $\int \dfrac{x \ln(x^2+7)}{x^2+7} \, dx$ using the substitution $u = \ln(x^2+7)$

b $\int \dfrac{4}{\sqrt{x}\sqrt{1-x}} \, dx$ using the substitution $x = \sin^2\theta$

Example 16 **Self Tutor**

Find $\int x\sqrt{x+2} \, dx$.

Let $u = x+2$ $\therefore \dfrac{du}{dx} = 1$

$$\therefore \int x\sqrt{x+2} \, dx = \int (u-2)\sqrt{u} \, \frac{du}{dx} \, dx$$

$$= \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \tfrac{2}{5}(x+2)^{\frac{5}{2}} - \tfrac{4}{3}(x+2)^{\frac{3}{2}} + c$$

14 Find $\int x^2\sqrt{x-16} \, dx$ using the substitution $u = x-16$.

15 Find:

a $\int x\sqrt{x-3} \, dx$ **b** $\int x\sqrt{1-x} \, dx$ **c** $\int x^2\sqrt{x+1} \, dx$

d $\int \dfrac{x}{\sqrt{x-3}} \, dx$ **e** $\int x^3\sqrt{3-x^2} \, dx$ **f** $\int t^3\sqrt{t^2+2} \, dt$

16 Find $\int \dfrac{\sqrt{x-1}}{x} \, dx$ using the substitution $u = \sqrt{x-1}$.



For your own experience, try to also find

$$\int \frac{\sqrt{x-1}}{x} \, dx$$

using the substitution $u = x-1$.

Example 17 **Self Tutor**

Find $\int \frac{\sqrt{x^2-9}}{x}\,dx$.

Let $x = 3\sec\theta$ $\quad\therefore\quad \frac{dx}{d\theta} = 3\sec\theta\tan\theta$

$$\therefore \int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta$$

$$= \int 3\sqrt{\sec^2\theta - 1}\,\tan\theta\,d\theta$$

$$= \int 3\tan^2\theta\,d\theta$$

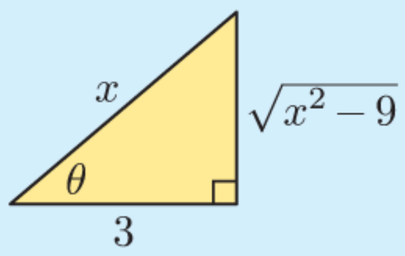
$$= \int (3\sec^2\theta - 3)\,d\theta$$

$$= 3\tan\theta - 3\theta + c$$

Since $\sec\theta = \frac{x}{3}$, $\quad\cos\theta = \frac{3}{x}$

$$\therefore \theta = \arccos\tfrac{3}{x}$$

and $\quad\tan\theta = \frac{\sqrt{x^2-9}}{3}$



So, $\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\arccos\frac{3}{x} + c$

17 Find:

a $\int \frac{1}{36+4x^2}\,dx$ using $x = 3\tan\theta$

b $\int \frac{\sqrt{4x^2-1}}{5x}\,dx$ using $x = \frac{1}{2}\sec\theta$.

18 Integrate with respect to x:

a $\frac{x^2}{9+x^2}$ **b** $\frac{x^2}{\sqrt{1-x^2}}$ **c** $\sqrt{9-x^2}$ **d** $\frac{4\ln x}{x(1+[\ln x]^2)}$

e $x^2\sqrt{1-x^2}$ **f** $\frac{\sqrt{x^2-4}}{x}$ **g** $\frac{1}{\sqrt{9-4x^2}}$ **h** $\frac{1}{x(9+4[\ln x]^2)}$

i $\frac{1-2x}{\sqrt{4-x^2}}$ **j** $\frac{x+4}{x^2+4}$ **k** $\frac{1}{x(x^2+16)}$ **l** $\frac{3}{x\sqrt{x^2-4}}$

m $\frac{1}{x^2\sqrt{16-x^2}}$ **n** $\frac{1}{x^2+2x+3}$ **o** $\frac{1}{x(1+x^2)}$ **p** $x^2\sqrt{4-x^2}$

19 Suppose ξ is a constant. Use the substitution $u = \sqrt{\frac{x-\xi}{x+\xi}}$ to find the following integrals, giving your answers in simplest form:

a $\int \frac{x-\xi}{(x+\xi)^3}\,dx$ **b** $\int (x-\xi)^{\frac{1}{2}}(x+\xi)^{-\frac{5}{2}}\,dx$

20 Find $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$ using the substitution $u = x^{\frac{1}{6}}$.

21 Find $\int \frac{1}{ax^2+bx+c}\,dx$.

Hint: You will need to consider the cases $b^2 < 4ac$, $b^2 > 4ac$, and $b^2 = 4ac$ separately.

G INTEGRATION BY PARTS

The method of **integration by parts** comes from the **product rule** of differentiation. In some cases it allows us to integrate a function which is written as a product.

Since $\frac{d}{dx}(uv) = u'v + uv'$, $\quad \int (u'v + uv')\,dx = uv$

$$\therefore \int u'v\,dx + \int uv'\,dx = uv$$

$$\therefore \boxed{\int \boldsymbol{uv'}\ \boldsymbol{dx} = \boldsymbol{uv} - \int \boldsymbol{u'v}\ \boldsymbol{dx}}$$

This rule allows us to find $\int uv'\,dx$ provided $\int u'v\,dx$ can be found.

Example 18 Self Tutor

Find:

a $\int xe^{-x}\,dx$ **b** $\int x\cos x\,dx$

a $\int xe^{-x}\,dx = -xe^{-x} - \int (-e^{-x})\,dx$ $\quad \begin{cases} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{cases}$

$= -xe^{-x} + (-e^{-x}) + c$

$= -e^{-x}(x+1) + c$

Check: $\frac{d}{dx}(-e^{-x}(x+1)+c) = e^{-x}(x+1) + -e^{-x}(1)$

$= xe^{-x} + \cancel{e^{-x}} - \cancel{e^{-x}}$

$= xe^{-x}$ ✓

b $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$ $\quad \begin{cases} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{cases}$

$= x\sin x - (-\cos x) + c$

$= x\sin x + \cos x + c$

Check: $\frac{d}{dx}(x\sin x + \cos x + c) = 1 \times \sin x + x\cos x - \sin x$

$= \cancel{\sin x} + x\cos x - \cancel{\sin x}$

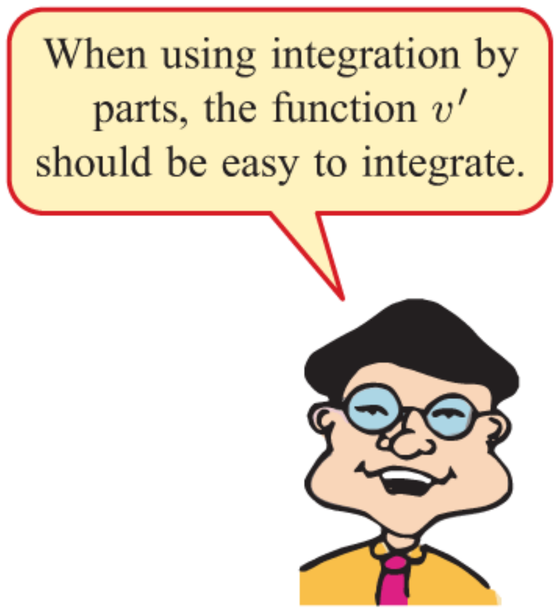
$= x\cos x$ ✓

EXERCISE 21G

1 Use integration by parts to find integrals of the following functions with respect to x:

a xe^x **b** $x \sin x$ **c** $x^2 \ln x$ **d** $\dfrac{\ln x}{x}$

e $x3^x$ **f** $x \sin 3x$ **g** $x \cos 2x$ **h** $x \sec^2 x$



2 **a** Find $\int \ln x \, dx$ by first writing $\ln x$ as $1 \times \ln x$.

b Hence find $\int (\ln x)^2 \, dx$.

3 Find $\int \arctan x \, dx$ by first writing $\arctan x$ as $1 \times \arctan x$.

Example 19 **Self Tutor**

Find $\int e^x \sin x \, dx$.

$$\int e^x \sin x \, dx$$

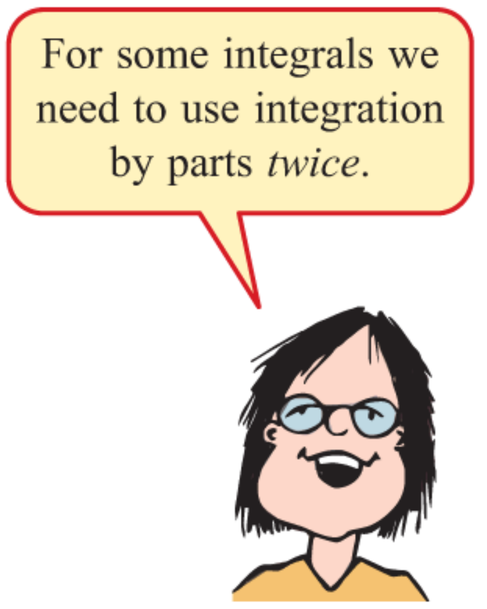
$$= e^x(-\cos x) - \int e^x(-\cos x) \, dx \quad \longleftarrow \begin{cases} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{cases}$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \quad \longleftarrow \begin{cases} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{cases}$$

$$\therefore \quad 2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\therefore \quad \int e^x \sin x \, dx = \tfrac{1}{2}e^x(\sin x - \cos x) + c$$



4 Integrate with respect to x:

a x^2e^{-x} **b** $e^x \cos x$ **c** $e^{-x} \sin x$ **d** $x^2 \sin x$ **e** $\dfrac{(\ln x)^2}{x^2}$

5 **a** Use integration by parts to find $\int u^2 e^u \, du$.

b Hence find $\int (\ln x)^2 \, dx$ using the substitution $u = \ln x$.

6 Find $\int \sin x \cos x \, dx$ using:

a a double angle formula **b** substitution **c** integration by parts.

7 Use integration by parts to find $\int \sin 4x \cos x \, dx$.

8 **a** Use integration by parts to find $\int u \sin u \, du$.

b Hence find $\int \sin \sqrt{2x} \, dx$ using the substitution $u^2 = 2x$.

9 Find $\int \cos \sqrt{3x} \, dx$ using the substitution $u^2 = 3x$.

REVIEW SET 21A

1 Find the derivative of $x^4 - x^2$, and hence find $\int (2x^3 - x) \, dx$.

2 Find the derivative of $\sin\left(\frac{\pi}{3} - 2x\right)$, and hence find $\int \cos\left(\frac{\pi}{3} - 2x\right) dx$.

3 Find:

a $\int \left(\sqrt{x} - \frac{2}{x^2}\right) dx$ **b** $\int \left(2x - \frac{3}{\sqrt[3]{x}}\right) dx$ **c** $\int \frac{6x+5}{\sqrt{x}} \, dx$

4 Integrate with respect to x:

a $\frac{4}{\sqrt{x}}$ **b** $\frac{3}{1-2x}$ **c** $\frac{1}{3}x^3 + 2x$ **d** $\frac{1-2x}{x^3}$

5 Find:

a $\int (-3x^4 + 6x^2) \, dx$ **b** $\int \frac{3x^3 - x^2 - 1}{x^2} \, dx$ **c** $\int (2x - \sqrt{x})^2 \, dx$

d $\int \left(4e^x - \frac{3}{x}\right) dx$ **e** $\int \sin(4x - 5) \, dx$ **f** $\int e^{4-3x} \, dx$

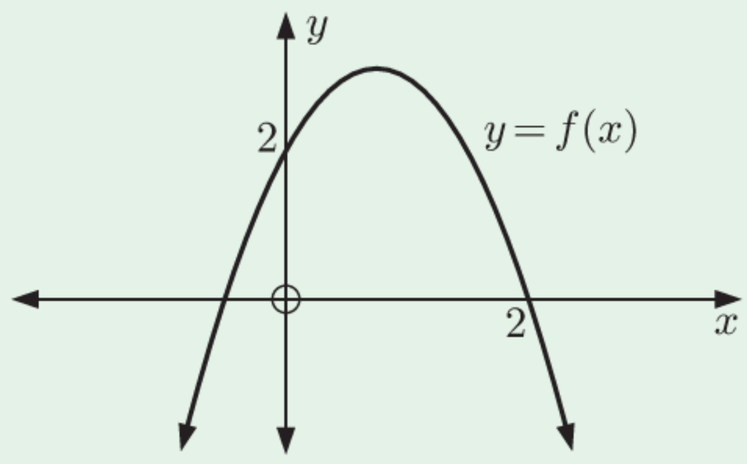
g $\int 5^{2-x} \, dx$ **h** $\int \sec^2\left(\frac{x}{2}\right) dx$ **i** $\int \operatorname{cosec}^2\left(x - \frac{\pi}{4}\right) dx$

6 Find y if:

a $\frac{dy}{dx} = 3e^{-x} - 2\sin\left(\frac{\pi}{2} - x\right)$ **b** $\frac{dy}{dx} = \cos 4x - \frac{1}{2}x^2$

7 Given that $f'(x) = 3x^2 - 4x + 1$ and $f(0) = 2$, find $f(x)$.

8 The curve $y = f(x)$ shown alongside has gradient function $f'(x) = ax + 3$.
Find the equation of the curve.



9 Given that $f'(x) = 3e^{2x}$ and $f(0) = 2$, find $f(x)$.

10 A curve has gradient function $f'(x) = a\cos 3x$ where a is a constant. $f(0) = -1$ and $f(\frac{\pi}{4}) = 1$. Find a and hence $f(x)$.

11 Find:

a $\int \frac{x^2 - 7}{x}\, dx$ **b** $\int \left(e^{2x-3} - \frac{2}{3x-1}\right) dx$

c $\int \left((4-3x)^3 + \sin(-2x)\right) dx$ **d** $\int \left(\frac{3}{1+x^2} - \frac{2}{5-x}\right) dx$

12 Find:

a $\int (1 - \sin x)^2\, dx$ **b** $\int \frac{x}{\sqrt{x^2-4}}\, dx$ **c** $\int x \sin\left(x^2 + \frac{\pi}{3}\right) dx$

13 **a** Show that $\frac{4}{x-2} + \frac{3}{x+1} = \frac{7x-2}{(x-2)(x+1)}$. **b** Hence find $\int \frac{7x-2}{x^2-x-2}\, dx$.

14 Integrate with respect to x:

a $\frac{x}{x^2-2x-8}$ **b** $\frac{3x-4}{2x^2-3x-5}$

15 By considering $\frac{d}{dx}(\tan^3 x)$, find $\int \sec^4 x\, dx$.

16 Integrate by substitution:

a $\int \frac{x+2}{x^2+4x}\, dx$ **b** $\int 2xe^{x^2-1}\, dx$ **c** $\int \sin^9 x \cos x\, dx$

d $\int \tan 2x\, dx$ **e** $\int e^{\sin x} \cos x\, dx$ **f** $\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$

17 Find:

a $\int x^2\sqrt{4-x}\, dx$ **b** $\int \frac{x^3}{(2-x)^3}\, dx$ **c** $\int \frac{x}{1+\sqrt{x+2}}\, dx$

18 Use integration by parts to find:

a $\int x^2 \ln(x^3)\, dx$ **b** $\int e^{2x} \sin x\, dx$

19 Find $\int \frac{x}{x^2-9}\, dx$ using:

a the substitution $u = x^2 - 9$ **b** partial fractions

c the substitution $x = 3\sin t$.

20 **a** Find $\int e^u \sin u\, du$. **b** Find $\int \sin(\ln x)\, dx$ using the substitution $u = \ln x$.

21 Find $\int \frac{1}{1+\cos^2 x}\, dx$ using the substitution $t = \tan x$. Give your answer in the form $a \arctan(b \tan x) + c$, where a and b are constants to be found.

REVIEW SET 21B

1 Find the derivative of e^{-2x}, and hence find $\int e^{-2x}\,dx$.

2 Find the derivative of $\ln(2x+1)$, and hence find $\int \frac{1}{2x+1}\,dx$.

3 Integrate with respect to x:

a $\frac{x^2-2}{x^2}$ **b** $(3x-4)^2$ **c** $4-2x^2$

4 Find:

a $\int (x^{\frac{1}{3}}+3)\,dx$ **b** $\int (3x^2-2)\,dx$ **c** $\int (3+2x)^2\,dx$

d $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$ **e** $\int \frac{5}{x\ln 2}\,dx$ **f** $\int \frac{-3}{\sqrt{1-x^2}}\,dx$

5 Given that $f'(x)=x^2-3x+2$ and $f(1)=3$, find $f(x)$.

6 Find y if:

a $\frac{dy}{dx}=(x^2-1)^2$ **b** $\frac{dy}{dx}=400-20x^{-\frac{1}{2}}$ **c** $\frac{dy}{dx}=\frac{5}{7-3x}+7^x$

7 Find:

a $\int (2x^3-5x+7)\,dx$ **b** $\int \left(3x-\frac{1}{x}\right)dx$ **c** $\int (1-x^2)^3\,dx$

d $\int (2e^{-x}+3)\,dx$ **e** $\int 4\cos 2x\,dx$ **f** $\int (3+e^{2x-1})^2\,dx$

8 Find $f(x)$ given $f'(x)=\frac{2}{x}-1$ and $f(2)=e$.

9 A curve has gradient function $\frac{dy}{dx}=ax^2+b\sqrt{x-1}$ where a, b are constants. Find the equation of the curve given that it contains the points $(1,\,4)$, $(2,\,4)$, and $(5,\,1)$.

10 Find $f(x)$ given $f'(x)=\frac{3}{\sqrt{4-3x}}$ and $f(-4)=0$.

11 Find:

a $\int \frac{1}{3-2x}\,dx$ **b** $\int \frac{4}{5x+1}\,dx$ **c** $\int (\sin x-\cos x)^2\,dx$

12 **a** Write $\frac{x+20}{(x-6)(2x+1)}$ as the sum of partial fractions.

b Hence find $\int \frac{x+20}{2x^2-11x-6}\,dx$.

13 Find $\int \frac{x+5}{x^2+x-2}\,dx$.

14 Show that $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c.$

15 **a** Perform the division $\frac{x^3 - 3x^2 + 5x + 4}{x - 3}$.

b Hence find $\int \frac{x^3 - 3x^2 + 5x + 4}{x - 3}\,dx.$

16 Integrate by substitution:

a $\int (3x^2 + x)^2(6x + 1)\,dx$

b $\int \frac{2x}{\sqrt{x^2 - 5}}\,dx$

c $\int \frac{\sin x}{\cos^4 x}\,dx$

d $\int 4xe^{-x^2}\,dx$

e $\int \sin^3 x\,dx$

f $\int \sin x \cos 2x\,dx$

g $\int \frac{1}{x\sqrt{x^2 - 16}}\,dx$

h $\int \frac{x^3}{\sqrt{9 - x^2}}\,dx$

17 Find $\int \frac{\sin x}{\sqrt{\cos^n x}}\,dx$. Comment on the existence of this integral.

18 Find $\int \frac{6\tan x}{\cos 2x}\,dx$ using the substitution $t = \tan x$.

19 Integrate with respect to x:

a $e^{-x}\cos x$

b $2^x \sin x$

c $\frac{\sqrt{x^2 - 4}}{x}$

20 Find:

a $\int \frac{5}{\sqrt{9 - x^2}}\,dx$

b $\int \frac{1}{9 + 4x^2}\,dx$

c $\int \frac{5 + 3x}{1 + x^2}\,dx$

21 Explain what is wrong with the following argument:

Let $I = \int \frac{1}{x}\,dx$

$\therefore\ I = \int 1 \times \frac{1}{x}\,dx$

$\therefore\ I = x\left(\frac{1}{x}\right) - \int x\left(-\frac{1}{x^2}\right)dx$ $\quad \begin{cases} u = \frac{1}{x} & v' = 1 \\ u' = -\frac{1}{x^2} & v = x \end{cases}$

$\therefore\ I = 1 + \int \frac{1}{x}\,dx$

$\therefore\ I = 1 + I$

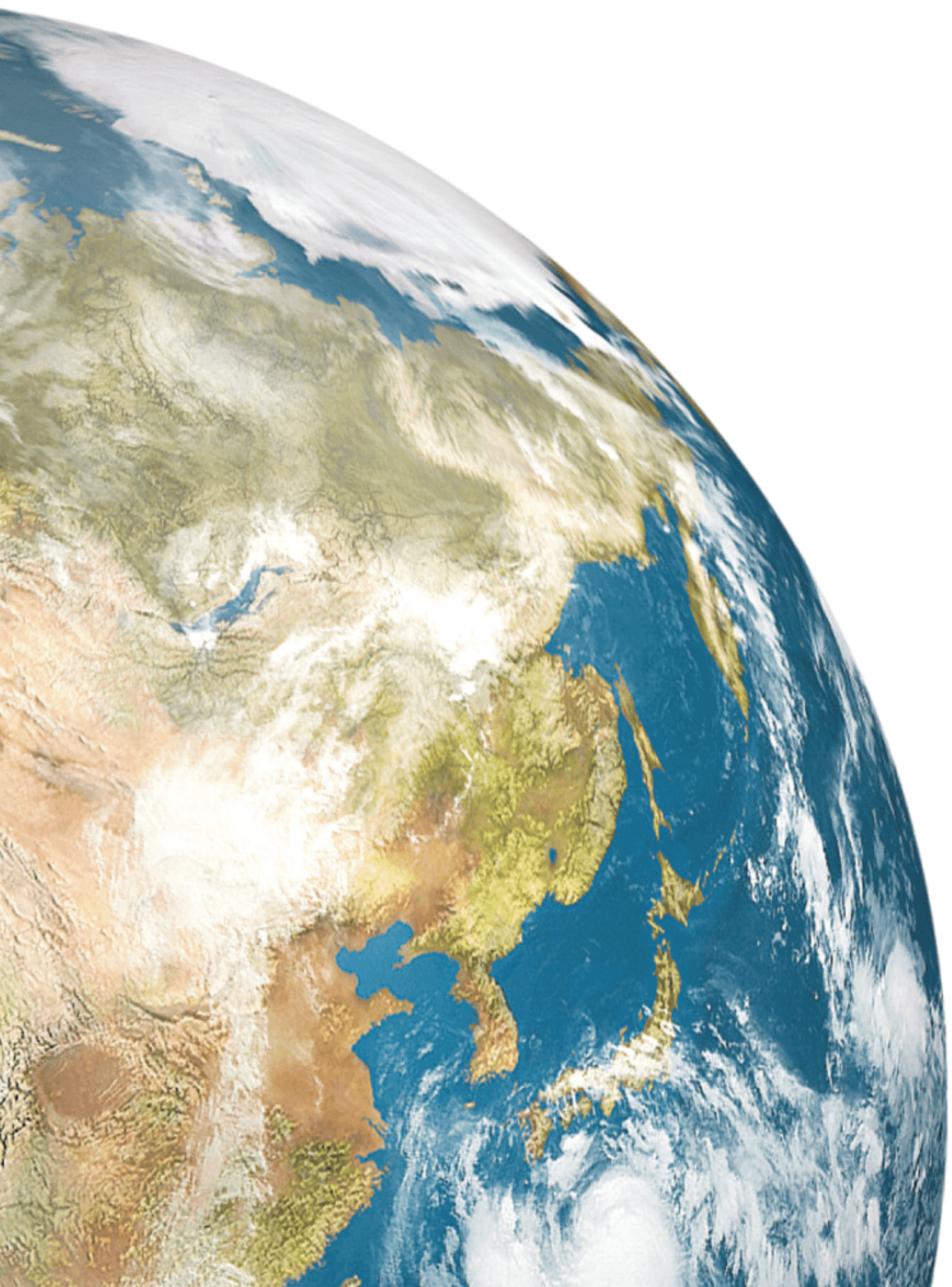
$\therefore\ 0 = 1$

Chapter 22

Definite integrals

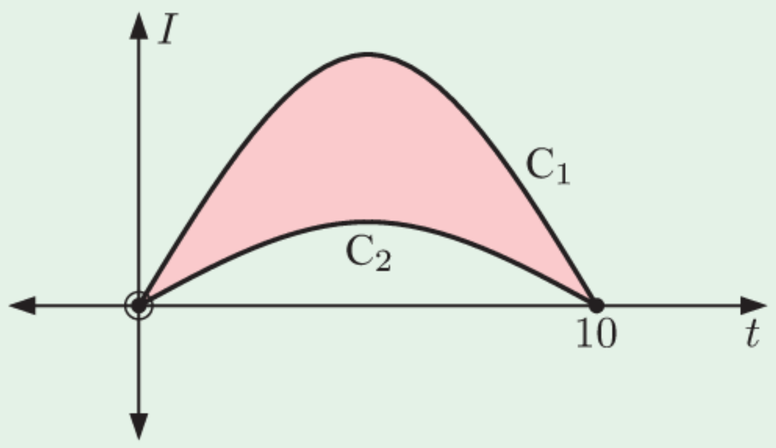
Contents:

A Definite integrals
B Definite integrals involving substitution
C The area under a curve
D The area above a curve
E The area between two functions
F The area between a curve and the y-axis
G Solids of revolution
H Problem solving by integration
I Improper integrals



OPENING PROBLEM

On August 8, the intensity of light entering a greenhouse is given by $I = 3\sin\frac{\pi t}{10}$ units per hour if the day is sunny, and $I = \sin\frac{\pi t}{10}$ units per hour if the day is overcast, where t is the number of hours after sunrise, $0 \leqslant t \leqslant 10$.



The graphs of these curves are shown alongside.

Things to think about:

a Can you identify each curve?

b Can you find the shaded area enclosed by C_1 and C_2 for $0 \leqslant t \leqslant 10$?

c If the intensity of light is regarded as its rate of energy transfer, what does the area in **b** mean about the energy entering the greenhouse?

We have already seen how the Fundamental Theorem of Calculus gives meaning to a **definite integral** over a particular domain:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

In this Chapter we explore definite integrals and how they are used to calculate areas.

A DEFINITE INTEGRALS

The definite integral

$\displaystyle\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x"
or "the integral from a to b of $f(x)$ with respect to x".

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

We write $F(b) - F(a)$ as $[F(x)]_a^b$, and so $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$.

For continuous functions, we can list the following properties of definite integrals:

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, k is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 1 — Self Tutor

Find:

a $\int_1^2 (x^3 - 4x + 5)\,dx$ **b** $\int_1^4 \left(2\sqrt{x} + \frac{3}{x}\right) dx$ **c** $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

a
$$\int_1^2 (x^3 - 4x + 5)\,dx$$
$$= \left[\tfrac{1}{4}x^4 - 2x^2 + 5x\right]_1^2$$
$$= \left(\tfrac{2^4}{4} - 2(2)^2 + 5(2)\right) - \left(\tfrac{1^4}{4} - 2(1)^2 + 5(1)\right)$$
$$= 6 - 3\tfrac{1}{4}$$
$$= \tfrac{11}{4}$$

We omit the constant of integration c for definite integrals.



b
$$\int_1^4 \left(2\sqrt{x} + \frac{3}{x}\right) dx$$
$$= \left[\tfrac{4}{3}x^{\frac{3}{2}} + 3\ln|x|\right]_1^4$$
$$= (\tfrac{4}{3} \times 8 + 3\ln 4) - (\tfrac{4}{3} + 3\ln 1)$$
$$= \tfrac{28}{3} + 6\ln 2$$

c
$$\int_0^{\frac{\pi}{3}} \sin 2x\,dx = \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$$
$$= (-\tfrac{1}{2}\cos\tfrac{2\pi}{3}) - (-\tfrac{1}{2}\cos 0)$$
$$= -\tfrac{1}{2}(-\tfrac{1}{2}) + \tfrac{1}{2}$$
$$= \tfrac{3}{4}$$

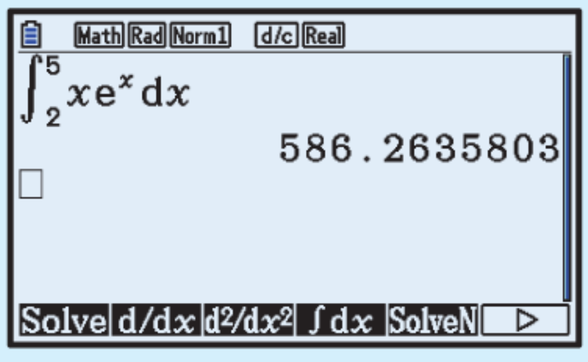
Some definite integrals are difficult or even impossible to evaluate analytically. In these cases you are expected to use a graphics calculator to evaluate the integral.



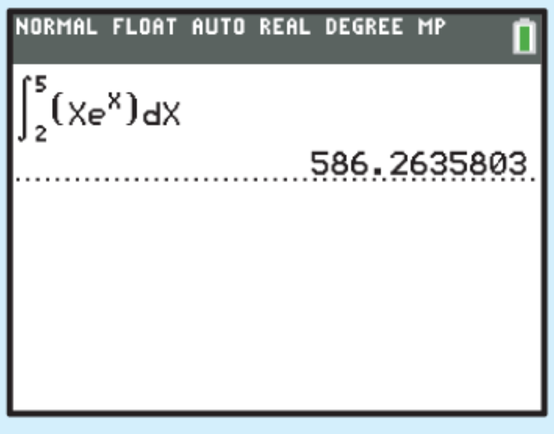
Example 2 — Self Tutor

Evaluate $\int_2^5 xe^x\,dx$ correct to 4 significant figures.

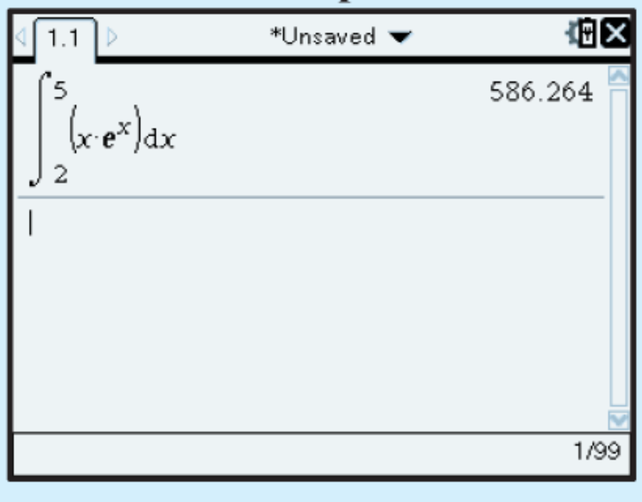
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



$$\int_2^5 xe^x\,dx \approx 586.3$$

EXERCISE 22A

1 Find:

a $\int_1^4 \sqrt{x}\,dx$ and $\int_1^4 (-\sqrt{x})\,dx$

b $\int_0^1 x^7\,dx$ and $\int_0^1 (-x^7)\,dx$

Use questions **1** to **4** to check the properties of definite integrals.



2 Find:

a $\int_0^1 x^2\,dx$ **b** $\int_1^2 x^2\,dx$ **c** $\int_0^2 x^2\,dx$ **d** $\int_0^1 3x^2\,dx$

3 Find:

a $\int_0^2 (x^3 - 4x)\,dx$ **b** $\int_2^3 (x^3 - 4x)\,dx$ **c** $\int_0^3 (x^3 - 4x)\,dx$

4 Find:

a $\int_0^1 x^2\,dx$ **b** $\int_0^1 \sqrt{x}\,dx$ **c** $\int_0^1 (x^2 + \sqrt{x})\,dx$

5 Evaluate:

a $\int_0^1 x^3\,dx$ **b** $\int_0^2 (x^2 - x)\,dx$ **c** $\int_0^2 (3x^2 - x + 6)\,dx$

d $\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$ **e** $\int_1^4 (x + 2\sqrt{x})\,dx$ **f** $\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

g $\int_1^3 \frac{1}{x^2}\,dx$ **h** $\int_1^2 (x+3)^2\,dx$ **i** $\int_1^4 \left(x^2 + \frac{1}{x}\right) dx$

6 Find:

a $\int_0^1 (3x+1)^4\,dx$ **b** $\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **c** $\int_{-3}^0 \sqrt{1-x}\,dx$

7 Evaluate:

a $\int_0^1 e^x\,dx$ **b** $\int_0^3 (2e^x - 3)\,dx$ **c** $\int_0^2 e^{3x}\,dx$

d $\int_0^1 e^{1-x}\,dx$ **e** $\int_0^{\ln 4} e^x(e^x - 2)\,dx$ **f** $\int_1^2 (e^{-x} + 1)^2\,dx$

8 Evaluate:

a $\int_0^{\frac{\pi}{6}} \cos x\,dx$ **b** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$

d $\int_0^{\frac{\pi}{6}} \sin 3x\,dx$ **e** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\left(x - \frac{\pi}{3}\right) dx$ **f** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(2x - \frac{\pi}{4}\right) dx$

g $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$ **h** $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ **i** $\int_0^{\frac{\pi}{6}} (\sin 3x - \cos x)\,dx$

9 Find:

a $\int_{-6}^{-2} \frac{1}{x}\,dx$ **b** $\int_{-1}^{5} \frac{1}{x+4}\,dx$ **c** $\int_{1}^{8} \frac{2}{3x+4}\,dx$ **d** $\int_{-4}^{0} \frac{4}{5-2x}\,dx$

10 Show that $\frac{4x+1}{x-1}$ may be written in the form $4 + \frac{5}{x-1}$.

Hence show that $\int_{3}^{5} \frac{4x+1}{x-1}\,dx = 8 + 5\ln 2$.

11 Find m such that:

a $\int_{m}^{2m} (2x-1)\,dx = 4$ **b** $\int_{m}^{-2} \frac{1}{4-x}\,dx = \ln\left(\frac{3}{2}\right)$

12 Evaluate:

a $\int_{2}^{4} 2^x\,dx$ **b** $\int_{-1}^{1} \frac{3}{x^2+1}\,dx$ **c** $\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

d $\int_{2}^{3} \frac{1}{x^2-1}\,dx$ **e** $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (\cos 2x - \sin x)^2\,dx$ **f** $\int_{0}^{2} \frac{-6}{\sqrt{4-x^2}}\,dx$

13 Find:

a $\int_{0}^{1} \frac{x+4}{x^2-5x+6}\,dx$ **b** $\int_{-2}^{3} \frac{3x-8}{x^2-x-42}\,dx$ **c** $\int_{1}^{4} \frac{2x+12}{3x^2+4x-4}\,dx$

14 Use integration by parts to evaluate:

a $\int_{0}^{1} (-xe^{-x})\,dx$ **b** $\int_{0}^{\frac{\pi}{2}} x\sin x\,dx$ **c** $\int_{1}^{3} \ln x\,dx$ **d** $\int_{1}^{e} x\ln x^2\,dx$

15 Evaluate using technology:

a $\int_{-1}^{1} e^{-x^2}\,dx$ **b** $\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x})\,dx$ **c** $\int_{0}^{1} x^x\,dx$

16 **a** Use integration by parts to find $\int_{0}^{\frac{\pi}{4}} x\sec^2 x\,dx$.

b **i** Show that if $\int_{0}^{a} x\sec^2 x\,dx = \frac{1}{2}$, $0 < a < \frac{\pi}{2}$ then a is a solution of $a\tan a + \ln(\cos a) = \frac{1}{2}$.

ii Hence find a using technology.

c Solve $\int_{0}^{a} x\sec^2 x = \frac{3}{4}$, $0 < x < \frac{\pi}{2}$ directly using technology.



17 Use the Fundamental Theorem of Calculus to prove that for continuous functions:

a $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$ **b** $\int_{a}^{b} k\,f(x)\,dx = k\int_{a}^{b} f(x)\,dx$, k is any constant

c $\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx = \int_{a}^{c} f(x)\,dx$

d $\int_{a}^{b} [f(x)+g(x)]\,dx = \int_{a}^{b} f(x)\,dx + \int_{a}^{b} g(x)\,dx$

18 Write as a single integral:

a $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$ **b** $\int_4^5 f(x)\,dx - \int_6^5 f(x)\,dx$

c $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$

19 **a** If $\int_1^3 f(x)\,dx = 2$ and $\int_1^6 f(x)\,dx = -3$, find $\int_3^6 f(x)\,dx$.

b If $\int_0^2 f(x)\,dx = 5$, $\int_4^6 f(x)\,dx = -2$, and $\int_0^6 f(x)\,dx = 7$, find $\int_2^4 f(x)\,dx$.

20 Suppose f is a function which is continuous on the interval $-a \leqslant x \leqslant a$. Prove that:

a if f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

b if f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx = 2\int_{-a}^{0} f(x)\,dx$.

21 Suppose $\int_{-1}^{1} f(x)\,dx = -4$. Find the value of:

a $\int_1^{-1} f(x)\,dx$ **b** $\int_{-1}^{1} (2 + f(x))\,dx$ **c** $\int_{-1}^{1} 2f(x)\,dx$

d k such that $\int_{-1}^{1} k\,f(x)\,dx = 7$.

22 If $g(2) = 4$ and $g(3) = 5$, calculate $\int_2^3 (g'(x) - 1)\,dx$.

B DEFINITE INTEGRALS INVOLVING SUBSTITUTION

When we evaluate a definite integral using substitution, we need to make sure the endpoints are converted to the new variable.

Example 3 Self Tutor

Evaluate: **a** $\int_2^3 \frac{x}{x^2-1}\,dx$ **b** $\int_0^1 \frac{6x}{(x^2+1)^3}\,dx$

a Let $u = x^2 - 1$ $\therefore \frac{du}{dx} = 2x$

When $x = 2$, $u = 2^2 - 1 = 3$

When $x = 3$, $u = 3^2 - 1 = 8$

$$\therefore \int_2^3 \frac{x}{x^2-1}\,dx = \int_2^3 \frac{1}{u}\left(\frac{1}{2}\frac{du}{dx}\right)dx$$

$$= \frac{1}{2}\int_3^8 \frac{1}{u}\,du$$

$$= \tfrac{1}{2}\Big[\ln|u|\Big]_3^8$$

$$= \tfrac{1}{2}(\ln 8 - \ln 3)$$

$$= \tfrac{1}{2}\ln\left(\tfrac{8}{3}\right)$$

b Let $u = x^2 + 1$ $\therefore$ $\dfrac{du}{dx} = 2x$

When $x = 0$, $u = 1$

When $x = 1$, $u = 2$

$$\therefore \int_0^1 \frac{6x}{(x^2+1)^3}\,dx = \int_0^1 \frac{1}{u^3}\left(3\frac{du}{dx}\right)dx$$

$$= 3\int_1^2 u^{-3}\,du$$

$$= 3\left[\frac{u^{-2}}{-2}\right]_1^2$$

$$= 3\left(\frac{2^{-2}}{-2} - \frac{1^{-2}}{-2}\right)$$

$$= \frac{9}{8}$$

EXERCISE 22B

1 Evaluate:

a $\displaystyle\int_1^2 2x(x^2-1)^3\,dx$

b $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\,dx$

c $\displaystyle\int_0^1 x^2 e^{x^3+1}\,dx$

d $\displaystyle\int_0^3 x\sqrt{x^2+16}\,dx$

e $\displaystyle\int_1^2 xe^{-2x^2}\,dx$

f $\displaystyle\int_2^3 \frac{x}{2-x^2}\,dx$

g $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

h $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx$

i $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$



Example 4 — Self Tutor

Evaluate: $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx$

Let $u = \sin x$ $\therefore$ $\dfrac{du}{dx} = \cos x$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{u}\,\frac{du}{dx}\,dx$$

When $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

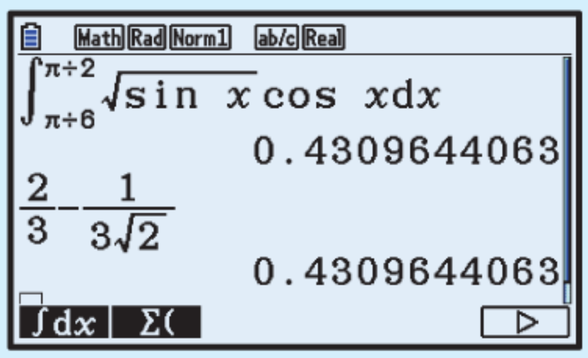
When $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$$= \int_{\frac{1}{2}}^1 u^{\frac{1}{2}}\,du$$

$$= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{2}}^1$$

$$= \tfrac{2}{3}(1)^{\frac{3}{2}} - \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{\frac{3}{2}}$$

$$= \frac{2}{3} - \frac{1}{3\sqrt{2}}$$

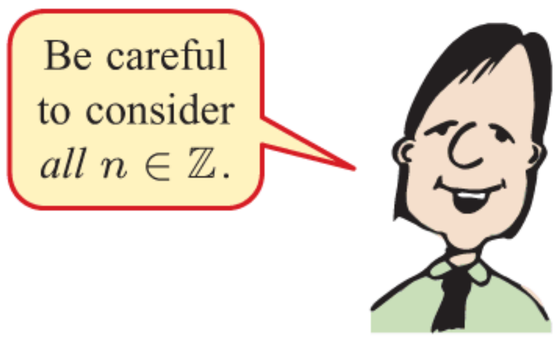


2 Evaluate:

a $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx$ **b** $\int_0^{\frac{\pi}{6}} \sin^2 x \cos x\,dx$ **c** $\int_0^{\frac{\pi}{4}} \tan x\,dx$

d $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx$ **e** $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin x}\,dx$ **f** $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x\,dx$

3 Evaluate for $n \in \mathbb{Z}$: $\int_0^1 (x^2+2x)^n(x+1)\,dx$



Example 5 — Self Tutor

Find the exact value of $\int_{-4}^{6} x\sqrt{x+4}\,dx$.

Let $u = x+4$ $\therefore \frac{du}{dx} = 1$

When $x = -4$, $u = 0$

When $x = 6$, $u = 10$

$\therefore \int_{-4}^{6} x\sqrt{x+4}\,dx = \int_0^{10} (u-4)\sqrt{u}\,du$

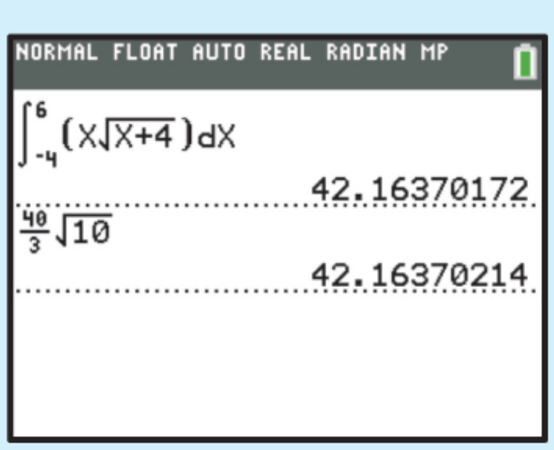
$= \int_0^{10} (u^{\frac{3}{2}} - 4u^{\frac{1}{2}})\,du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_0^{10}$

$= \frac{2}{5} \times 10^{\frac{5}{2}} - \frac{8}{3} \times 10^{\frac{3}{2}}$

$= 40\sqrt{10} - \frac{80}{3}\sqrt{10}$

$= \frac{40}{3}\sqrt{10}$



4 Find the exact value of:

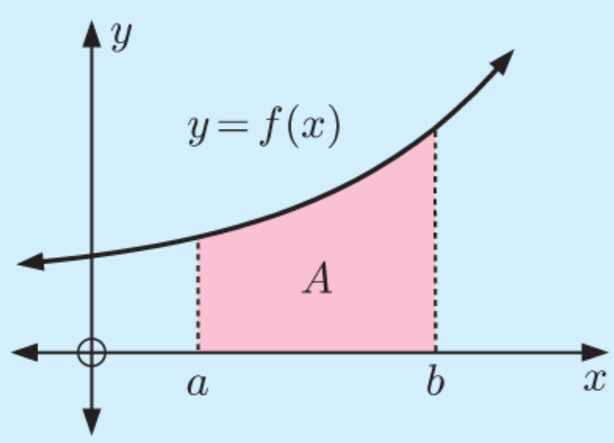
a $\int_3^4 x\sqrt{x-1}\,dx$ **b** $\int_0^3 x\sqrt{x+6}\,dx$ **c** $\int_2^5 x^2\sqrt{x-2}\,dx$

Check your answers using technology.

C THE AREA UNDER A CURVE

We have already established in **Chapter 20** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

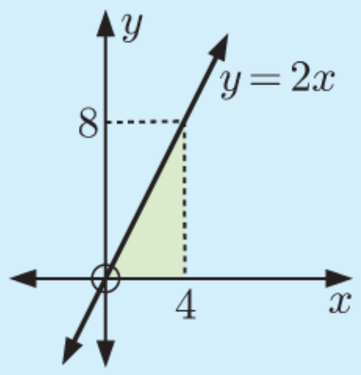


Example 6 — Self Tutor

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$, and $x = 4$ using:

a a geometric argument **b** integration.

a

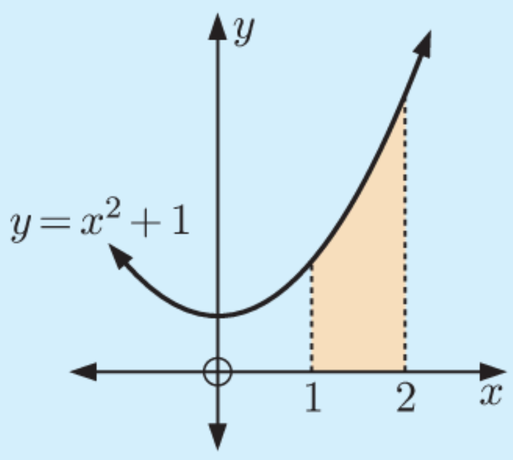


$$\text{Area} = \tfrac{1}{2} \times 4 \times 8$$
$$= 16 \text{ units}^2$$

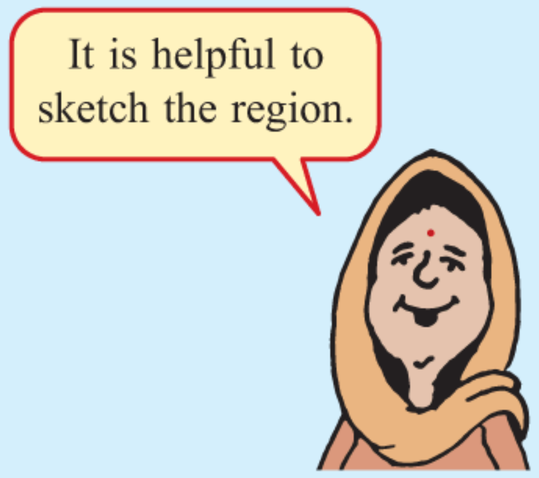
b
$$\text{Area} = \int_0^4 2x \, dx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0^2$$
$$= 16 \text{ units}^2$$

Example 7 — Self Tutor

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$, and $x = 2$.

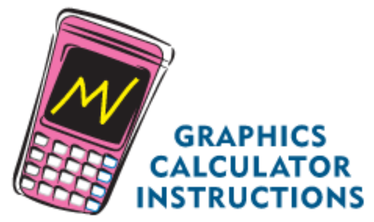


$$\text{Area} = \int_1^2 (x^2 + 1) \, dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$



We can check this result using technology.



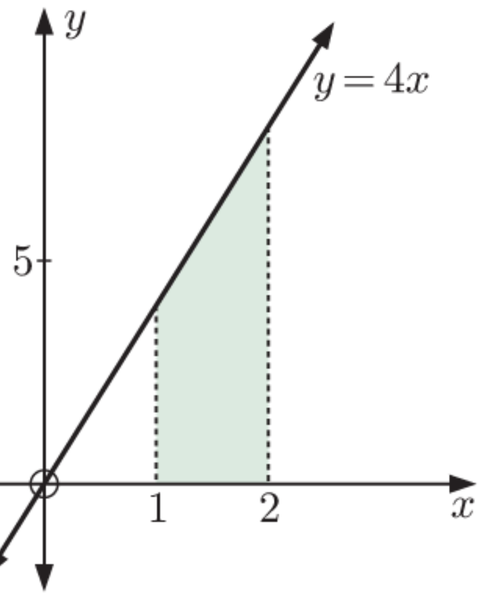


EXERCISE 22C

1 Find the shaded area using:

a a geometric argument

b integration.



2 Find the area of each region described below using:

i a geometric argument **ii** integration

a the region enclosed by $y = 5$, the x-axis, $x = -6$, and $x = 0$

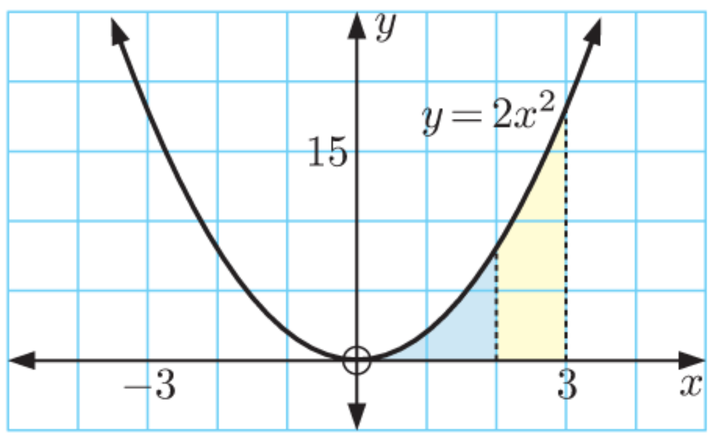
b the region enclosed by $y = x$, the x-axis, $x = 4$, and $x = 5$

c the region enclosed by $y = -3x$, the x-axis, $x = -3$, and $x = 0$.

3 Find the exact area of:

a the blue shaded region

b the yellow shaded region.



4 Find the area of the region bounded by:

a $y = x^2$, the x-axis, and $x = 1$

b $y = x^3$, the x-axis, $x = 1$, and $x = 4$

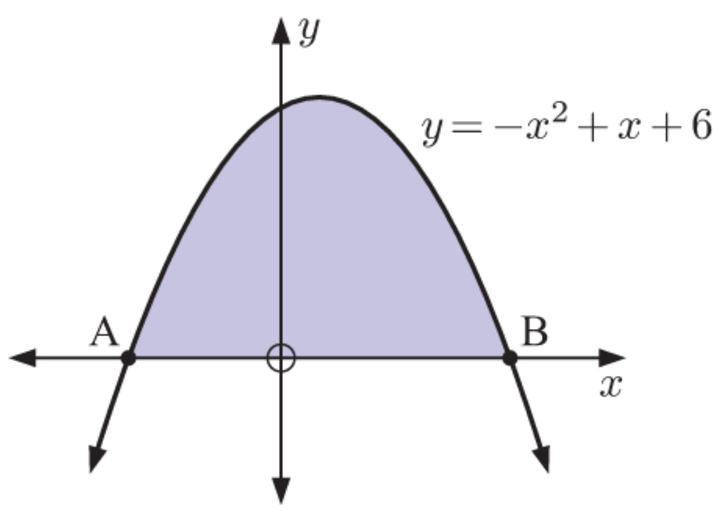
c $y = \frac{1}{2}x^2 - 1$, the x-axis, $x = 2$, and $x = 3$

d $x = \sqrt{y-2}$, the axes, and $x = 2$.

5 The graph of $y = -x^2 + x + 6$ is shown alongside.

a Find the coordinates of A and B.

b Hence find the shaded area.



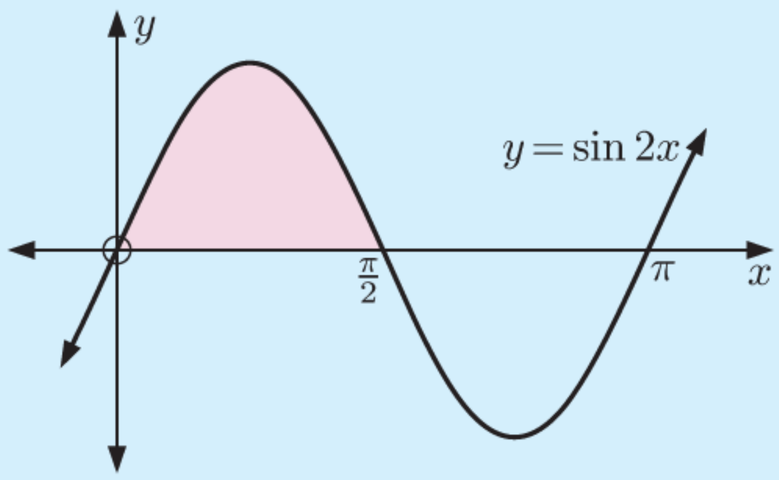
6 Find the area enclosed by each curve and the x-axis:

a $y = -x^2 + 7x - 10$ **b** $y = -2x^2 + 2x + 4$ **c** $y = 3 - x^2$

Example 8 **Self Tutor**

Find the area enclosed by one arch of the curve $y = \sin 2x$ and the x-axis.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, so the first positive x-intercept is $\frac{\pi}{2}$.



$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x \, dx$$

$$= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$$

$$= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= -\tfrac{1}{2}(\cos \pi - \cos 0)$$

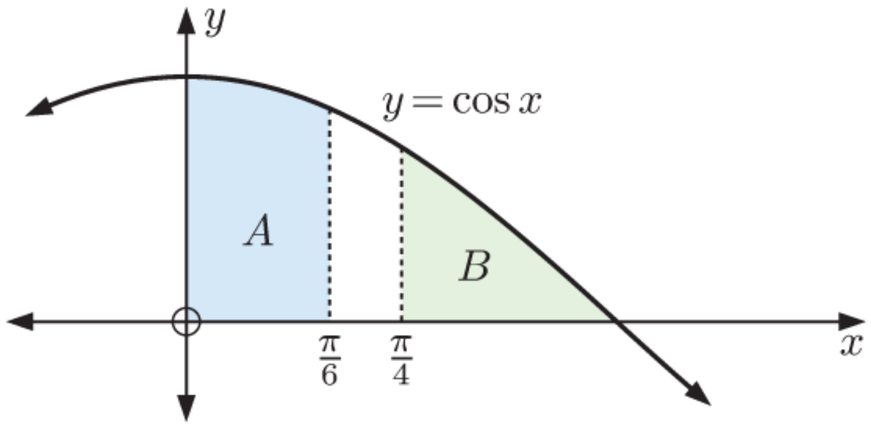
$$= 1 \text{ unit}^2$$

7 Show that the area enclosed by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ is 2 units^2.

8 Find the area of the region bounded by $y = \cos x$, the x-axis, $x = 0$, and $x = \frac{\pi}{2}$.

9 Find the area enclosed by one arch of the curve $y = \cos 3x$ and the x-axis.

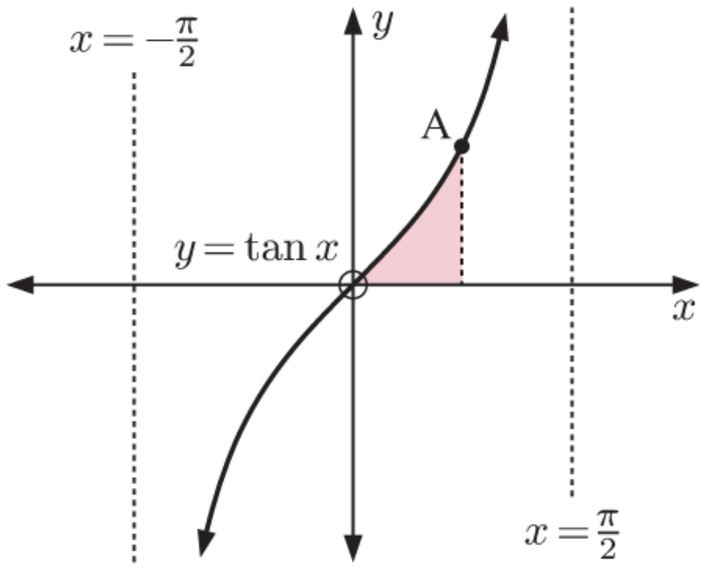
10



a Which of the shaded regions appears to be larger?

b Calculate the area of each region, and check your answer to **a**.

11



The graph alongside shows $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
A is the point on the graph with y-coordinate 1.

a Find the x-coordinate of A.

b Find $\int \tan x \, dx$.

c Hence find the shaded area.

12 Find the area of the region bounded by:

a $y = \dfrac{1}{x^2}$, the x-axis, $x = 1$, and $x = 2$

b $y = e^x$, the axes, and $x = 1$

c $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis, and $x = 4$

d the axes and $y = \sqrt{9 - x}$

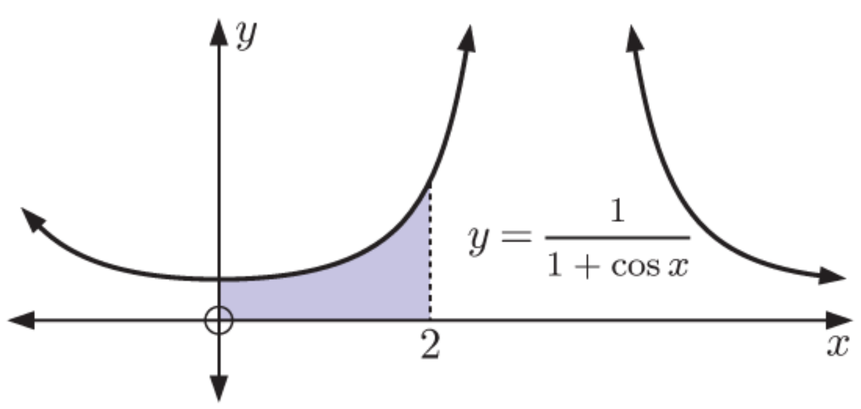
e $y = e^x + e^{-x}$, the x-axis, $x = -1$, and $x = 1$

f $y = 3^x$, the x-axis, $x = 0$, and $x = \ln 3$

g $y = \dfrac{1}{\sqrt{4 - x^2}}$, the x-axis, $x = 0$, and $x = 1$

h $y = \dfrac{8}{x^2 + 16}$, the x-axis, $x = -2$, and $x = 2$.

13



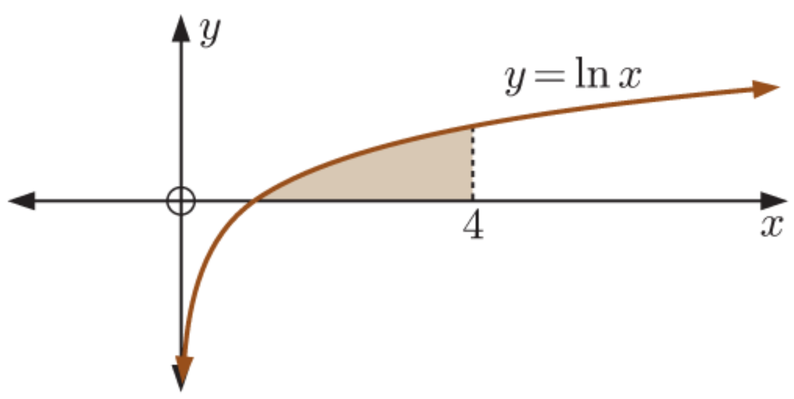
a Show that $\dfrac{d}{dx}\left(\dfrac{\sin x}{1 + \cos x}\right) = \dfrac{1}{1 + \cos x}$.

b Hence find the area of the region bounded by $y = \dfrac{1}{1 + \cos x}$, the x-axis, $x = 0$, and $x = 2$.

14 **a** Use integration by parts to show that

$$\int \ln x \, dx = x \ln x - x + c.$$

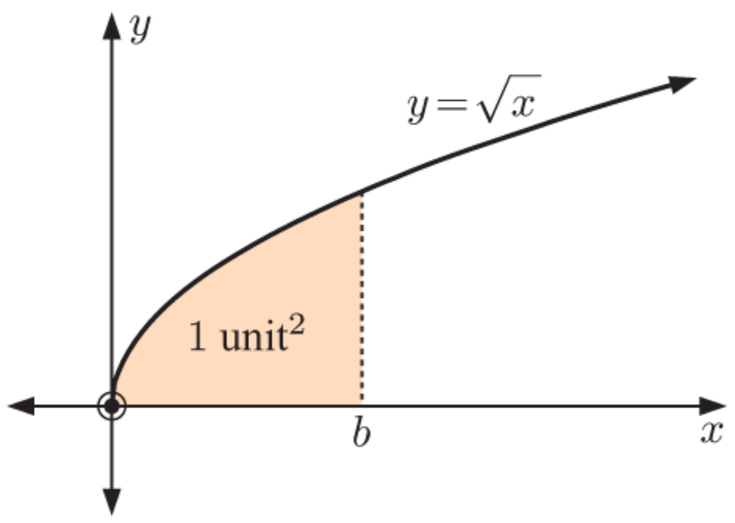
b Hence find the shaded area.



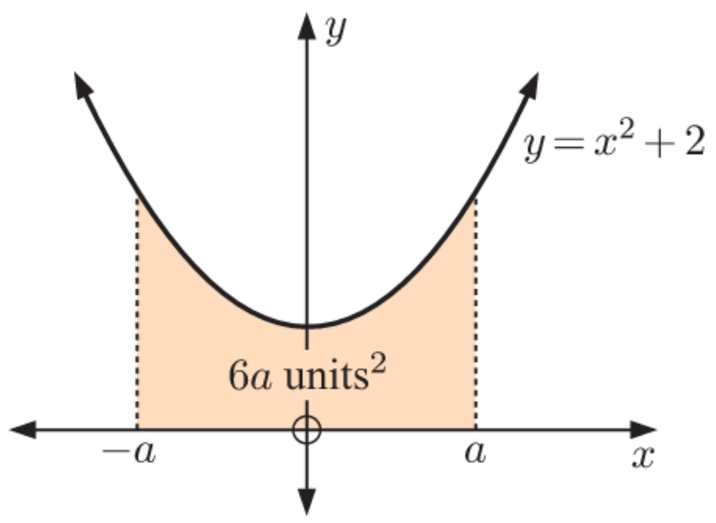
15 **a** Use integration by parts to find $\int x \sin x \, dx$.

b Hence find the exact area of the region bounded by $y = x \sin x$, the x-axis, $x = 1$, and $x = \frac{\pi}{2}$.

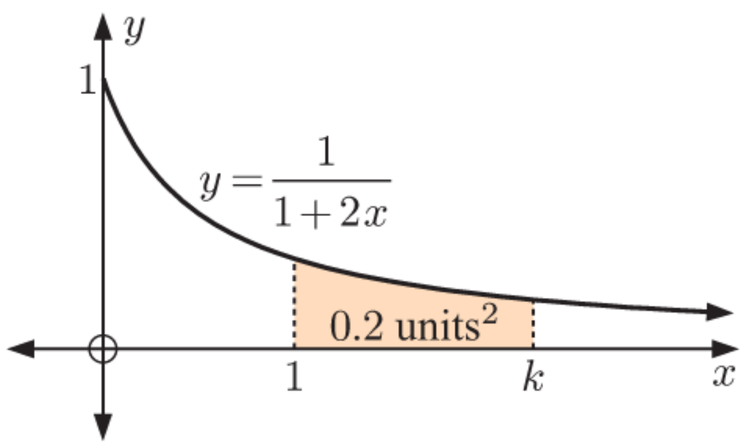
16 **a** Find b, correct to 4 decimal places.



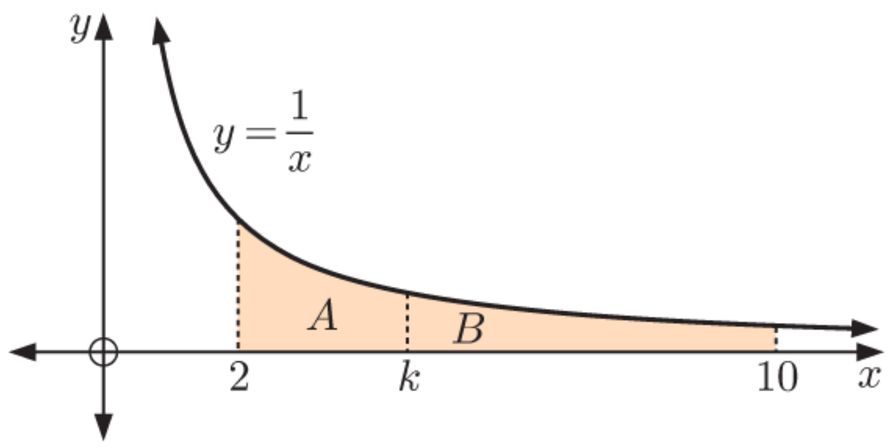
b Find the exact value of a.



c Find k, correct to 4 decimal places.



d Find k such that area A and area B are equal.



INVESTIGATION 1 $\int_a^b f(x)\,dx$ AND AREAS

Can $\int_a^b f(x)\,dx$ always be interpreted as an area?

What to do:

1 **a** Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.

b Using a graph, explain why the first integral in **a** gives an area, whereas the second integral does not.

c Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.

d Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

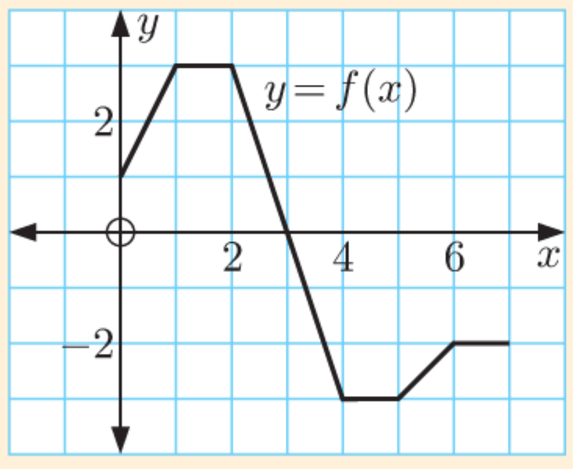
e Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.

2 Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Write an expression for the area between the function and the x-axis for $a \leqslant x \leqslant b$.

3 Evaluate using area interpretation:

a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

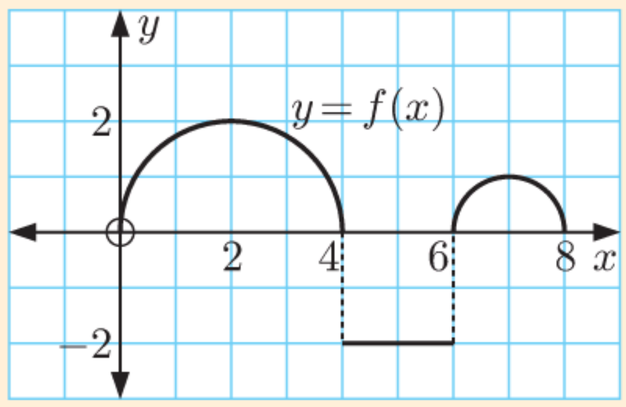
c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$



4 Evaluate using area interpretation:

a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

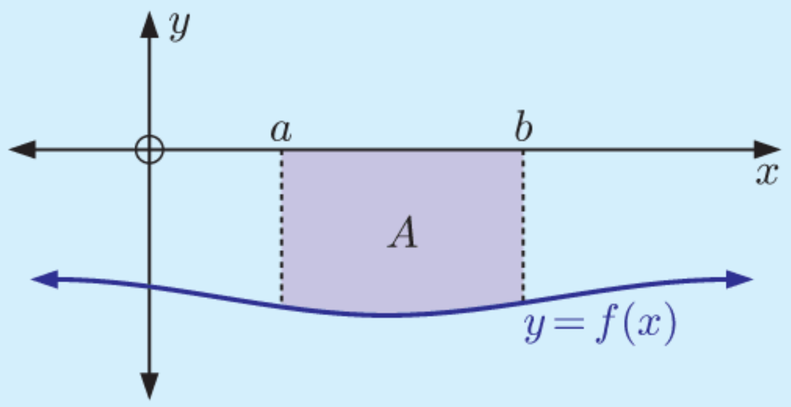
c $\int_6^8 f(x)\,dx$ **d** $\int_0^8 f(x)\,dx$



D THE AREA ABOVE A CURVE

If $f(x)$ is negative and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = -\int_a^b f(x)\,dx \quad \text{or} \quad -\int_a^b y\,dx.$$

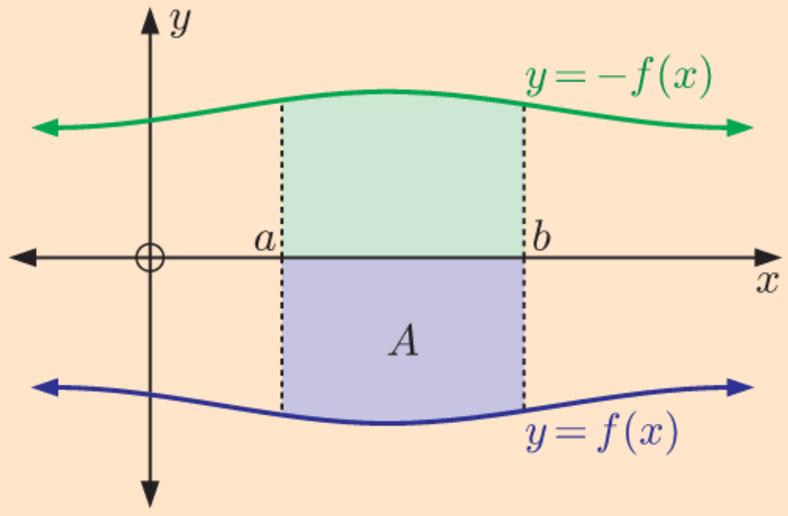


Proof:

The reflection of $y = f(x)$ in the x-axis is $y = -f(x)$.

By symmetry, $-f(x) \geqslant 0$ on the interval $a \leqslant x \leqslant b$, and the area bounded by $y = -f(x)$, the x-axis, $x = a$, and $x = b$ is also A.

$$\therefore \quad A = \int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$$



Example 9 **Self Tutor**

Find the area bounded by the x-axis and $y = x^2 - 2x$.

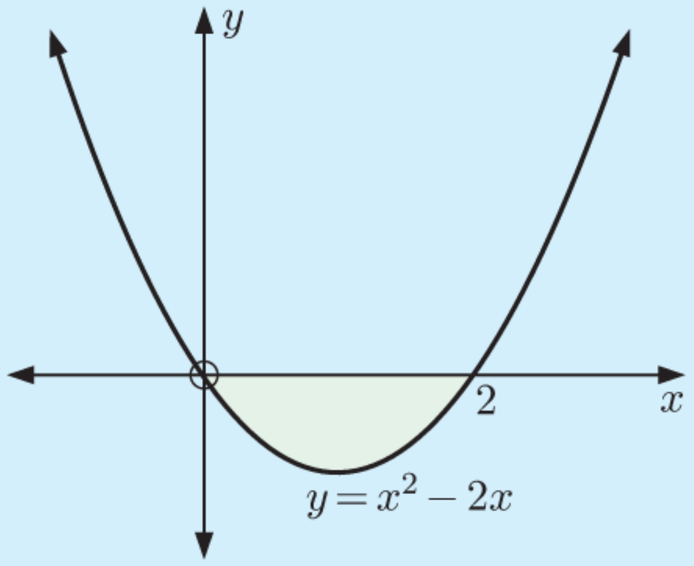
The curve cuts the x-axis when $y = 0$

$\therefore \quad x^2 - 2x = 0$

$\therefore \quad x(x - 2) = 0$

$\therefore \quad x = 0$ or 2

$\therefore$ the x-intercepts are 0 and 2.



$$\text{Area} = -\int_0^2 (x^2 - 2x)\,dx$$

$$= -\left[\frac{x^3}{3} - x^2\right]_0^2$$

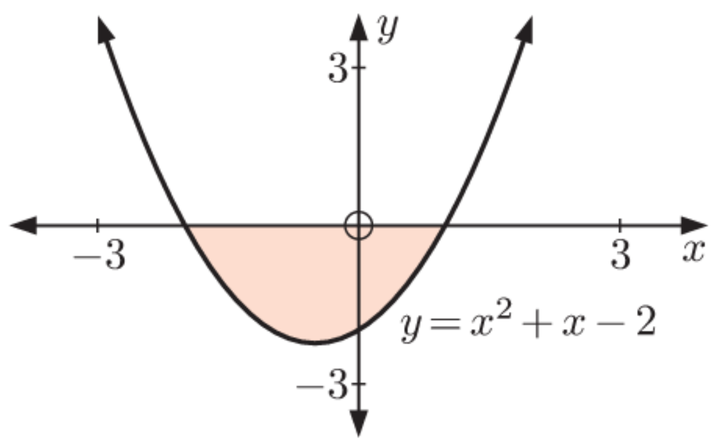
$$= -\left[\left(\tfrac{8}{3} - 4\right) - (0)\right] = \tfrac{4}{3}$$

$\therefore$ the area is $\frac{4}{3}$ units^2.

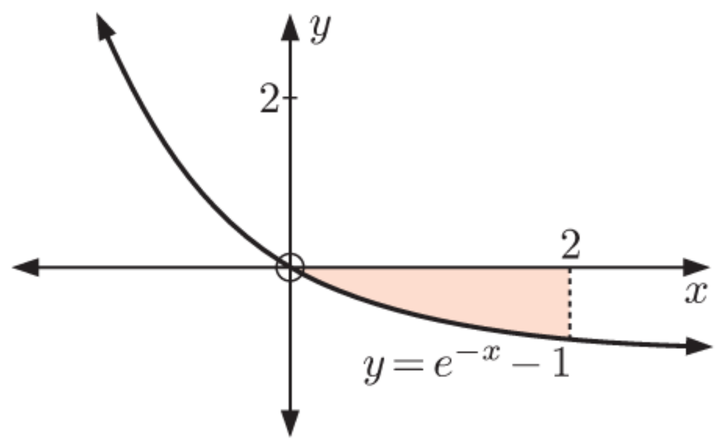
EXERCISE 22D

1 Find the shaded area:

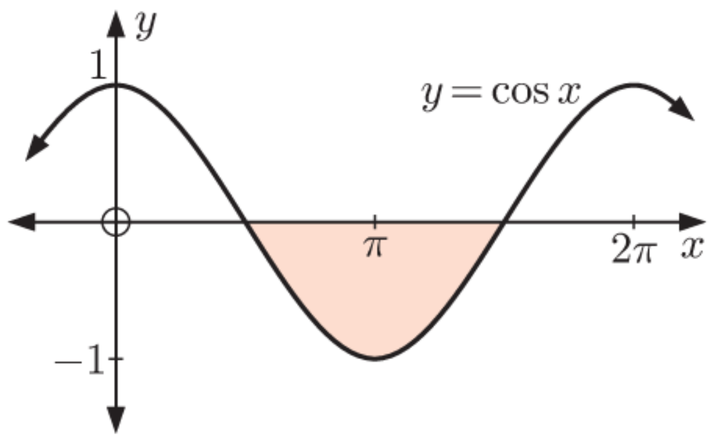
a



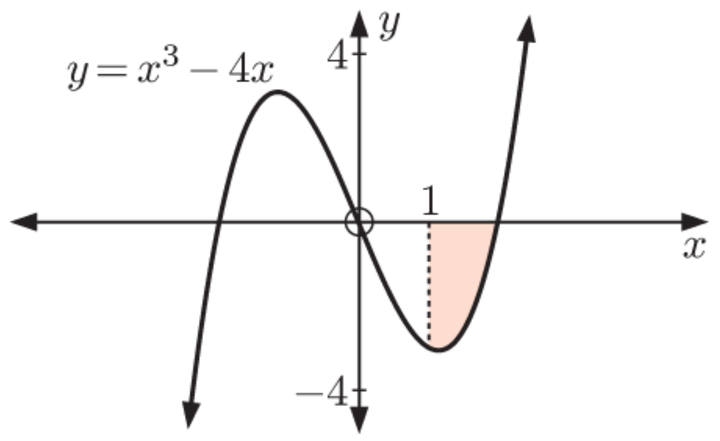
b



c



d

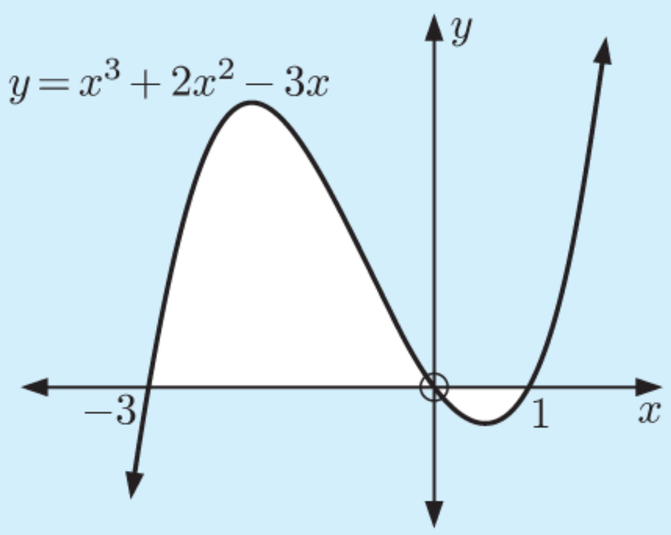


Example 10 **Self Tutor**

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x-1)(x+3) \end{aligned}$$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 1, and -3.



Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\,dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}$$

$$= \left(0 - \left(-11\tfrac{1}{4}\right)\right) - \left(-\tfrac{7}{12} - 0\right)$$

$$= 11\tfrac{5}{6} \text{ units}^2$$

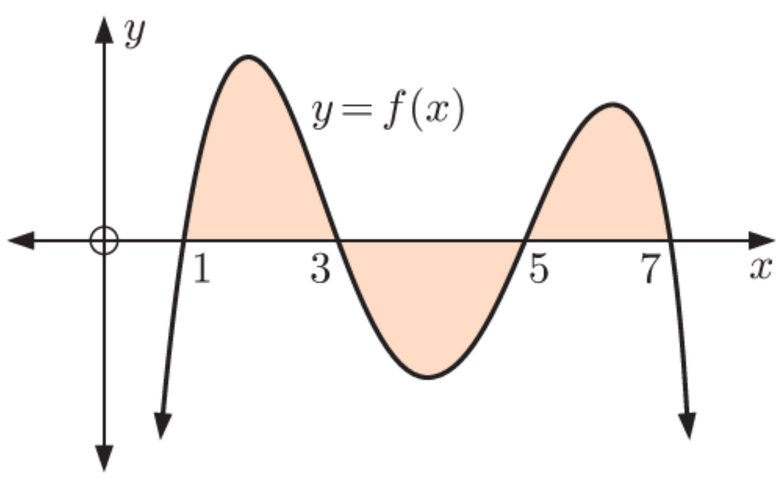


2 Find the total area enclosed by the function $y = f(x)$ and the x-axis for:

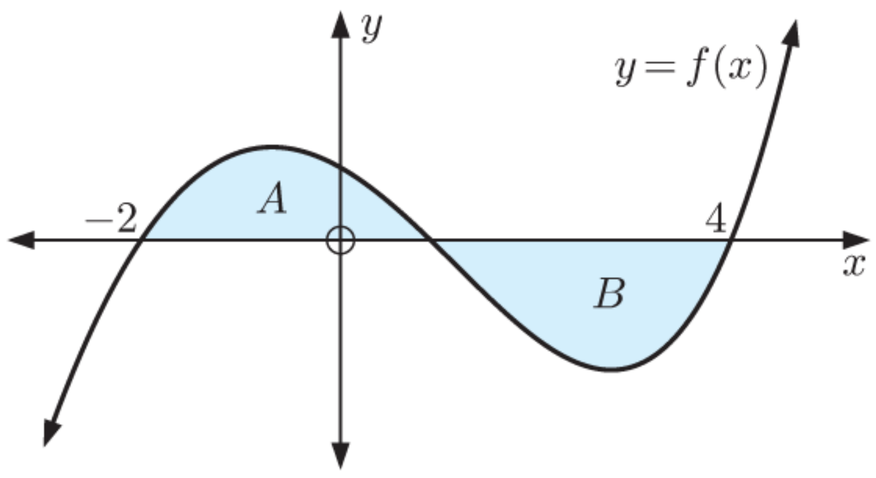
a $f(x) = x^3 - 9x$ **b** $f(x) = -x(x-2)(x-4)$ **c** $f(x) = x^4 - 5x^2 + 4$.

3 **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\,dx$.

b Write an expression for the total shaded area in terms of integrals.



4



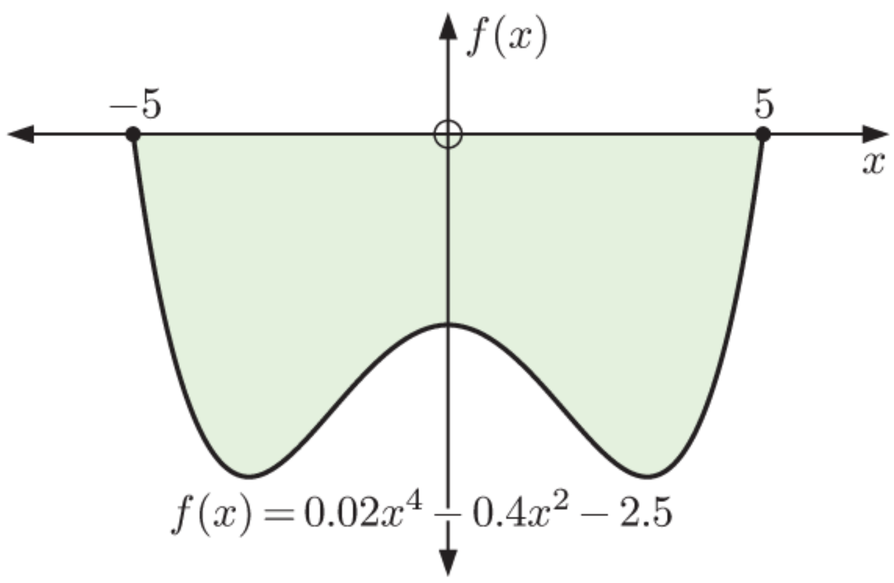
For the graph of $y = f(x)$ alongside, $\int_{-2}^{4} f(x)\,dx = -6$. Which region is larger, A or B? Explain your answer.

5 The area of the region bounded by $f(x) = -\frac{9}{x}$, the x-axis, $x = 3$, and $x = k$, is $9\ln 2$ units^2. Find the value of k.

6 **a** Sketch the graph of $y = 2\sin x + 1$ for $0 \leqslant x \leqslant 2\pi$.

b Find the area between the x-axis and the part of $y = 2\sin x + 1$ that is below the x-axis on $0 \leqslant x \leqslant 2\pi$.

7

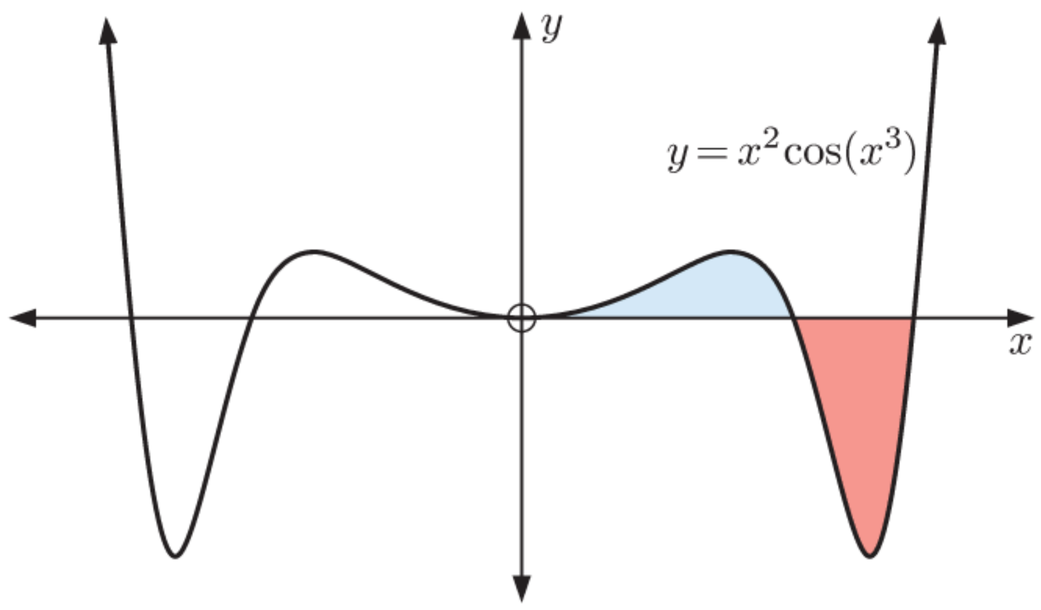


The cross-section of a roof gutter is defined by $f(x) = 0.02x^4 - 0.4x^2 - 2.5$, for $-5 \leqslant x \leqslant 5$ cm.

a Find the cross-sectional area of the gutter.

b The gutter is 20 m long. How much water can it hold in total?

8 Show that the red shaded region is twice as large as the blue shaded region.

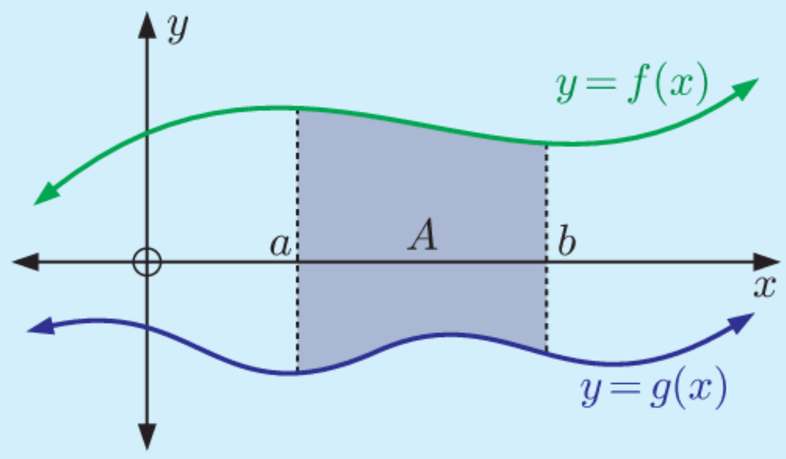


E THE AREA BETWEEN TWO FUNCTIONS

Consider two functions $f(x)$ and $g(x)$ where $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$.

The area between the two functions on the interval $a \leqslant x \leqslant b$ is given by

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



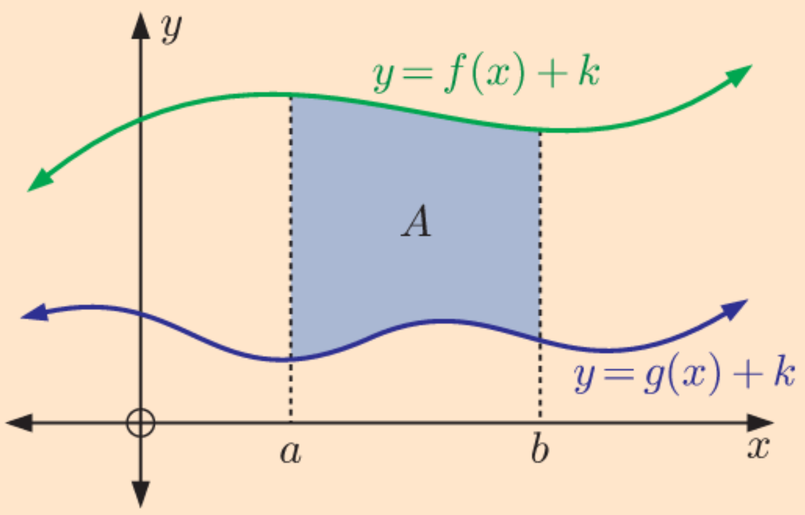
Proof:

If necessary, we translate each curve k units upwards until they are both above the x-axis on the interval $a \leqslant x \leqslant b$.

$\therefore \quad A = (\text{area below } \ y = f(x) + k) - (\text{area below } \ y = g(x) + k)$

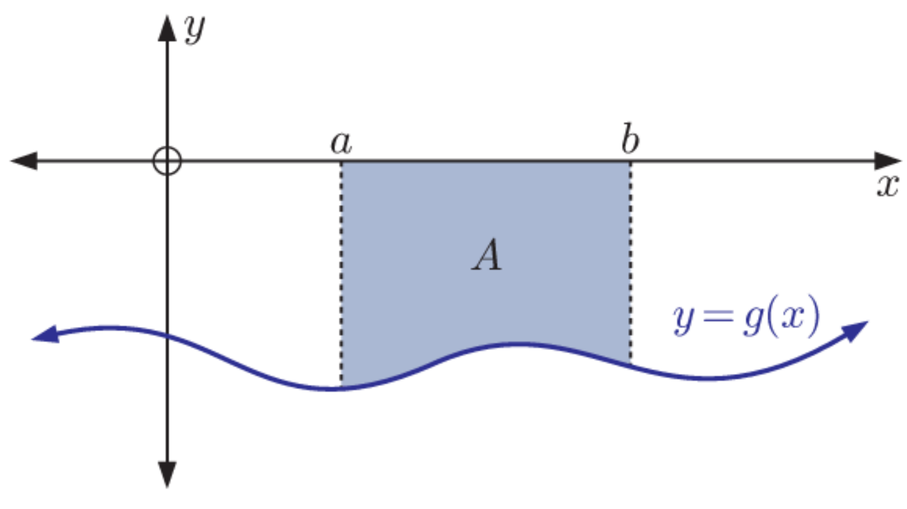
$$= \int_a^b [f(x) + k]\,dx - \int_a^b [g(x) + k]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$



Notice that if $y = g(x)$ is negative on the interval $a \leqslant x \leqslant b$, we can let $f(x) = 0$ and hence derive the formula we saw in the last Section:

$$A = -\int_a^b g(x)\,dx$$



Example 11 — **Self Tutor**

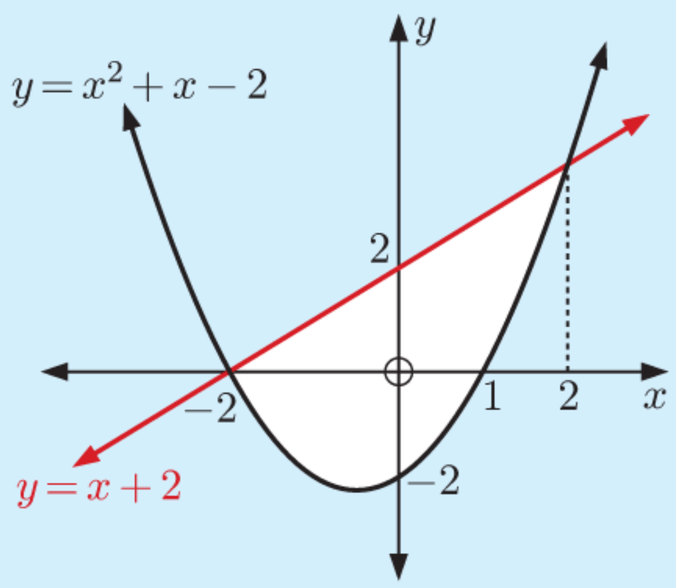
Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

$y = x + 2$ meets $y = x^2 + x - 2$ where $x^2 + x - 2 = x + 2$

$$\therefore \quad x^2 - 4 = 0$$

$$\therefore \quad (x + 2)(x - 2) = 0$$

$$\therefore \quad x = \pm 2$$



Since $x + 2 \geqslant x^2 + x - 2$ on the interval $-2 \leqslant x \leqslant 2$,

$$\text{area} = \int_{-2}^{2} [(x + 2) - (x^2 + x - 2)]\,dx$$

$$= \int_{-2}^{2} (4 - x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$

$$= 10\tfrac{2}{3} \text{ units}^2$$

$\therefore$ the area is $10\frac{2}{3}$ units^2.

EXERCISE 22E

1 Find the area of the region enclosed by:

a $y = x^2 - 2x$ and $y = 3$

b $y = x - 3$ and $y = x^2 - 3x$

c $y = 2x$ and $y = 4x^2$

d $y = \sqrt{x}$ and $y = x^2$

2 Find the area of the region bounded by $y = 2e^x$, $y = e^{2x}$, and $x = 0$.

3 Find the area of the region enclosed by:

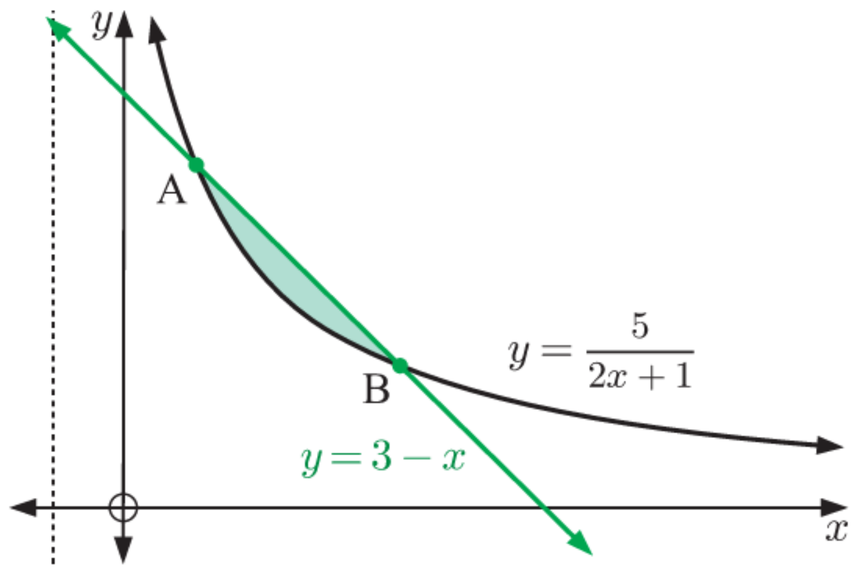
a $y = x^2 - 3x + 4$ and $y = 2x^2 + 3x + 4$

b $y = -\frac{1}{2}x^2 - 2x + 1$ and $y = 2x^2 - 3x - 3$

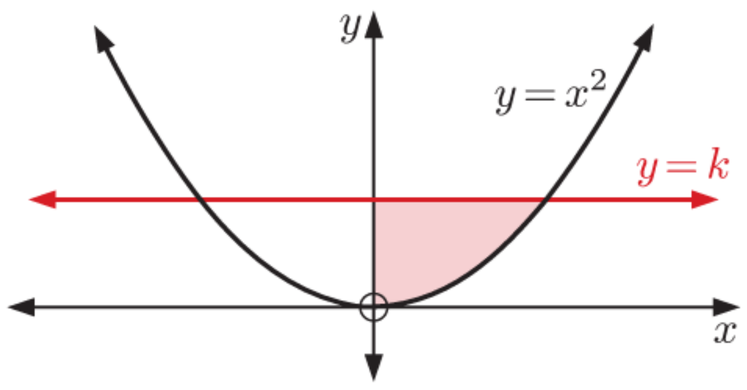
4 The graphs of $y = \dfrac{5}{2x+1}$ and $y = 3 - x$ are shown alongside.

a Find the coordinates of A and B.

b Find the exact value of the shaded area.



5 The shaded area is 2.4 units^2.
Find k, correct to 4 decimal places.

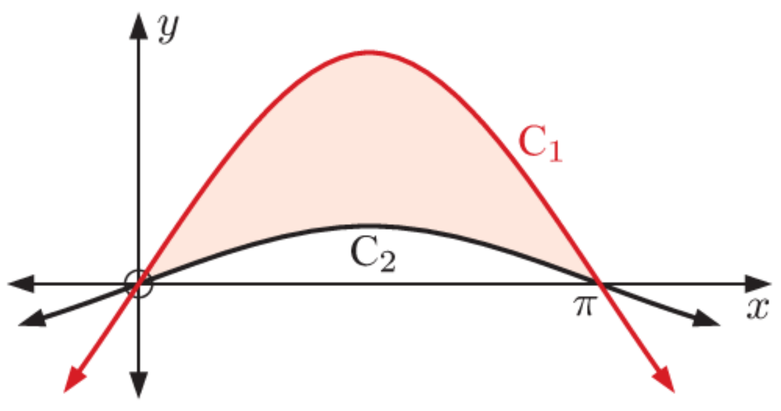


6 A region with $x \geqslant 0$ has boundaries defined by $y = \sin x$, $y = \cos x$, and the y-axis. Sketch the region and find its area.

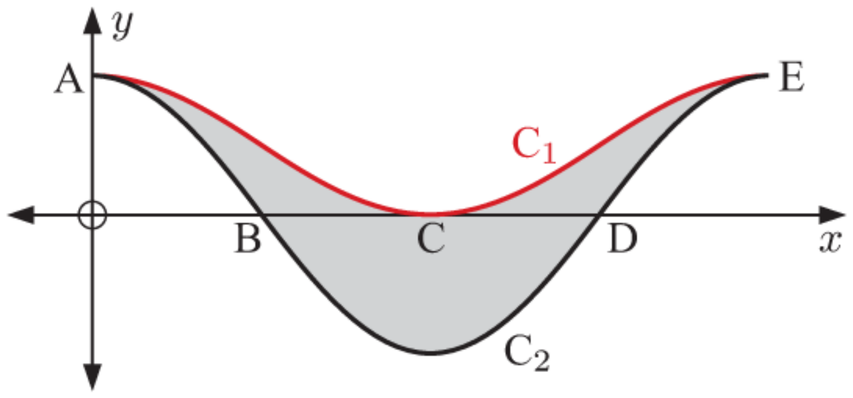
7 The illustrated curves are the graphs of $y = \sin x$ and $y = 4\sin x$.

a Identify each curve.

b Calculate the shaded area.



8



The illustrated curves are $y = \cos 2x$ and $y = \cos^2 x$.

a Identify each curve.

b Determine the coordinates of A, B, C, D, and E.

c Show that the shaded area is $\frac{\pi}{2}$ units^2.

9 **a** On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$. Show all axes intercepts and asymptotes.

b Find the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.

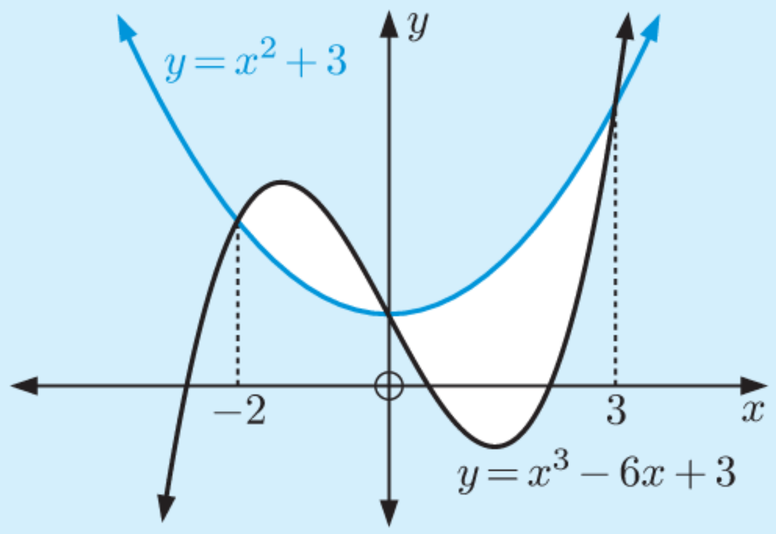
c Find the area of the region enclosed by the two curves.

Example 12 **Self Tutor**

Find the total area of the regions enclosed by $y = x^3 - 6x + 3$ and $y = x^2 + 3$.

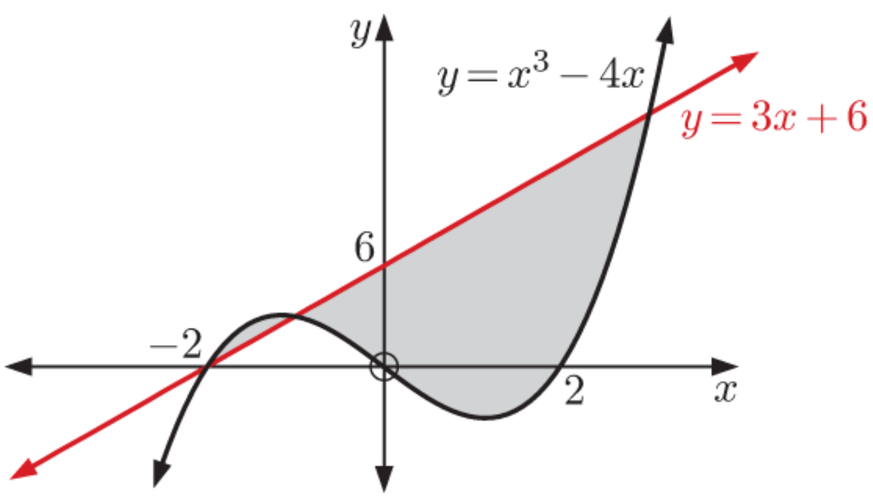
$y = x^3 - 6x + 3$ and $y = x^2 + 3$ meet where

$$x^3 - 6x + 3 = x^2 + 3$$
$$\therefore \quad x^3 - x^2 - 6x = 0$$
$$\therefore \quad x(x^2 - x - 6) = 0$$
$$\therefore \quad x(x+2)(x-3) = 0$$
$$\therefore \quad x = 0,\ -2, \text{ or } 3$$



$$\text{Total area} = \int_{-2}^{0} ((x^3 - 6x + 3) - (x^2 + 3))\,dx + \int_{0}^{3} ((x^2 + 3) - (x^3 - 6x + 3))\,dx$$
$$= \int_{-2}^{0} (x^3 - x^2 - 6x)\,dx + \int_{0}^{3} (-x^3 + x^2 + 6x)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} + \frac{x^3}{3} + 3x^2\right]_{0}^{3}$$
$$= \left(0 - \tfrac{16}{4} + \tfrac{-8}{3} + 12\right) + \left(-\tfrac{81}{4} + \tfrac{27}{3} + 27 - 0\right)$$
$$= 21\tfrac{1}{12} \text{ units}^2$$

10



a Write the shaded area as the sum of two definite integrals.

b Find the total shaded area.

11 **a** Sketch the graphs of $y = x^3 - 5x$ and $y = 2x^2 - 6$ on the same set of axes.

b Find the x-coordinates of their intersection points.

c Hence find the area enclosed by $y = x^3 - 5x$ and $y = 2x^2 - 6$.

12 Find the area enclosed by:

a $y = -x^3 + 3x^2 + 6x - 8$ and $y = 5x - 5$

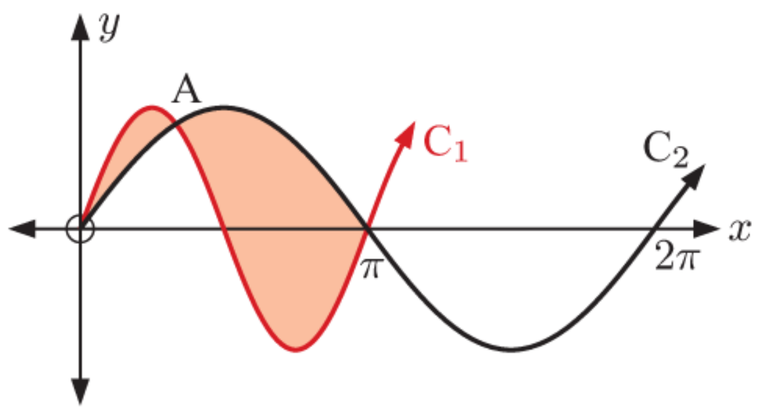
b $y = 2x^3 - 3x^2 + 18$ and $y = x^3 + 10x - 6$

13 The illustrated curves are those of $y = \sin x$ and $y = \sin 2x$.

a Identify each curve.

b Find algebraically the coordinates of A.

c Find the total area enclosed by C_1 and C_2 for $0 \leqslant x \leqslant \pi$.



14 On the same set of axes, draw the graphs of $y = 2x$ and $y^2 = 4x$. Find the area of the region enclosed by these relations.

15 **a** Sketch the graphs of $y = 2xe^x$ and $y = xe^{2x}$ on the same set of axes.

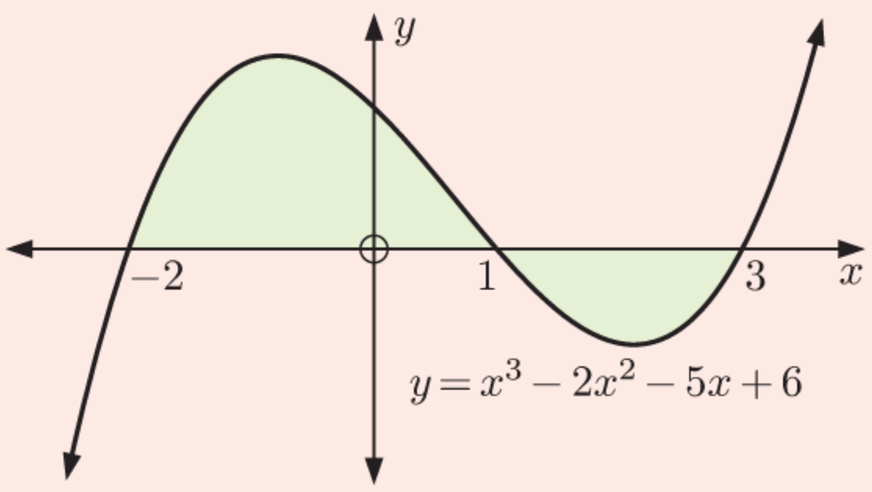
b Find the x-coordinates of their points of intersection.

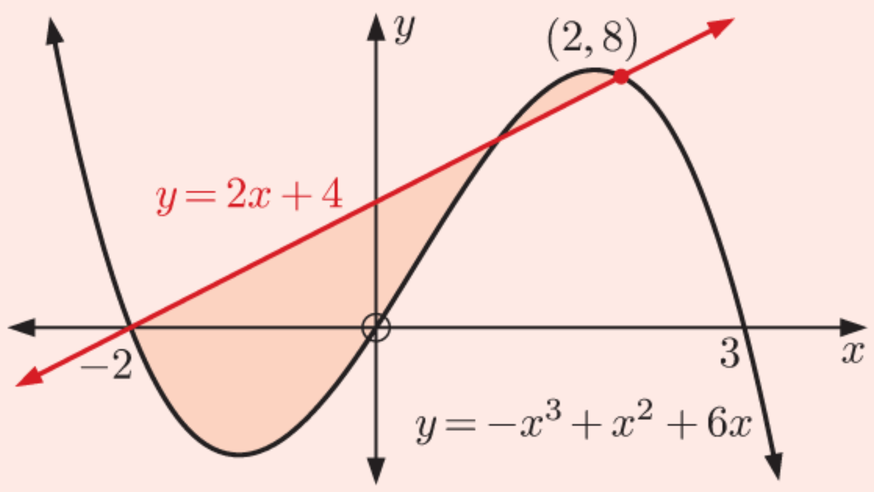
c Hence find the area enclosed by the graphs.

ACTIVITY 1 CALCULATING AREAS USING TECHNOLOGY

We have seen many instances where we have had to add several areas together in order to find the total area between a function and the x-axis, or the total area between two functions.

For example:





This has required us to find every point of intersection, and take care as to whether we are adding or subtracting a particular integral.

If we are using technology, we can save some work by using an absolute value function.

What to do:

1 **a** Copy the graph of $y = x^3 - 2x^2 - 5x + 6$, and on the same set of axes, plot $y = \left| x^3 - 2x^2 - 5x + 6 \right|$.

b Explain why the total area enclosed between $y = x^3 - 2x^2 - 5x + 6$ and the x-axis is equal to $\displaystyle\int_{-2}^{3} \left| x^3 - 2x^2 - 5x + 6 \right| dx$.

c Calculate this area using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

2 Consider the total area enclosed between $y = -x^3 + 3x^2 + x - 3$ and $y = 2x + 4$ on the interval $-2 \leqslant x \leqslant 2$.

a Explain why the total area is equal to

$$\int_{-2}^{2} \left| (-x^3 + 3x^2 + x - 3) - (2x + 4) \right| dx$$

$$= \int_{-2}^{2} \left| -x^3 + 3x^2 - x - 7 \right| dx$$

b Calculate this area using technology.

3 Explain why the area between the functions $f(x)$ and $g(x)$ on the interval $a \leqslant x \leqslant b$ is $A = \displaystyle\int_a^b |\, f(x) - g(x) \,|\, dx$.

Use this formula to check your answers to some of the questions in the previous Exercise.

ACTIVITY 2 NUMERICAL INTEGRATION

There are many functions that do not have indefinite integrals. However, definite integrals can still be determined by **numerical methods** such as the upper and lower rectangles which we saw in **Chapter 20**.

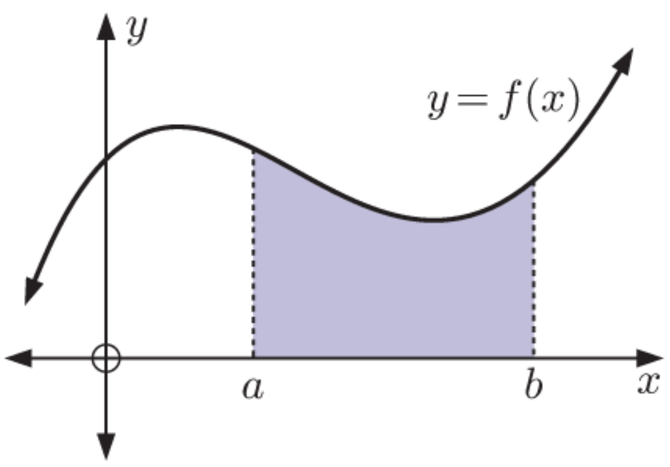
In this online Activity we consider numerical integration using the **midpoint rule**.



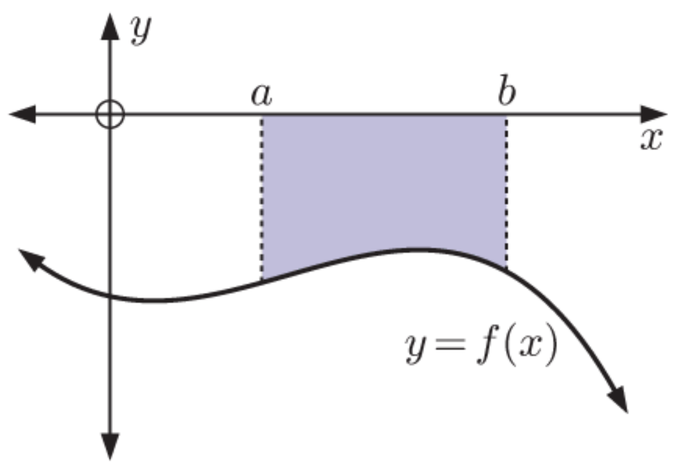
F THE AREA BETWEEN A CURVE AND THE y-AXIS

We have seen that the area between the curve $y = f(x)$ and the x-axis on the interval $a \leqslant x \leqslant b$ is:

- For $y = f(x) > 0$,
  $\text{area} = \int_a^b f(x)\,dx.$



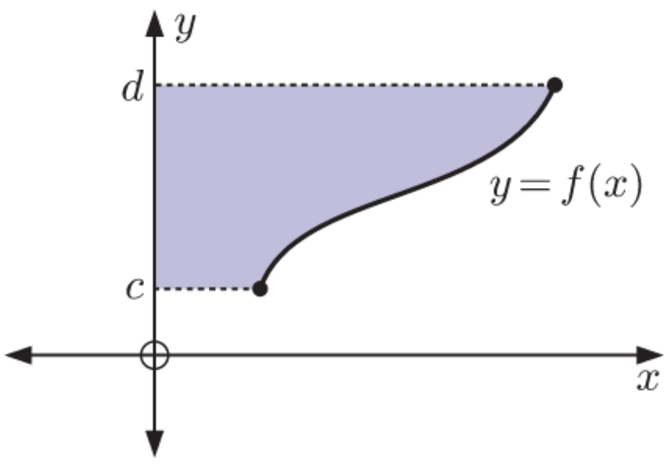
- For $y = f(x) < 0$,
  $\text{area} = -\int_a^b f(x)\,dx.$



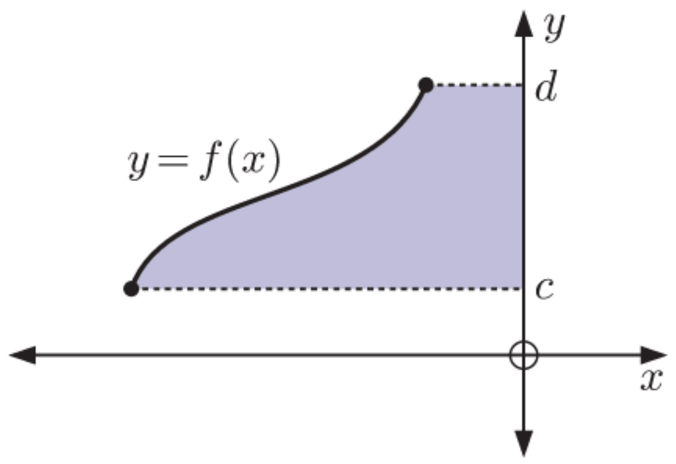
If f is invertible so that $x = f^{-1}(y)$, then we can write analogous integrals for the area between the curve and the y-axis.

The area between the curve $y = f(x)$ and the y-axis on the interval $c \leqslant y \leqslant d$ is:

- For $x = f^{-1}(y) > 0$,
  $\text{area} = \int_c^d f^{-1}(y)\,dy.$



- For $x = f^{-1}(y) < 0$,
  $\text{area} = -\int_c^d f^{-1}(y)\,dy.$



Example 13 Self Tutor

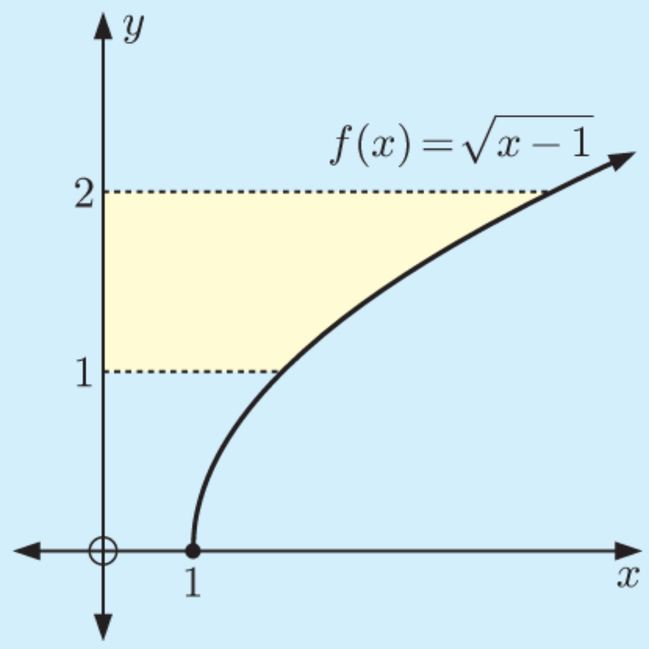
Find the area between the curve:

a $f(x) = \sqrt{x-1}$ and the y-axis between $y = 1$ and $y = 2$

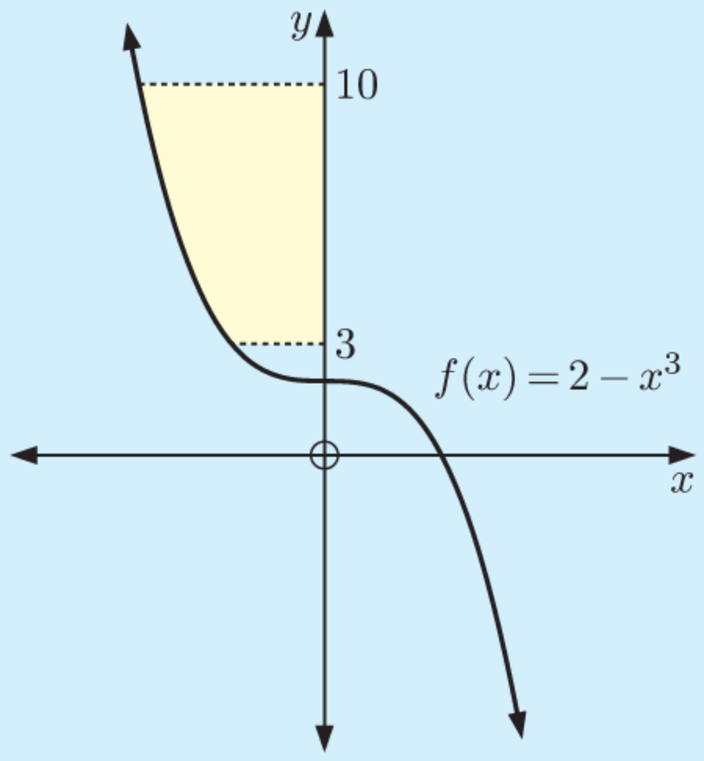
b $f(x) = 2 - x^3$ and the y-axis between $y = 3$ and $y = 10$.

a

$$y = \sqrt{x-1}$$
$$\therefore \quad y^2 = x - 1$$
$$\therefore \quad x = y^2 + 1$$
$$\therefore \quad f^{-1}(y) = y^2 + 1$$
$$\therefore \quad \text{area} = \int_1^2 (y^2+1)\,dy$$
$$= \left[\frac{y^3}{3} + y\right]_1^2$$
$$= \tfrac{8}{3} + 2 - \tfrac{1}{3} - 1$$
$$= \tfrac{10}{3} \text{ units}^2$$



b

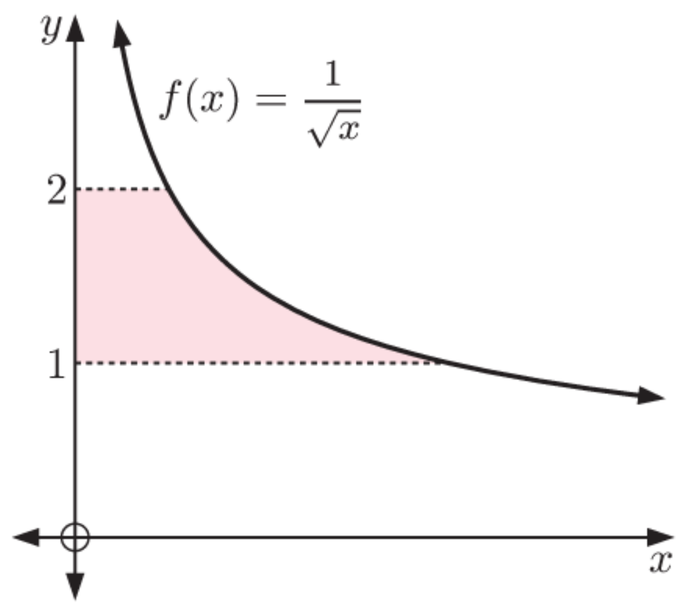


$$y = 2 - x^3$$
$$\therefore \quad x^3 = 2 - y$$
$$\therefore \quad x = \sqrt[3]{2-y}$$
$$\therefore \quad f^{-1}(y) = \sqrt[3]{2-y}$$
$$\therefore \quad \text{area} = -\int_3^{10} (2-y)^{\frac{1}{3}}\,dy$$
$$= -\left[-\tfrac{3}{4}(2-y)^{\frac{4}{3}}\right]_3^{10}$$
$$= -\left(-\tfrac{3}{4}(16) + \tfrac{3}{4}(1)\right)$$
$$= \tfrac{45}{4} \text{ units}^2$$

EXERCISE 22F

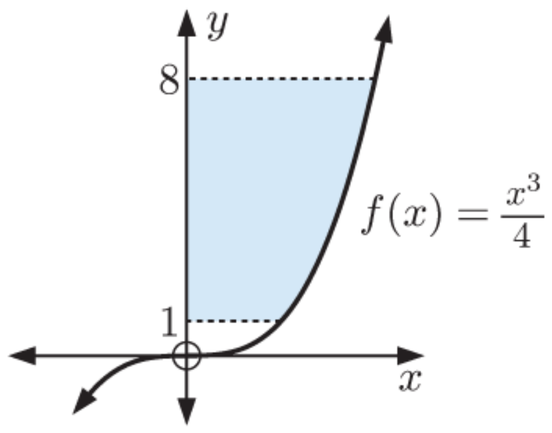
1 The graph of $f(x) = \dfrac{1}{\sqrt{x}}$ is shown alongside.

a Find $f^{-1}(y)$. **b** Hence find the shaded area.

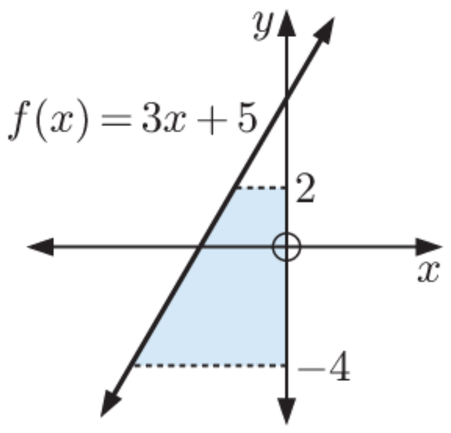


2 Find the shaded area:

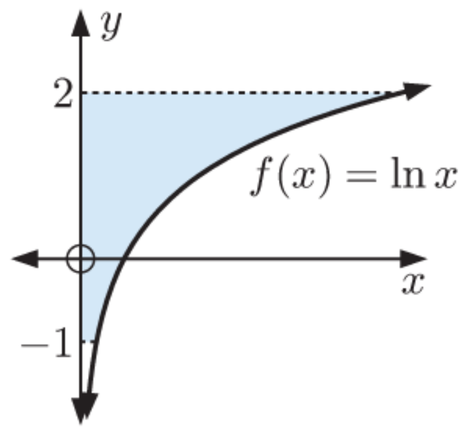
a



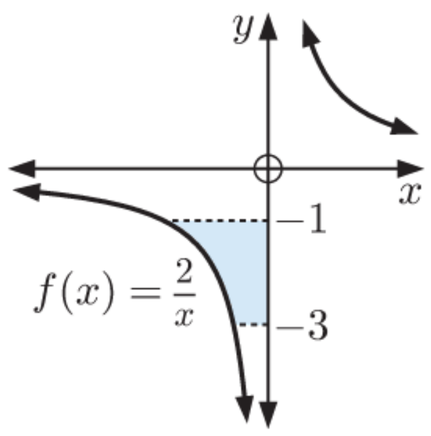
b



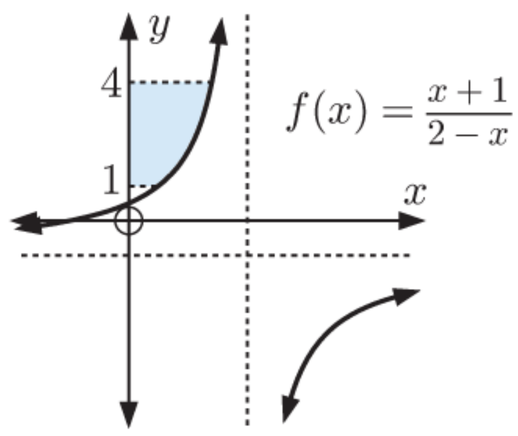
c



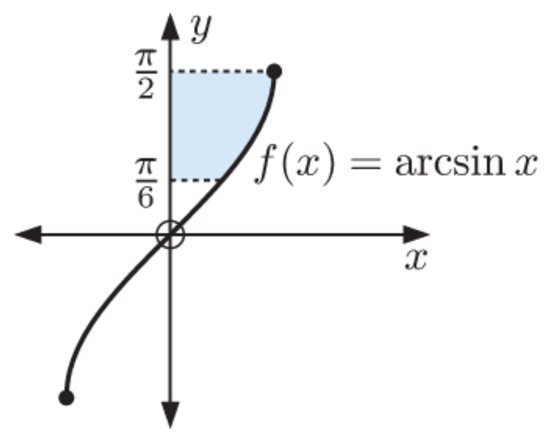
d



e



f



3 Find the area of the region bounded by:

a $x = y^2 + 1$, the y-axis, and the lines $y = 1$ and $y = 2$

b $x = \sqrt{y+5}$, the y-axis, and the lines $y = -1$ and $y = 4$.

4 Find the exact area of the region bounded by:

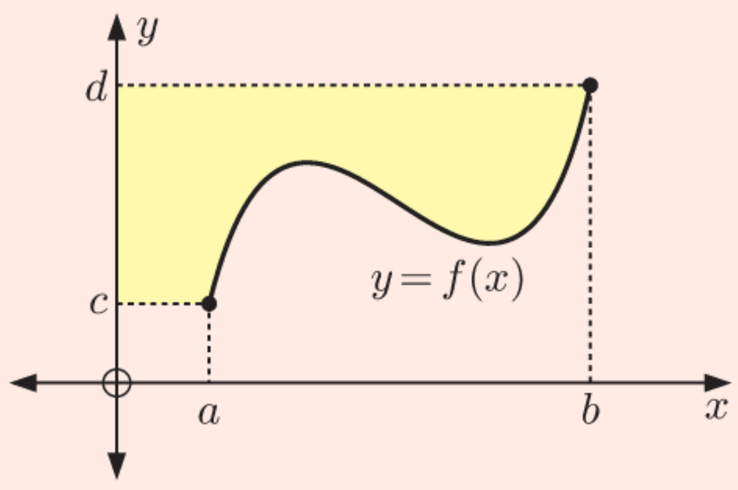
a $y = e^x$, the y-axis, and the lines $y = 2$ and $y = 3$

b $y = \arctan x$, the y-axis, and the lines $y = -\frac{\pi}{6}$ and $y = -\frac{\pi}{3}$

c $y = \sqrt{\dfrac{x^2-1}{x^2}}$ with domain $x \geqslant 1$, and the lines $x = 0$, $y = 0$, and $y = \frac{1}{2}$.

DISCUSSION

- Why did we require f to be invertible?
- How could you find the shaded area for this function?

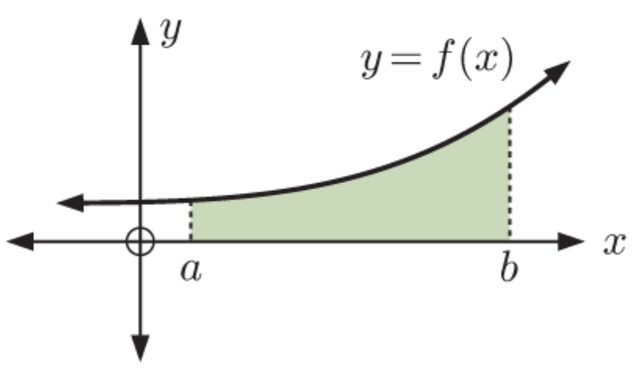


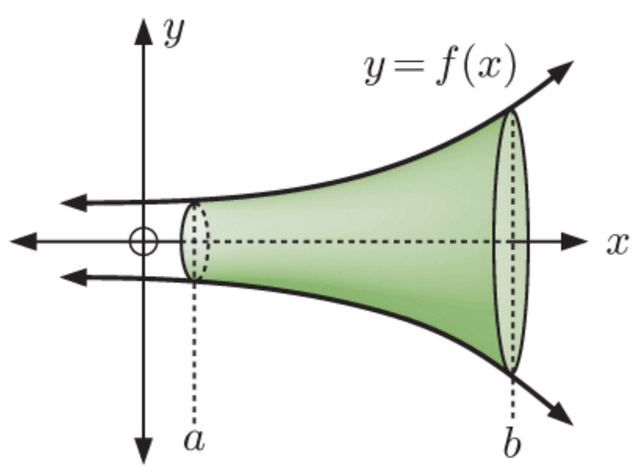
G SOLIDS OF REVOLUTION

SOLIDS OF REVOLUTION ABOUT THE x-AXIS

Consider the curve $y = f(x)$ for $a \leqslant x \leqslant b$.

If the shaded area below is revolved about the x-axis through $360°$ or 2π, a 3-dimensional solid will be formed. This solid is called a **solid of revolution**.

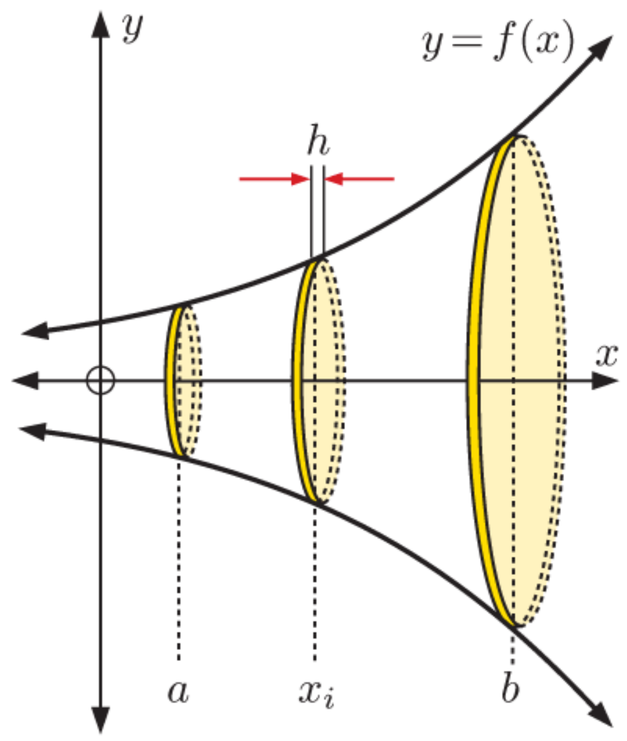






To find the volume of the solid, we can think of it as being made up of n thin cylindrical discs, where the radius of each disc is the value of the function at the left end $x = x_i$, and the height is h. The volume of the ith disc is $\pi[f(x_i)]^2 h$, so the total volume of the discs is

$$\sum_i \pi[f(x_i)]^2 h.$$



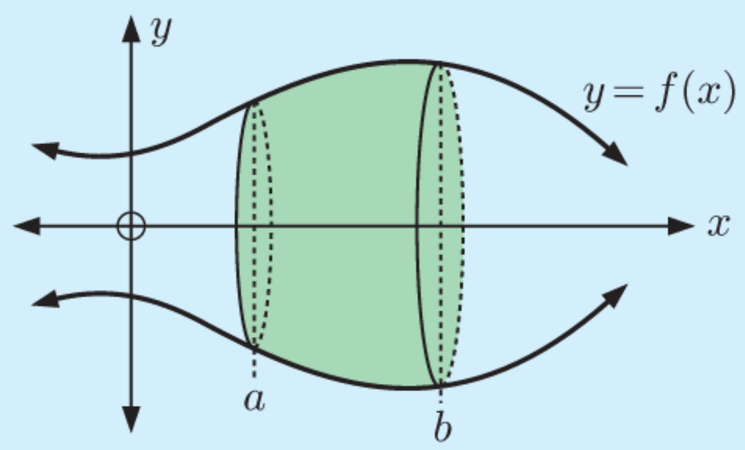
To find the volume *exactly*, we need infinitely many discs. We therefore take the limit as $h \to 0$.

$$\text{Volume of revolution} = \lim_{h\to 0} \sum_i \pi[f(x_i)]^2 h$$
$$= \int_a^b \pi[f(x)]^2 \, dx$$
$$= \pi \int_a^b [f(x)]^2 \, dx$$

When the region enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is revolved through 2π about the x-axis to generate a solid, the

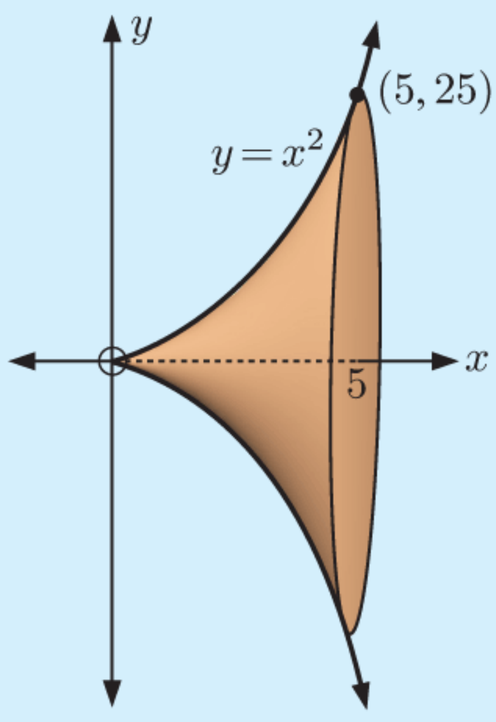
$$\textbf{Volume of revolution} = \boldsymbol{\pi \int_a^b [f(x)]^2 \, dx}$$

$$\text{or} \quad \boldsymbol{\pi \int_a^b y^2 \, dx}.$$



Example 14 **Self Tutor**

Find the volume of the solid formed when the graph of the function $y = x^2$ for $0 \leqslant x \leqslant 5$ is revolved through 2π about the x-axis.



$$\text{Volume of revolution} = \pi \int_a^b y^2 \, dx$$
$$= \pi \int_0^5 (x^2)^2 \, dx$$
$$= \pi \int_0^5 x^4 \, dx$$
$$= \pi \left[\frac{x^5}{5}\right]_0^5$$
$$= \pi(625 - 0)$$
$$= 625\pi \text{ units}^3$$



GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 22G.1

1 Find the volume of the solid formed when the following are revolved through 2π about the x-axis:

a $y = 2x$ for $0 \leqslant x \leqslant 3$

b $y = \sqrt{x}$ for $0 \leqslant x \leqslant 4$

c $y = x^3$ for $1 \leqslant x \leqslant 2$

d $y = x^{\frac{3}{2}}$ for $1 \leqslant x \leqslant 4$

e $y = x^2$ for $2 \leqslant x \leqslant 4$

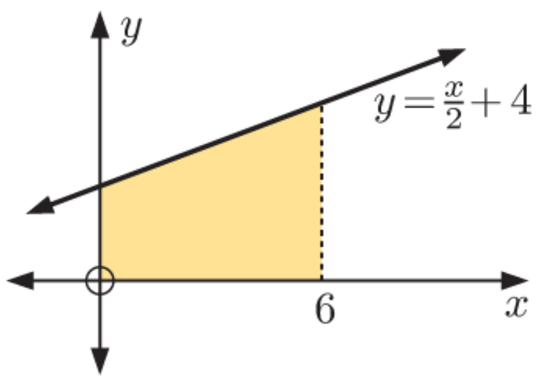
f $y = \sqrt{25 - x^2}$ for $0 \leqslant x \leqslant 5$

g $y = \dfrac{1}{x-1}$ for $2 \leqslant x \leqslant 3$

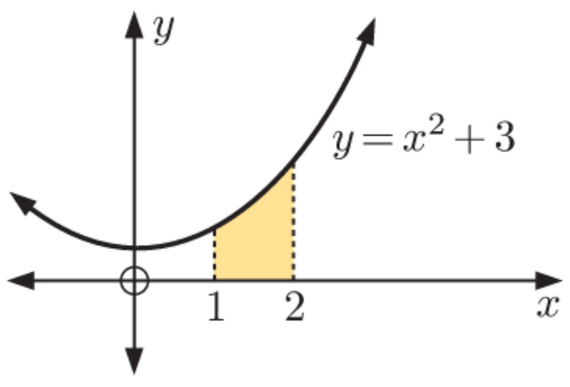
h $y = x + \dfrac{1}{x}$ for $1 \leqslant x \leqslant 3$

2 Find the volume of revolution when the shaded region is revolved through 2π about the x-axis.

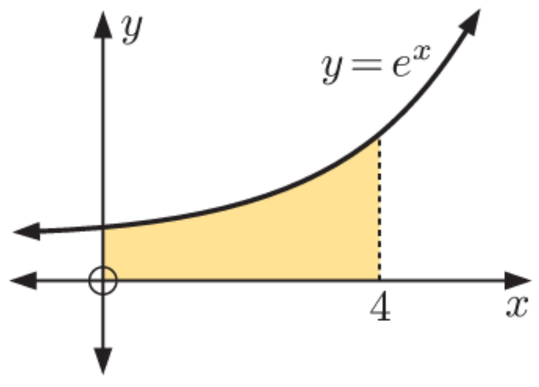
a



b



c

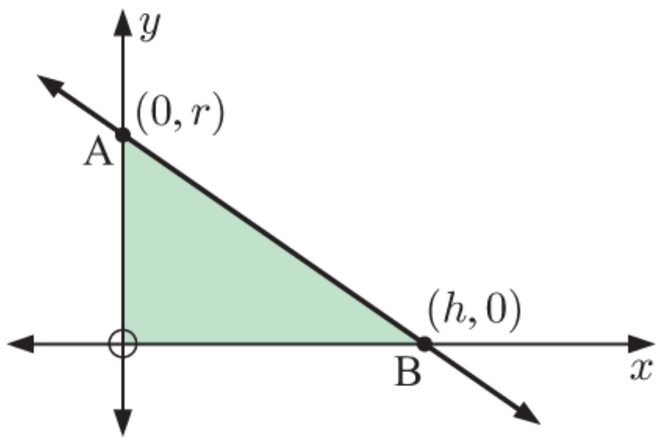


3 **a** Describe the solid of revolution when the shaded region is revolved about the x-axis.

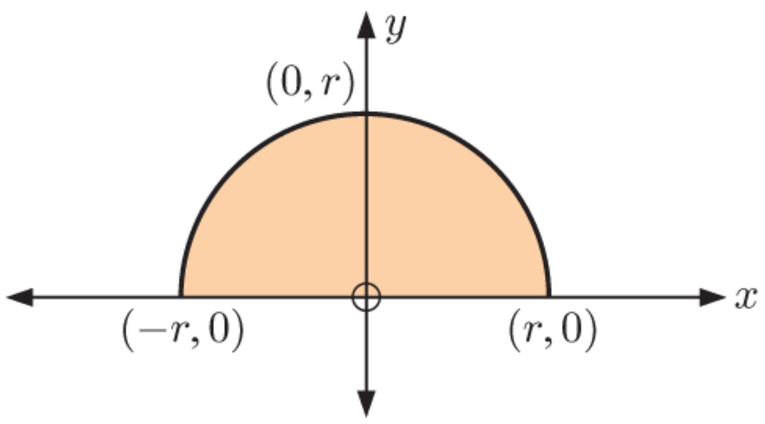
b Find the equation of the line (AB) in the form $y = mx + c$.

c Find a formula for the volume of the solid using

$$V = \pi \int_a^b y^2 \, dx.$$



4



A circle with centre $(0,\ 0)$ and radius r units has equation $x^2 + y^2 = r^2$.

a If the shaded region is revolved about the x-axis, what solid is formed?

b Show that the volume of revolution is $\frac{4}{3}\pi r^3$.

5 Find the volume of revolution when the following regions are revolved through 2π about the x-axis:

a $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$

b $y = \cos 2x$ for $0 \leqslant x \leqslant \frac{\pi}{4}$

c $y = \sqrt{\sin x}$ for $0 \leqslant x \leqslant \pi$

d $y = \dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant \frac{\pi}{3}$

e $y = \sec 3x$ for $0 \leqslant x \leqslant \frac{\pi}{12}$

f $y = \tan \frac{x}{2}$ for $0 \leqslant x \leqslant \frac{\pi}{2}$

6 **a** Sketch the graph of $y = \sin x + \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

b Hence find the volume of revolution of the shape bounded by $y = \sin x + \cos x$, the x-axis, $x = 0$, and $x = \frac{\pi}{4}$ when it is rotated through 2π about the x-axis.

7 **a** Sketch the graph of $y = 4 \sin 2x$ for $0 \leqslant x \leqslant \frac{\pi}{4}$.

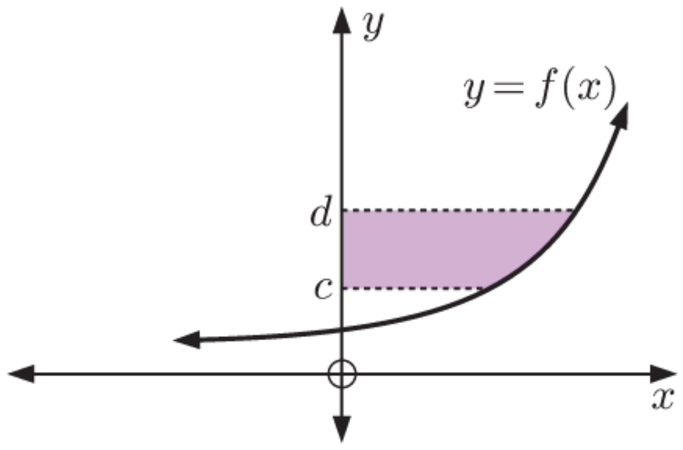
b Hence find the volume of revolution of the shape bounded by $y = 4 \sin 2x$, the x-axis, $x = 0$, and $x = \frac{\pi}{4}$ when it is rotated through 2π about the x-axis.

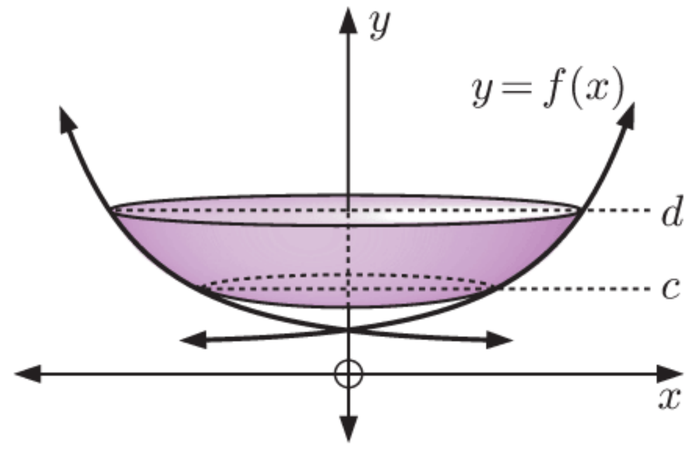
8 Use technology to find, correct to 3 significant figures, the volume of the solid of revolution formed when these functions are rotated through 360° about the x-axis:

a $y = \dfrac{x^3}{x^2+1}$ for $1 \leqslant x \leqslant 3$ **b** $y = e^{\sin x}$ for $0 \leqslant x \leqslant 2$.

SOLIDS OF REVOLUTION ABOUT THE y-AXIS

If the function $f(x)$ is invertible, we can also define a solid of revolution about the y-axis.



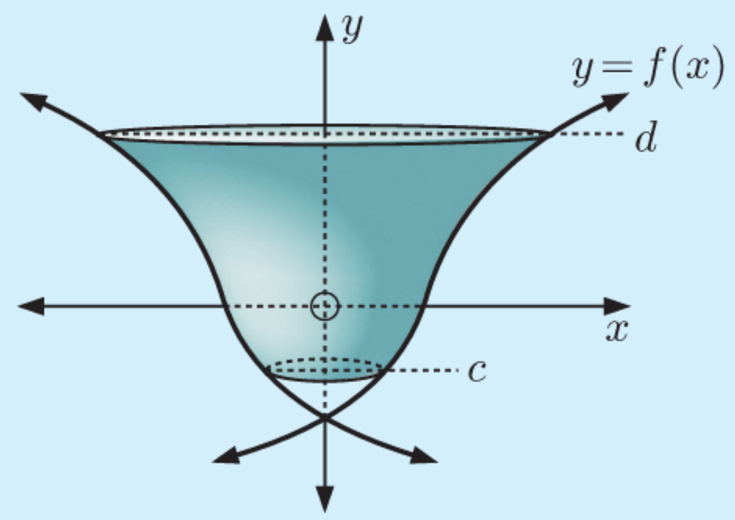


DEMO

When the region enclosed by the invertible function $y = f(x)$, the y-axis, and the horizontal lines $y = c$ and $y = d$ is revolved through 2π about the y-axis to generate a solid, the

$$\textbf{Volume of revolution} = \pi \int_c^d \left[f^{-1}(y)\right]^2 \, dy$$

$$\text{or} \quad \pi \int_c^d x^2 \, dy.$$



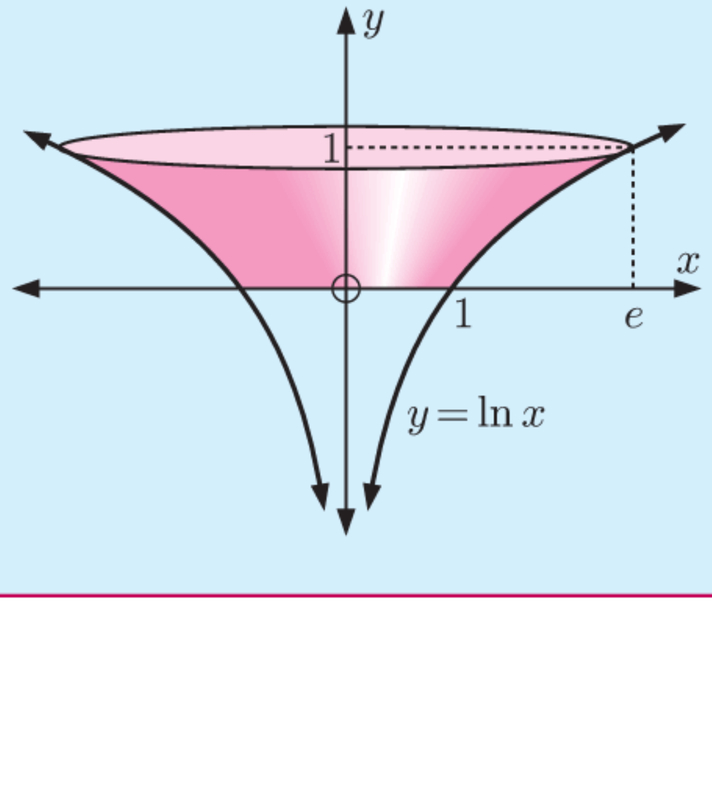
Example 15 **Self Tutor**

The graph of $y = \ln x$, $1 \leqslant x \leqslant e$ is revolved through 2π about the y-axis. Find the volume of revolution.

When $x = 1$, $y = 0$

When $x = e$, $y = 1$

$\therefore$ we rotate the function for $0 \leqslant y \leqslant 1$.



Volume of revolution

$$= \pi \int_0^1 x^2 \, dy$$

$$= \pi \int_0^1 (e^y)^2 \, dy$$

$$= \pi \int_0^1 e^{2y} \, dy$$

$$= \pi \left[\tfrac{1}{2}e^{2y}\right]_0^1$$

$$= \pi \left(\tfrac{1}{2}e^2 - \tfrac{1}{2}e^0\right)$$

$$= \tfrac{\pi}{2}(e^2 - 1) \text{ units}^3$$

EXERCISE 22G.2

1 Find the volume of the solid formed when the following are revolved through 2π about the y-axis:

a $y = x^2,\ x \geqslant 0$, between $y = 0$ and $y = 4$ **b** $y = \sqrt{x}$ between $y = 1$ and $y = 4$

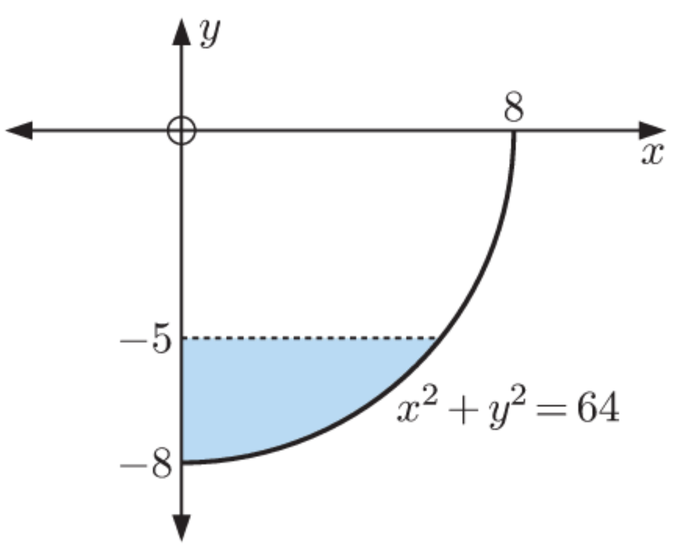
c $y = \ln x$ between $y = 0$ and $y = 2$ **d** $y = \sqrt{x - 2}$ between $x = 2$ and $x = 11$

e $y = (x - 1)^3$ between $x = 1$ and $x = 3$.

2 Find exactly the volume of the solid of revolution formed by revolving the relation $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, $x \geqslant 0$ through $360°$ about the y-axis.

3 A wooden bowl is made in the shape of a *paraboloid* by revolving the curve $y = \frac{1}{4}x^2,\ x \geqslant 0$, between $y = 0$ and $y = 4$ through 2π about the y-axis. Find the capacity of the bowl.

4



The shaded region is rotated through $360°$ about the y-axis.

a Find the volume of revolution.

b A hemispherical bowl of radius 8 cm contains water to a depth of 3 cm.
Find the volume of water in the bowl.

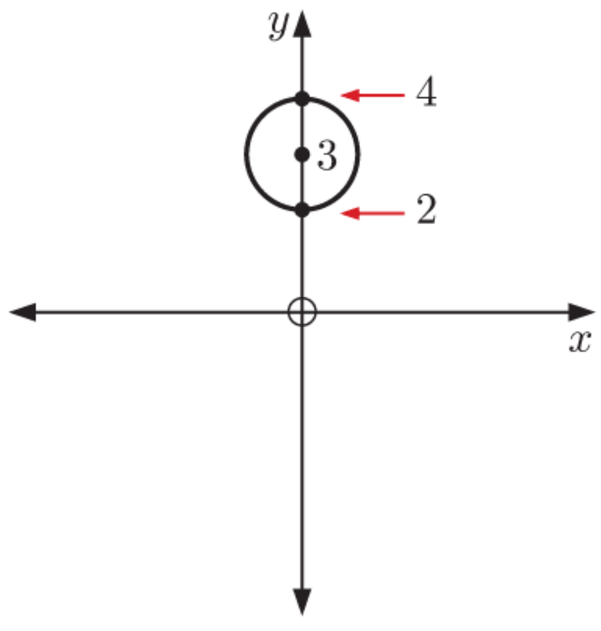
VOLUMES FOR TWO DEFINING FUNCTIONS

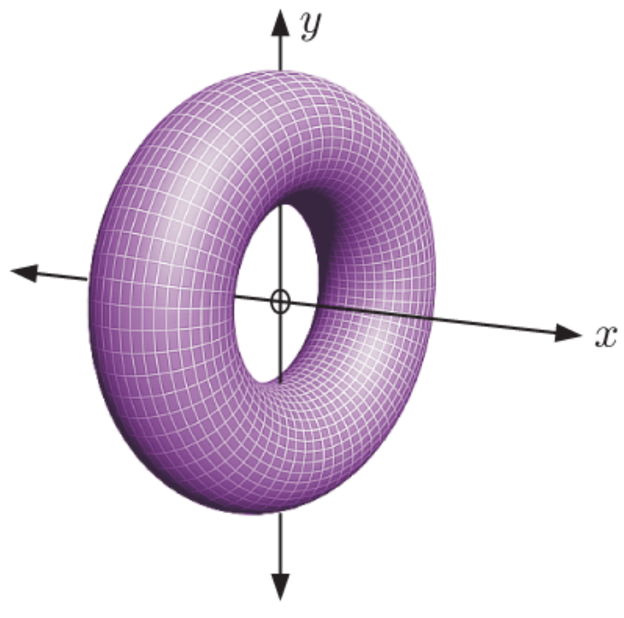
Consider the circle with centre $(0, 3)$ and radius 1 unit.

When this circle is revolved about the x-axis, we obtain a doughnut shape or *torus*.

DEMO





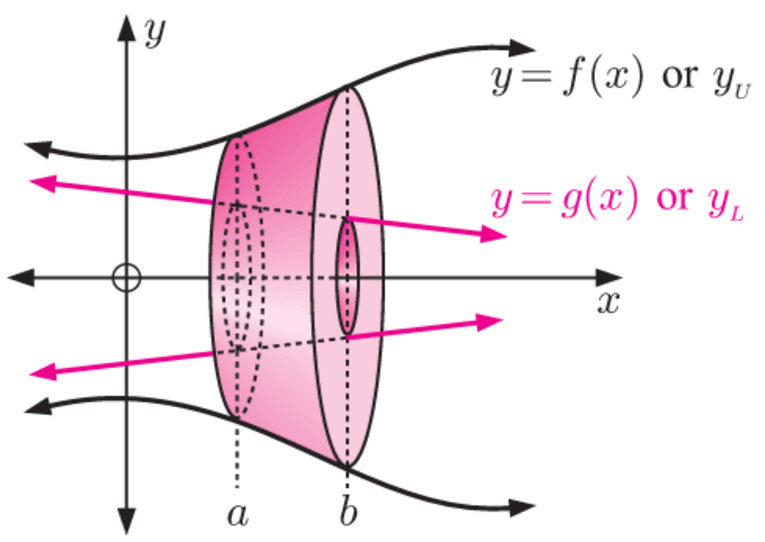


Suppose $y_U = f(x)$ and $y_L = g(x)$, where $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$.

If the region bounded by the upper function $y_U = f(x)$ and the lower function $y_L = g(x)$, and the lines $x = a$, $x = b$ is revolved about the x-axis, then its volume of revolution is given by:

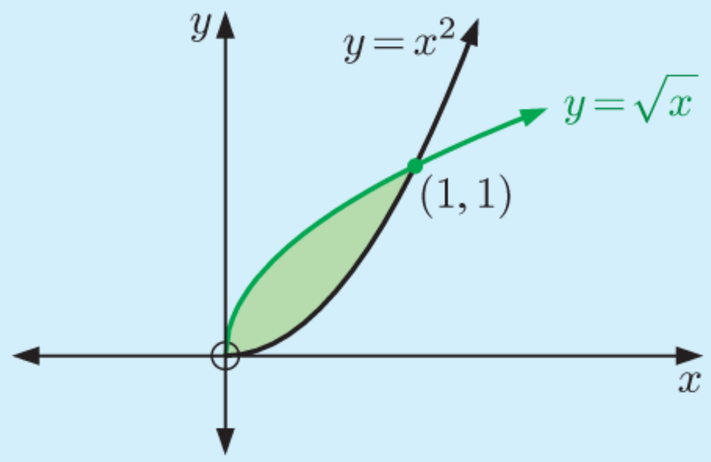
$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx \quad \text{or}$$

$$V = \pi \int_a^b \left(y_U^{\,2} - y_L^{\,2} \right) dx$$

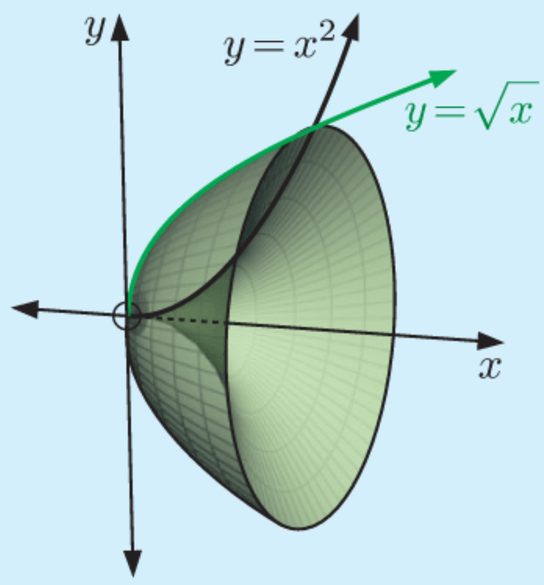


Example 16 **Self Tutor**

Find the volume of revolution generated by revolving the region between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.



$$\text{Volume} = \pi \int_0^1 \left(y_U^{\,2} - y_L^{\,2}\right) dx$$
$$= \pi \int_0^1 \left(\left(\sqrt{x}\right)^2 - (x^2)^2\right) dx$$
$$= \pi \int_0^1 (x - x^4)\, dx$$
$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$
$$= \pi\left(\left(\tfrac{1}{2} - \tfrac{1}{5}\right) - (0)\right)$$
$$= \tfrac{3\pi}{10} \text{ units}^3$$

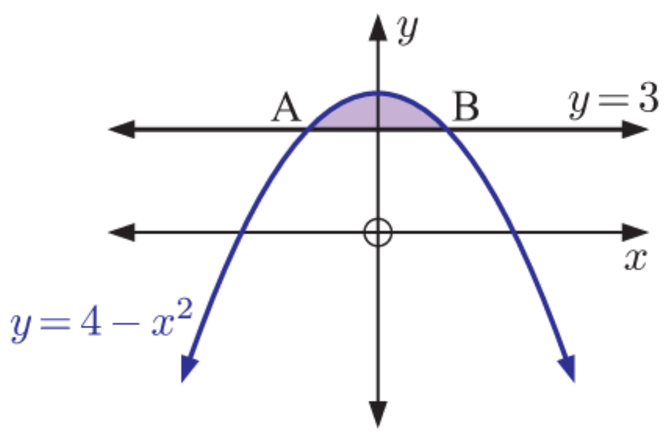


EXERCISE 22G.3

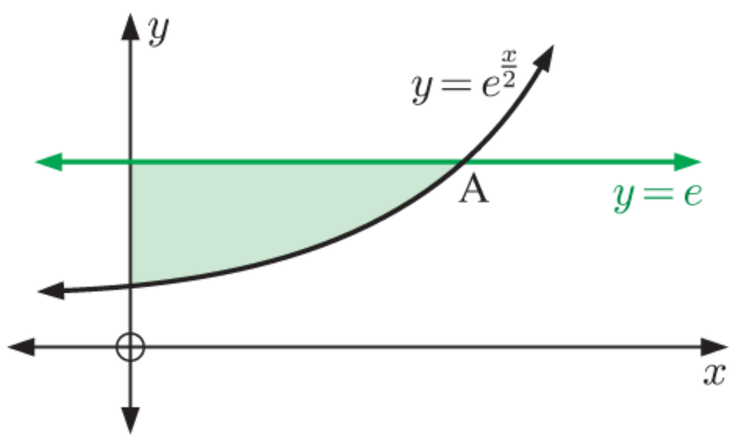
1 The shaded region between $y = 4 - x^2$ and $y = 3$ is revolved about the x-axis.

a Find the coordinates of A and B.

b Find the volume of revolution.



2



The shaded region is revolved about the x-axis.

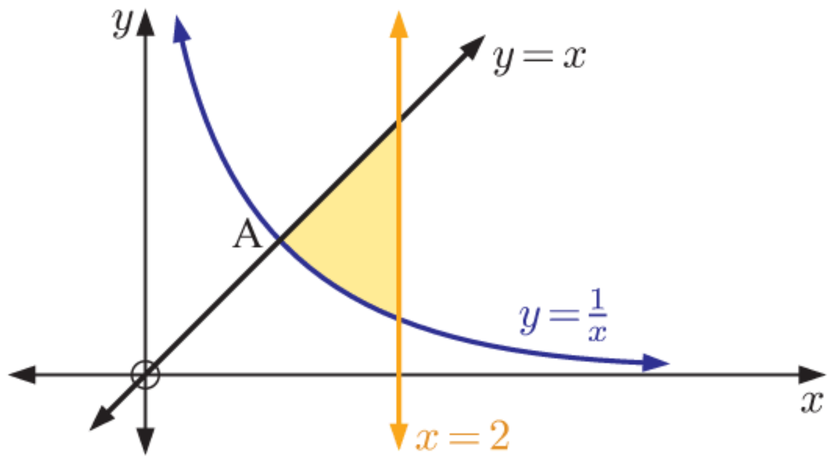
a Find the coordinates of A.

b Find the volume of revolution.

3 The shaded region between $y = x$, $y = \dfrac{1}{x}$, and $x = 2$ is revolved about the x-axis.

a Find the coordinates of A.

b Find the volume of revolution.

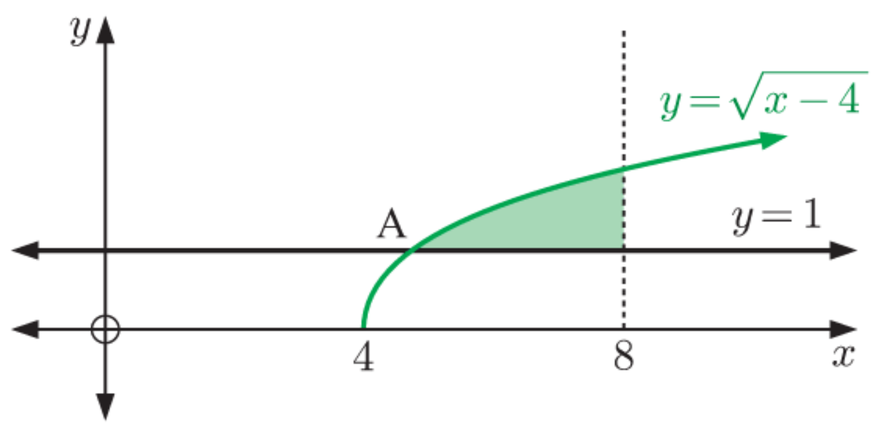


4 Find exactly the volume of the solid of revolution generated by rotating the region enclosed by $y = x^2 - 4x + 6$ and $x + y = 6$ through $360°$ about the x-axis.

5 The shaded region is revolved about the x-axis.

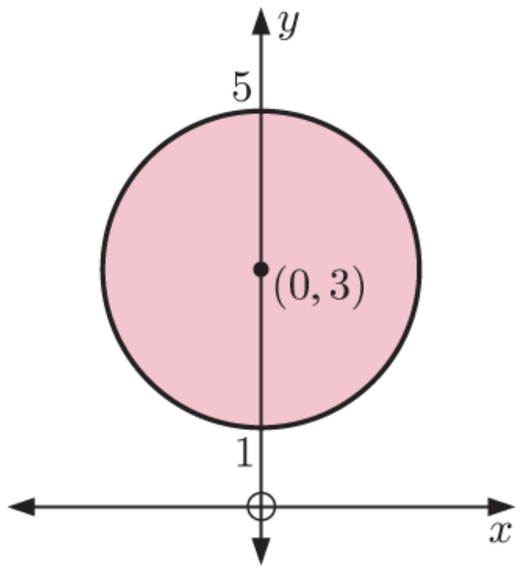
a State the coordinates of A.

b Find the volume of revolution.



6 The illustrated circle has equation $x^2 + (y-3)^2 = 4$.

Find the volume of the torus generated by revolving the circle through 2π about the x-axis.



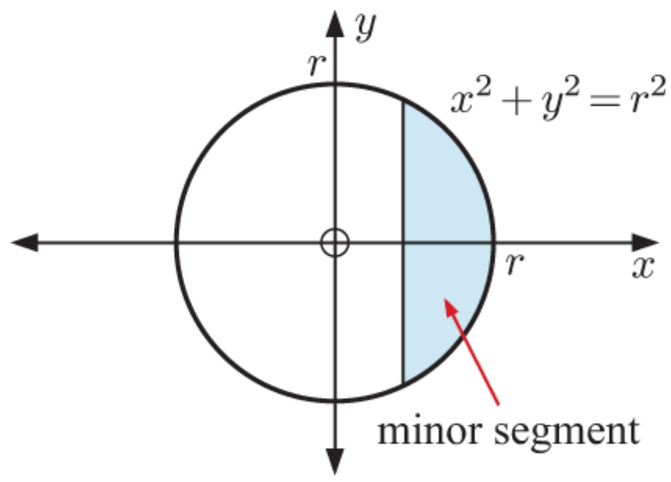
7 A circle has equation $x^2 + y^2 = r^2$. A minor segment is cut off by a vertical chord, then revolved about the y-axis to generate a solid of revolution.

a Suppose the chord has length r.

Prove that the solid has volume $V = \dfrac{\pi r^3}{6}$.

b Now suppose $r > 3$ and the chord has length 6 units.

Show that the volume of the solid is independent of r.



H PROBLEM SOLVING BY INTEGRATION

When we studied differential calculus, we saw how to find the rate of change of a function by differentiation.

Since integration is the reverse process of differentiation, integration can be used to measure the change in a quantity from its rate of change.

When we introduced integration, we separated the area under a curve into rectangles. The area of each rectangle is the rate of change of the quantity times the subinterval width, which tells us the change in the quantity corresponding to that subinterval.

So, the area under a rate function for a particular interval tells us the overall change in the quantity over that interval.

Example 17 **Self Tutor**

The rate of power consumption by the city of Bristol can be modelled by the function

$$E(t) = 0.3\sin\left(\tfrac{(t-10)\pi}{12}\right) + 0.1\cos\left(\tfrac{(t-6)\pi}{6}\right) + 0.775 \text{ GW}$$

where t is the number of hours after midnight each day, $0 \leqslant t \leqslant 24$.

Find the following quantities and explain what they represent:

a $\int_0^{12} E(t)\,dt$ **b** $\int_{12}^{24} E(t)\,dt$ **c** $\int_0^{24} E(t)\,dt$

$$E(t) = 0.3\sin\left(\tfrac{(t-10)\pi}{12}\right) + 0.1\cos\left(\tfrac{(t-6)\pi}{6}\right) + 0.775 \text{ GW}$$

$$\therefore \int E(t)\,dt = 0.3\left(-\cos\left(\tfrac{(t-10)\pi}{12}\right)\right)\left(\tfrac{12}{\pi}\right) + 0.1\sin\left(\tfrac{(t-6)\pi}{6}\right)\left(\tfrac{6}{\pi}\right) + 0.775t + c$$

$$= -\tfrac{3.6}{\pi}\cos\left(\tfrac{(t-10)\pi}{12}\right) + \tfrac{0.6}{\pi}\sin\left(\tfrac{(t-6)\pi}{6}\right) + 0.775t + c$$

a $$\int_0^{12} E(t)\,dt = \left(-\tfrac{3.6}{\pi}\cos\tfrac{\pi}{6} + \tfrac{0.6}{\pi}\cancel{\sin\pi} + 9.3\right) - \left(-\tfrac{3.6}{\pi}\cos\left(-\tfrac{5\pi}{6}\right) + \tfrac{0.6}{\pi}\cancel{\sin(-\pi)} + \cancel{0}\right)$$
$$\approx 7.315 \text{ GWh}$$

The morning power consumption of Bristol is about 7.32 GWh.

b $$\int_{12}^{24} E(t)\,dt = \left(-\tfrac{3.6}{\pi}\cos\tfrac{7\pi}{6} + \tfrac{0.6}{\pi}\cancel{\sin 3\pi} + 18.6\right) - \left(-\tfrac{3.6}{\pi}\cos\tfrac{\pi}{6} + \tfrac{0.6}{\pi}\cancel{\sin\pi} + 9.3\right)$$
$$\approx 11.285 \text{ GWh}$$

The afternoon power consumption of Bristol is about 11.29 GWh.

c $$\int_0^{24} E(t)\,dt = \int_0^{12} E(t)\,dt + \int_{12}^{24} E(t)\,dt$$
$$\approx 18.6 \text{ GWh}$$

The total daily power consumption of Bristol is about 18.6 GWh.

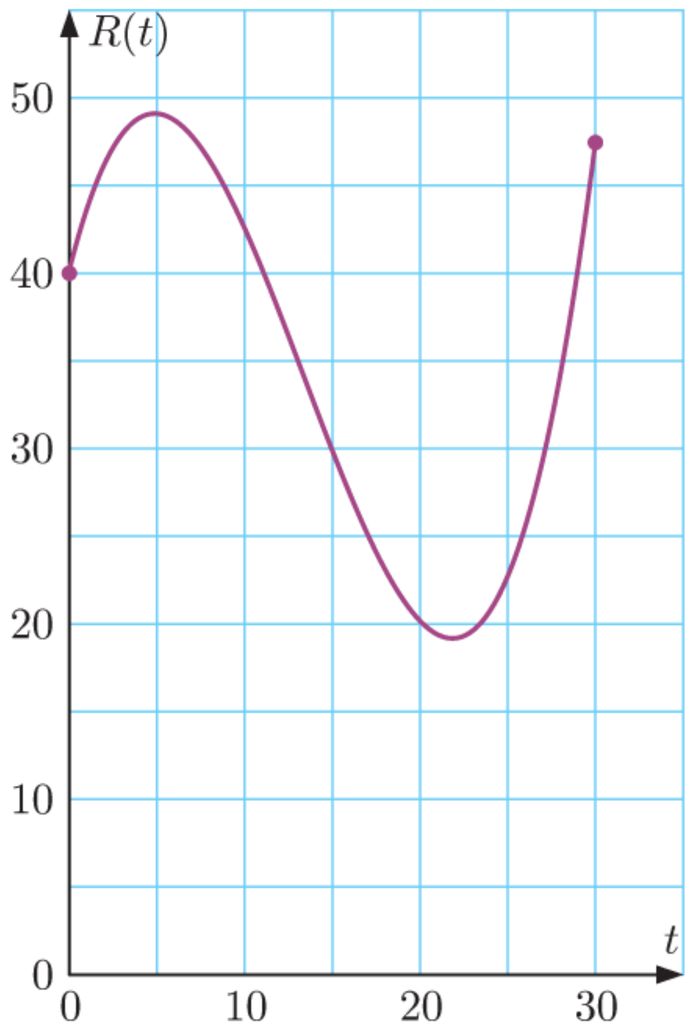
EXERCISE 22H

1 The rate of traffic flow past a pedestrian crossing between 8 am and 8:30 am is given by

$$R(t) = \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40 \text{ cars per minute,}$$

where t is the number of minutes after 8 am, $0 \leqslant t \leqslant 30$.

The graph of $R(t)$ against t is shown alongside.



a Find the rate of traffic flow at 8:20 am.

b Use the graph to estimate the time at which the traffic flow was greatest.

c Copy the graph, and shade the region corresponding to $\int_{10}^{15} R(t)\,dt$. Explain what this region represents.

d How many cars passed the crossing between 8 am and 8:30 am?

PRINTABLE GRAPH

2 Evan is happily paddling until his kayak strikes a sharp rock. Water begins to leak in at the rate $R_1(t) = 5 - 5e^{-0.2t}$ litres per minute, where t is the time in minutes. Evan tries to bail the water out of the kayak, removing the water at the rate $R_2(t) = 6 - 6e^{-0.1t}$ litres per minute.



a After 2 minutes, at what rate is water:

i leaking into the kayak

ii being bailed from the kayak?

b Is the amount of water in the kayak increasing or decreasing after 2 minutes? Explain your answer.

c Evaluate the following integrals, and interpret their meaning:

i $\int_0^3 R_1(t)\,dt$ **ii** $\int_2^5 R_2(t)\,dt$ **iii** $\int_0^8 [R_1(t) - R_2(t)]\,dt$

d How much water is in the kayak after 10 minutes?

3 Answer the **Opening Problem** on page **590**.

4 The rate of power consumption of the United Kingdom can be modelled by the function

$$E(t) = 13\sin\left(\tfrac{(t+3)\pi}{3}\right) + 70\cos\left(\tfrac{(t-1)\pi}{6}\right) + 196 \text{ TWh per month}$$

where t is the number of months after January 1st, $0 \leqslant t \leqslant 12$.

a Use technology to help sketch the function.

b Find the following quantities and explain what they represent:

i $\int_3^4 E(t)\,dt$ **ii** $\int_5^8 E(t)\,dt$ **iii** $\int_0^{12} E(t)\,dt$

ACTIVITY 3 BUFFON'S NEEDLE PROBLEM

The French naturalist and mathematician **Georges-Louis Leclerc, Comte de Buffon** (1707 - 1788) is best known for his challenges to previously held views of natural history. While ultimately incorrect, they gave spark to new and better models from those who followed.

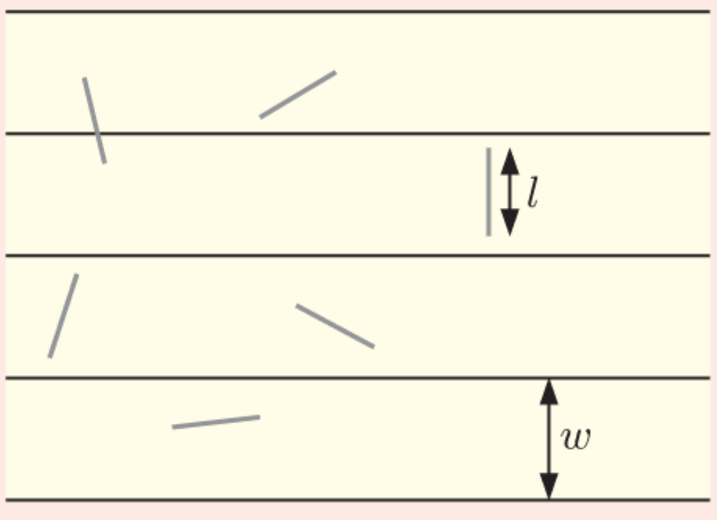
For example, he proposed that the planets were formed from the collision of a comet with the Sun, and thus estimated the planets must be around 70 000 years old.

In mathematics, Buffon was the first person to apply calculus ideas in probability.

Buffon's needle problem is named after him:

> *Given a series of parallel planks w units apart, what is the probability that a needle of length l units, when tossed onto the planks, will lie on a line between two planks?*

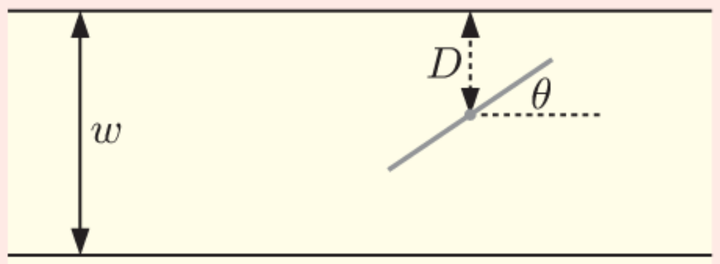
To solve this problem, we must separately consider the "short needle" case where $l \leqslant w$, and the "long needle" case where $l > w$.



CASE 1: THE SHORT NEEDLE

Suppose a needle of length $l \leqslant w$ is tossed onto the planks.

Let θ be the angle between the needle and the plank lines (measured anticlockwise as shown), and D be the distance between the centre of the needle and the nearest plank line.



What to do:

1 Determine the range of possible values of: **a** θ **b** D

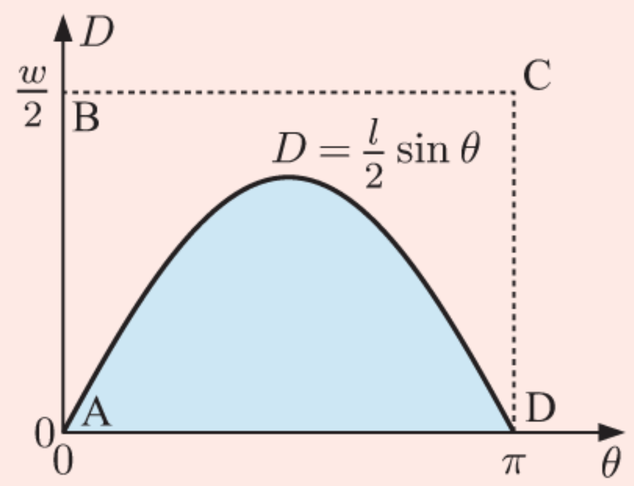
2 Is it reasonable to assume that θ and D will take values in their ranges with equal probability? Explain your answer.

3 Explain why the needle will lie on a line if $D \leqslant \frac{l}{2}\sin\theta$.

4 Use **1** to **3** to explain why the probability of the needle lying on a line is equal to the probability that a randomly chosen point from rectangle ABCD lies in the shaded area.

5 Find the shaded area, in terms of l.

6 Hence show that the probability that the needle lies on a line is $\dfrac{2l}{w\pi}$.



CASE 2: THE LONG NEEDLE

Now suppose the needle has length $l > w$.

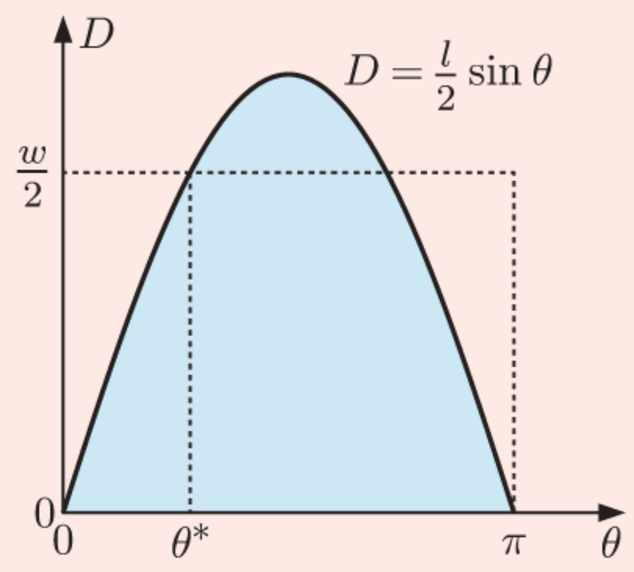
What to do:

1 Use the diagram alongside to explain why the reasoning for $l \leqslant w$ is not valid for $l > w$.

2 Show that $\theta^* = \sin^{-1}\left(\frac{w}{l}\right)$.

3 **a** Explain why the probability that a needle of length $l > w$ lies on a line is given by

$$\frac{(\pi - 2\theta^*)\frac{w}{2} + 2\displaystyle\int_0^{\theta^*}\left(\frac{l}{2}\sin\theta\right)d\theta}{\frac{w\pi}{2}}.$$



b Hence show that this probability is $1 - \dfrac{2}{\pi}\sin^{-1}\left(\frac{w}{l}\right) + \dfrac{2l}{w\pi}\left(1 - \dfrac{\sqrt{l^2 - w^2}}{l}\right)$.

4 Verify that the short needle and long needle formulae give the same probability for the boundary case $l = w$.

I IMPROPER INTEGRALS

An integral is described as **improper** if one of the following is true:

- The function being integrated approaches ∞ or $-\infty$ for one or more points on the domain of integration.

  For example, $\int_{-1}^{2} \frac{1}{x^2}\,dx$ is improper since $\frac{1}{x^2}$ is discontinuous at $x = 0$. As $x \to 0$, $f(x) \to \infty$.

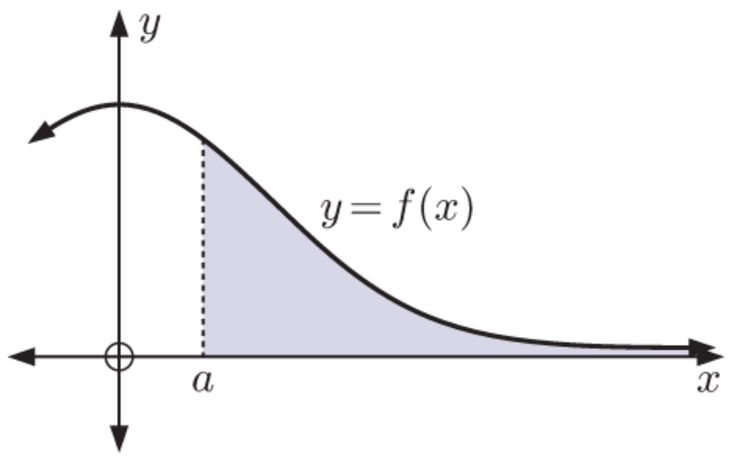
- The integral has the form $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$, or $\int_{-\infty}^{\infty} f(x)\,dx$ where $a, b \in \mathbb{R}$.

In this course we will consider improper integrals of the form $\int_a^{\infty} f(x)\,dx$.

We define $$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^{b} f(x)\,dx$$

provided the limit exists.

For this improper integral to exist, $f(x)$ *must* approach 0 as $x \to \infty$.



DISCUSSION

If the improper integral $\int_a^{\infty} f(x)\,dx$ is to exist, *why* must $\lim_{x\to\infty} f(x) = 0$?

Example 18 **Self Tutor**

Evaluate $\int_1^{\infty} \frac{1}{x^3}\,dx$.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^3}\,dx &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^3}\,dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^{b} \\ &= \lim_{b\to\infty} \left(-\frac{1}{2b^2}\right) + \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

INVESTIGATION 2 $\int_1^{\infty} x^p\,dx$

Suppose $f(x)$ is a continuous function for all $x \geqslant a$.

The condition that $\lim_{x\to\infty} f(x) = 0$ is *necessary* for $\int_a^{\infty} f(x)\,dx$ to exist.

But does this condition *guarantee* that $\int_a^{\infty} f(x)\,dx$ exists?

In this Investigation we will consider the existence of $\int_1^\infty x^p\,dx$.

What to do:

1 Explain why $\int_1^\infty x^p\,dx$ does not exist for $p \geqslant 0$.

2 Evaluate, if possible:

a $\int_1^\infty x^{-2}\,dx$ **b** $\int_1^\infty x^{-\frac{3}{2}}\,dx$ **c** $\int_1^\infty x^{-1}\,dx$ **d** $\int_1^\infty x^{-\frac{1}{2}}\,dx$

3 **a** Write $\int_1^b x^p\,dx$ in terms of b and p.

b For what values of p does $\int_1^\infty x^p\,dx = \lim_{b\to\infty} \int_1^b x^p\,dx$ exist?

From the **Investigation**, you should have found that $\int_1^\infty x^p\,dx$ exists $\Leftrightarrow$ $p < -1$.

For $-1 \leqslant p < 0$, $\int_1^\infty x^p\,dx$ does not exist, even though $\lim_{x\to\infty} x^p = 0$.

So the condition $\lim_{x\to\infty} f(x) = 0$ is *necessary but not sufficient* for $\int_a^\infty f(x)\,dx$ to exist.

EXERCISE 22I

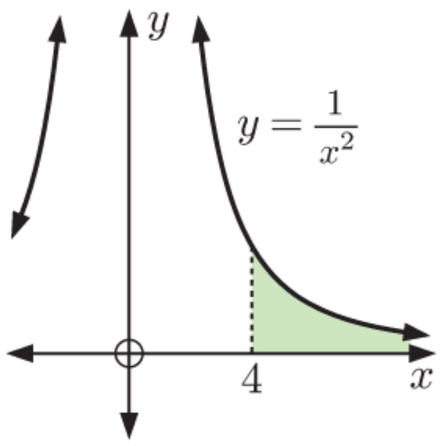
1 Evaluate, if possible:

a $\int_2^\infty \frac{1}{x^4}\,dx$ **b** $\int_1^\infty x^2\,dx$ **c** $\int_8^\infty \frac{1}{\sqrt[3]{x}}\,dx$

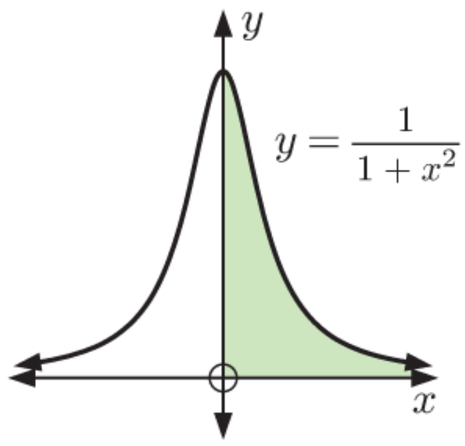
d $\int_3^\infty \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$ **e** $\int_1^\infty e^{-x}\,dx$ **f** $\int_0^\infty xe^{-x^2}\,dx$

2 Find the shaded area:

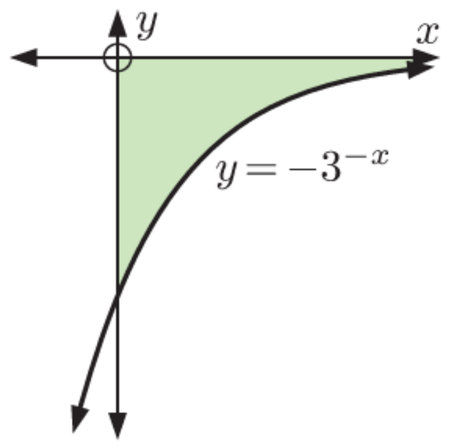
a



b



c



3 Find:

a $\int_1^\infty \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+3}}\right) dx$ **b** $\int_{\frac{1}{\pi}}^\infty \frac{1}{x^2} \sin\frac{1}{x}\,dx$ **c** $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

4 Evaluate $\int_{\ln\sqrt{3}}^\infty \frac{1}{e^x + e^{-x}}\,dx$ using the substitution $u = e^x$.

Example 19 **Self Tutor**

Evaluate $\int_a^\infty xe^{-x}\,dx$ for any constant $a \in \mathbb{R}$.

$$\int_a^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_a^b xe^{-x}\,dx$$

$$= \lim_{b\to\infty} \left(\left[-xe^{-x}\right]_a^b - \int_a^b -e^{-x}\,dx \right) \quad \{\text{integration by parts}\}$$

$$= \lim_{b\to\infty} \left(-be^{-b} + ae^{-a} - \left[e^{-x}\right]_a^b\right)$$

$$= \lim_{b\to\infty} \left(-be^{-b} + ae^{-a} - e^{-b} + e^{-a}\right)$$

$$= e^{-a}(a+1) + \lim_{b\to\infty} \left(e^{-b}(-1-b)\right)$$

$$= e^{-a}(a+1) + \lim_{b\to\infty} \left(\frac{-1-b}{e^b}\right)$$

$$= e^{-a}(a+1) + \lim_{b\to\infty} \frac{-1}{e^b} \quad \{\text{l'Hôpital's rule}\}$$

$$= e^{-a}(a+1)$$

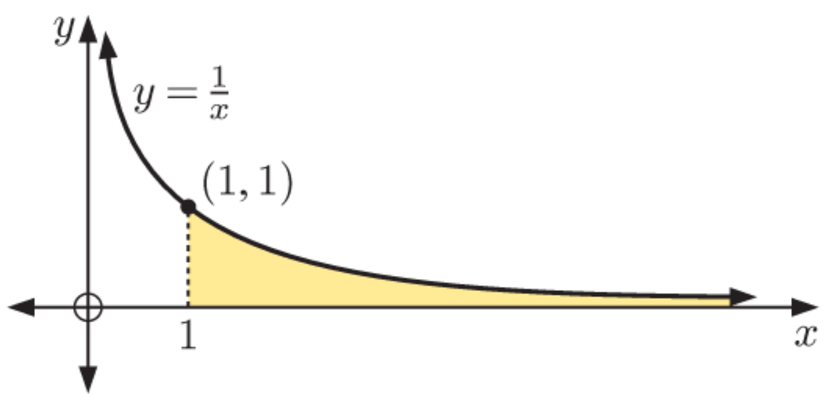
5 **a** Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2, 3$.

b Predict the value of $\int_0^\infty x^n e^{-x}\,dx$ for $n \in \mathbb{Z}^+$.

c Prove your prediction using mathematical induction.

6 Find the total area between $f(x) = e^{-x}\sin^2 x$ and the x-axis for $x \geqslant 0$.

7 **Gabriel's horn**, also known as **Torricelli's trumpet** after the Italian mathematician **Evangelista Torricelli** (1608 - 1647), is the solid of revolution formed when the curve $y = \frac{1}{x}$ on the interval $x \geqslant 1$ is revolved about the x-axis.



Prove that the shaded area under $y = \frac{1}{x}$ is infinite, and yet Gabriel's horn has finite volume.



ACTIVITY 4 SINGULARITIES

In this Activity we consider improper integrals for which the function being integrated is undefined at one or more points on the domain of integration.

When the function approaches a vertical asymptote, we say there is a **singularity** at that point.



REVIEW SET 22A

1 Find the exact value of:

a $\int_{-2}^{0} (1-3x)\,dx$ **b** $\int_{0}^{\frac{1}{2}} \left(x-\sqrt{x}\right) dx$ **c** $\int_{1}^{2} (x^2+1)^2\,dx$

2 Find b such that:

a $\int_{0}^{b} (x-b)^2\,dx = 9$ **b** $\int_{0}^{b} \left(x^2+\frac{1}{2}x\right) dx = 3$ **c** $\int_{0}^{b} e^{1-2x}\,dx = \frac{e}{4}$

3 Find the exact value of:

a $\int_{-5}^{-1} \sqrt{1-3x}\,dx$ **b** $\int_{0}^{\frac{\pi}{2}} \cos\frac{x}{2}\,dx$ **c** $\int_{2}^{6} \frac{2}{x}\,dx$

d $\int_{0}^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right) dx$ **e** $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\sec^2\left(\frac{x}{2}\right) dx$

4 Find $\frac{d}{dx}(e^{-2x}\sin x)$ and hence evaluate $\int_{0}^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$.

5 Find exactly:

a $\int_{3}^{4} \frac{x}{\sqrt{2x+1}}\,dx$ **b** $\int_{0}^{1} x^2 e^{x+1}\,dx$ **c** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot\theta\,d\theta$

d $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$ **e** $\int_{0}^{\frac{\pi}{6}} \frac{\cos x}{(1+\sin x)^3}\,dx$ **f** $\int_{-2}^{2} \frac{60}{x^2-16}\,dx$

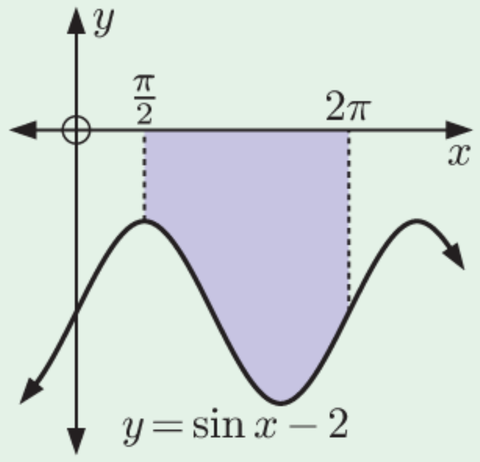
6 If $\int_{1}^{4} f(x)\,dx = 3$, determine:

a $\int_{1}^{4} (f(x)+1)\,dx$ **b** $\int_{1}^{2} f(x)\,dx - \int_{4}^{2} f(x)\,dx$

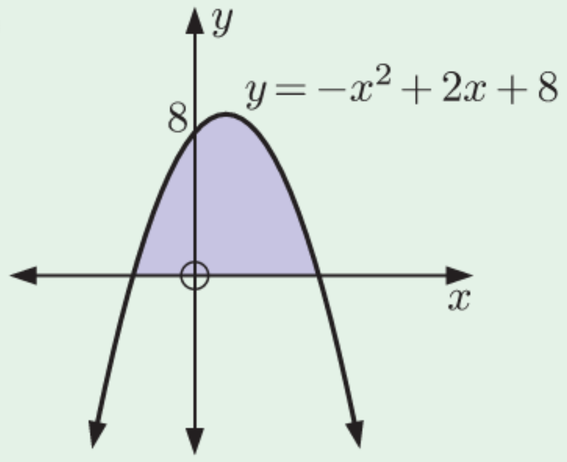
c k given that $\int_{4}^{1} k\,f(x)\,dx = 5$.

7 Find the shaded area:

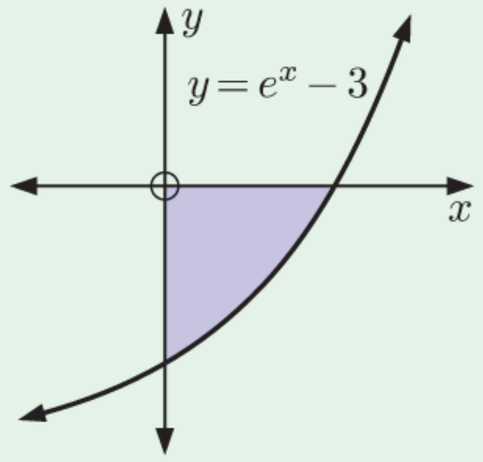
a



b



c



8 Find the area of the region bounded by:

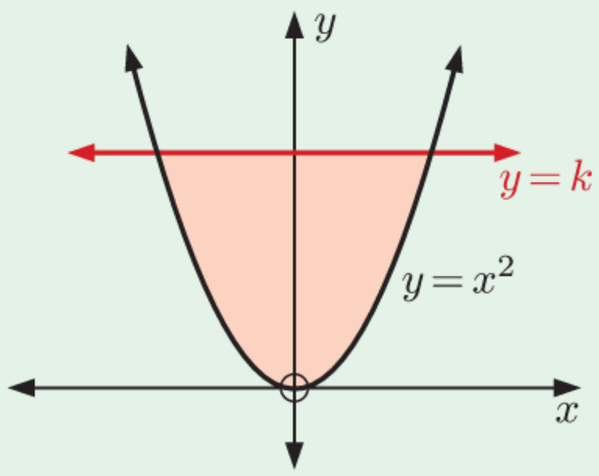
a $y = x^2$, the x-axis, $x = 2$, and $x = 5$

b $y = \sqrt{5-x}$, the x-axis, $x = 1$, and $x = 2$

c $y = \sin\frac{x}{3}$, the x-axis, $x = \pi$, and $x = 2\pi$.

9 Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

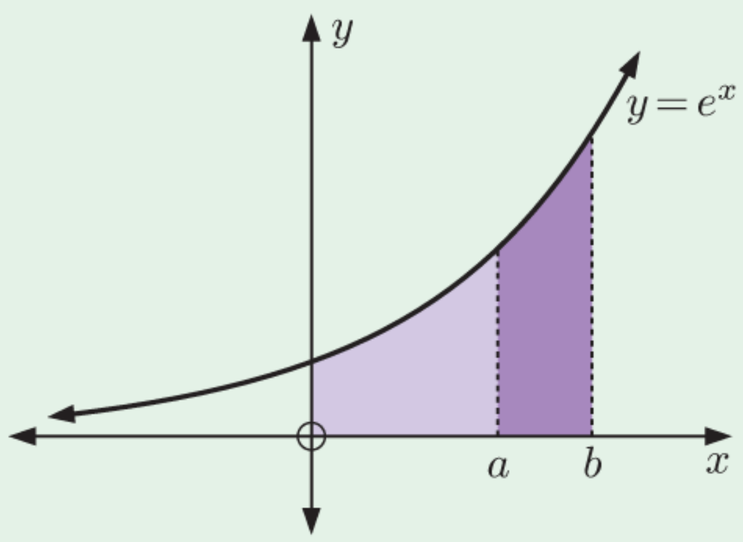
10



Determine k given that the enclosed region has area $5\frac{1}{3}$ units^2.

11 **a** Find a given that the area of the region between $y = e^x$ and the x-axis from $x = 0$ to $x = a$ is 2 units^2.

b Hence determine b such that the area of the region from $x = a$ to $x = b$ is also 2 units^2.

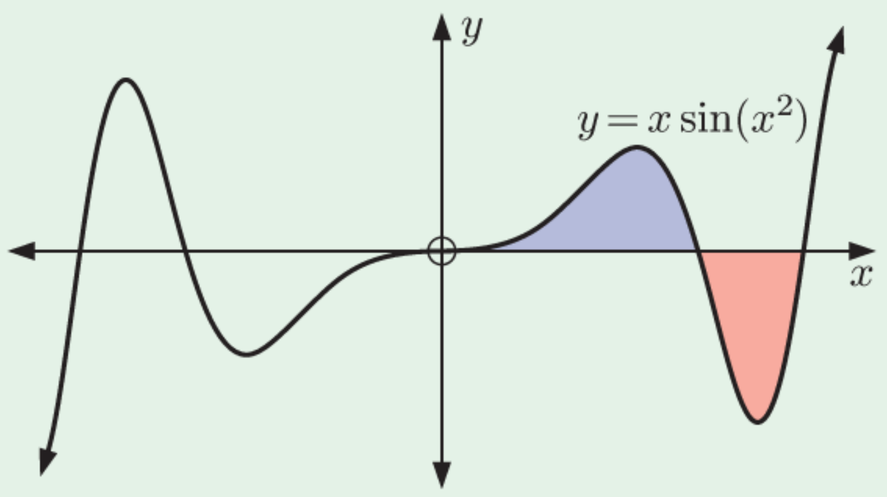


12 Find the area of the region enclosed by $y = x^3 + x^2 + 2x + 6$ and $y = 7x^2 - x - 4$.

13 **a** Sketch the graphs of $y = \sin^2 x$ and $y = \sin x$ on the same set of axes for $0 \leqslant x \leqslant \pi$.

b Find the exact value of the area enclosed by these curves for $0 \leqslant x \leqslant \frac{\pi}{2}$.

14 Show that the shaded regions have equal area.

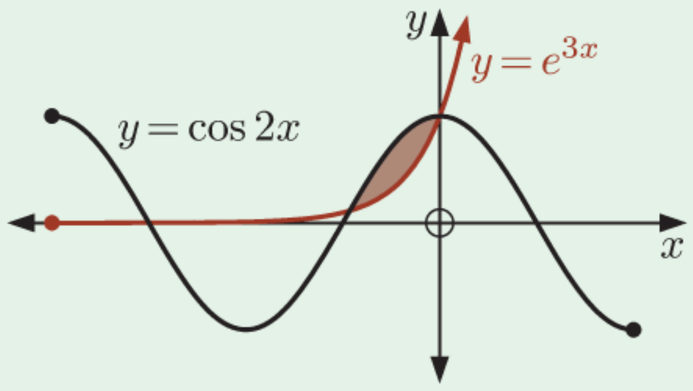


15 The figure shows the graphs of $y = \cos 2x$ and $y = e^{3x}$ for $-\pi \leqslant x \leqslant \frac{\pi}{2}$.

Find correct to 4 decimal places:

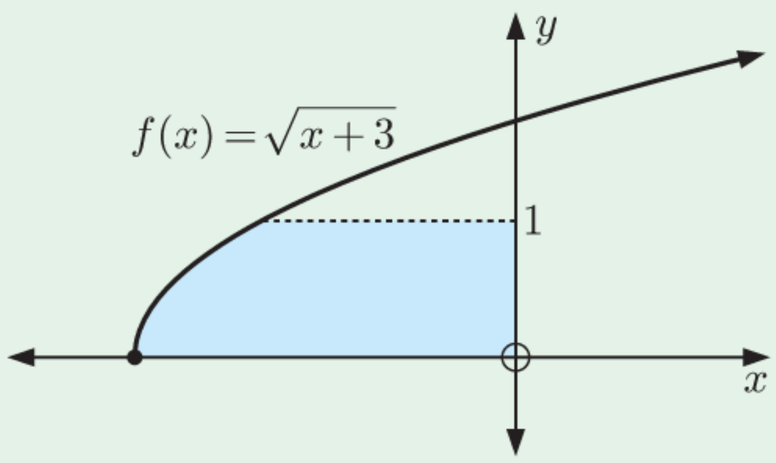
a the x-coordinates of their points of intersection

b the area of the shaded region.

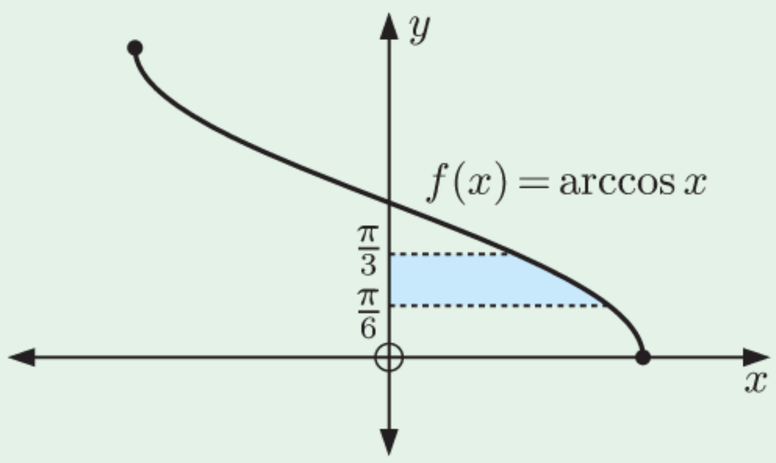


16 Find the shaded area:

a



b



17 Consider the function $f(x) = \sqrt{\frac{1}{x} - 1}$.

a Use technology to help sketch $y = f(x)$.

b Find the exact area bounded by $y = f(x)$ and the y-axis, between $y = 1$ and $y = \sqrt{3}$.

18 Find the volume of the solid of revolution formed when the following curves are revolved through 360° about the x-axis:

a $y = x$ between $x = 4$ and $x = 10$

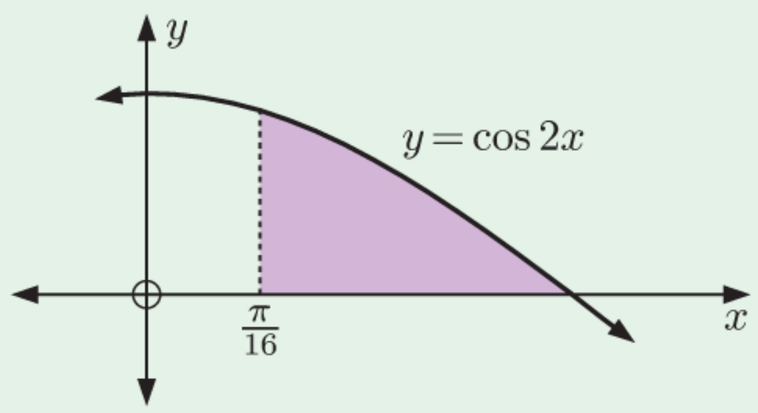
b $y = x + 1$ between $x = 4$ and $x = 10$

c $y = \sqrt{\sin x}$ for $0 \leqslant x \leqslant \pi$

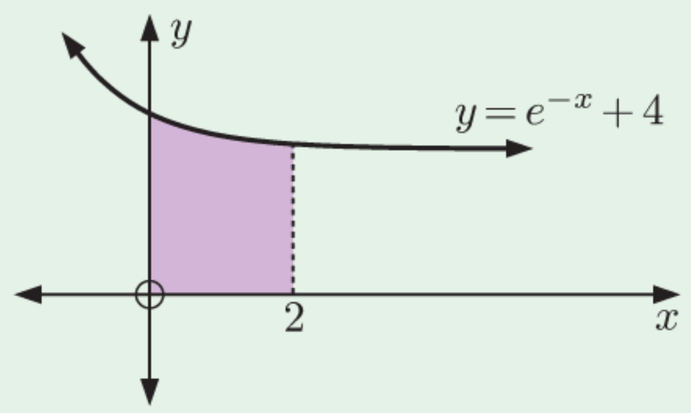
d $y = 1 - \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

19 Find the volume of the solid of revolution formed when the shaded region is revolved through 360° about the x-axis:

a



b

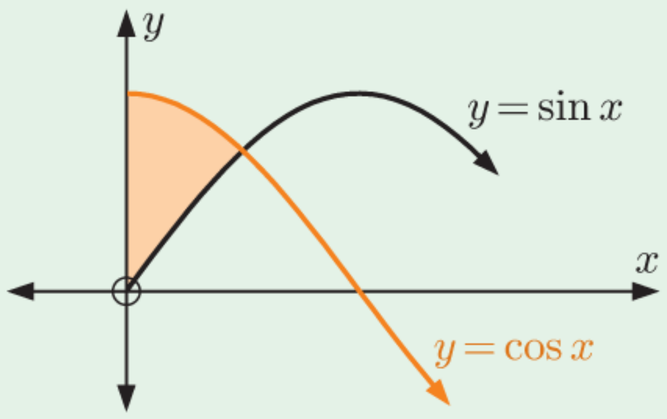


20 Find the volume of the solid of revolution formed when the following are rotated through 2π about the y-axis:

a $y = \sqrt{x}$ between $y = 1$ and $y = 2$

b $y = \sqrt[3]{x^2}$ between $y = 2$ and $y = 3$.

21



Find the volume of the solid of revolution obtained when the shaded region is rotated through 360° about the x-axis.

22 Bettina is filling a watering can with water from a tap.
The water enters at the rate $R_1(t) = 6.4$ litres per minute.
Bettina's watering can starts empty, and has capacity 16 litres. Unfortunately it has a hole in the bottom, so it leaks water at the rate $R_2(t) = 2.5 - 1.25e^{-0.2t}$ litres per minute, where t is the time in minutes.

a Evaluate the following integrals, and interpret their meaning:

i $\int_0^{\frac{1}{2}} R_2(t)\,dt$

ii $\int_0^1 [R_1(t) - R_2(t)]\,dt$

b To the nearest second, how long will it take for the watering can to be filled?

23 Evaluate, if possible:

a $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$

b $\int_2^\infty \frac{x}{\sqrt{x^2 - 1}}\,dx$

c $\int_{-1}^\infty 2^{-x}\,dx$

24 Find the volume of the solid formed when $y = \frac{1}{x^2}$, $x \geqslant 1$ is revolved about the x-axis.

25 The curve $f(x) = \frac{1}{\pi}\sqrt{\frac{a}{x^2+a^2}}$ is revolved about the x-axis.

a Explain why the volume of revolution is given by $V = 2\pi \int_0^\infty [f(x)]^2 \, dx$.

b Find the volume of revolution.

REVIEW SET 22B

1 Find the exact value of:

a $\int_2^3 \frac{1}{\sqrt{3x}} \, dx$ **b** $\int_1^4 (x - \frac{1}{2}x^2) \, dx$ **c** $\int_0^1 (x^2 + \frac{1}{3})^2 \, dx$

2 Find a such that:

a $\int_0^a (x^2 - \frac{1}{2}x) \, dx = \frac{9}{16}$ **b** $\int_0^a \cos x \, dx = \frac{1}{\sqrt{2}}, \ 0 < a < \pi$

3 Find the exact value of:

a $\int_2^3 \frac{1}{\sqrt{3x-4}} \, dx$ **b** $\int_{2e}^{3e} \frac{4}{x+e} \, dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin x + 1) \, dx$

d $\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx$ **e** $\int_0^{\frac{\pi}{4}} \tan x \, dx$ **f** $\int_0^1 (e^x + 2)^3 \, dx$

4 Evaluate correct to 4 significant figures:

a $\int_{-2}^0 4e^{-x^2} \, dx$ **b** $\int_0^1 \frac{10x}{\sqrt{3x+1}} \, dx$

5 **a** Show that $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$.

b Hence find $\int_0^{\frac{\pi}{4}} (\sin\theta - \cos\theta)^2 \, d\theta$.

6 **a** Show that $\frac{4}{x-2} + \frac{3}{x+1} = \frac{7x-2}{(x-2)(x+1)}$.

b Hence find $\int_3^4 \frac{7x-2}{x^2-x-2} \, dx$.

7 Find the exact value of:

a $\int_0^1 \frac{4x^2}{(x^3+2)^3} \, dx$ **b** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^5 x \cos x \, dx$ **c** $\int_1^2 3x^2\sqrt{x^3-1} \, dx$

d $\int_{-1}^2 -x(1+x^2)^3 \, dx$ **e** $\int_0^1 x^2 e^{1-x^3} \, dx$

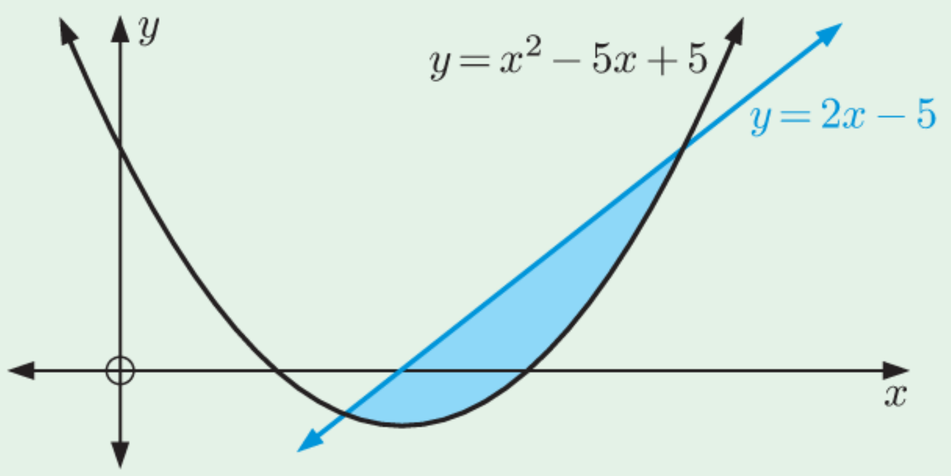
8 Find the area of the region bounded by:

a $y = \frac{1}{x}$, the x-axis, $x = 3$, and $x = 9$

b $y = e^{2x}$, the x-axis, $x = 0$, and $x = 2$

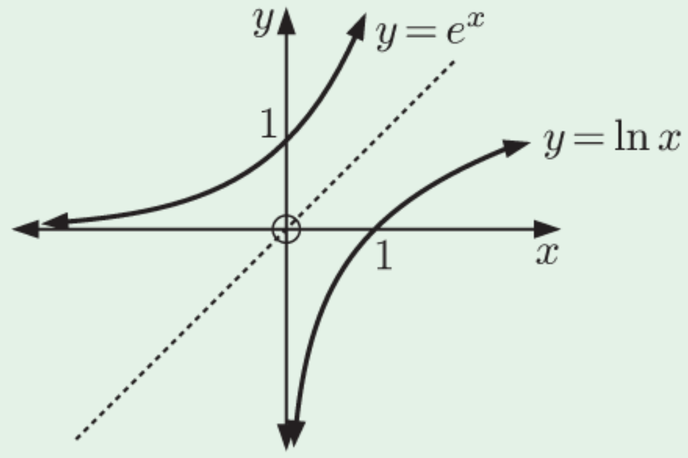
c $y = -2 - 2\cos 3x$ and the x-axis from $x = -\frac{\pi}{3}$ to $x = \frac{\pi}{3}$.

9 Find the shaded area.



10 By appealing to geometrical evidence only, explain why

$$\int_0^1 e^x \, dx + \int_1^e \ln x \, dx = e.$$

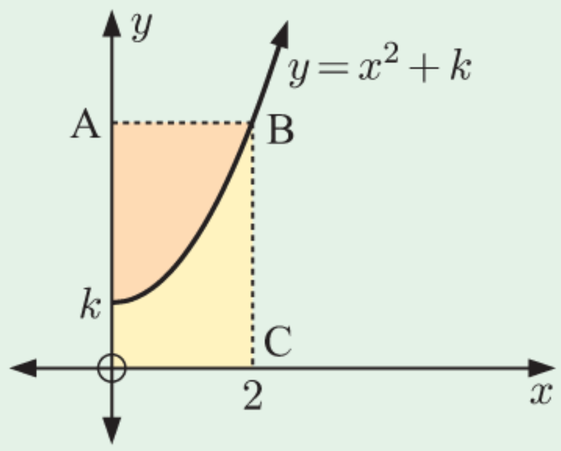


11 Determine the area enclosed by:

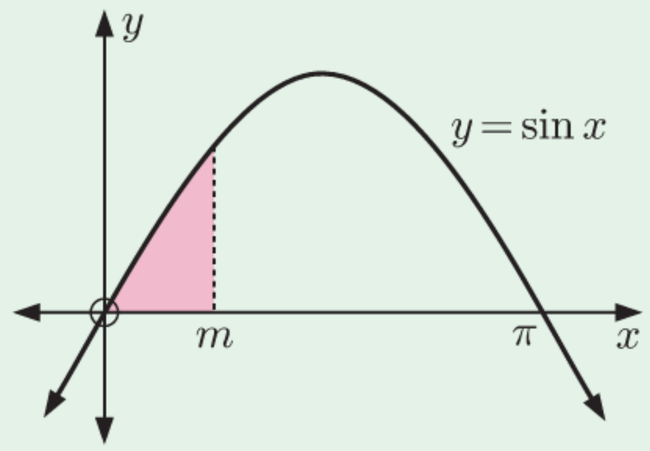
a the axes and $y = 4e^x - 1$

b $y = x^4 - 12$ and $y = -x^2$.

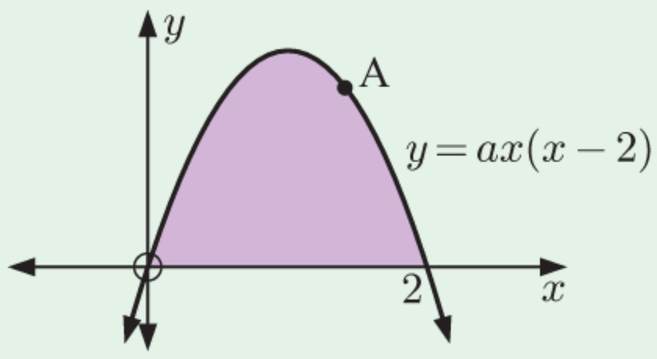
12 OABC is a rectangle and the two shaded regions are equal in area. Find k.



13 The shaded region has area $\frac{1}{2}$ unit^2. Find the value of m.



14



a Find a given that the shaded area is 4 units^2.

b Find the x-coordinate of A given [OA] divides the shaded region into equal areas.

15 Determine the total area enclosed by:

a $y = x^3$ and $y = 7x^2 - 10x$

b $y = x$, $y = \sin x$, and $x = \pi$

c $y = \frac{2}{\pi}x$ and $y = \sin x$

d $y = \tan x$, $x = \frac{\pi}{3}$, and $y = 0$

16 Consider $f(x) = 2 - \sec^2 x$ for $-4 \leqslant x \leqslant 4$.

a Use technology to help sketch the graph of the function.

b Find the equations of the function's vertical asymptotes.

c Find the axes intercepts.

d Find the area of the region bounded by one arch of the function and the x-axis.

17 Find the area enclosed by:

a the y-axis, the line $y = 3$, and the curve $x = \ln\left(\frac{y+3}{2}\right)$

b $y = x^3 + 2$, the y-axis, $y = 3$, and $y = 6$.

18 **a** Find $\frac{d}{dx}[\ln(\tan x + \sec x)]$ and hence find $\int \sec x \, dx$.

b Consider $y = \sec 2x$ for $0 \leqslant x \leqslant \pi$.

i Sketch the graph of the function on the given domain.

ii Find the area of the region bounded by $y = \sec 2x$, the y-axis, and the line $y = 3$.

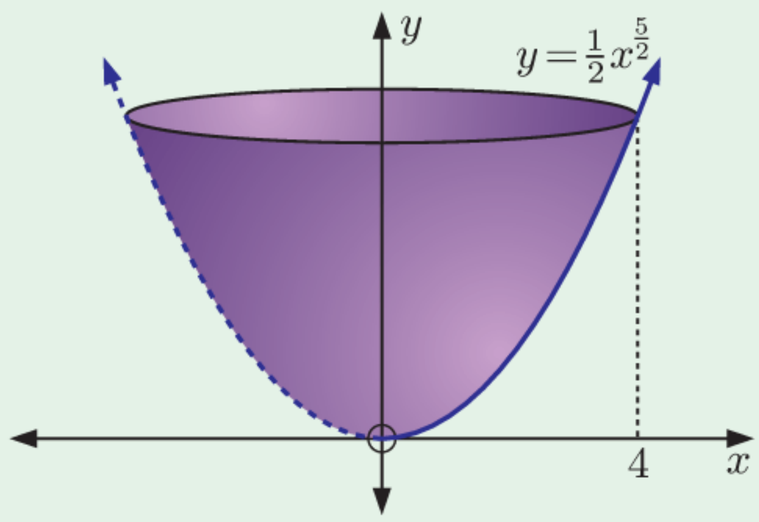
19 Find the volume of the solid of revolution formed when the following curves are rotated about the x-axis:

a $y = \sin x$ between $x = 0$ and $x = \pi$ **b** $y = \sqrt{9 - x^2}$, $0 \leqslant x \leqslant 3$.

20 **a** Find the derivative of $\cot x$.

b Find the volume of revolution when $y = \operatorname{cosec} x$ is rotated through $360°$ about the x-axis for $\frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4}$.

21 The shape of a vase is found by revolving the curve $y = \frac{1}{2}x^{\frac{5}{2}}$, $0 \leqslant x \leqslant 4$ through $360°$ about the y-axis. The units are in centimetres.



a Find the height of the vase.

b Find the volume of the vase.

c The vase is filled with water to a height of 10 cm. Find the amount of water in the vase.

22 Over the course of a day, the rate of solar energy being transferred into Callum's solar panels is given by $E(t) = 2\sin\left(\frac{t-5}{5}\right) + \frac{1}{2}\sin\left(\frac{t-5}{4}\right)$ kW where t is the time in hours after midnight, $5 \leqslant t \leqslant 20$.

Find the following quantities and explain what they represent:

a $\int_5^{12} E(t)\,dt$ **b** $\int_{12}^{20} E(t)\,dt$ **c** $\int_5^{20} E(t)\,dt$

23 **a** Find $\int_0^{\infty} ae^{-ax}\,dx$, $a > 0$.

b Explain your result by sketching the cases $a = 1, 2, 3$ on the same set of axes.

24 **a** Prove by mathematical induction that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{Z}^+$.

b Let $\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx$ for all $n \in \mathbb{Z}^+$.

i Evaluate $\Gamma(1)$.

ii Use integration by parts to show that $\Gamma(n) = (n-1)\Gamma(n-1)$ for all $n \in \mathbb{Z}^+$, $n > 1$.

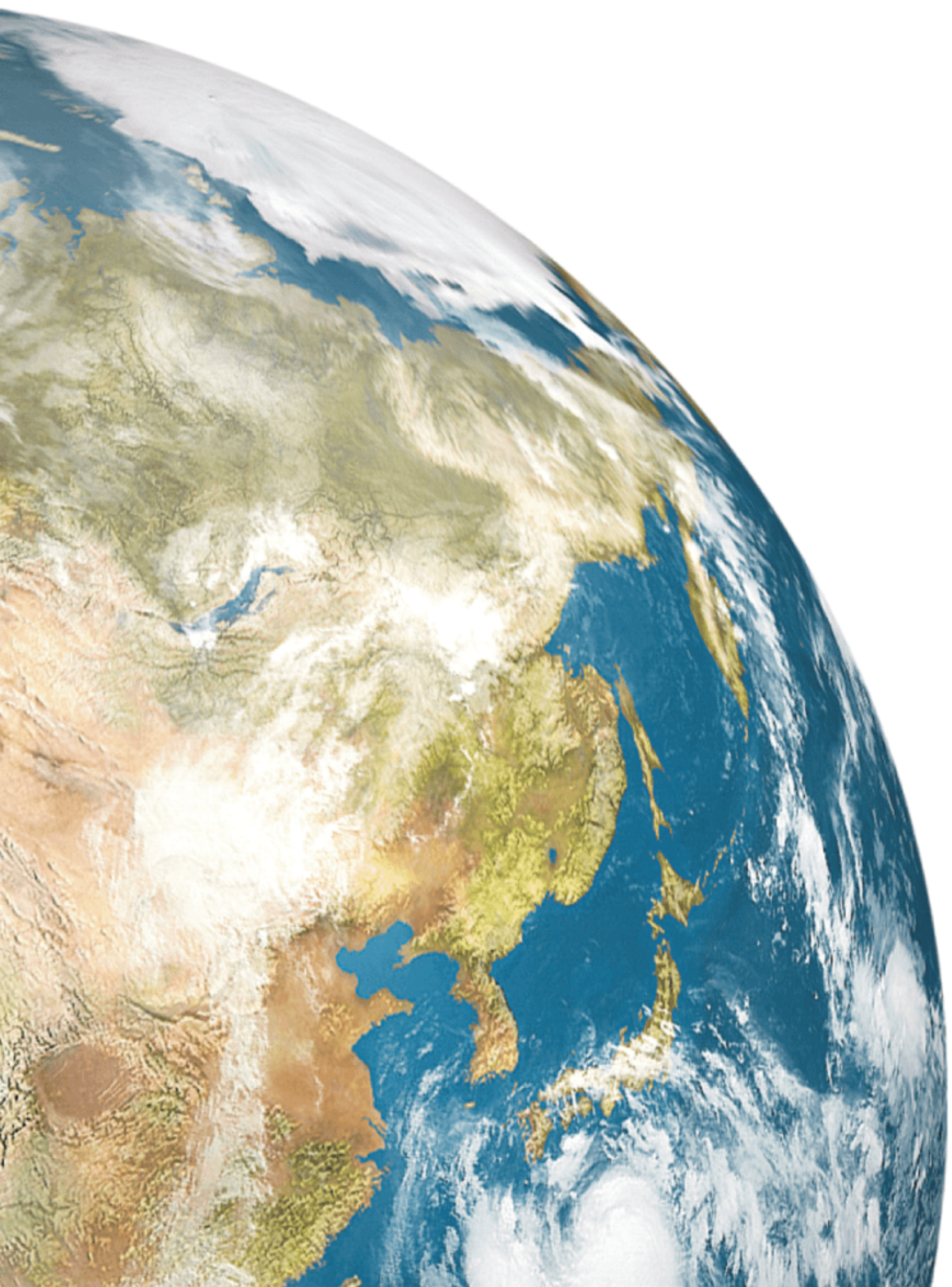
iii Hence show that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{Z}^+$.

Chapter 23

Kinematics

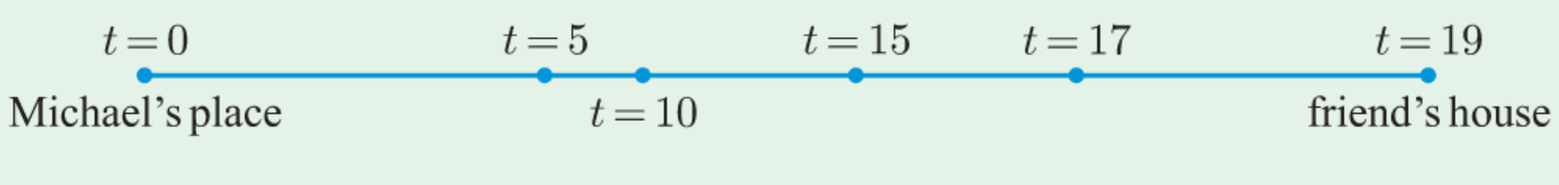
Contents:

A Displacement
B Velocity
C Acceleration
D Speed



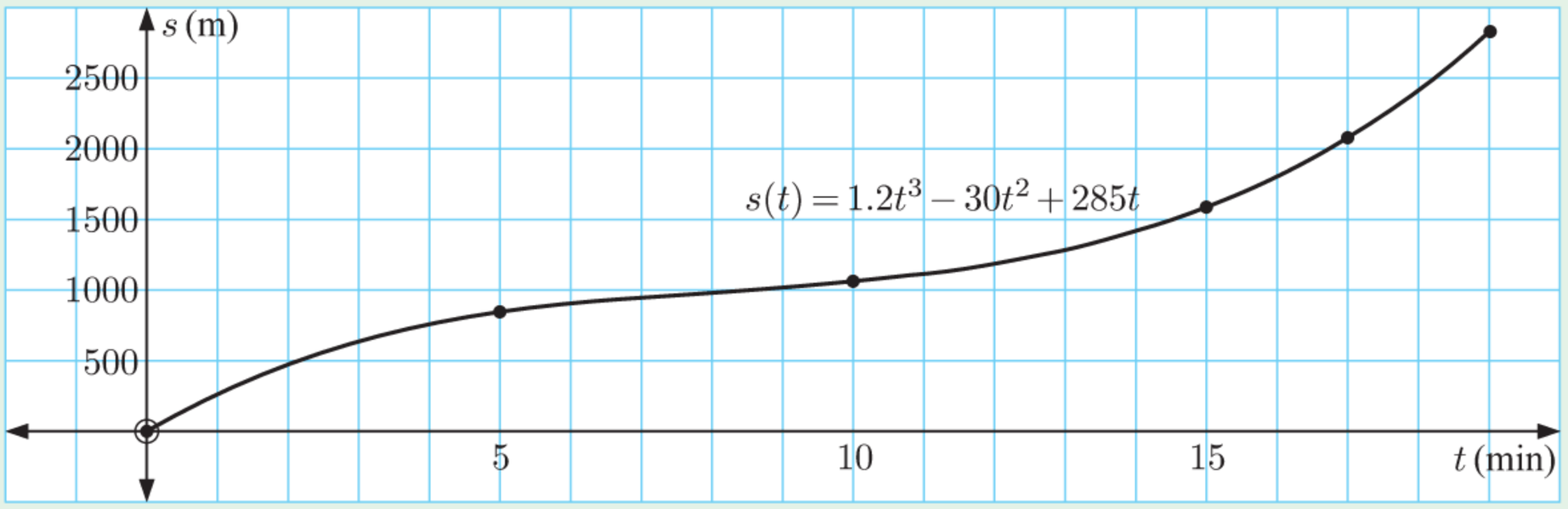
OPENING PROBLEM

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.





The distance Michael has travelled at various times is given by the function $s(t) = 1.2t^3 - 30t^2 + 285t$ metres for $0 \leqslant t \leqslant 19$ minutes.



Things to think about:

a At what point do you think the hill was steepest? How far had Michael travelled to this point?

b How can we find Michael's *average* speed between $t = 5$ and $t = 15$ minutes?

c How can we find Michael's speed at a particular instant? Can we write a *function* which will tell us Michael's speed at any time t?



Kinematics is the study of motion.

In previous years you should have studied **travel graphs** which plot distance travelled against time taken.

In this Chapter we consider kinematics more formally, taking into account *direction* as well as distance and speed. This will lead us to functions for **displacement**, **velocity**, and **acceleration**, and how they are linked by calculus.

THE LANGUAGE OF MOTION

Consider an object P which is in motion along a straight line.

- The **distance** it travels accumulates over time, irrespective of its direction of travel.
- The **speed** S of P is how fast it is travelling.
- The **displacement** s of P is its position relative to a fixed origin O.
- The **velocity** v of P is its rate of change of displacement.
- The **acceleration** a of P is its rate of change of velocity.

DISCUSSION

1 What is the difference between:

a distance travelled and displacement **b** speed and velocity?

2 Each of the following expressions refers to a zero of one of the variables time t, displacement s, velocity v, or acceleration a. Discuss which variable each expression refers to and what it means.

a initial conditions **b** at the origin

c stationary **d** reverses direction

e maximum or minimum displacement **f** constant velocity

g maximum or minimum velocity

3 Does a zero displacement correspond to:

a zero distance travelled **b** zero speed?

4 How should we interpret the sign of:

a displacement **b** velocity **c** acceleration?

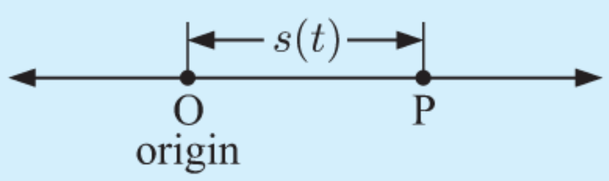
QUICK REFERENCE SUMMARY



A DISPLACEMENT

For an object P which is in motion along a straight line:

- The **displacement** s of P is its position relative to a fixed origin O.
- The **displacement function** $s(t)$ gives the displacement of the object at any time $t \geqslant 0$.
  - When $s(t) > 0$, P is to the right of the origin.
  - When $s(t) < 0$, P is to the left of the origin.



Example 1

Self Tutor

A lighthouse is situated on the top of a cliff. A stone is thrown from the top of the lighthouse. The stone falls past the base of the lighthouse, and lands after 6 seconds in the water below.



The displacement of the stone above ground level $s(t) = -5t^2 + 20t + 25$ m for $0 \leqslant t \leqslant 6$ s.

a Find the initial displacement of the stone.

b Use quadratic theory to graph $s(t)$.

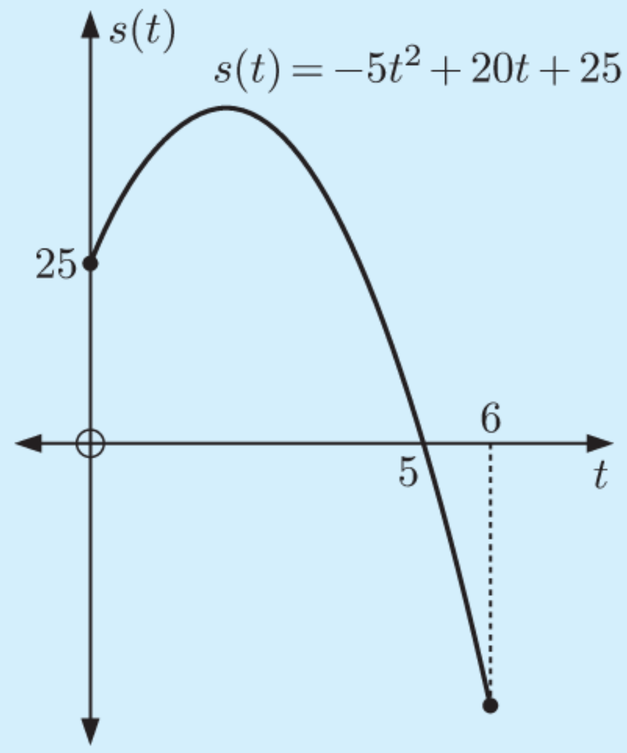
c At what time and place does the stone change direction?

d For what time interval is the stone above ground level?

a $s(0) = 25$

$\therefore$ the initial displacement of the stone is 25 m.

b $s(t) = -5t^2 + 20t + 25$
$= -5(t^2 - 4t - 5)$
$= -5(t - 5)(t + 1)$



c The stone changes direction at the turning point of $s(t)$.

This occurs when $t = \dfrac{-20}{2(-5)} = 2$ s.

$s(2) = -5(2)^2 + 20(2) + 25 = 45$ m

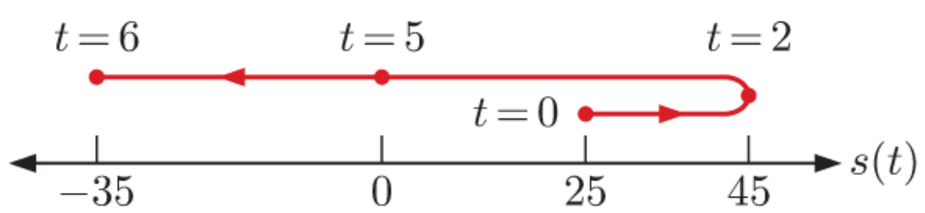
The stone changes direction after 2 seconds, when it is 45 m above ground level.

d The stone is above ground level whenever $s(t) > 0$.

This occurs for $0 \leqslant t \leqslant 5$ s.

A **motion diagram** is used to plot the position of an object on a single number line. It can be helpful in reminding us that we are looking at motion in one dimension only.

For the object in **Example 1**, the motion diagram is:



Click on the icon to explore motion diagrams for other displacement functions.



EXERCISE 23A

1 An object travels with displacement function $s(t) = 5 - t$ cm for $0 \leqslant t \leqslant 10$ s.

a Find the initial displacement of the object.

b Find the displacement of the object at time:

i $t = 3$ s **ii** $t = 10$ s.

c At what time does the object reach the origin?

d Does the object ever change direction? Explain your answer.

e Draw a motion diagram showing the information you have found.

2 An object travels with displacement function $s(t) = 10t^2 - 7t + 1$ m for $0 \leqslant t \leqslant 1$ s.

a Find the initial displacement of the object.

b Use quadratic theory to graph $s(t)$.

c At what time and place does the object change direction?

d For what time interval is the object to the right of the origin?

e Draw a motion diagram showing the information you have found.

3 A mass on a spring oscillates with displacement $s(t) = 8\sin 8\pi t + 6$ cm where $0 \leqslant t \leqslant 0.25$ s.

a At what times is the displacement 6 cm?

b Plot the graph of $s(t)$ for $0 \leqslant t \leqslant 0.25$ s.

c Find where and when the mass changes direction.

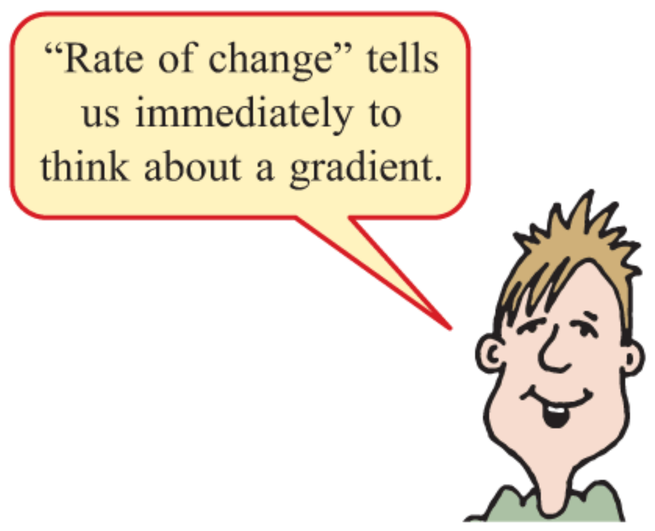
d Draw a motion diagram showing the information you have found.

B VELOCITY

The **velocity** of an object is its rate of change of displacement.

AVERAGE VELOCITY

In many real-world scenarios, we only know certain data points for where an object was at a particular time. In these cases, we can only calculate the *average* velocity between data points.

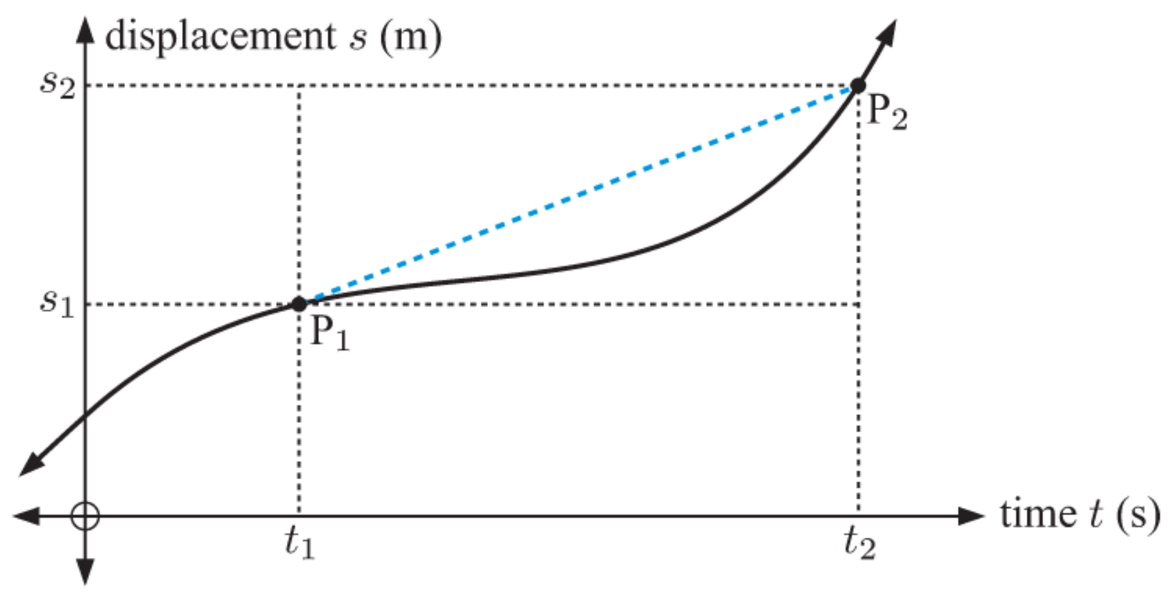


The **average velocity** of an object moving in the time interval from $t = t_1$ to $t = t_2$ is given by

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

On a graph of $s(t)$ against t, the average velocity is the gradient of the chord through the points $P_1(t_1, s(t_1))$ and $P_2(t_2, s(t_2))$.



INSTANTANEOUS VELOCITY

If we know the complete function $s(t)$, we can find the instantaneous velocity of the object at any time.

The **instantaneous velocity** or **velocity function** of the object at time t is $\boldsymbol{v(t) = s'(t)}$.
It is the gradient of the tangent to the function $s(t)$ at any given time.

Notice that:

- When $v(t) > 0$, the object is moving to the right.
- When $v(t) < 0$, the object is moving to the left.
- When $v(t) = 0$, the object is instantaneously at rest. A change in the sign of $v(t)$ at this time indicates that the object has changed direction.

When we take a derivative with respect to t, we calculate a rate per unit of time. So, for a displacement in metres and time in seconds, the units of velocity are m s^{-1}.

Example 2 **Self Tutor**

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity for the time interval from $t = 2$ to $t = 5$ seconds

b the instantaneous velocity at $t = 2$ seconds.

a average velocity $= \dfrac{s(5) - s(2)}{5 - 2}$

$= \dfrac{(15 - 25) - (6 - 4)}{3}$

$= \dfrac{-10 - 2}{3}$

$= -4 \text{ m s}^{-1}$

b $s(t) = 3t - t^2$

$\therefore \ v(t) = s'(t) = 3 - 2t$

$\therefore \ v(2) = 3 - 2(2)$

$= -1 \text{ m s}^{-1}$

The particle is travelling at 1 m s^{-1} to the left.

EXERCISE 23B.1

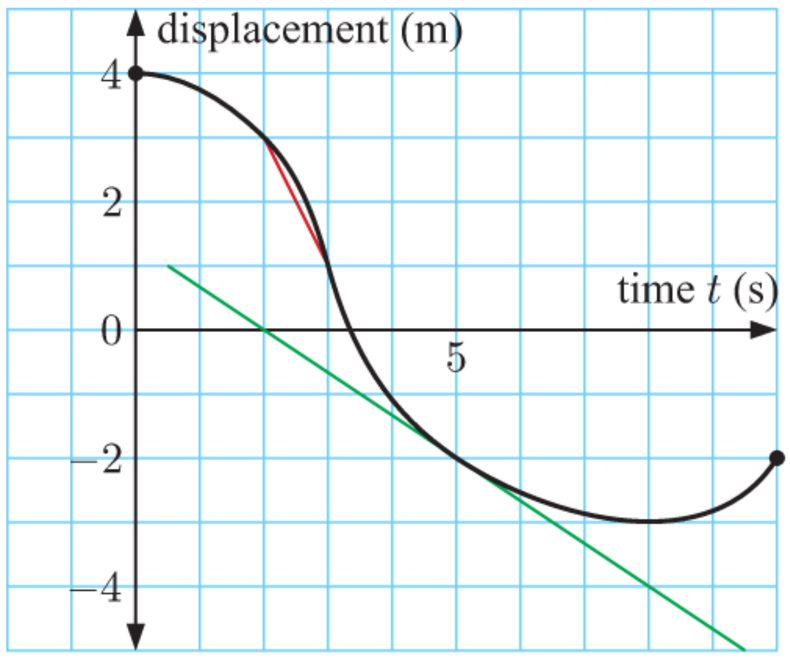
1 Consider the displacement-time graph alongside.

a Find the displacement of the object when:

i $t = 2$ seconds **ii** $t = 8$ seconds.

b Find the average velocity of the object from $t = 2$ to $t = 3$ seconds.

c Find the instantaneous velocity of the object when $t = 5$ seconds.



2 A particle moves in a straight line with displacement from O given by $s(t) = t^2 - 6t + 1$ metres at time t seconds, $t \geqslant 0$.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find $v(t)$.

c Hence find the instantaneous velocity at:

i $t = 1$ second **ii** $t = 5$ seconds.

3 Consider the displacement-time graph alongside.

a Describe the initial position of the object.

b At what time is the object at the origin?

c In which direction is the object moving when $t = 5$ seconds?

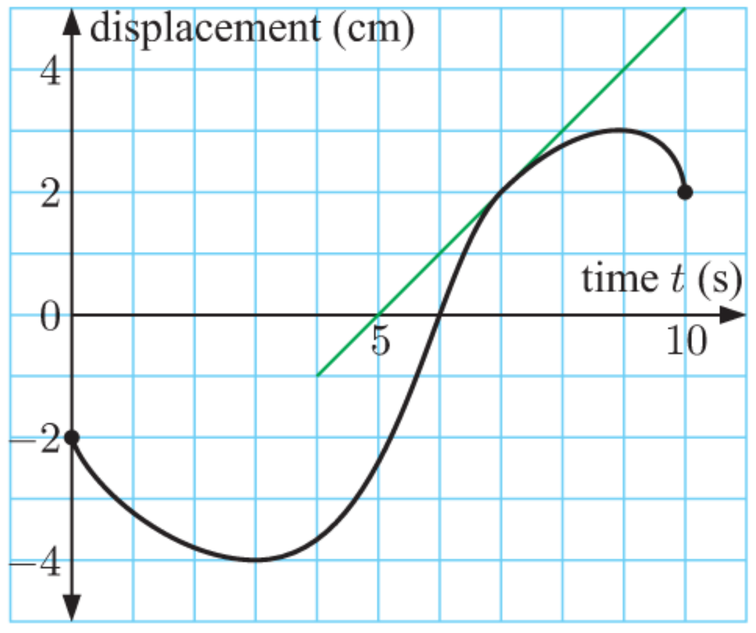
d At what times does the object change direction?

e Draw a sign diagram for:

i the displacement function $s(t)$

ii the velocity function $v(t)$.

f Find the instantaneous velocity of the object when $t = 7$ seconds.



4 An object moves in a straight line with displacement function $s(t) = 2\sqrt{t} + 3$ cm, where t is in seconds, $t \geqslant 0$.

a Find the average velocity from $t = 1$ to $t = 4$ seconds.

b Find the initial position of the object.

c Find $v(t)$.

d Hence find the instantaneous velocity at:

i $t = 4$ seconds **ii** $t = 16$ seconds.

5 An object moves in a straight line with position given by $s(t) = t^3 - 11t^2 + 24t$ m from O, where t is in seconds, $t \geqslant 0$.

a Find the velocity function $v(t)$.

b Describe the initial conditions of the object.

c Draw a sign diagram for $s(t)$ and $v(t)$.

d At what times is the object at O?

e At which times does the object reverse direction? Find the position of the object at each of these times.

f Describe in words the motion of the object.

g Draw a motion diagram for the object.

6 A shell is accidentally fired vertically from a mortar at ground level. Its height above the ground after t seconds is given by $s(t) = bt - 4.9t^2$ metres where b is constant.

a Show that the initial velocity of the shell is b m s^{-1} upwards.

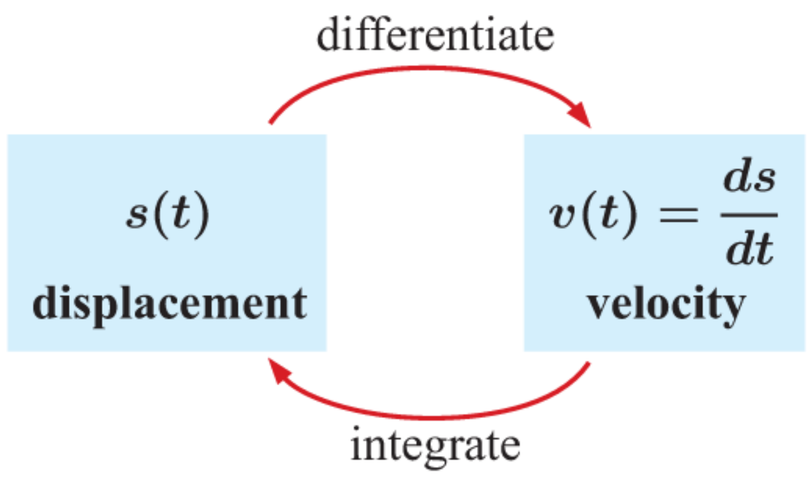
b The shell reaches its maximum height after 7.1 seconds.

i Find the initial velocity of the shell.

ii Find the maximum height reached by the shell.

FINDING DISPLACEMENT FROM VELOCITY

Since the velocity function is the derivative of the displacement function, we can use integration to reverse the process.



To find the constant of integration, we would need to know the displacement at a particular time. Most commonly, this would be the initial displacement $s(0)$.

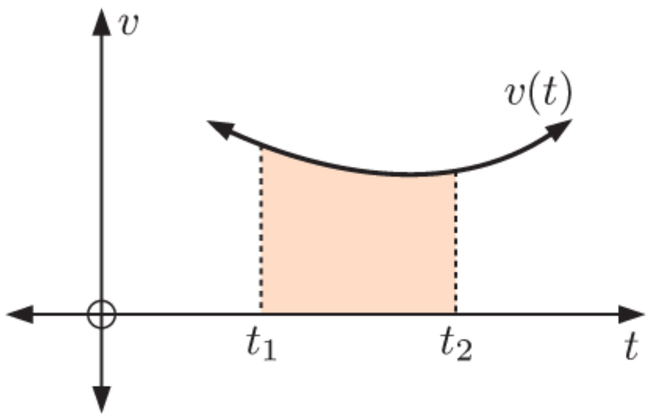
CHANGE IN DISPLACEMENT

We can determine the change in displacement in a time interval $t_1 \leqslant t \leqslant t_2$ using the integral:

$$\textbf{Change in displacement} = s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt$$

DISTANCE TRAVELLED

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$, the distance travelled will simply be the change in displacement $\int_{t_1}^{t_2} v(t)\,dt$.





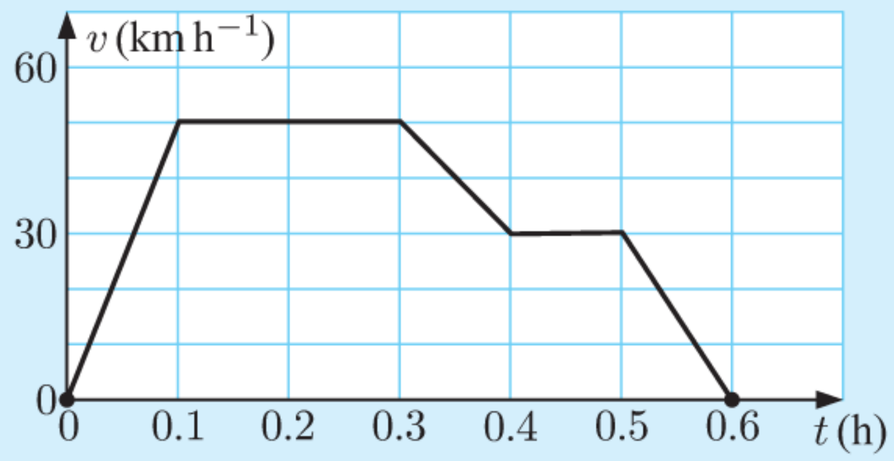
If we have a change of direction within the time interval then the velocity will change sign. To find the total distance travelled, we therefore need to add the components of area above and below the t-axis.

Alternatively, we can integrate the absolute value of the velocity, which is the object's **speed**.

$$\textbf{Distance travelled} = \int_{t_1}^{t_2} |\,v(t)\,|\,dt$$

Example 3 — Self Tutor

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.



Total distance travelled

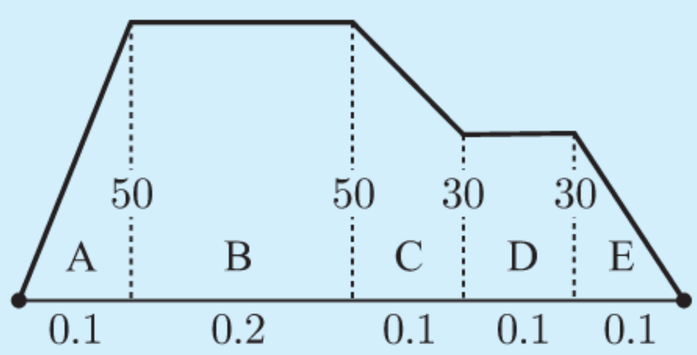
= total area under the graph

= area A + area B + area C + area D + area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30 + \frac{1}{2}(0.1)30$

$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km



Example 4 — Self Tutor

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ m s^{-1}.

a Find the times when P reverses direction.

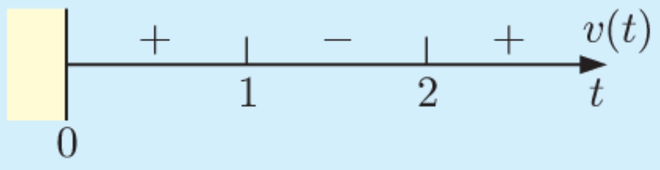
b Hence find the distance P travels in the first 4 seconds of motion.

Check your answer by evaluating $\int_0^4 |\,v(t)\,|\,dt$.

c Find the change in displacement of P in the first 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$= (t-1)(t-2)$

$\therefore$ the sign diagram of v is:



Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

b Now $s(t) = \int (t^2 - 3t + 2)\,dt$

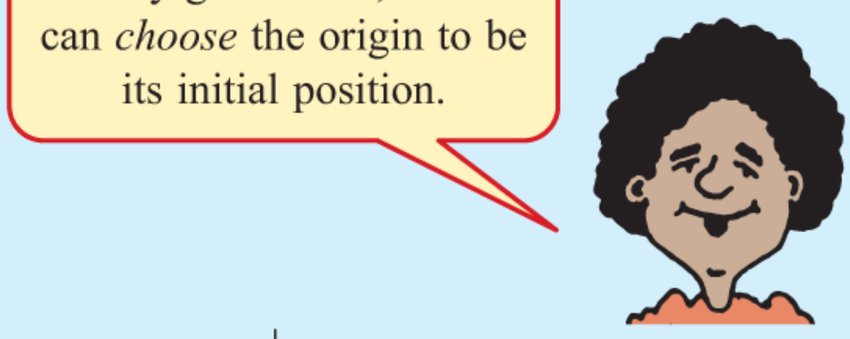
$= \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

We choose the initial displacement to be zero, so $s(0) = c = 0$.

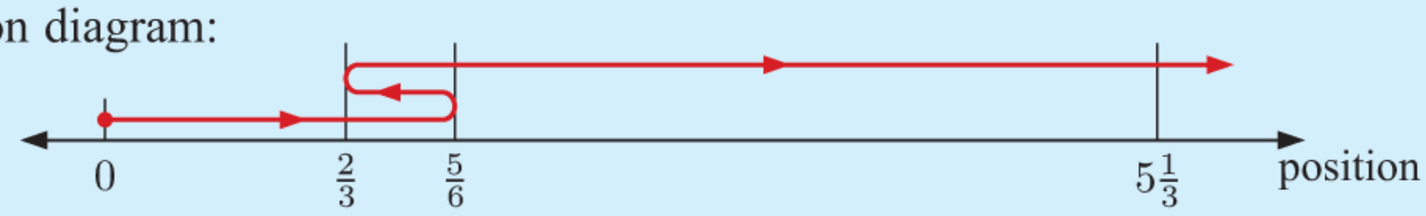
$\therefore \quad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = \frac{2}{3}$

$s(4) = \frac{64}{3} - 24 + 8 + c = 5\frac{1}{3}$

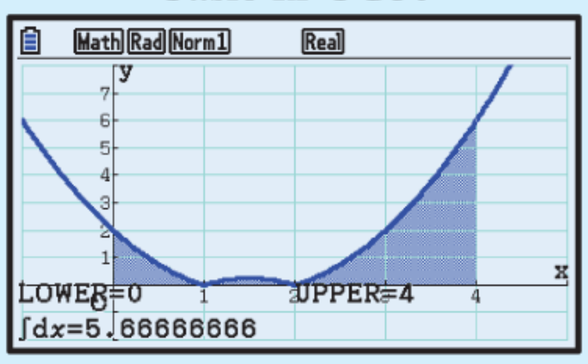


Motion diagram:

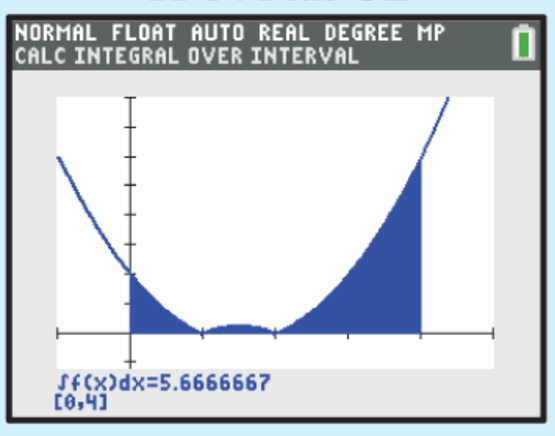


$\therefore$ total distance travelled $= \frac{5}{6} + (\frac{5}{6} - \frac{2}{3}) + (5\frac{1}{3} - \frac{2}{3}) = 5\frac{2}{3}$ m

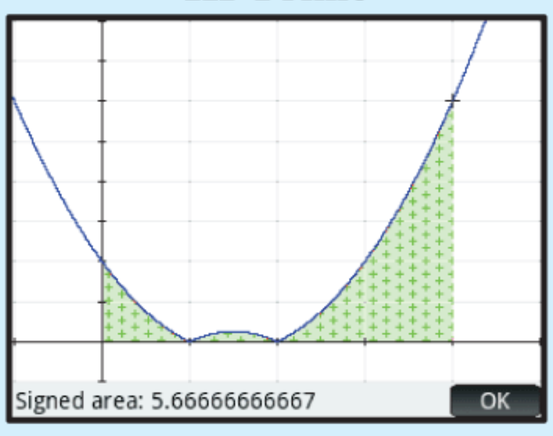
Casio fx-CG50



TI-84 Plus CE



HP Prime

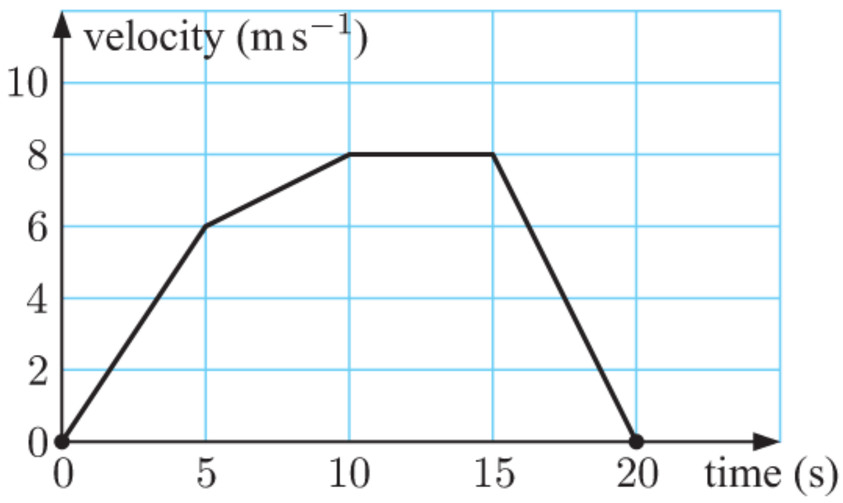


c Change in displacement = final position − original position
$= s(4) - s(0)$
$= 5\frac{1}{3}$ m

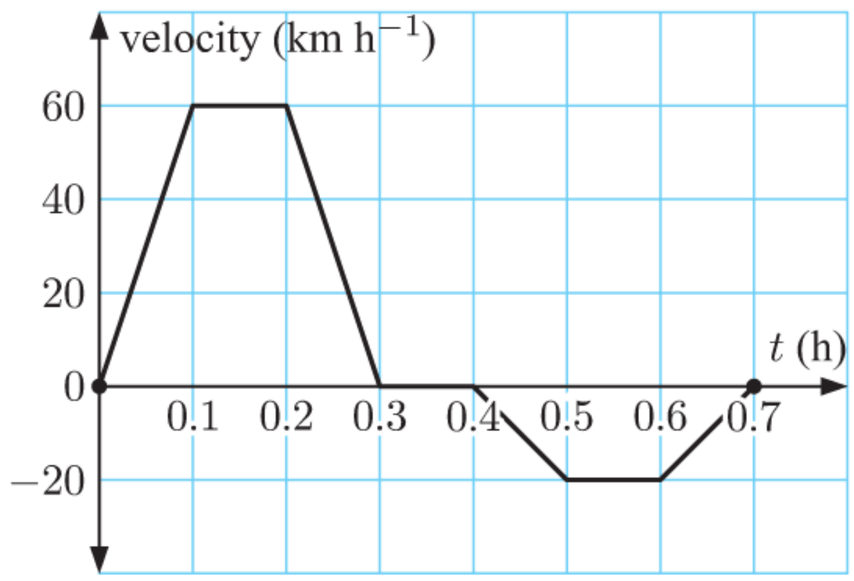
So, the displacement is $5\frac{1}{3}$ m to the right.

EXERCISE 23B.2

1 A runner has the velocity-time graph shown. Find the total distance travelled by the runner.



2



A car travels along a straight road with the velocity-time graph illustrated.

a What is the significance of the graph:

i above the t-axis

ii below the t-axis?

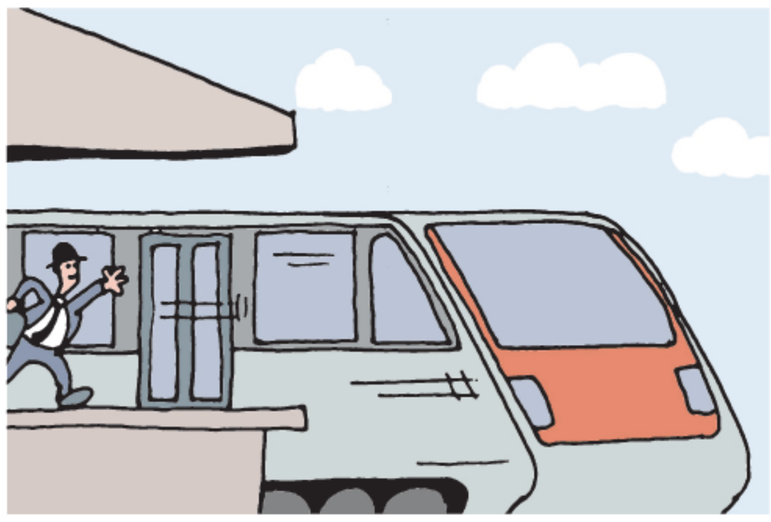
b Find the total *distance* travelled by the car.

c Find the final *displacement* of the car from its starting point.

3 After leaving a station, a train accelerates at a constant rate for 40 seconds until its speed reaches 15 m s^{-1}. The train then travels at this speed for 160 seconds. On its approach to the next station, the train slows down at a constant rate for 80 seconds until it is at rest.

a Draw a graph to show the train's motion.

b How far did the train travel between the stations?



4 A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h^{-1}. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate over one minute to 20 km h^{-1}, then continues at this speed for 5 minutes. At the top of the hill she reduces her speed uniformly, and she is stationary 2 minutes later.

a Draw a graph to show the cyclist's motion.

b How far has the cyclist travelled?

5 A particle has velocity function $v(t) = 1 - 2t$ cm s^{-1} as it moves in a straight line.

a Find the time when the particle reverses direction.

b Hence find the total distance travelled in the first second of motion.

Check your answer by evaluating $\int_0^1 |v(t)|\,dt$.

c Find the change in displacement of the particle in the first second.

6 Particle P is initially at the origin O. It moves with the velocity function $v(t) = t^2 - t - 2$ cm s^{-1}.

a Write a formula for the displacement function $s(t)$.

b Find the total distance travelled in the first 3 seconds of motion.

c Find the particle's displacement from its starting position after three seconds.

7 A ball is thrown from 1 m above ground level. Its velocity is $v(t) = 29.4 - 9.8t$ m s^{-1}.

a Find the displacement function $s(t)$.

b Find the maximum height reached by the ball.



8 The velocity of a moving object is given by $v(t) = 32 + 4t$ m s^{-1}.

a If $s = 16$ m when $t = 0$ seconds, find the displacement function.

b Explain why the change in displacement of the object and its total distance travelled in the interval $0 \leqslant t \leqslant \tau$, can both be represented by the definite integral $\int_0^\tau (32 + 4t)\, dt$.

c Find the distance travelled by the object in the first 4 seconds.

9 An object has velocity function $v(t) = \cos 2t$ m s^{-1}.

a Show that the particle oscillates between two points, and find the distance between them.

b If $s(\frac{\pi}{4}) = 1$ m, determine $s(\frac{\pi}{3})$ exactly.

10 When a pendulum is released, its velocity along the arc of motion after t seconds is given by $v(t) = 20 + 5\sin 4t$ cm s^{-1}.

a Sketch the graph of $v(t)$ against t for $0 \leqslant t \leqslant 6$.

b Find the pendulum's velocity after 4.5 seconds.

c Find the total distance travelled by the tip of the pendulum in the first 2 seconds.

11 The velocity of a moving object is given by $v(t) = -4 + \sqrt{t}$ m s^{-1}. Suppose the object is initially at the origin. Find:

a the displacement function

b the time when the object changes direction

c the change in displacement after the first 30 seconds

d the total distance travelled in the first 30 seconds.

12 A stunt motorcyclist rides towards a ramp. His velocity after t seconds is given by $v(t) = 10\sqrt{t}$ m s^{-1}.



a Find the motorcyclist's velocity after:

i 1 second **ii** 2 seconds.

b Write a function $s(t)$ for the displacement of the motorcyclist after t seconds.

c Find $\int_0^2 v(t)\, dt$, and interpret your answer.

d In order to perform his stunt, the motorcyclist needs to be travelling at 20 m s^{-1} or faster when he reaches the ramp.

i How long will he take to reach a speed of 20 m s^{-1}?

ii The motorcyclist started his approach 55 m from the ramp. Has he given himself enough distance to reach the required speed?

13 Use technology and the formula distance travelled $= \int_{t_1}^{t_2} |v(t)|\, dt$ to find the total distance travelled in the following scenarios:

a a skydiver has velocity $v(t) = -54(1 - e^{-\frac{t}{6}})$ m s^{-1} for the first 15 seconds

b a mass on a spring has velocity $v(t) = e^{-t}\cos 16t$ cm s^{-1} for 10 seconds.

C ACCELERATION

The **acceleration** of an object is its rate of change of velocity.

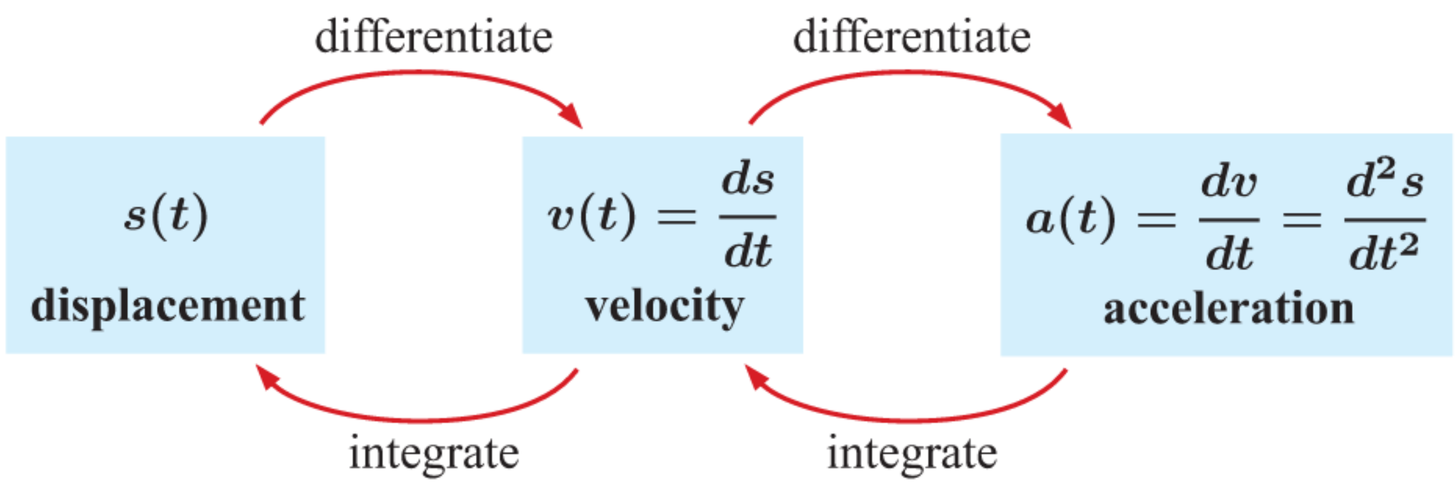
If an object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, then:

- The **average acceleration** for the time interval from $t = t_1$ to $t = t_2$ is given by

$$\textbf{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

- The **acceleration function** $a(t)$ of the object is given by $a(t) = v'(t) = s''(t)$.

The displacement, velocity, and acceleration functions are therefore connected as follows:



To obtain acceleration from velocity, we must calculate a rate per unit of time. So, for displacement in metres and time in seconds, the units of acceleration are m s^{-2}.

Example 5 **Self Tutor**

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 6t^2 + 9t - 1$ cm, where t is the time in seconds, $t \geqslant 0$.

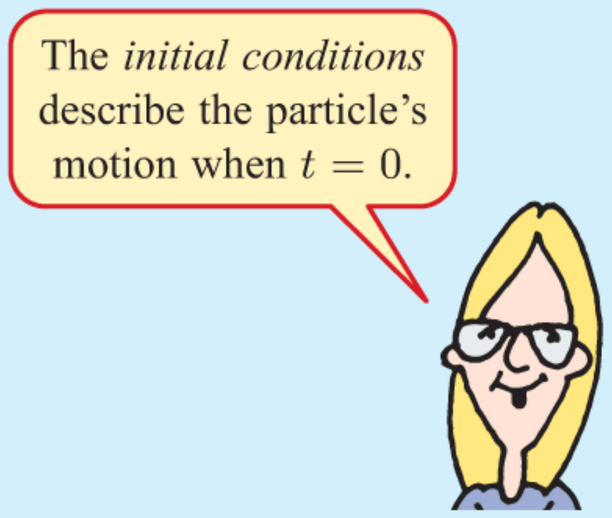
a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Find the position of the particle when it changes direction.

d Draw a motion diagram for the particle.

e Find the total distance travelled in the first 3 seconds.



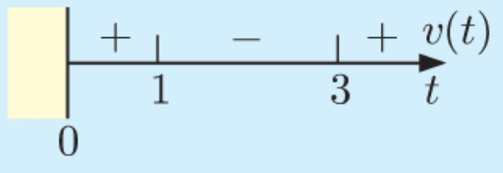
a $s(t) = t^3 - 6t^2 + 9t - 1$ cm

$\therefore\ v(t) = 3t^2 - 12t + 9 \qquad \{v(t) = s'(t)\}$

$= 3(t^2 - 4t + 3)$

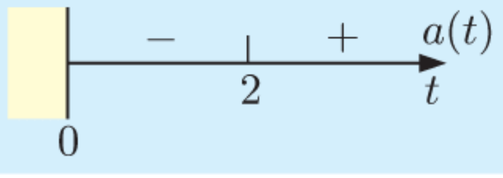
$= 3(t-1)(t-3)\ \text{cm s}^{-1}$

which has sign diagram:



and $a(t) = 6t - 12\ \text{cm s}^{-2} \qquad \{a(t) = v'(t)\}$

which has sign diagram:

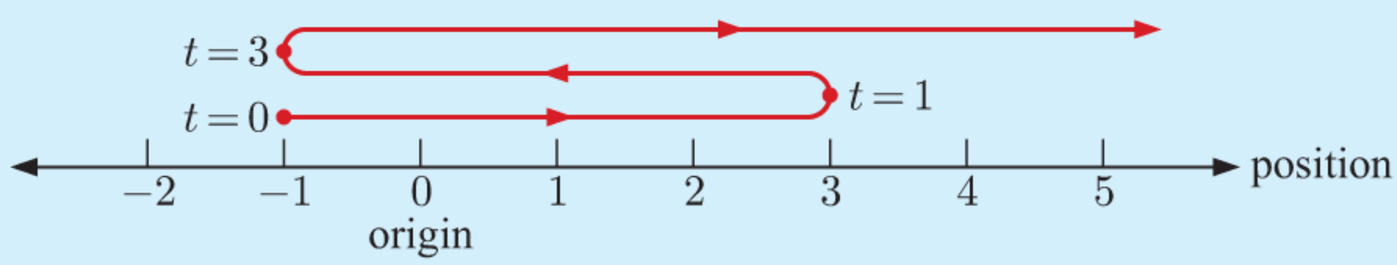


b When $t = 0$, $s(0) = -1$ cm
$v(0) = 9$ cm s^{-1}
$a(0) = -12$ cm s^{-2}

$\therefore$ the particle is 1 cm to the left of O, moving to the right with speed 9 cm s^{-1}, and has acceleration -12 cm s^{-2}.

c Since $v(t)$ changes sign when $t = 1$ and $t = 3$, a change of direction occurs at these instants.
$s(1) = 1 - 6 + 9 - 1 = 3$, and $s(3) = 27 - 54 + 27 - 1 = -1$.
So, the particle changes direction when it is 3 cm to the right of O, and when it is 1 cm to the left of O.

d



e The total distance travelled $= 4 + 4 = 8$ cm.

EXERCISE 23C

1 A particle moves with velocity function $v(t) = 10t - t^2$ cm s^{-1}, $t \geqslant 0$. Find:

a the velocity of the particle when $t = 2$ seconds

b the average acceleration of the particle from $t = 1$ to $t = 3$ seconds

c the acceleration function $a(t)$

d the instantaneous acceleration of the particle when $t = 3$ seconds.

2 An object moves in a straight line with displacement function $s(t) = t^3 - t^2 - 5$ metres at time t seconds, $t \geqslant 0$.

a Find the object's displacement, velocity, and acceleration when $t = 2$ seconds.

b Find the time at which the object has zero acceleration.

3 A stone is fired from a catapult so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.

a Find the velocity and acceleration functions for the stone, and draw sign diagrams for each function.

b Find the initial position and velocity of the stone.

c Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.

d Find the maximum height reached by the stone.

e Find the time taken for the stone to hit the ground.

4 A particle P moves in a straight line with displacement function $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Discuss the velocity of P as $t \to \infty$.

d Discuss the acceleration of P as $t \to \infty$.

5 An object has displacement function $s(t) = t - \ln(2t+1)$ cm, where t is in seconds, $t \geqslant 0$.

a Show that the object is initially at the origin.

b Find the velocity function.

c Over what time interval is the object moving:

i to the right **ii** to the left?

d Show that the object's acceleration is positive for all $t \geqslant 0$.

e Find the acceleration of the object after 2 seconds.

f Find the distance travelled by the object in the first 2 seconds.

6 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, where $t \geqslant 0$, t in seconds.

a State the initial velocity of the particle.

b Find the velocity of the particle after 3 seconds.

c How long will it take for the particle's velocity to reach 45 m s^{-1}?

d Discuss $v(t)$ as $t \to \infty$.

e Show that the particle's acceleration is always positive.

f Find the exact time at which the acceleration of the particle is 2 m s^{-2}.

g Draw the graph of $v(t)$ against t.

h Find the total distance travelled by the particle in the first 3 seconds of motion.

Example 6

Self Tutor

A train is initially at rest at a station. It begins to accelerate according to the function $a(t) = \frac{1}{2}e^{-\frac{t}{100}}$ m s^{-2}.

a Find the velocity function of the train, and sketch its graph.

b How long will it take for the train to reach the speed 40 m s^{-1}?

c How far will the train travel in this time?



a $a(t) = \frac{1}{2}e^{-\frac{t}{100}}$ m s^{-2}

$\therefore\ v(t) = \int \frac{1}{2}e^{-\frac{t}{100}}\,dt$

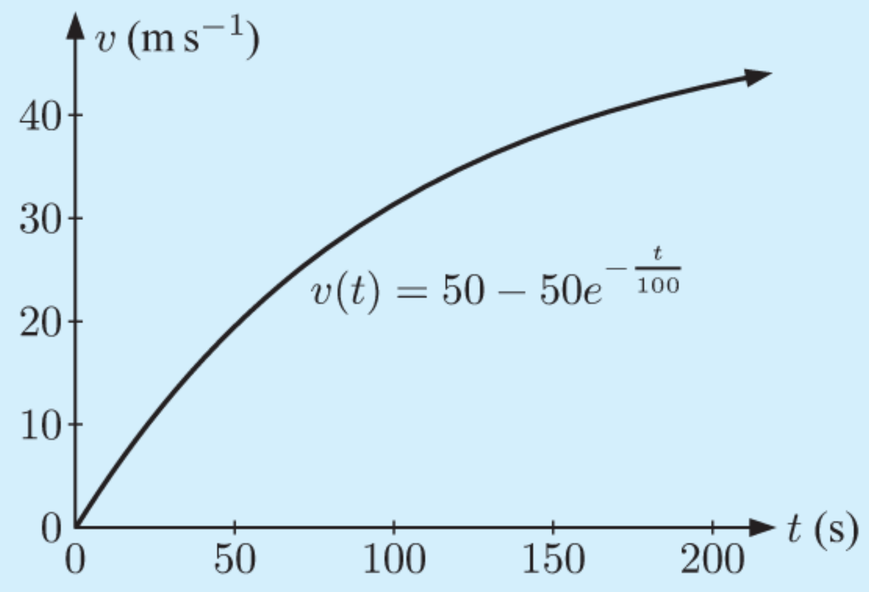
$= -50e^{-\frac{t}{100}} + c$ m s^{-1}

The train was initially stationary.

$\therefore\ -50e^{0} + c = 0$

$\therefore\ c = 50$

$\therefore\ v(t) = 50 - 50e^{-\frac{t}{100}}$ m s^{-1}



b If $v = 40$ then

$$50 - 50e^{-\frac{t}{100}} = 40$$

$$\therefore \quad 50e^{-\frac{t}{100}} = 10$$

$$\therefore \quad e^{-\frac{t}{100}} = \tfrac{1}{5}$$

$$\therefore \quad -\frac{t}{100} = \ln\left(\tfrac{1}{5}\right) = -\ln 5$$

$$\therefore \quad t = 100 \ln 5 \approx 160.9 \text{ s}$$

c The train does not change direction.

$\therefore$ distance travelled

$$= \int_0^{100 \ln 5} (50 - 50e^{-\frac{t}{100}})\, dt$$

$$= \left[50t + 5000e^{-\frac{t}{100}}\right]_0^{100 \ln 5}$$

$$= 50(100 \ln 5) + 5000e^{-\frac{100 \ln 5}{100}} - 5000e^0$$

$$= 5000 \ln 5 + \tfrac{5000}{5} - 5000$$

$$= 5000 \ln 5 - 4000$$

$$\approx 4047 \text{ m}$$

7 A train moves along a straight track with acceleration $a(t) = \dfrac{t}{10} - 3$ m s^{-2}. The initial velocity of the train is 45 m s^{-1}.

a Determine the velocity function $v(t)$.

b Evaluate $\displaystyle\int_0^{60} v(t)\, dt$ and explain what this value represents.

8 An object has initial velocity 20 m s^{-1} as it moves in a straight line with acceleration function $a(t) = 4e^{-\frac{t}{20}}$ m s^{-2}.

a Show that as t increases, the object approaches a limiting velocity.

b Find the total distance travelled in the first 10 seconds of motion.

9 A particle is initially stationary at the origin. It accelerates according to the function $a(t) = \dfrac{2}{(t+1)^3}$ m s^{-2}.

a Find the velocity function $v(t)$ for the particle.

b Find the displacement function $s(t)$ for the particle.

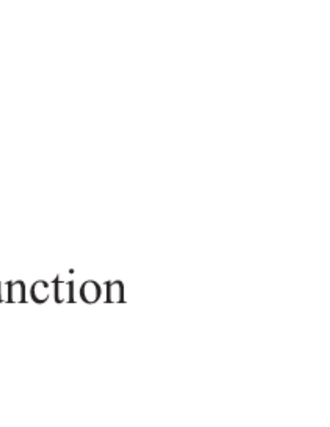
c Describe the motion of the particle at the time $t = 2$ seconds.

10 When a tyre hits a bump in the road, the end point P of the car's suspension is 10 cm from its top T.

The velocity of P relative to T is given by

$$v(t) = 10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t \text{ cm s}^{-1}$$

where t is the time in seconds.



a Find the velocity of P relative to T after 0.3 seconds.

b Find the acceleration of P relative to T after 0.3 seconds.

c Use integration by parts to find $s(t)$.

d Hence find the distance by which the suspension has contracted after 0.1 seconds.

D SPEED

We have seen that velocities have size (magnitude) and sign (direction). In contrast, speed simply measures *how fast* something is travelling, regardless of the direction of travel. Speed is a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** S of an object at any instant is the magnitude of the object's velocity.

$$S = |v|$$

For example, an object with velocity -5 m s^{-1} is moving to the left with speed 5 m s^{-1}.

To determine when the speed of an object P with displacement $s(t)$ is increasing or decreasing, we use a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the speed of P is increasing.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the speed of P is decreasing.

DISCUSSION

Discuss why the sign test for speed works by considering each case below:

- The object is travelling to the right $(v > 0)$ and the acceleration $a > 0$.
- The object is travelling to the right $(v > 0)$ and the acceleration $a < 0$.
- The object is travelling to the left $(v < 0)$ and the acceleration $a > 0$.
- The object is travelling to the left $(v < 0)$ and the acceleration $a < 0$.

Example 7 Self Tutor

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion of the particle at this instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the position of the particle when it changes direction.

e Draw a motion diagram for the particle.

f For what time interval is the particle's speed increasing?

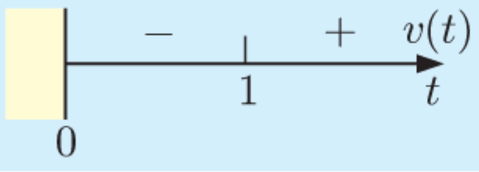
a $s(t) = t^3 - 3t + 1$ cm

$\therefore\ v(t) = 3t^2 - 3 \qquad \{v(t) = s'(t)\}$

$= 3(t^2 - 1)$

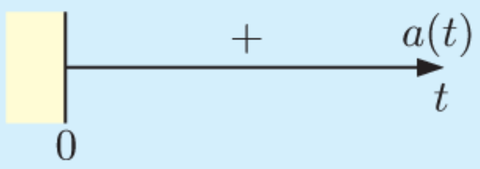
$= 3(t + 1)(t - 1)$ cm s^{-1}

which has sign diagram:



$\therefore \quad a(t) = 6t \text{ cm s}^{-2} \qquad \{a(t) = v'(t)\}$

which has sign diagram:



b When $t = 0$, $s(0) = 1$ cm
$v(0) = -3 \text{ cm s}^{-1}$
$a(0) = 0 \text{ cm s}^{-2}$

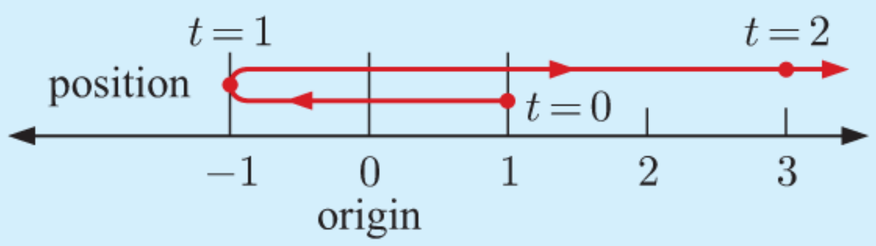
$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm
$v(2) = 12 - 3 = 9 \text{ cm s}^{-1}$
$a(2) = 12 \text{ cm s}^{-2}$

$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}.
Since a and v have the same sign, the speed of the particle is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

e



f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

EXERCISE 23D

1 An object moves in a straight line with position from O given by $s(t) = t^2 - 6t + 7$ m, where t is in seconds, $t \geqslant 0$.

a Find expressions for the object's velocity and acceleration, and show sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Find the position of the object when it changes direction.

d Draw a motion diagram for the object.

e During which time interval is the object's speed decreasing?

2 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres, where t is the time in seconds.

a From what distance above the ground was the ball released?

b Find $s'(t)$ and state what it represents.

c Find the maximum height reached by the ball.

d Find the ball's speed:

i when released **ii** at $t = 2$ s **iii** at $t = 5$ s.

3 A particle moves in a straight line with displacement function $s(t) = 12t - 2t^3 - 1$ cm where t is in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for the particle's motion.

b Find the initial conditions, and interpret their meaning.

c Find the times and positions when the particle reverses direction.

d At what times is the particle's:

i speed increasing **ii** velocity increasing?

e Draw a motion diagram for the particle.

4 A particle has displacement function $s(t) = 4 - \sqrt{t+1}$ m, where t is in seconds, $t \geqslant 0$.

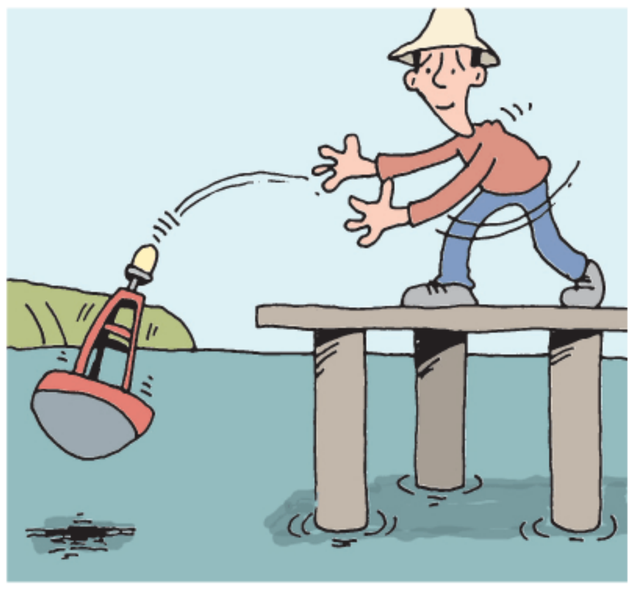
a Find the velocity and acceleration functions for the particle's motion, and draw sign diagrams for each of them.

b Find the initial conditions and interpret their meaning.

c Describe the motion of the particle after 3 seconds.

d Describe what is happening to the speed of the particle.

5 A flotation device is thrown from a jetty into the water below. It takes k seconds for the device to reach the water. The height of the device above sea level is given by $s(t) = -4.9t^2 + 4.9t + 8$ m, where t is in seconds, $0 \leqslant t \leqslant k$ seconds.



a Find the value of k.

b Draw sign diagrams for the device's velocity and acceleration functions.

c Was the speed of the device increasing or decreasing after:

i 0.2 seconds **ii** 1 second?

6 A particle P moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm, $t \geqslant 0$ seconds.

a State the initial position, velocity, and acceleration of P.

b Describe the motion of P when $t = \frac{\pi}{4}$ seconds.

c Find the times when the particle reverses direction on $0 < t < 2\pi$, and find the position of the particle at these instants.

d When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

7 A dog paces back and forth along a fence, guarding its owner's property. The dog's horizontal displacement relative to its kennel is given by $s(t) = 8\sin\frac{t}{2}$ m, $t \geqslant 0$ seconds.

a Is the dog to the left or the right of its kennel after:

i 3 seconds **ii** 7 seconds?

b Find the velocity function $v(t)$.

c Is the dog moving to the left or the right after:

i 4 seconds **ii** 10 seconds?

d Find the acceleration function $a(t)$.

e Is the dog's speed increasing or decreasing after 2 seconds?

f Show that the dog's speed is maximised when it is moving past its kennel.

8 The velocity of an object after t seconds is given by $v(t) = 25te^{-2t}$ cm s^{-1}, $t \geqslant 0$.

a Use technology to help sketch the velocity function.

b Show that the object's acceleration at time t is given by $a(t) = 25(1-2t)e^{-2t}$ cm s^{-2}, $t \geqslant 0$.

c When is the velocity of the object increasing?

d When is the speed of the object decreasing?

9 A lion is chasing a zebra. The zebra notices the lion when the lion is 40 m away. From this time, the lion approaches with speed $v_1(t) = 15e^{-0.1t}$ m s^{-1}, and the zebra runs away with speed $v_2(t) = 20 - 20e^{-0.1t}$ m s^{-1}.



a Find the speed of each animal after 1 second.

b Show that the lion's speed decreases over time, whereas the zebra's speed increases over time.

c Find $\int_0^3 v_1(t)\,dt$, and interpret your answer.

d Find $\int_0^3 [v_1(t) - v_2(t)]\,dt$, and interpret your answer.

e Explain why the lion will be closest to the zebra when $v_1(t) = v_2(t)$.

f Find the time at which $v_1(t) = v_2(t)$.

g Did the lion catch the zebra? If not, how close did the lion get to the zebra?

INVESTIGATION PROJECTILE MOTION

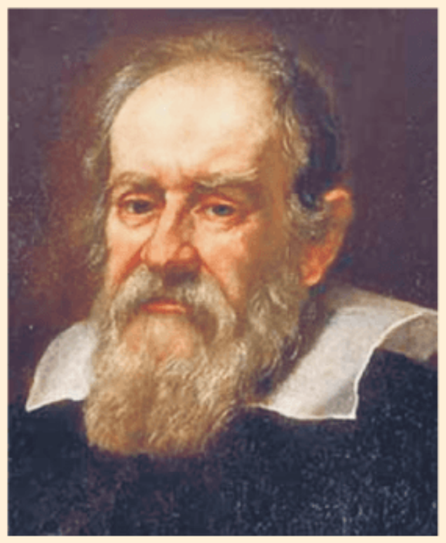
A **projectile** is an object upon which the only force acting is gravity. If we ignore air resistance, balls and missiles travel through the air with **projectile motion**.

For example, the trebuchet at Warwick Castle in England is claimed to be the largest working siege machine in the world. It is constructed mainly of oak, with the long throwing arm made of ash, a more flexible wood. It is 8.7 m high, increasing to 18 m high with the arm fully extended. Using a counterweight of approximately 6 tonnes, it is capable of throwing an 18 kg projectile a horizontal distance of up to 242 m.



The theory of projectile motion was developed in Europe in the 14th century, driven by the desire to improve guns and cannons. At that time, scientists were still using Aristotle's theory of motion which suggested that forces gave rise to momentum. This would mean that as soon as you stopped pushing something (even an object on wheels) it would stop moving.

It was **Galileo Galilei** (1564 - 1642) who first suggested that in the absence of resistance, a projectile would move in a quadratic curve.



Galileo

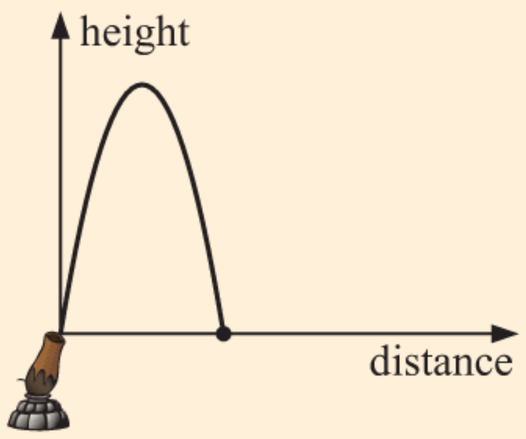
Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor. In the second, a horizontal shelf was placed at the end of the ramp, and the ball travelled along this shelf before descending to the floor.

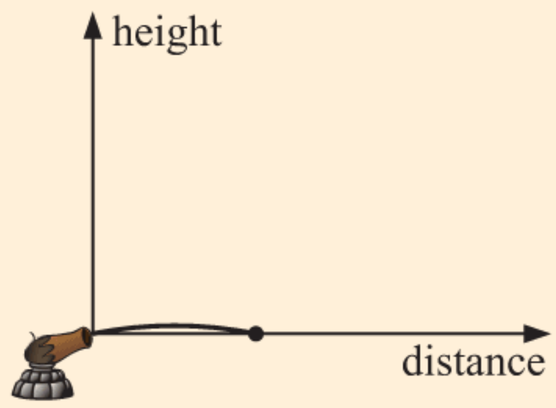
In both experiments Galileo found that once the ball left the ramp or shelf, its path was *parabolic* and could therefore be modelled by a quadratic function.

In this Investigation we suppose you are operating a cannon. Your task is to fire the cannonball as far as possible. At what angle to the ground should you fire the cannonball?

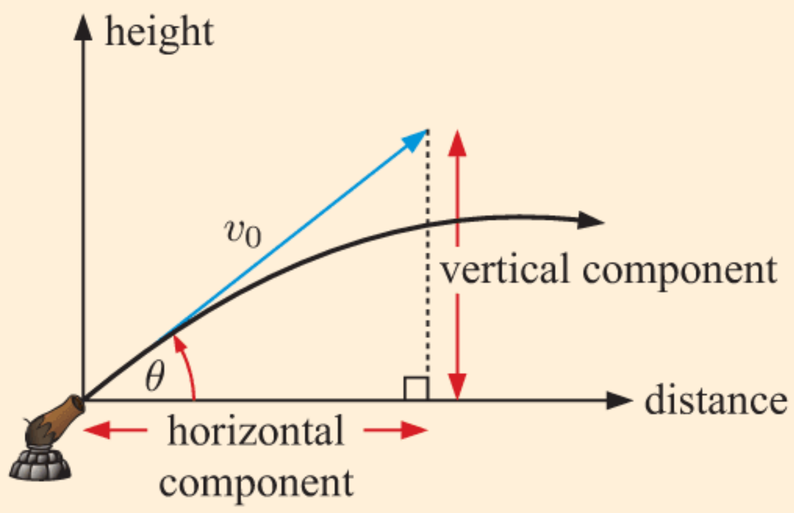
- If the angle is too high, the cannonball will go high into the air, but will not travel very far horizontally.



- If the angle is too low, the cannonball will reach its maximum height too soon, and will not travel very far.



Suppose the cannonball is fired at an angle θ to the ground, with initial velocity v_0 m s^{-1}. The motion of the cannonball has both a **vertical component** and a **horizontal component**. We will consider these components separately.



What to do:

1 **a** If the cannon is fired from ground level, what is the initial vertical height of the cannonball?

b Use the diagram above to show that the initial vertical velocity of the cannonball is $v_0 \sin\theta$ m s^{-1}.

c The path of the cannonball is affected by gravity, which acts downwards on the cannonball at a rate of 9.8 m s^{-2}.
Explain why the vertical acceleration of the cannonball is given by $a(t) = -9.8$ m s^{-2}.

d Show that the vertical displacement function $s(t) = -4.9t^2 + [v_0 \sin\theta]t$ satisfies the properties in **a**, **b**, and **c**.

e Hence show that the cannonball takes $\dfrac{v_0 \sin\theta}{4.9}$ seconds to hit the ground.

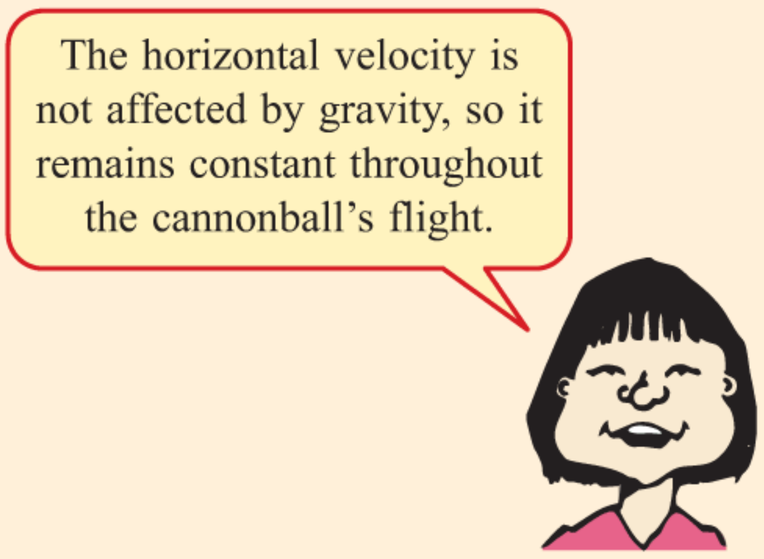
2 **a** Show that the horizontal velocity of the cannonball is $v_0 \cos\theta$ m s^{-1}.

b Hence show that the horizontal distance travelled by the cannonball before it hits the ground is $\dfrac{v_0^2 \sin 2\theta}{9.8}$ metres.

c Suppose a cannonball is fired with initial velocity 200 m s^{-1}. Find the horizontal distance travelled by the cannonball if it is fired at an angle of:

i $20°$ **ii** $50°$ **iii** $80°$

d Find the angle θ which maximises the range of the cannonball.



3 Click on the icon to run a cannon simulation.
Change the initial velocity and angle of trajectory, and observe the effect these have on the path of the cannonball.
Use the simulation to check that your answer to **2 d** is correct.

SIMULATION



REVIEW SET 23A

1 An object travels with displacement function $s(t) = 12 - 2t$ m for $0 \leqslant t \leqslant 10$ s.

a Find the initial displacement of the object.

b Find the displacement of the object at time: **i** $t = 1$ s **ii** $t = 3$ s.

c At what time does the object reach the origin?

d Does the object ever change direction? Explain your answer.

e Draw a motion diagram showing the information you have found.

2 A particle moves in a straight line with displacement from O given by $s(t) = 2t^2 + t - 5$ cm at time t seconds, $t \geqslant 0$.

a Find the average velocity from $t = 1$ to $t = 5$ seconds.

b Find the instantaneous velocity at: **i** $t = 2$ seconds **ii** $t = 4$ seconds.

c Find the acceleration function $a(t)$.

3 A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions, and draw a sign diagram for each.

b Find the initial conditions.

c Describe the motion of the particle at time $t = 2$ seconds.

d Find the times and positions where the particle changes direction.

e Draw a diagram to illustrate the motion of P.

f Determine the time intervals when the particle's speed is increasing.

4 A particle moves in a straight line with velocity $v(t) = t^2 - 6t + 8$ m s^{-1}, $t \geqslant 0$ seconds.

a Draw a sign diagram for $v(t)$.

b Describe what happens to the particle in the first 5 seconds of motion.

c After 5 seconds, how far is the particle from its original position?

d Find the total distance travelled in the first 5 seconds of motion.

5 When a kayaker stops paddling, the velocity of the kayak in the following 6 seconds is given by $v(t) = 2.75 - t + 0.5t^{1.2}$ m s^{-1}, where t is the time in seconds.



a Find the velocity of the kayak:

i when the kayaker stops paddling

ii after 3 seconds.

b Show that the kayak's speed is decreasing during the 6 second period.

c Find $\int_0^2 v(t)\,dt$ and interpret your answer.

6 A particle P moves in a straight line with position from O given by $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for P's motion.

b Describe the motion of P at $t = 3$ seconds.

c For what values of t is the particle's speed increasing?

7 A spotlight moves back and forth across a stage at a concert. Its position after t seconds is given by $x(t) = 3 + 2\sin \pi t$ m.

a Find the initial position, velocity, and acceleration of the spotlight.

b Find the times when the spotlight changes direction during $0 \leqslant t \leqslant 5$ seconds.

c Find the total distance travelled by the spotlight in the first 5 seconds.

8 A particle has velocity function $v(t) = 2\cos 4t$ m s^{-1}.

a Show that the particle oscillates between two points, and find the distance between them.

b Given that $s\left(\frac{\pi}{12}\right) = 6$ m, determine $s\left(\frac{\pi}{6}\right)$.

c Find the total distance travelled by the particle in the first π seconds.

9 When an aeroplane lands, its velocity is 65 m s^{-1}. Its acceleration t seconds after landing is given by $a(t) = -2$ m s^{-2}.

a Find the velocity $v(t)$ of the plane t seconds after landing.

b Find the displacement $s(t)$ of the plane t seconds after landing.

c **i** How long will it take the plane to reduce its speed to 3 m s^{-1}?

ii How far would the plane have travelled along the runway at this time?

10 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time t seconds thereafter is given by $v(t) = \dfrac{100}{(t+2)^2}$ m s^{-1}.

a Find the initial velocity of the boat, and its velocity after 3 seconds.

b Discuss $v(t)$ as $t \to \infty$.

c Sketch the graph of $v(t)$ against t.

d Find $\int_0^2 v(t)\,dt$, and interpret your answer.

e From when the engine is turned off, how long will it take for the boat to travel 30 metres?

REVIEW SET 23B

1 An object moves in a straight line with displacement function $s(t) = t^2 + 4t + 1$ m, where t is in seconds, $t \geqslant 0$.

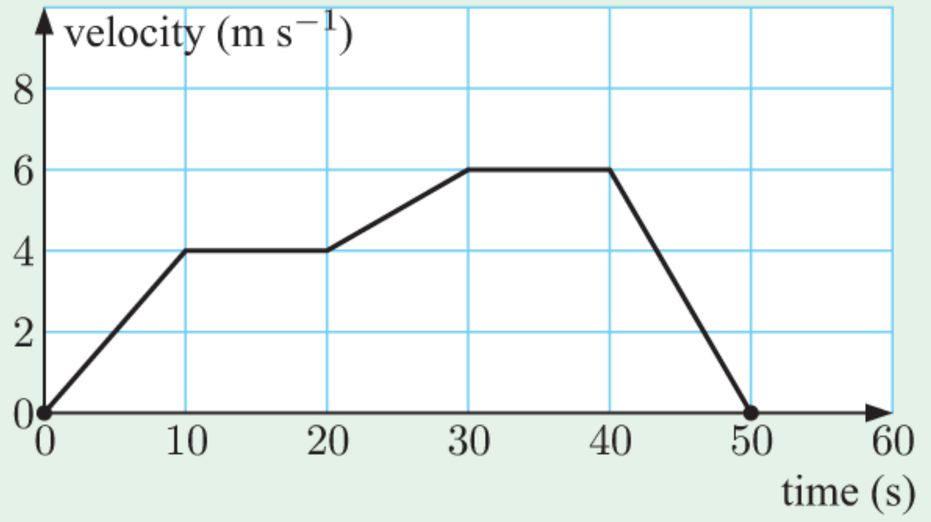
a Find the initial position of the object.

b Find the average velocity from $t = 1$ to $t = 3$ seconds.

c Find $v(t)$.

d Find the instantaneous velocity at $t = 1$ second.

2 A jogger has the velocity-time graph shown. Find the total distance travelled by the jogger.



3 At time $t = 0$ a particle passes through the origin with velocity 27 cm s^{-1}. Its acceleration t seconds later is $6t - 30$ cm s^{-2}.

a Write an expression for the particle's velocity.

b Calculate the displacement from the origin after 6 seconds.

4 A particle moves along the x-axis with position relative to origin O given by $x(t) = 3t - t\sqrt{t}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions, and draw a sign diagram for each.

b Find the initial position and velocity of the particle and hence describe the motion at that instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the time and position when the particle reverses direction.

e Determine the time interval when the particle's speed is decreasing.

f Draw a motion diagram for the particle.

g Find the distance travelled by the particle in the first 6 seconds of motion.

5 The velocity of a human cannonball is given by $v(t) = 4.8t^2 - 0.8t^3$ m s^{-1} for $0 \leqslant t \leqslant 6$ seconds.

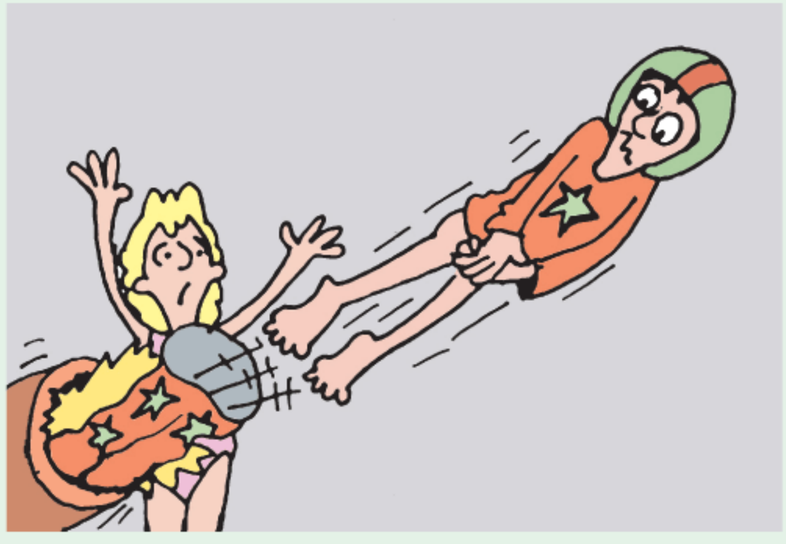
a Find the acceleration of the human cannonball after:

i 1 second **ii** 2 seconds

iii 4 seconds **iv** 5 seconds.

b Find $\int_0^3 v(t)\,dt$ and interpret your answer.

c How long does it take for the human cannonball to travel 30 m?



6 A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m, $t \geqslant 0$ seconds.

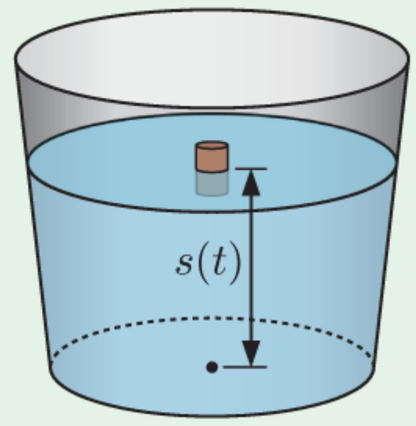
a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Sketch the graph of the velocity function.

d Find the exact time when the velocity is -44 m s^{-1}.

7 A cork bobs up and down in a bucket of water. The distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos \pi t$ cm, $t \geqslant 0$ seconds.



a Find the cork's velocity at times $t = 0$, $\frac{1}{2}$, 1, $1\frac{1}{2}$, and 2 seconds.

b Find the time intervals when the cork is falling.

8 A skier is travelling down a hill. Her velocity after t seconds is given by $v(t) = \dfrac{\left(t^{1.1} + 3t\right)^{1.5}}{10}$ m s^{-1}.

a Find the velocity of the skier after 4 seconds.

b Write an expression for the acceleration of the skier after t seconds.

c Find the acceleration of the skier after 2 seconds.

d Use technology to find the total distance travelled by the skier in the first 10 seconds.

9 A feather is falling with velocity function $v(t) = -\frac{1}{24}t^3 - \frac{1}{12}t$ m s^{-1}. The feather is initially 2 metres above the ground.

a Find the displacement function $s(t)$.

b Find the time taken for the feather to reach the ground.

10 Tyson and Maurice are competing in a 100 m sprint. Tyson runs with speed $v_1(t) = 10(1 - e^{-1.25t})$ m s^{-1}, and Maurice runs with speed $v_2(t) = 10.5(1 - e^{-t})$ m s^{-1}, for $t \geqslant 0$ seconds.

a Who is travelling faster after 2 seconds?

b Find $\displaystyle\int_0^5 v_1(t)\,dt$, and interpret your answer.

c Find displacement functions $s_1(t)$ and $s_2(t)$ for each sprinter.

d Who is winning the race after 3 seconds?

e Show that Tyson completes the 100 m in approximately 10.8 seconds.

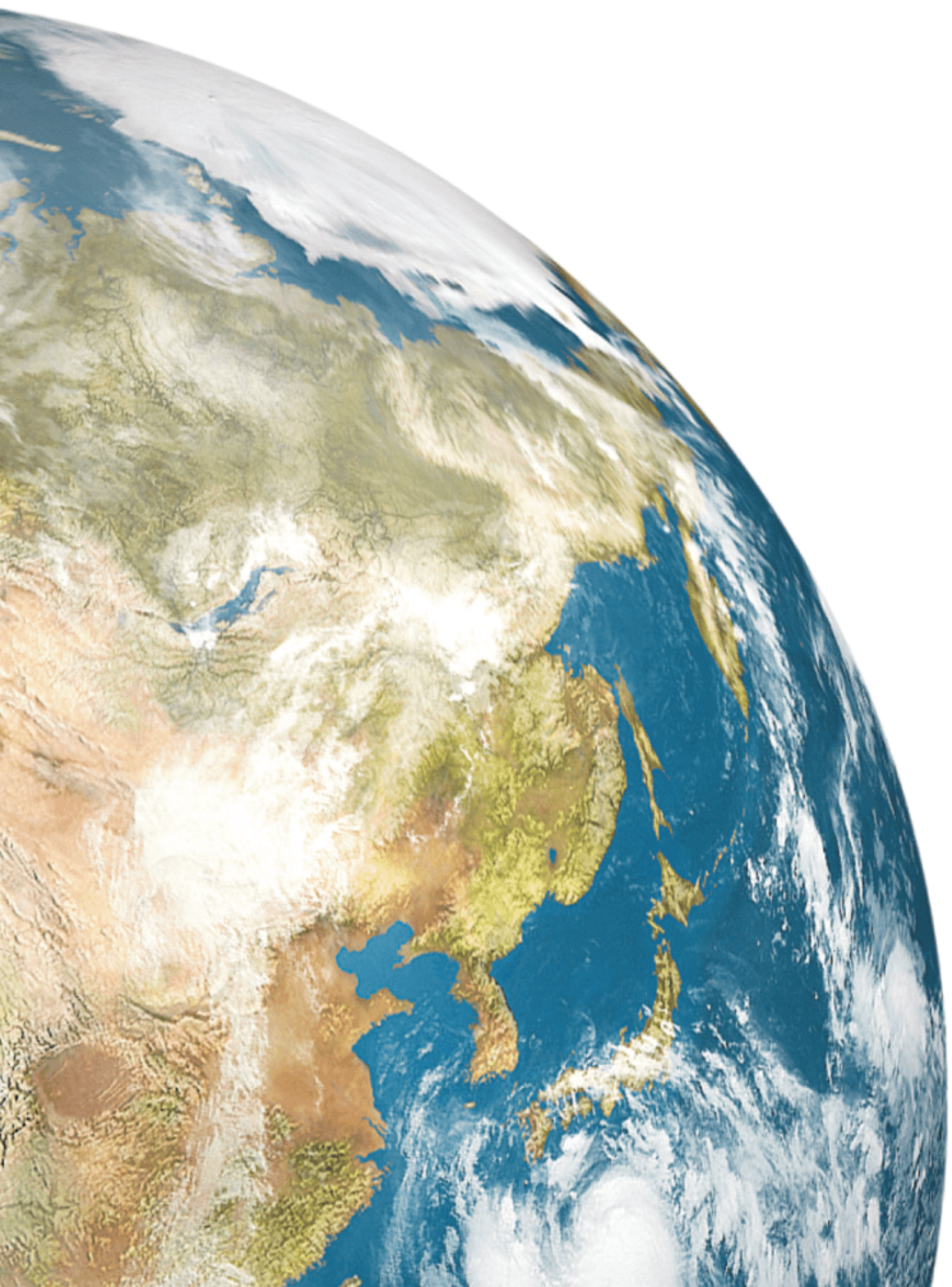
f Who will win the race?

Chapter 24

Maclaurin series

Contents:

A Maclaurin series
B Convergence
C Composite functions
D Addition and subtraction
E Differentiation and integration
F Multiplication
G Division



OPENING PROBLEM

Earlier in the course, we defined the *tangent* to a curve as the best straight line approximation to the curve at that point.

For a curve which is differentiable at $x = a$, the gradient of the tangent is $f'(a)$, and the tangent has equation $y = f(a) + f'(a)(x - a)$.

Consider the curve $f(x) = e^x$.

Things to think about:

a What is the tangent to the curve at $x = 0$?

b What is the best *quadratic* approximation to the curve at $x = 0$, having the same gradient and *curvature* (second derivative)?

c What is the best *cubic* approximation to the curve at $x = 0$?

d Could you continue this process to form an nth order polynomial approximation to the curve?

e Can we write an *exact* expression for e^x in the form $e^x = \sum_{k=0}^{\infty} c_n x^k$ where $c_n \in \mathbb{R}$?

f What uses could there be in having a polynomial approximation to a curve?

HISTORICAL NOTE

Prior to Archimedes, philosophers like **Zeno of Elea** and **Plato** had struggled with the mathematical idea that an infinite series could sum to a finite result. We have already seen several examples where Archimedes used an essentially infinite number of pieces to form a whole. Another is his approximation of π using the method of exhaustion to compute the areas of regular polygons with more and more sides.

In the 3rd century AD in China, **Liu Hui** independently developed similar methods. He made important commentary on the *Jiuzhang suanshu* or "Nine Chapters on the Mathematical Art" including early ideas about limits and infinitesimals, and calculations of volume formulae. Considering proof most important and modestly recognising when his own results were imperfect, he wrote "Let us leave the problem to whoever can tell the truth".



Liu Hui

In the 14th century, the Indian mathematician **Mādhava of Sangamagrāma** discovered infinite series expansions for a number of trigonometric functions, as well as infinite series for calculating π such as $\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2n+1}$.

The Scottish mathematician **James Gregory** (1638 - 1675) was probably unaware of Mādhava's work. He discovered infinite series representations for several trigonometric functions himself, and is regarded as the discoverer of the **Taylor series** representation for functions. He also published the first proof for the Fundamental Theorem of Calculus.

The Taylor series representation is named after **Brook Taylor** (1685 - 1731) who formalised the idea in 1715.

INVESTIGATION 1 TAYLOR SERIES

Consider a continuous function f which can be differentiated infinitely many times.

The Taylor series representation for f has the form

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$$
$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots.$$

This is a **power series** which generates an infinite polynomial representation for f.

What to do:

1 Explain why $c_0 = f(a)$.

2 Find $f'(x)$. Hence write c_1 in terms of $f'(a)$. What do you notice about the first two terms of the Taylor series expansion?

3 Find $f''(x)$. Hence write c_2 in terms of $f''(a)$.

4 **a** Prove that $f^{(n)}(x) = n!c_n + (n+1)!c_{n+1}(x-a) + \dfrac{(n+2)!}{2!}c_{n+2}(x-a)^2 + \ldots.$

$$= \sum_{k=0}^{\infty} \frac{(n+k)!}{k!}c_{n+k}(x-a)^k$$

b Hence write c_n in terms of $f^{(n)}(a)$.

c Write the Taylor series formula for f in terms of its function value and derivatives at $x = a$.

5 In the Taylor series we claim that a function can be represented everywhere using only information about the curve generated at $x = a$.

a Explain how information about the rest of the curve is established in the formula.

b Is it reasonable to expect that the sum of polynomial terms can exactly produce:

i a horizontal asymptote **ii** a vertical asymptote?

c Do you therefore expect that a Taylor series expansion will converge for *every* value of x in the domain of *every* function?

From the **Investigation** you should have established that

The **Taylor series expansion of $f(x)$ about $x = a$** is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots. = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Functions defined in this way may look awkward. However, for values of x where they converge, power series can be added, differentiated, and integrated just like ordinary polynomials. This makes them extremely useful.

In this course we consider the special case of the Taylor series where $a = 0$. This case is called the **Maclaurin series**, named after **Colin Maclaurin** (1698 - 1746) who made extensive use of it in the 18th century.

A MACLAURIN SERIES

Consider a continuous function f which can be differentiated infinitely many times.
The **Maclaurin series expansion** of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

For any values of x where the series converges, it is an exact representation of the function.

However, if we consider just the first n terms of the Maclaurin series expansion, we are left with an nth order polynomial approximation for the function.

The ***n*th degree Maclaurin polynomial** approximation to f is $M_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)x^k}{k!}$.

Example 1 **Self Tutor**

a Find the Maclaurin series representation for $f(x) = \frac{1}{2+x}$.

b Hence find $M_4(x)$, the 4th order Maclaurin polynomial approximation for $\frac{1}{2+x}$. Compare $M_4(1)$ with $f(1)$.

a
$$f(x) = \frac{1}{2+x} = (2+x)^{-1}$$
$$\therefore\ f'(x) = (-1)(2+x)^{-2}$$
$$\therefore\ f''(x) = (-1)(-2)(2+x)^{-3}$$
$$\vdots$$
$$f^{(k)}(x) = (-1)^k k!(2+x)^{-(k+1)}$$
$$\therefore\ f^{(k)}(0) = \frac{(-1)^k k!}{2^{k+1}} \text{ for all } k \in \mathbb{Z}^+.$$

Since $f(0) = \frac{1}{2}$, the Maclaurin series representation for $f(x)$ is

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k \cancel{k!}}{\cancel{k!}\,2^{k+1}} x^k$$
$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k$$

b The 4th order Maclaurin approximation for $\frac{1}{2+x}$ is

$$M_4(x) = \sum_{k=0}^{4} \frac{(-1)^k}{2^{k+1}} x^k$$
$$= \frac{1}{2} + \frac{(-1)^1}{2^2}x + \frac{(-1)^2}{2^3}x^2 + \frac{(-1)^3}{2^4}x^3 + \frac{(-1)^4}{2^5}x^4$$
$$= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \tfrac{1}{32}x^4$$
$$\therefore\ M_4(1) = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{32} = \tfrac{11}{32} \quad \text{compared with} \quad f(1) = \tfrac{1}{3} = \tfrac{11}{33}.$$

EXERCISE 24A

1 Consider the function $f(x) = e^x$.

a Find the nth derivative $f^{(n)}(x)$ for $n \in \mathbb{Z}^+$.

Hence explain why $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \ldots.$

b The first two Maclaurin polynomial approximations for e^x are $M_1(x) = 1 + x$ and $M_2(x) = 1 + x + \frac{x^2}{2}$.

Write down the next *four* Maclaurin polynomial approximations.

c Graph $y = e^x$ and the first *four* polynomial approximations on the same set of axes. Discuss how the accuracy of the approximation is affected by the addition of the extra terms.

d Copy and complete this table of values:

n	1	2	3	4	5	6
$M_n(1)$						

Compare your results with $f(1) = e$.

2 Derive these Maclaurin series representations:

a $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. = \sum_{k=0}^{\infty} x^k$

b $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots. = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$

3 Explain why there is no Maclaurin series representation for $\ln x$.

4 Find an infinite series polynomial representation for $\frac{1}{x+1}$ using:

a the binomial theorem **b** Maclaurin series.

5 **a** Show that the Maclaurin series representation for $\cos x$ is

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots. = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$.

b Explain why there is no 3rd order Maclaurin polynomial in this case. What feature of $\cos x$ has caused this?

c Find $M_4(x)$, the fourth order Maclaurin polynomial approximation of $\cos x$.

Compare $M_4\left(\frac{\pi}{3}\right)$ with $\cos\frac{\pi}{3} = \frac{1}{2}$.

6 **a** Show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots. = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

b Explain why there is no 2nd order Maclaurin polynomial in this case. What feature of $\sin x$ has caused this?

c Find $M_5(x)$ and compare $M_5\left(\frac{\pi}{6}\right)$ with $\sin\frac{\pi}{6} = \frac{1}{2}$.

7 Derive these Maclaurin series representations:

a $\ln\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1}$ **b** $\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} x^{2k-1}$

INVESTIGATION 2 — $\int x^k\,dx$

We have previously seen that, for $k \neq -1$, $\int x^k\,dx = \dfrac{x^{k+1}}{k+1} + c$.

In this Investigation, we will consider the idea that a "better" formula might be
$\int x^k\,dx = \dfrac{x^{k+1}-1}{k+1} + c$.

The expressions only differ by the constant $\dfrac{1}{k+1}$, and for $k \neq -1$ the derivative of both expressions is x^k. So, at first glance there may seem little meaningful difference between them.

To see why $\int x^k\,dx = \dfrac{x^{k+1}-1}{k+1} + c$ might be considered a "better" result, we will consider the limit of $\dfrac{x^{k+1}-1}{k+1}$ as $k \to -1$.

What to do:

1 We first establish the result $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$ discovered by **Leonhard Euler**.

a Show that in the expansion of $\left(1+\dfrac{x}{n}\right)^n$, the coefficient of x^i can be written as
$\dfrac{1}{i!} \times \dfrac{n}{n} \times \dfrac{n-1}{n} \times \ldots \times \dfrac{n-i+1}{n}$.

b Hence find the limit of the coefficient of x^i as $n \to \infty$.

c Compare $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$ with the Maclaurin series expansion of e^x.

2 Suppose $y = \dfrac{x^{k+1}-1}{k+1}$ for $x > 0$.

a By letting $n = \dfrac{1}{k+1}$, show that $x = \left(1+\dfrac{y}{n}\right)^n$.

b Show that in the limit as $n \to \infty$, $x = e^y$.

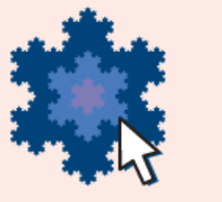
c Hence show that $\lim\limits_{k\to -1}\dfrac{x^{k+1}-1}{k+1} = \ln x$.

3 Can you explain why the formula $\int x^k\,dx = \dfrac{x^{k+1}-1}{k+1} + c$ might be considered "better" than $\int x^k\,dx = \dfrac{x^{k+1}}{k+1} + c$?

ACTIVITY — THE REMAINDER TERM

The **remainder term** is the difference between a function and its nth order Maclaurin polynomial approximation.

The remainder term is used to give us a maximum bound for the possible error in using the approximation. This is particularly useful if we do not have an accurate value to compare our approximation with.



B CONVERGENCE

DISCUSSION

Given your previous study of infinite geometric series, what condition(s) do you think might be necessary for the convergence of a Maclaurin series?

We have seen that the nth order Maclaurin polynomial approximation $M_n(x)$ can be used to approximate a function $f(x)$. We can say that $M_n(x)$ is the "nth order polynomial of best fit" for the curve.

Since $M_n(x)$ is generated using the values of f and its derivatives at $x = 0$, $M_n(x)$ will generally be most accurate near to $x = 0$.

As we get further from $x = 0$, we need to consider whether increasing the order of the polynomial will be sufficient to accurately model the shape of the function.

For example, it is not possible for any polynomial approximation to model a vertical asymptote, no matter what order polynomial is used.

The infinite Maclaurin series for $f(x)$ **converges** for any value of x such that

$$\lim_{n\to\infty} M_n(x) = f(x).$$

The set of values of x for which the infinite Maclaurin series converges is called the **interval of convergence**.

It should be clear that any interval of convergence for a Maclaurin series must include $x = 0$, since $M_n(0) = f(0)$ for all n.

From our previous study of infinite geometric series, we can also deduce that the infinite Maclaurin series will only converge for values of x such that the terms of the series are getting smaller.

For example, consider these series from **Exercise 24A**:

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots.$
  The terms decrease in size provided $|x| < 1$, and indeed the series is convergent for $|x| < 1$.

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots.$
  It might seem like the fractions would help the terms to reduce in size. However, for any $|x| > 1$, the terms will certainly increase.
  For example, using l'Hôpital's rule, $\lim\limits_{n\to\infty} \dfrac{2^n}{n} = \lim\limits_{n\to\infty} \dfrac{2^n \ln 2}{1}$ which does not exist.
  For the case $x = -1$, $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots.$ does not converge, which is reasonable since $\ln 0$ is undefined.
  For the case $x = 1$, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ does converge.
  So, the Maclaurin series is convergent for $-1 < x \leqslant 1$.

- $e^x = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4 + \ldots.$
  In this case, the fractions *do* help the series to converge. For any value of x we pick, we can find a value of n for which $n!$ grows faster than x^n. Hence $\dfrac{x^n}{n!}$ will reduce in size. This series converges for all $x \in \mathbb{R}$.

EXERCISE 24B

1 Given the following Maclaurin series, discuss the values for which you think they converge:

a $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots.$

b $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots.$

c $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots.$

d $\ln\left(\dfrac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \ldots.$

2 Consider the binomial expansion for $(1+x)^p$, $p \in \mathbb{R}$.

a Show that the Maclaurin series for $(1+x)^p$, $p \in \mathbb{R}$, is $\displaystyle\sum_{k=0}^{\infty} \frac{p(p-1)\ldots(p-k+1)}{k!}\,x^k$.

Does this agree with the result previously stated as the **binomial theorem**?

b Explain why this series is only guaranteed to converge for $|x| < 1$.

3 **a** Show that the Maclaurin series for $\tan x$ is $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots.$

b Explain why this series is only convergent for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

4 **a** Show that the nth order Maclaurin polynomial for $\dfrac{1}{1-2x}$ is

$M_n(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots. + 2^n x^n$.

b **i** Copy and complete this table:

n	1	2	3	4	5	6
$M_n(1)$						

ii Do you think this Maclaurin series converges when $x = 1$?

c **i** Copy and complete this table:

n	1	2	3	4	5	6
$M_n(0.1)$						

ii Do you think this Maclaurin series converges when $x = 0.1$?

d For what values of x do you think the Maclaurin series converges?

5 **a** Show that the Maclaurin series for $\sqrt{x+1}$ is

$\sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \ldots.$

State the maximum domain on which this series *might* converge.

b Copy and complete the table:

n	1	2	3	4	5
$M_n(1)$					

Discuss whether you think the Maclaurin series is convergent for $x = 1$.

c Find the first six terms in the Maclaurin series for $\sqrt{x+1.8}$.

Using a similar table to **b** above, calculate $M_n(0.2)$ for the new series.

Compare the convergence of the two series when evaluating $\sqrt{2}$.

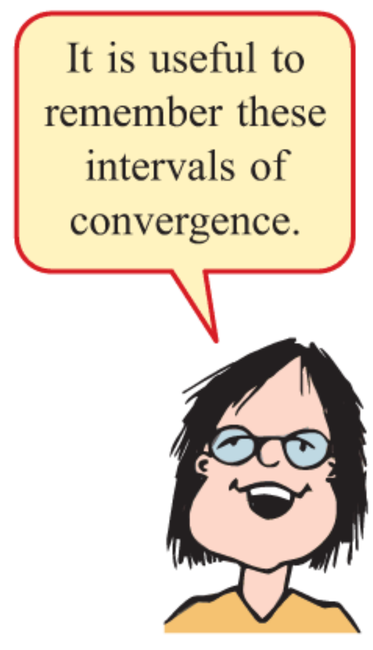
DISCUSSION

Suppose you want to use a Maclaurin polynomial $M_n(x)$ to approximate the value of a function at a particular value $x = a$.

Does it matter if the Maclaurin series is not *convergent* when $x = a$?

SUMMARY OF IMPORTANT MACLAURIN SERIES

Function	*Maclaurin series*	*Interval of convergence*
e^x	$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots.$	$x \in \mathbb{R}$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots.$	$x \in \mathbb{R}$
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots.$	$x \in \mathbb{R}$
$\arctan x$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots.$	$\lvert x \rvert \leqslant 1$
$\ln(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots.$	$-1 < x \leqslant 1$
$(1+x)^p, \quad p \in \mathbb{R}$	$\displaystyle\sum_{k=0}^{\infty} \frac{p(p-1)\ldots.(p-k+1)}{k!}\, x^k$	$\lvert x \rvert < 1$



C COMPOSITE FUNCTIONS

Having a standard list of common Maclaurin series is useful because we can use them to generate infinite series representations for other functions. We achieve this using **composite functions**.

Note that:

- the infinite series generated will only converge if the interval of convergence of the original Maclaurin series is obeyed
- the infinite series generated is only itself a Maclaurin series if it is a **polynomial**.

Example 2 **Self Tutor**

Use the Maclaurin series $\dfrac{1}{2+x} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}}\, x^k$ to find the Maclaurin series for $\dfrac{1}{2-x^2}$.

Write down the terms of the series up to x^6.

$$\frac{1}{2+x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}}\, x^k$$

$$\therefore \quad \frac{1}{2-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}}\, (-x^2)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k} x^{2k}}{2^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{k+1}}$$

$$= \frac{1}{2} + \frac{x^2}{4} + \frac{x^4}{8} + \frac{x^6}{16} + \ldots.$$

EXERCISE 24C

1 Use the Maclaurin series $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ to find the Maclaurin series for the following.

In each case, write down the terms of the series up to x^4.

a $\dfrac{1}{1-x^2}$ **b** $\dfrac{1}{1+x}$ **c** $\dfrac{1}{1-2x}$ **d** $\dfrac{2}{1+x}$

2 Use the Maclaurin series $\cos x = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} x^{2k}$ to find the Maclaurin series for the following.

In each case, write down the terms of the series up to x^6.

a $\cos 2x$ **b** $\cos(-x)$ **c** $\cos(x^2)$ **d** $x\cos x$

3 Use the Maclaurin series $\sin x = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)!} x^{2k+1}$ to find the Maclaurin series for:

a $\sin(-x)$ **b** $\sin 3x$ **c** $\sin(x^2)$ **d** $x\sin x$

4 Use the Maclaurin series $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$ to find the Maclaurin series for the following.

In each case, write down the terms of the series up to x^4.

a e^{3x} **b** e^{x^2} **c** 2^x

5 **a** Use the Maclaurin series $\sin x = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)!} x^{2k+1}$ to find an infinite series representation for $\sin\left(x + \frac{\pi}{2}\right)$.

b Use the binomial theorem to write out the terms of the series up to x^4. Explain your result.

Example 3 Self Tutor

a Use the Maclaurin series expansion for e^x to write down expansions for:

i e^{2x} **ii** e^{4x} **iii** e^{nx}, $n \in \mathbb{Z}^+$.

b Use the Maclaurin series expansion for $\cos x$ to find the first two non-zero terms of the Maclaurin series expansion of $\cos(e^x)$.

c Hence show that for x near 0, $\cos(e^x) \approx \cos(1) - \sin(1)x$.

d Find, by direct calculation, the Maclaurin polynomial of degree 2 of $\cos(e^x)$.

a $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots.$

i $e^{2x} = \sum_{k=0}^{\infty} \dfrac{(2x)^k}{k!} = 1 + 2x + \dfrac{4x^2}{2!} + \dfrac{8x^3}{3!} + \ldots.$

ii $e^{4x} = \sum_{k=0}^{\infty} \dfrac{(4x)^k}{k!} = 1 + 4x + \dfrac{16x^2}{2!} + \dfrac{64x^3}{3!} + \ldots.$

iii $e^{nx} = \sum_{k=0}^{\infty} \dfrac{(nx)^k}{k!} = 1 + nx + \dfrac{n^2x^2}{2!} + \dfrac{n^3x^3}{3!} + \ldots.$

b $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots.$

$$\therefore \quad \cos(e^x) = 1 - \frac{e^{2x}}{2!} + \frac{e^{4x}}{4!} - \frac{e^{6x}}{6!} + \ldots.$$

$$= 1 - \frac{1}{2!}\left(1 + 2x + \frac{4x^2}{2!} + \ldots.\right) + \frac{1}{4!}\left(1 + 4x + \frac{16x^2}{2!} + \ldots.\right) + \ldots.$$

$$+ \frac{(-1)^k}{(2k)!}\left(1 + (2k)x + \frac{(2k)^2x^2}{2!} + \ldots.\right) + \ldots. \qquad \{\text{using } \textbf{a iii}\}$$

$$= 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\right) + \left(\sum_{k=1}^{\infty} \frac{(-1)^k(2k)}{(2k)!}\right)x + \ldots.$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!}\,x + \ldots.$$

c For x close to 0, higher order terms involving x^2, x^3, ... tend to 0.

$$\therefore \quad \cos(e^x) \approx 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!}\,x$$

Now $\cos(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots. = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}$ {using known series for $\cos x$ and $\sin x$}

and $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \ldots. = -\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!}$

$\therefore \quad \cos(e^x) \approx \cos(1) - \sin(1)x$

d Let $f(x) = \cos(e^x)$.

$\therefore \quad f'(x) = -\sin(e^x) \times e^x$

$\therefore \quad f''(x) = -e^x \sin(e^x) - \cos(e^x)e^{2x}$

$\therefore \quad f(0) = \cos(e^0) = \cos(1)$

$\therefore \quad f'(0) = -\sin(1) \times 1 = -\sin(1)$

$\therefore \quad f''(0) = -\sin(1) - \cos(1)$

$\therefore \quad M_2(x) = \cos(1) - \sin(1)x - (\sin(1) + \cos(1))\frac{x^2}{2!}$

6 **a** Use the Maclaurin series expansion for e^x to write down expansions for:

i e^{-x} **ii** e^{-3x} **iii** $e^{-(2n+1)x}$, $n \in \mathbb{Z}^+$.

b Use the Maclaurin series expansion for $\sin x$ to find the first two non-zero terms of the Maclaurin series expansion for $\sin(e^{-x})$.

c Hence find an approximation for $\sin(e^{-x})$ for x near 0.

d Find, by direct calculation, the Maclaurin polynomial of degree 2 of $\sin(e^{-x})$.

D ADDITION AND SUBTRACTION

Maclaurin series can be added and subtracted like normal polynomials. The resulting series will converge on the intersection of the intervals of convergence of the individual series.

EXERCISE 24D

1 Use $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$ to write the terms of the Maclaurin series for the following up to x^7:

a $\sin x + \cos x$ **b** $\sin x + \sin 2x$ **c** $\sin(x^2) - \cos 3x$

2 **a** Use $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to write the Maclaurin series for e^{-x}.

b The **hyperbolic sine function** is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$.

Show that $\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

c The **hyperbolic cosine function** is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$.

Find the Maclaurin series expansion for $\cosh x$.

3 Use the Maclaurin series expansion for e^x to find the series for $e^{i\theta}$ where $i = \sqrt{-1}$, $\theta \in \mathbb{R}$.

By separating the series into real and imaginary parts, prove **Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$.

E DIFFERENTIATION AND INTEGRATION

One advantage of polynomial representations for functions is that they can be differentiated and integrated term by term.

This is particularly useful for integrating functions for which we otherwise cannot write an indefinite integral.

Example 4 **Self Tutor**

The Maclaurin series for $\cos x$ is $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$. Find the Maclaurin series for $\sin x$ by:

a differentiation **b** integration.

a $\frac{d}{dx}(\cos x) = -\sin x$

$\therefore \sin x = -\frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right)$

$= -\frac{d}{dx}\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right)$

$= -\frac{d}{dx}(1) - \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \frac{d}{dx}(x^{2k})$

$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} (2k)x^{2k-1}$

$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(2(n+1)-1)!} x^{2(n+1)-1}$ {letting $n = k-1$}

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$



b $\int \cos x\,dx = \sin x$

$\therefore \int_0^x \cos t\,dt = \big[\sin t\big]_0^x$

$= \sin x - \sin 0$

$= \sin x$

$\therefore \sin x = \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}\right) dt$

$= \sum_{k=0}^{\infty} \left(\int_0^x \frac{(-1)^k t^{2k}}{(2k)!}\,dt\right)$

$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\right]_0^x$

$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all $x \in \mathbb{R}$

EXERCISE 24E

1 Show that the derivative of the Maclaurin series for e^x is itself.

2 The Maclaurin series of $\sin x$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$. Find the Maclaurin series for $\cos x$ by:

a differentiation **b** integration.

3 **a** Use the Maclaurin series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots.$ to find the Maclaurin series for $\frac{1}{1+x}$.

b *Hence* find the Maclaurin series for $\frac{1}{1+x^2}$.

c *Hence* show that the Maclaurin series for $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$

Example 5 Self Tutor

a Use the Maclaurin series $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ to find the Maclaurin series for $\cos(x^2)$.

b Hence estimate $\int_0^1 \cos(x^2)\,dx$ correct to 4 decimal places.

a $\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^2)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k}$

b $\int_0^1 \cos(x^2)\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k}\,dx$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^1 x^{4k}\,dx$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[\frac{x^{4k+1}}{4k+1}\right]_0^1$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,(4k+1)}$

$= 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} + \frac{1}{685\,440} - \ldots.$ where $\frac{1}{685\,440} \approx 0.000\,001\,46$

Using the first 4 terms, $\int_0^1 \cos(x^2)\,dx \approx 0.9045$

4 **a** Use the Maclaurin series $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ to find the Maclaurin series for $\sin(x^2)$.

b Hence estimate $\int_0^1 \sin(x^2)\,dx$ correct to 4 decimal places.

5 **a** Use the Maclaurin series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to find the Maclaurin series for $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

b Hence estimate to 4 decimal places:

i $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$ **ii** $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$ **iii** $\int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$.

6 **a** Use the Maclaurin series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ to find the Maclaurin series for $\frac{1}{1+x^3}$. State the values of x for which you would expect the series to converge.

b Hence esimate $\int_0^{\frac{1}{3}} \frac{1}{1+x^3}\,dx$ to 4 decimal places.

Example 6 — Self Tutor

a Use the Maclaurin series for $(1+x)^p$ to find the Maclaurin series for $(1-x^2)^{-\frac{1}{2}}$.

b Use the fact that $\arccos x - \arccos(0) = \int_0^x \frac{-1}{\sqrt{1-t^2}}\,dt$ to find a Maclaurin series for $\arccos x$. For which values of x is this valid?

c Hence find the first three non-zero terms of $\arccos(x^2)$, and use them to estimate $\int_0^{0.4} \arccos(x^2)\,dx$.

a $(1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\ldots.(p-k+1)}{k!} x^k$ for all $|x| < 1$

$$\therefore\ (1-x^2)^{-\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots.\left(\frac{1}{2}-k\right)}{k!}(-x^2)^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots.\left(\frac{-2k+1}{2}\right)(-1)^k}{k!} x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{2k} \times 1 \times 3 \times 5 \times \ldots. \times (2k-1)}{2^k\, k!} x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1 \times 3 \times 5 \times \ldots. \times (2k-1)}{2^k (k!)} x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1 \times 3 \times 5 \times \ldots. \times (2k-1)}{2^k k!} \times \frac{2 \times 4 \times 6 \times \ldots. \times 2k}{2 \times 4 \times 6 \times \ldots. \times 2k} \times x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!\, 2^k k!} x^{2k}$$

$$\therefore\ (1-x^2)^{-\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2} x^{2k} \quad \text{for all } |x^2| < 1, \text{ which is for all } |x| < 1.$$

b $\arccos x - \arccos(0) = \displaystyle\int_0^x \frac{-1}{\sqrt{1-t^2}}\,dt$

$$\therefore \quad \arccos x - \tfrac{\pi}{2} = -\int_0^x \left(1 + \sum_{k=1}^{\infty} \frac{(2k)!}{4^k(k!)^2}\,t^{2k}\right) dt$$

$$= -\left(x + \sum_{k=1}^{\infty} \frac{(2k)!}{4^k(k!)^2(2k+1)}\,x^{2k+1}\right)$$

$$\therefore \quad \arccos x = \tfrac{\pi}{2} - x - \sum_{k=1}^{\infty} \frac{(2k)!}{4^k(k!)^2(2k+1)}\,x^{2k+1}$$

This is valid provided $|x| < 1$. These values of x all lie in the domain of $\arccos x$, so the series is valid for $-1 < x < 1$.

c $\arccos x \approx \tfrac{\pi}{2} - x - \tfrac{1}{6}x^3$ {using the first 3 terms}

$\therefore \quad \arccos(x^2) \approx \tfrac{\pi}{2} - x^2 - \tfrac{1}{6}x^6$ for $|x| < 1$

$$\therefore \quad \int_0^{0.4} \arccos\left(x^2\right) dx \approx \int_0^{0.4} \left(\tfrac{\pi}{2} - x^2 - \frac{x^6}{6}\right) dx \quad \text{since} \quad |0.4| < 1$$

$$= \left[\tfrac{\pi}{2}x - \frac{x^3}{3} - \frac{x^7}{7}\right]_0^{0.4}$$

$$\approx 0.607$$

7 **a** Use the Maclaurin series for $(1+x)^p$ to find the Maclaurin series for $(1+x^2)^{-1}$.

b Use the fact that $\arctan x - \arctan(0) = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$ to obtain the Maclaurin series for $\arctan x$. For which values of x is this valid?

c Hence find the first four non-zero terms of $\arctan(x^2)$, and use them to estimate $\displaystyle\int_0^1 \arctan(x^2)\,dx$.

8 **a** Use the Maclaurin series for $(1-x^2)^{-\frac{1}{2}}$ found in **Example 6** part **a**, and the fact that $\arcsin x - \arcsin(0) = \displaystyle\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$, to obtain the Maclaurin series for $\arcsin x$. For which values of x is this valid?

b By considering $\arcsin \frac{1}{2}$, show that $\dfrac{\pi}{6} = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{2^{4k+1}(k!)^2(2k+1)}$.

F MULTIPLICATION

If we multiply two Maclaurin series together, we will be faced with collecting infinitely many "like" terms. This is generally not possible. However, by multiplying the first few terms of each Maclaurin series, we can generate the first few terms of the product.

Example 7 **Self Tutor**

The Maclaurin series expansion for $\cos 2x$ is $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots.$

a Use the first few terms of $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots.$ to find the terms of $\sin^2 x$ up to x^6.

b Compare your result with the expansion of $\frac{1}{2} - \frac{1}{2}\cos 2x$.

a
$$\begin{aligned}\sin^2 x &= \left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \ldots.\right)\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \ldots.\right)\\ &= x^2 - \tfrac{1}{6}x^4 + \tfrac{1}{120}x^6 - \ldots.\\ &\quad - \tfrac{1}{6}x^4 + \tfrac{1}{36}x^6 - \ldots.\\ &\quad + \tfrac{1}{120}x^6 - \ldots.\\ &= x^2 - \tfrac{1}{3}x^4 + \tfrac{2}{45}x^6 - \ldots.\end{aligned}$$

b
$$\begin{aligned}\tfrac{1}{2} - \tfrac{1}{2}\cos 2x &= \tfrac{1}{2} - \tfrac{1}{2}\left(1 - 2x^2 + \tfrac{2}{3}x^4 - \tfrac{4}{45}x^6 + \ldots.\right)\\ &= \tfrac{1}{2} - \tfrac{1}{2} + x^2 - \tfrac{1}{3}x^4 + \tfrac{2}{45}x^6 - \ldots.\\ &= x^2 - \tfrac{1}{3}x^4 + \tfrac{2}{45}x^6 - \ldots.\end{aligned}$$

which matches the first terms of the expansion of $\sin^2 x$.

EXERCISE 24F

1 By multiplication of the Maclaurin series for e^x and $\sin x$, find the first five non-zero terms in the Maclaurin series for $e^x \sin x$.

2 **a** Use the first few terms of $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots.$ and $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots.$ to find the terms of $2\sin x \cos x$ up to x^5.

b Compare your result with the expansion of $\sin 2x$.

3 **a** Use $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots.$ and $\arccos x = \frac{\pi}{2} - x - \frac{1}{6}x^3 - \ldots.,\ |x| < 1$ to find the first four non-zero terms in the Maclaurin series for $e^x \arccos x$.

b For which values of x will the Maclaurin series for $e^x \arccos x$ converge? Explain your answer.

4 **a** Use $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to find the Maclaurin series for e^{-x^3}.

b Hence, or otherwise, find the terms in the Maclaurin series for $e^{-x^3}\cos x$ up to x^9.

G DIVISION

To perform the division of two Maclaurin series $f(x)$ and $g(x)$ we assume the result is another Maclaurin series $p(x) = \dfrac{f(x)}{g(x)}$ where $p(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\{a_k\}$ are unknowns.

Multiplying both sides by $g(x)$, $p(x)g(x) = f(x)$.

We can perform the multiplication $p(x)g(x)$ for as many terms as we like, then equate coefficients with $f(x)$ to solve for the unknowns $a_0, a_1, a_2, \ldots$.

Example 8 **Self Tutor**

Let $\sum_{k=0}^{\infty} a_k x^k = \dfrac{x}{\sin x}$ be the power series expansion of $\dfrac{x}{\sin x}$, $x \neq n\pi$, $n \in \mathbb{Z}$.

Use the Maclaurin series for $\sin x$ to find a_0, a_1, a_2 such that $\dfrac{x}{\sin x} \approx a_0 + a_1 x + a_2 x^2$ for x near 0.

Since $\dfrac{x}{\sin x} = \sum_{k=0}^{\infty} a_k x^k$, we find

$$x = \left(\sum_{k=0}^{\infty} a_k x^k\right) \sin x$$

$$= (a_0 + a_1 x + a_2 x^2 + \ldots.)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.\right)$$

$$= a_0 x + a_1 x^2 + \left(a_2 - \frac{a_0}{3!}\right) x^3 + \ldots.$$

Equating coefficients, $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{6}$

$\therefore \dfrac{x}{\sin x} \approx 1 + \frac{1}{6}x^2$ for x near 0.

EXERCISE 24G

1 Use known Maclaurin series to find the first three non-zero terms in the expansion of $\dfrac{x}{\cos x}$.

2 Use the Maclaurin series for $\sin x$ and $\cos x$ to determine the first three non-zero terms of the Maclaurin series for $\tan x = \dfrac{\sin x}{\cos x}$.

3 **a** Find the Maclaurin series for: **i** $e^{2x} - 1$ **ii** $\sin 4x$

b Hence find $\lim_{x \to 0} \dfrac{e^{2x} - 1}{\sin 4x}$.

4 **a** Write $\dfrac{\sin x}{\ln(1 + x)}$ as the quotient of two Maclaurin series.

b Hence find $\lim_{x \to 0} \dfrac{\sin x}{\ln(1 + x)}$.

5 Suppose f and g are continuous functions where $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$.

Use Maclaurin series to show that $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \dfrac{f'(0)}{g'(0)}$, provided $g'(0) \neq 0$.

6 **a** By direct calculation, find the Maclaurin series for $e^{\cos x}$ up to and including the x^4 term.

b Use the Maclaurin series for $\sin x$ and e^{2x} to find the Maclaurin series for $\sin(e^{2x}-1)$ up to and including the x^4 term.

c Hence find $\lim\limits_{x\to 0} \dfrac{e^{\cos x}-e}{\sin(e^{2x}-1)}$.

7 **a** Find *all* the zeros of $\sin x$ and of $\dfrac{\sin x}{x}$ for $x \in \mathbb{R}$.

b Given that $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots.$ for all $x \in \mathbb{R}$, find a polynomial expansion for $\dfrac{\sin x}{x}$.

c Find all the zeros of $\left(1-\dfrac{x}{\pi}\right)\left(1+\dfrac{x}{\pi}\right)\left(1-\dfrac{x}{2\pi}\right)\left(1+\dfrac{x}{2\pi}\right)\ldots.$

d Show that

$$\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots. = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots.$$

and comment on Euler's claim that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots. = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots.$$

e By equating the coefficients of x^2 in this last equation, prove Euler's result that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots. = \frac{\pi^2}{6}.$$

f By writing $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^2} = \underbrace{\sum\limits_{r=1}^{\infty}\dfrac{1}{(2r)^2}}_{\text{even } k} + \underbrace{\sum\limits_{r=1}^{\infty}\dfrac{1}{(2r-1)^2}}_{\text{odd } k}$, find the exact value of $\sum\limits_{k=1}^{\infty}\dfrac{1}{(2k-1)^2}$.

HISTORICAL NOTE

Euler was able to derive a method to sum **all** series of the form $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^{2n}}$, $n \in \mathbb{Z}^+$.

However, the exact value of $\sum\limits_{k=1}^{\infty}\dfrac{1}{k^{2n+1}}$, for **any** $n \in \mathbb{Z}^+$ is still an open problem.

REVIEW SET 24A

1 **a** Find the Maclaurin series representation for $f(x) = \dfrac{1}{3-x}$.

b Hence find $M_3(x)$, the 3rd order Maclaurin polynomial approximation for $\dfrac{1}{3-x}$. Compare $M_3(1)$ with $f(1)$.

2 The Maclaurin series for $\cos x$ is $\cos x = \sum\limits_{k=0}^{\infty}\dfrac{(-1)^k}{(2k)!}x^{2k}$.

Use this Maclaurin series to estimate the value of $\cos\frac{\pi}{90}$ correct to 5 decimal places.

3 Given the Maclaurin series $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots.$, discuss the values of x for which the series converges.

4 **a** Derive the Maclaurin series representation $\ln(2-x) = \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k2^k} x^k$.

State the values of x for which you expect the series to converge.

b Hence show that $\sum_{k=1}^{\infty} \frac{1}{k2^k} = \ln 2$.

5 **a** Use the Maclaurin series for $\ln(1+x)$ to derive the Maclaurin series for $\ln(1+3x)$.

b Copy and complete the table:

n	1	2	3	4	5
$M_n(1)$					
$M_n\left(-\frac{1}{4}\right)$					

Hence determine whether the Maclaurin series is convergent for:

i $x = 1$ **ii** $x = -\frac{1}{4}$.

c For which values of x do you think the series will converge?

6 Use an appropriate Maclaurin series to evaluate $\int_0^1 e^{x^2}\,dx$ correct to 3 decimal places.

7 Use the Maclaurin series for $\cos x$ to find the first three non-zero terms in the Maclaurin series for $\sec x$.

8 **a** Find the Maclaurin series for $\frac{1}{1+x^3}$.

b Hence find the Maclaurin series for $\frac{x^2}{(1+x^3)^2}$.

9 **a** Use the Maclaurin series $e^x = 1 + x + \frac{x^2}{2!} + \ldots.$ and $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots.$ to write the first three non-zero terms in the Maclaurin series for:

i $\tan 2x$ **ii** $e^{\tan x}$

b Hence find $\lim_{x \to 0} \frac{\tan 2x}{e^{\tan x} - 1}$.

REVIEW SET 24B

1 Find the Maclaurin series for xe^x up to the term x^4.

2 The Maclaurin series representation for $\frac{1}{x^2-4}$ is $\frac{1}{x^2-4} = -\frac{1}{4} - \frac{x^2}{16} - \frac{x^4}{64} - \ldots. = \sum_{k=0}^{\infty} -\frac{x^{2k}}{4^{k+1}}$.

a Copy and complete this table:

n	2	4	6
$M_n(0.1)$			
$M_n(1)$			
$M_n(2)$			

b For what values of x do you think the Maclaurin series converges?

3 Use the Maclaurin series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ to find the Maclaurin series for:

a $\arctan \frac{x}{2}$ **b** $\arctan(x^2)$ **c** $\frac{1}{1+x^2}$

4 Using the (known) Maclaurin series for e^x and $(1+x)^p$ or otherwise, find the Maclaurin series for:

a $(1+x)^{\frac{3}{2}}$ **b** $(1-x)e^{1-x}$ **c** $\frac{1}{(1+e^{2x})^2}$

5 Using an appropriate Maclaurin series, evaluate $\int_0^1 \sin 2x \, dx$ to 3 decimal places.

6 **a** Find the first four terms of the Maclaurin series for $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$, $|x| < 1$, by letting $\sqrt{1+x} = \sum_{k=0}^{\infty} a_k x^k$ and using $(\sqrt{1+x})^2 = 1+x$.

b Verify that your answer agrees with the $(1+x)^p$ series with $p = \frac{1}{2}$.

7 Let $\sum_{k=0}^{\infty} a_k x^k$ be the Maclaurin series for $\frac{e^{-x}}{x^2+1}$.

a Use the Maclaurin series expansion for e^{-x} to find a_0, a_1, a_2, and a_3.

b Check your answer using direct calculation.

8 **a** Write down the terms up to x^6 of the Maclaurin series for:

i $\cos^2 x$ **ii** $\sin^2 x$

b Show that your series are consistent with the identity:

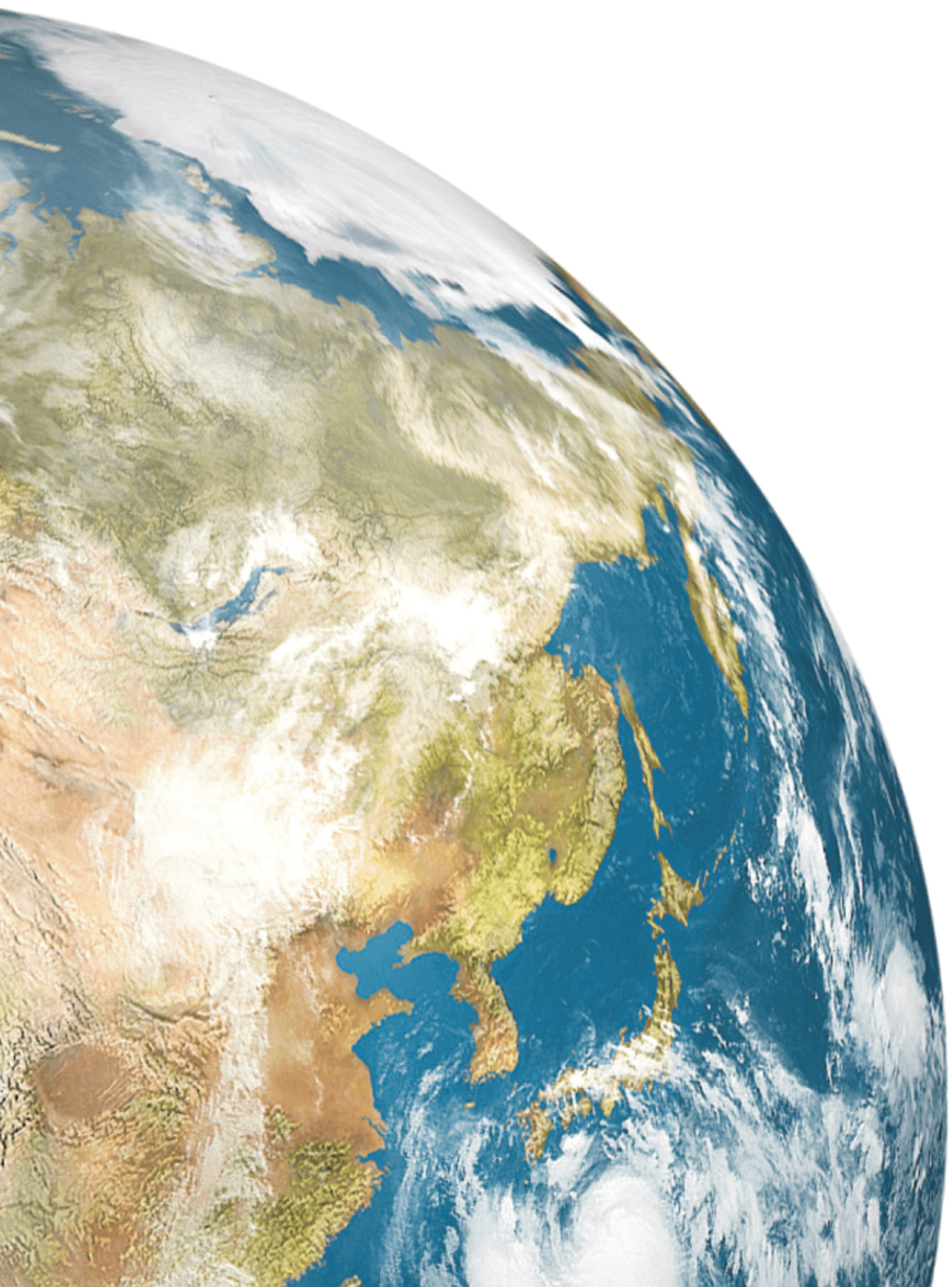
i $\cos^2 x + \sin^2 x = 1$ **ii** $\cos^2 x - \sin^2 x = \cos 2x$

Chapter 25

Differential equations

Contents:

A Differential equations
B Euler's method for numerical integration
C Differential equations of the form $\frac{dy}{dx} = f(x)$
D Separable differential equations
E Logistic growth
F Homogeneous differential equations $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
G The integrating factor method
H Maclaurin series developed from a differential equation



OPENING PROBLEM

The population of rodents on an island is currently 500. Its growth rate is expected to be given by $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{3000}\right)$, where t is the time in years from now.



Things to think about:

a How would we describe the relationship between the variables P and t?

b What would the graph of P against t look like?

c How can we write P as a function of t?

d Can you find:

i the expected population after 8 years

ii the expected time taken for the population to increase to 2000

iii the maximum population which the island can sustain?

In the **Opening Problem**, the equation $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{3000}\right)$ describes the relationship between the function $P(t)$ and its derivative. This is an example of a **differential equation**.

A DIFFERENTIAL EQUATIONS

A **differential equation** is an equation involving a *derivative* of a function.

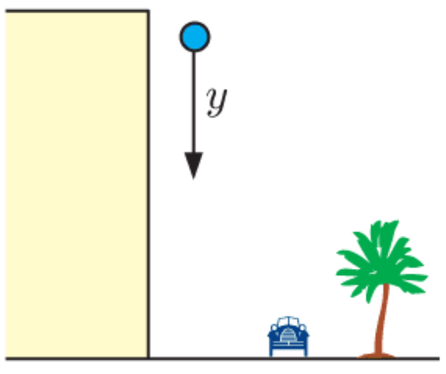
Suppose y is a function of x, so $y = y(x)$. Examples of differential equations for this function are:

- $\frac{dy}{dx} = \frac{x^2}{y}$
- $\frac{dy}{dx} = -0.075y^3$
- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = 0$

Differential equations not only arise in pure mathematics, but are also used to model and solve problems in applied mathematics, physics, engineering, economics, and other subjects.

For example:

A falling object



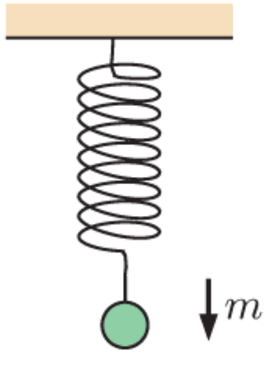
$$\frac{d^2y}{dt^2} = 9.8$$

A parachutist



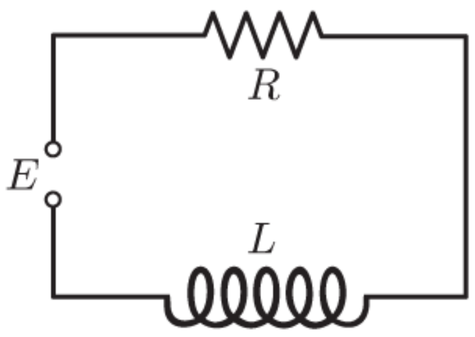
$$m\frac{dv}{dt} = mg - av^2$$

Object on a spring



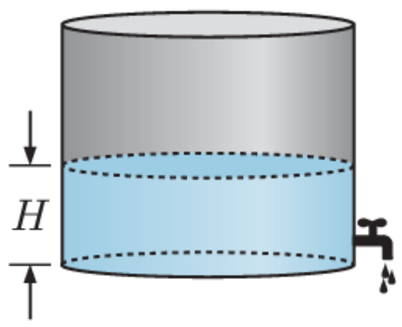
$$m\frac{d^2y}{dt^2} = -ky$$

Current in an RL Circuit



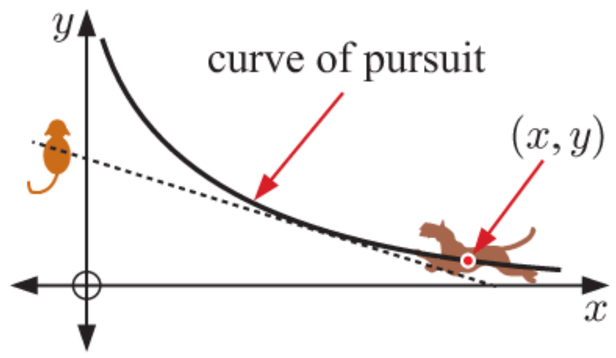
$$L\frac{dI}{dt} + RI = E$$

Water leaking from a tank



$$\frac{dH}{dt} = -a\sqrt{H}$$

A dog pursuing a cat



$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

SOLUTIONS OF DIFFERENTIAL EQUATIONS

A **solution** of a differential equation is a function y in terms of x, which satisfies the differential equation.

For example:

- a solution to the differential equation $\frac{dy}{dx} = 2xy$ is $y = e^{x^2}$
- a solution to the differential equation $\frac{dy}{dx} = y^2$ is $y = -\frac{1}{x}$.

Example 1 **Self Tutor**

Verify that $y = e^{x^2}$ is a solution to the differential equation $\frac{dy}{dx} = 2xy$.

If $y = e^{x^2}$, then $\frac{dy}{dx} = 2x(e^{x^2})$

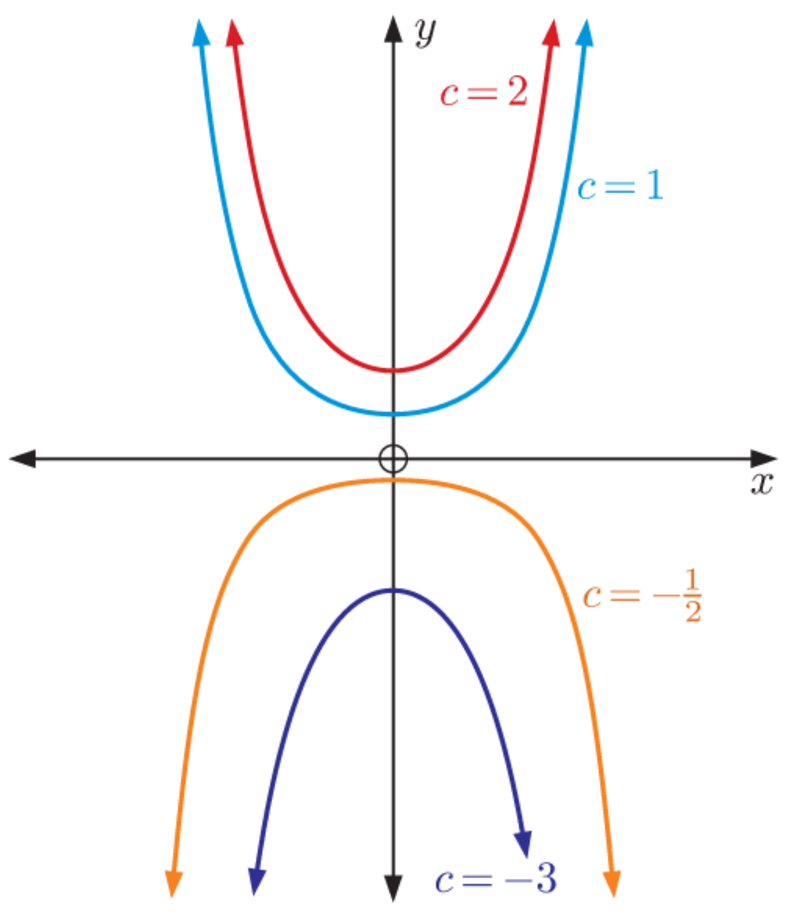
$\therefore \frac{dy}{dx} = 2xy$ as required.

THE GENERAL SOLUTION

In addition to the solution $y = e^{x^2}$, the differential equation $\frac{dy}{dx} = 2xy$ has infinitely many other solutions, including $y = 2e^{x^2}$, $y = 3e^{x^2}$, and $y = -7e^{x^2}$.

All of these solutions have the form $y = ce^{x^2}$ where c is a constant, so we say that $y = ce^{x^2}$ is the **general solution** to the differential equation.

If we are given **initial conditions** for the problem, such as a value of y or $\frac{dy}{dx}$ for a specific value of x, then we can evaluate c. This gives us a **particular solution** to the problem, which is one particular curve from the family of curves described by the general solution.



Example 2 **Self Tutor**

Consider the differential equation $\frac{dy}{dx} - 3y = 3$.

a Show that $y = ce^{3x} - 1$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0, \pm1, \pm2, \pm3$.

c Find the particular solution which passes through $(0, 2)$.

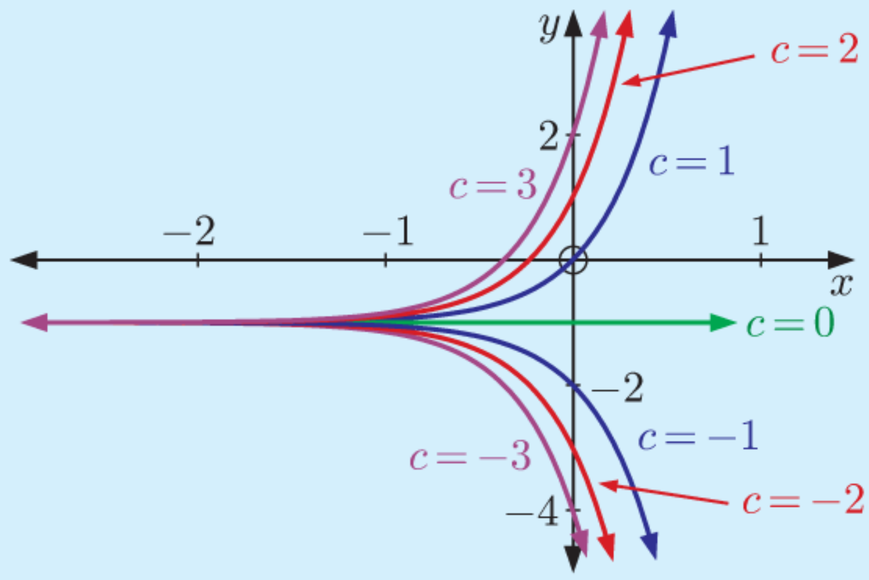
d Find the equation of the tangent to the particular solution at $(0, 2)$.

a If $y = ce^{3x} - 1$ then $\frac{dy}{dx} = 3ce^{3x}$.

$$\therefore \quad \frac{dy}{dx} - 3y = 3ce^{3x} - 3(ce^{3x} - 1)$$
$$= 3ce^{3x} - 3ce^{3x} + 3$$
$$= 3$$

$\therefore$ the differential equation is satisfied for all $x \in \mathbb{R}$ for any constant c.

b The solution curves for $c = 0, \pm1, \pm2, \pm3$ are shown alongside.



c $y = ce^{3x} - 1$ is a general solution to the differential equation.

The particular solution passes through $(0, 2)$, so $2 = ce^{3\times0} - 1$

$$\therefore \quad c = 3$$

$$\therefore \quad \text{the particular solution is} \quad y = 3e^{3x} - 1$$

d $\frac{dy}{dx} = 3 + 3y$

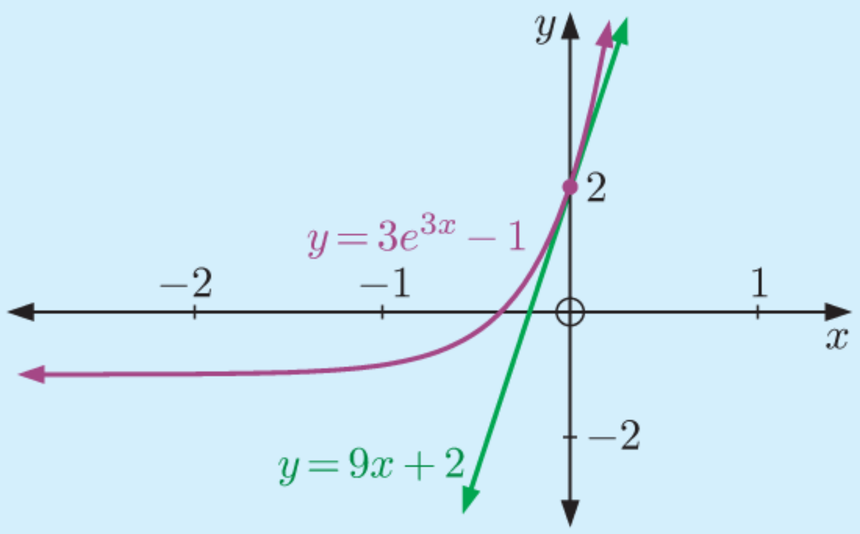
$\therefore$ at the point $(0, 2)$, $\frac{dy}{dx} = 3 + 3 \times 2 = 9$

$\therefore$ the gradient of the tangent to the particular solution $y = 3e^{3x} - 1$ at $(0, 2)$, is 9.

$\therefore$ the equation of the tangent is

$$y = 9(x - 0) + 2$$

$$\therefore \quad y = 9x + 2$$



EXERCISE 25A

1 Verify that:

a $y = x^4$ is a solution to $\frac{dy}{dx} = 4x^3$

b $y = 5e^{2x}$ is a solution to $\frac{dy}{dx} = 2y$

c $y = \sqrt{x^2 + 1}$ is a solution to $\frac{dy}{dx} = \frac{x}{y}$

d $y = -\frac{1}{x}$ is a solution to $\frac{dy}{dx} = y^2$

e $y = 3e^{\frac{x^2}{2}+x}$ is a solution to $\frac{dy}{dx} - y = xy$.

2 Match each differential equation with the correct solution:

a $\frac{dy}{dx} = \frac{1}{2y}$ **b** $\frac{dy}{dx} = -\frac{2y}{x}$ **c** $\frac{dy}{dx} = \frac{2x}{3y^2}$

A $y = \sqrt[3]{x^2 + 1}$ **B** $y = \sqrt{x + 3}$ **C** $y = \frac{1}{x^2}$

3 Verify that:

a $y = x^3 + c$ is the general solution to $\frac{dy}{dx} = 3x^2$

b $y = ce^{-x}$ is the general solution to $\frac{dy}{dx} = -y$

c $y = -\frac{2}{x^2 + c}$ is the general solution to $\frac{dy}{dx} = xy^2$.

4 Consider the differential equation $\frac{dy}{dx} = 4x$.

a Show that $y = 2x^2 + c$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0, \pm 1, \pm 2$.

c Find the particular solution which passes through $(1, \frac{1}{2})$.

d Find the equation of the tangent to the particular solution at $(1, \frac{1}{2})$.

5 Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a Show that $y = 2x - 2 + ce^{-x}$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0, \pm 1, \pm 2$.

c Find the particular solution which passes through $(0, 1)$.

d Find the equation of the tangent to the particular solution at $(0, 1)$.

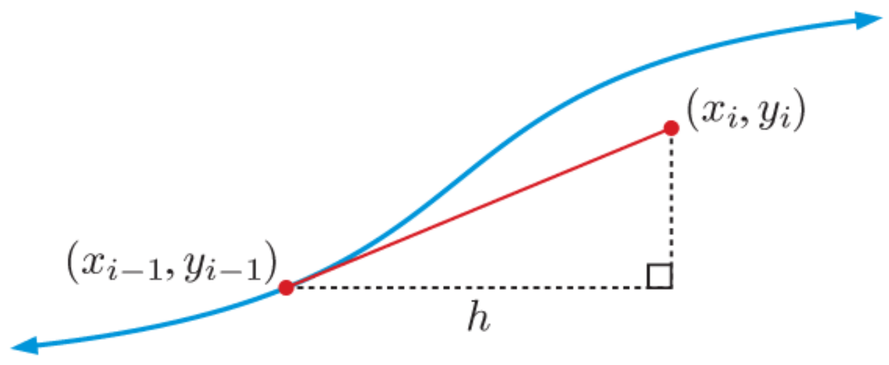
B EULER'S METHOD FOR NUMERICAL INTEGRATION

In the book *Institutionum calculi integralis*, **Leonhard Euler** (1707 - 1783) described a method which allows us to numerically approximate the solution curve to a differential equation where $\frac{dy}{dx} = f(x, y)$ with particular solution (x_0, y_0).

The particular solution gives us initial values for x_0 and y_0. At each iteration of Euler's method, we generate another point for our solution curve.

We choose a step size h and let $x_i = x_{i-1} + h$ be the x-coordinate of the ith point.

Since $\frac{dy}{dx}$ gives us the gradient of the tangent to the curve, we follow the tangent to find the next value of y.



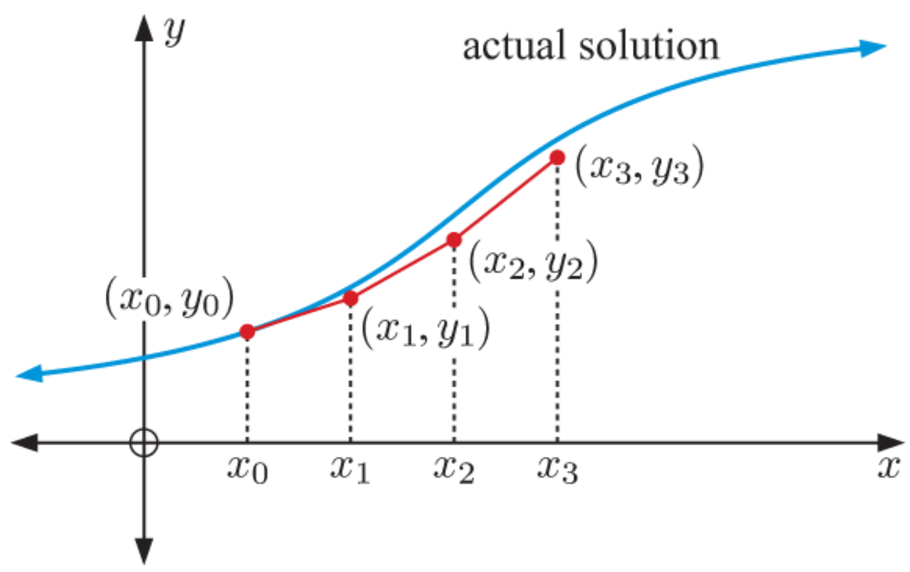
Now $\frac{y_i - y_{i-1}}{h} = f(x_{i-1}, y_{i-1})$

$\therefore\ y_i - y_{i-1} = h\,f(x_{i-1}, y_{i-1})$

$\therefore\ y_i = y_{i-1} + h\,f(x_{i-1}, y_{i-1})$

The iterative procedure $\begin{cases} x_i = x_{i-1} + h \\ y_i = y_{i-1} + h\,f(x_{i-1},\, y_{i-1}) \end{cases}$ creates a polygonal approximation to the solution curve.

We correct our direction at each iteration by recalculating the gradient of the tangent to the curve.



Suppose we apply Euler's method for n iterations to reach $x = b$.

If we let $n \to \infty$ so we use infinitely many steps, Euler's method gives $y(b) = y(x_0) + \int_{x_0}^{b} \frac{dy}{dx}\,dx$, which is a statement of the Fundamental Theorem of Calculus.

Example 3 **Self Tutor**

Consider the differential equation $\frac{dy}{dx} = e^x + 1$ with $y(0) = 1$.

a Estimate $y(0.5)$ by applying Euler's method with:

i $h = 0.25$ for two steps **ii** $h = 0.1$ for five steps.

b Find $y(0.5)$ exactly using the Fundamental Theorem of Calculus. Comment on your results.

$y(0) = 1$ gives us $x_0 = 0$ and $y_0 = 1$.

a **i**

Iteration	x_{n-1}	y_{n-1}	$\frac{dy}{dx}$	x_n	y_n
1	0	1	2	0.25	1.5
2	0.25	1.5	2.2840	0.5	2.0710

$\therefore\ y(0.5) \approx 2.0710$

ii

Iteration	x_{n-1}	y_{n-1}	$\frac{dy}{dx}$	x_n	y_n
1	0	1	2	0.1	1.2
2	0.1	1.2	2.1052	0.2	1.4105
3	0.2	1.4105	2.2214	0.3	1.6327
4	0.3	1.6327	2.3500	0.4	1.8676
5	0.4	1.8677	2.4918	0.5	2.1168

$\therefore\ y(0.5) \approx 2.1168$

b Using the Fundamental Theorem of Calculus,

$$\begin{aligned} y(0.5) &= y(0) + \int_0^{0.5} \frac{dy}{dx}\,dx \\ &= 1 + \int_0^{0.5} (e^x + 1)\,dx \\ &= 1 + \big[e^x + x\big]_0^{0.5} \end{aligned}$$

$$= 1 + (e^{0.5} + 0.5) - (e^0 + 0)$$
$$= e^{0.5} + 0.5$$
$$\approx 2.148\,72$$

The accuracy of Euler's method was improved by decreasing the step size.

EXERCISE 25B

1 Consider the differential equation $\frac{dy}{dx} = xy - 1$ with initial point $(0, 3)$.

a Let $h = 0.1$ be the step size. Copy and complete:

Iteration	x_{n-1}	y_{n-1}	$\frac{dy}{dx}$	x_n	y_n
1	0	3	-1	0.1	2.9
2	0.1	2.9			
3					
4					
5					

b Plot the solution curve for $0 \leqslant x \leqslant 0.5$.

2 Consider the differential equation $\frac{dy}{dx} = 3e^{2x} - 1$ with $y(0) = 2$.

a Estimate $y(0.5)$ by applying Euler's method with:

i $h = 0.25$ for two steps **ii** $h = 0.1$ for five steps.

b Find $y(0.5)$ exactly using the Fundamental Theorem of Calculus. Comment on your results.

3 Use Euler's method with step size 0.2 to estimate $y(1)$ given $\frac{dy}{dx} = 1 + 2x - 3y$, $y(0) = 1$.

4 Consider the differential equation $\frac{dy}{dx} = -\cos x$ with $y(0) = 0$.

a Estimate $y(0.5)$ using Euler's method with step size 0.1.

b Use technology to apply Euler's method with step size 0.001 for 500 steps.

c Find the exact value of $y(0.5)$ using the Fundamental Theorem of Calculus. Comment on your results.





5 Consider the differential equation $\frac{dy}{dx} = x\cos y$ with initial point $(0, 0)$.

Use Euler's method to estimate the value of y when $x = 2$, using:

a $h = 1$ **b** $h = 0.1$ **c** $h = 0.01$

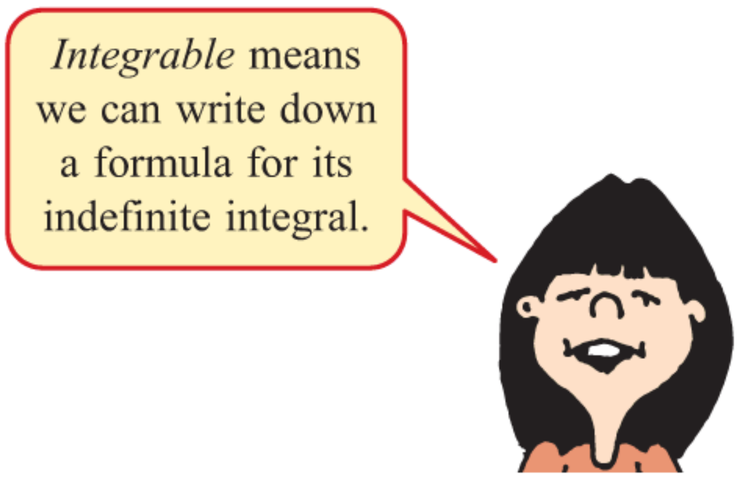
DISCUSSION

- What features of Euler's method are most useful?
- For a given step size h, what happens to the accuracy of the solution as the number of iterations increases?
- What things can go wrong with Euler's method? Can they all be avoided simply by using a smaller step size?

C DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

For many differential equations, we can find analytical solutions.

The first form we consider are differential equations of the form $\frac{dy}{dx} = f(x)$ where $f(x)$ is integrable. In this case we simply integrate with respect to x.



Example 4 — Self Tutor

Find the general solution to:

a $\frac{dy}{dx} = e^{2x}$ b $\frac{dy}{dx} = \frac{2x}{x^2+1}$ c $\frac{dy}{dx} + 1 = \cos x$

a $\frac{dy}{dx} = e^{2x}$

$\therefore\ y = \int e^{2x}\,dx$

$\therefore\ y = \frac{1}{2}e^{2x} + c$

b $\frac{dy}{dx} = \frac{2x}{x^2+1}$

$\therefore\ y = \int \frac{2x}{x^2+1}\,dx$

$\therefore\ y = \ln|\,x^2+1\,| + c$

c $\frac{dy}{dx} + 1 = \cos x$

$\therefore\ \frac{dy}{dx} = \cos x - 1$

$\therefore\ y = \int (\cos x - 1)\,dx$

$\therefore\ y = \sin x - x + c$

Example 5 — Self Tutor

Find the particular solution to $\frac{dy}{dx} = \sin^2 x$ given $y(0) = \frac{1}{2}$.

$\frac{dy}{dx} = \sin^2 x$

$\therefore\ y = \int \sin^2 x\,dx$

$\therefore\ y = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$\therefore\ y = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Now $y(0) = \frac{1}{2}$

$\therefore\ \frac{1}{2} = \frac{1}{2}(0) - \frac{1}{4}\sin 0 + c$

$\therefore\ c = \frac{1}{2}$

So, the solution is $y = \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}$.

EXERCISE 25C

1 Find the general solution to:

a $\frac{dy}{dx} = 4x^3$ **b** $\frac{dy}{dx} = x^2 + 6x$ **c** $\frac{dy}{dx} = e^{3x} + 4$

d $\frac{dy}{dx} = \cos x + \sin 2x$ **e** $\frac{dx}{dt} = \cos^2 t$ **f** $\frac{dy}{dx} = \frac{1}{x+4} - \frac{2}{3x-5}$

g $\frac{dM}{dt} = \frac{3t^2}{t^3 - 4}$ **h** $\frac{dy}{dx} = \frac{x}{\sqrt{25 - x^2}}$ **i** $f'(t) = te^{-t^2+1} + 2$

j $\frac{dS}{dt} = \frac{\sqrt{\ln t}}{t}$ **k** $\frac{dy}{dx} + \cos 3x = 1$ **l** $\frac{dy}{dx} + \frac{2}{x} = \sqrt{x}$

2 Find the particular solution to:

a $\frac{dy}{dx} = 3x - 2$ given $y(0) = 5$ **b** $\frac{dy}{dx} = e^{3x} + 1$ given $y(0) = 0$

c $\frac{dy}{dx} = \frac{1}{x}$ given $y(2) = \ln 12$

3 Find the particular solution to:

a $\frac{dy}{dt} = 2e^{2t} - e^{-t}$ given that $y = 2.5$ when $t = \ln 2$

b $\frac{dM}{d\alpha} = \cos 2\alpha - 3\sin\alpha$ given that $M = 2$ when $\alpha = \frac{\pi}{2}$

c $\frac{dP}{dx} = 2x\cos x$ given that $P = \sqrt{3}$ when $x = \frac{\pi}{6}$.

4 A function $f(x)$ has gradient function $f'(x) = 2x - 5$, and the point $(2, -18)$ lies on the curve. Find the value of $f(-2)$.

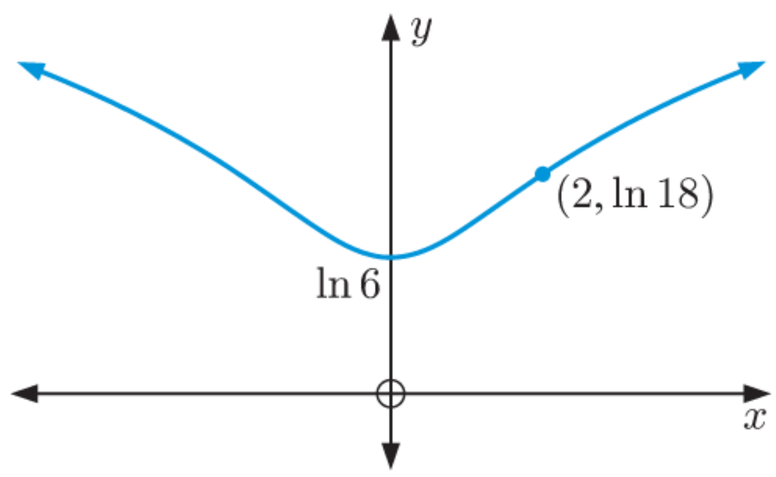
5 **a** Find the particular solution to the differential equation $\frac{dy}{dx} = e^x - e^{-x}$ given $y(0) = 1$.

b Sketch the graph of the particular solution.

c Find the equation of the tangent to the solution curve at the point where $x = \ln 2$.

6 A curve has gradient function $f'(x) = ax + bx^{-2}$ where a and b are constants. It passes through $(-1, -2)$ and has a turning point at $(1, 0)$. Find the function $f(x)$.

7 The curve alongside has gradient function $\frac{dy}{dx} = \frac{2x}{x^2 + k}$ for some positive constant k.
Find the equation of the curve.



8 Find the particular solution to:

a $f''(x) = 6x - 4$ given $f'(1) = 3$ and $f(2) = 7$

b $\frac{d^2y}{dx^2} = \sin 2x$ given $y(0) = 0$ and $y(\frac{\pi}{2}) = 2\pi$.

Example 6 Self Tutor

The marginal cost of producing x urns per week is given by

$\frac{dC}{dx} = 2.15 - 0.02x + 0.000\,36x^2$ pounds per urn provided $0 \leqslant x \leqslant 120$.

The initial costs before production starts are £185.

Find the total cost of producing 100 urns per day.

The marginal cost is $\frac{dC}{dx} = 2.15 - 0.02x + 0.000\,36x^2$ pounds per urn

$$\therefore \quad C(x) = \int (2.15 - 0.02x + 0.000\,36x^2)\,dx$$

$$= 2.15x - 0.02\,\frac{x^2}{2} + 0.000\,36\,\frac{x^3}{3} + c \text{ pounds}$$

$$= 2.15x - 0.01x^2 + 0.000\,12x^3 + c \text{ pounds}$$

But $C(0) = 185$ pounds, so $c = 185$

$$\therefore \quad C(x) = 2.15x - 0.01x^2 + 0.000\,12x^3 + 185 \text{ pounds}$$

$$\therefore \quad C(100) = 2.15(100) - 0.01(100)^2 + 0.000\,12(100)^3 + 185 \text{ pounds}$$

$$= 420 \text{ pounds}$$

$\therefore$ the total cost is £420.

9 The marginal cost per day of producing x gadgets is $C'(x) = 3.15 + 0.004x$ pounds per gadget. Find the total daily production cost for 800 gadgets given that the fixed costs before production commences are £450 per day.

10 The marginal profit for producing x dinner plates per week is given by $P'(x) = 15 - 0.03x$ pounds per plate. If no plates are made then a loss of £650 each week occurs.

a Find the profit function $P(x)$.

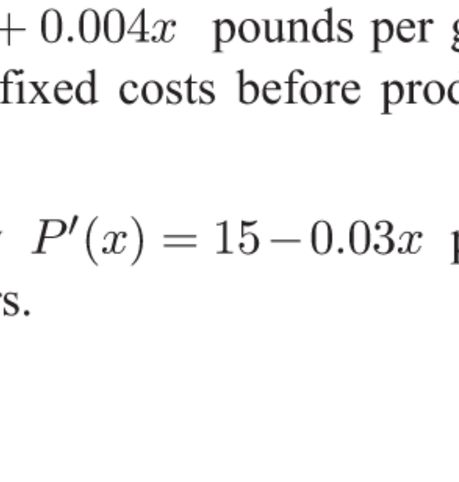
b What is the maximum profit, and when does it occur?

c What production levels enable a profit to be made?

Example 7 Self Tutor

A metal water pipe has an annulus cross-section as shown. The outer radius is 4 cm and the inner radius is 2 cm. Within the pipe, the water temperature is maintained at 100°C. Within the metal, the temperature drops off from inside to outside according to $\frac{dT}{dx} = -\frac{10}{x}$, where x is the distance from the central axis and $2 \leqslant x \leqslant 4$.

Find the temperature of the outer surface of the pipe.



tube cross-section

Since $\frac{dT}{dx} = -\frac{10}{x}$, $\quad T = \int -\frac{10}{x}\,dx$

$$\therefore \quad T = -10\ln x + c, \quad x > 0$$

But when $x = 2, \quad T = 100$

$$\therefore \quad 100 = -10\ln 2 + c$$

$$\therefore \quad c = 100 + 10\ln 2$$

Thus $T = -10\ln x + 100 + 10\ln 2$

$$\therefore \quad T = 100 + 10\ln\left(\tfrac{2}{x}\right)$$

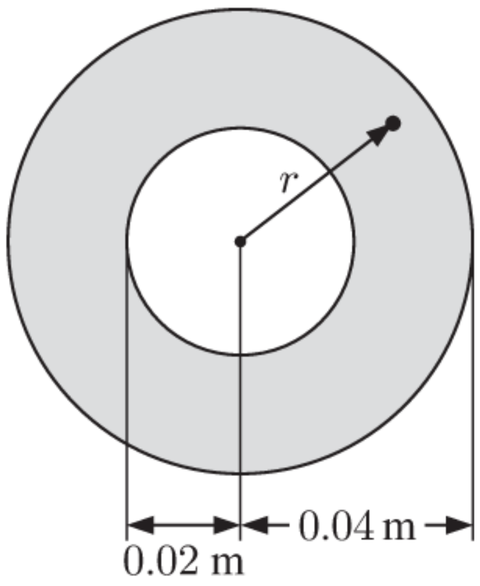
When $x = 4, \quad T = 100 + 10\ln\left(\tfrac{1}{2}\right) \approx 93.1$

The outer surface temperature is 93.1°C.

11 An insulation tube has inner radius 0.02 m and outer radius 0.04 m. Fluid flowing through the tube maintains the temperature on the inner wall at 600°C.

Heat is lost through each metre of tube length according to *Fourier's law* $\dfrac{dT}{dr} = -\dfrac{q}{2\pi kr}$ where $q = 680\ \text{W m}^{-1}$ is the heat transfer rate per metre of length, $k = 0.2\ \text{W m}^{-1}\,^\circ\text{C}$ is the thermal conductivity constant, r is the radius from the centre of the tube, and T is the temperature in °C.

Calculate the external temperature of the tube.

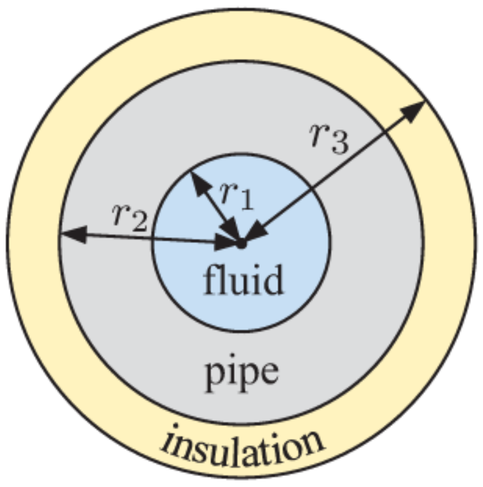


12 Fluid flows through a stainless steel pipe with thermal conductivity $k = 19\ \text{W m}^{-1}\,^\circ\text{C}$.

The pipe has inner radius $r_1 = 0.14$ m and outer radius $r_2 = 0.20$ m. The inner wall temperature is maintained at 400°C.

To meet occupational health and safety standards, the stainless steel pipe needs to be insulated with urethane foam with $k = 0.018\ \text{W m}^{-1}\,^\circ\text{C}$ so that the temperature on the outside of the foam is not more than 50°C.

The heat loss $q = 60\ \text{W m}^{-1}$ is constant throughout the pipe and the insulation. Use Fourier's law $\dfrac{dT}{dr} = -\dfrac{q}{2\pi kr}$ to find:



a the temperature on the outer surface of the pipe

b the outer radius r_3 of the insulation

c the thickness of insulation needed.

13 A wooden plank is supported only at its ends O and P, which are 4 m apart. The plank sags under its own weight by y m at the distance x m from O.

The differential equation $\dfrac{d^2y}{dx^2} = 0.01\left(2x - \dfrac{x^2}{2}\right)$ relates the variables x and y, for $0 \leqslant x \leqslant 4$.

a Find the function $y(x)$ which measures the sag from the horizontal at any point along the plank.

b Find the maximum sag from the horizontal. Does the position of the point of maximum sag seem reasonable?

c Find the sag 1 m away from P.

d Find the angle the plank makes with the horizontal at the point 1 m from P.

D SEPARABLE DIFFERENTIAL EQUATIONS

Differential equations of the form $\frac{dy}{dx} = f(x)\,g(y)$ are called **separable differential equations**.

Examples of separable differential equations include:

- $\frac{dy}{dx} = xy^2 \qquad \{f(x) = x,\ \ g(y) = y^2\}$
- $\frac{dy}{dx} = y^2 + 3 \quad \{f(x) = 1,\ \ g(y) = y^2 + 3\}$
- $\frac{dy}{dx} = \frac{x+2}{y^3} \qquad \{f(x) = x + 2,\ \ g(y) = \frac{1}{y^3}\}$

Differential equations of the forms $\frac{dy}{dx} = f(x)$ and $\frac{dy}{dx} = g(y)$ are special cases of separable differential equations.



These equations are called separable differential equations because we can *separate* the variables so that everything involving y is on one side, and everything involving x is on the other side.

Since $\frac{dy}{dx} = f(x)\,g(y)$,

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrating both sides of this equation with respect to x, $\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$

$$\therefore \text{ by the chain rule, } \int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

The variables are now separated, so we can now find the two integrals separately.

Example 8 — Self Tutor

Solve the following differential equations:

a $\frac{dy}{dx} = xy$ **b** $\frac{dy}{dx} = \frac{1}{y^2}$

a

$$\frac{dy}{dx} = xy$$
$$\therefore \frac{1}{y}\frac{dy}{dx} = x$$
$$\therefore \int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$
$$\therefore \int \frac{1}{y}\,dy = \int x\,dx$$
$$\therefore \ln|y| = \tfrac{1}{2}x^2 + c$$
$$\therefore y = \pm e^{\frac{1}{2}x^2 + c}$$
$$\therefore y = Ae^{\frac{1}{2}x^2} \qquad \{A = \pm e^c\}$$

b

$$\frac{dy}{dx} = \frac{1}{y^2}$$
$$\therefore y^2\frac{dy}{dx} = 1$$
$$\therefore \int y^2\frac{dy}{dx}\,dx = \int 1\,dx$$
$$\therefore \int y^2\,dy = \int 1\,dx$$
$$\therefore \frac{y^3}{3} = x + c$$
$$\therefore y^3 = 3x + c$$
$$\therefore y = \sqrt[3]{3x + c}$$

EXERCISE 25D

1 Solve the following separable differential equations:

a $\frac{dy}{dx} = \frac{x}{y^2}$ **b** $\frac{dy}{dx} = \frac{2x}{e^y}$ **c** $\frac{dy}{dx} = 3xy$

d $\frac{dy}{dx} = 2x\sqrt{y}$ **e** $\frac{dy}{dx} = y \sin x$ **f** $\frac{dy}{dx} = -x\sqrt{y+1}$

g $\frac{dy}{dx} = \frac{y}{x}$ **h** $\frac{dy}{dx} = 3x^2 e^y$ **i** $\frac{dy}{dx} = \frac{y+2}{x-1}$

2 Solve:

a $\frac{dy}{dx} = y$ **b** $\frac{dy}{dx} = \frac{1}{y}$ **c** $\frac{dy}{dt} = y - 4$

d $\frac{dP}{dt} = 3\sqrt{P}$ **e** $\frac{dQ}{dt} = 2Q + 3$ **f** $\frac{dQ}{dt} = \frac{1}{2Q+3}$

3 Solve:

a $\frac{dy}{dx} = \frac{y}{3x+1}$ **b** $4 + \frac{dy}{dx} = 2y$ **c** $(x^2+5)\frac{dy}{dx} = \frac{2x}{y^2}$

d $\sqrt{4-x}\,\frac{dy}{dx} = 1 - y$ **e** $\frac{dy}{dx} = xy^2 - 2y^2$ **f** $y\frac{dy}{dx} = \frac{6x\sqrt{y}}{x^2+5}$

4 Solve the differential equation $\frac{dy}{dx} = y^2$. Discuss the values of x for which the solution is defined.

Example 9 **Self Tutor**

Solve the differential equation $\frac{dy}{dx} = e^{2y}\cos 3x$ given that $y(0) = 0$.

$$\frac{dy}{dx} = e^{2y}\cos 3x$$

$$\therefore \quad \frac{1}{e^{2y}}\frac{dy}{dx} = \cos 3x$$

$$\therefore \int \frac{1}{e^{2y}}\frac{dy}{dx}\,dx = \int \cos 3x\,dx$$

$$\therefore \int e^{-2y}\,dy = \int \cos 3x\,dx$$

$$\therefore \quad -\tfrac{1}{2}e^{-2y} = \tfrac{1}{3}\sin 3x + c$$

$$\therefore \quad e^{-2y} = -\tfrac{2}{3}\sin 3x + c$$

$$\therefore \quad -2y = \ln\left(-\tfrac{2}{3}\sin 3x + c\right)$$

$$\therefore \quad y = -\tfrac{1}{2}\ln\left(-\tfrac{2}{3}\sin 3x + c\right)$$

But $y(0) = 0$, so $0 = -\frac{1}{2}\ln c$

$$\therefore \quad \ln c = 0$$

$$\therefore \quad c = 1$$

The particular solution to the differential equation is $y = -\frac{1}{2}\ln\left(-\frac{2}{3}\sin 3x + 1\right)$.

5 Find the particular solution to:

a $\frac{dy}{dx} = \frac{3x}{y^2}$ given that $y(0) = 1$

b $\frac{dy}{dx} = \frac{\sqrt{y}}{3}$ given that $y(44) = 9$

c $\frac{dy}{dx} = y + yx^2$ given that $y(0) = 1$

d $\frac{dy}{dx} = \frac{3x}{\cos y}$ given that $y(1) = 0$

e $\frac{dy}{dx} = \frac{6\cos 2x}{\sqrt{y}}$ given that $y(0) = 3$

f $e^y(2x^2 + 4x + 1)\frac{dy}{dx} = (x+1)(e^y + 3)$ given that $y(0) = 2$

6 Find the particular solution to $\frac{dy}{dx} = 2\sqrt{y}$ given that $y(0) = 9$. Find the values of x for which the solution is defined.

7 **a** Show that $\frac{3-x}{x^2-1} = \frac{1}{x-1} - \frac{2}{x+1}$.

b Find the particular solution to $\frac{dy}{dx} = \frac{3y - xy}{x^2 - 1}$ given that $y(0) = 1$.

8 **a** Write $\frac{8x+3}{x^2 - x - 12}$ as a sum of partial fractions.

b Find the general solution to $\frac{dy}{dx} = \frac{8x\sqrt{y} + 3\sqrt{y}}{x^2 - x - 12}$.

9 Find the particular solution to $\frac{dy}{dx} = \frac{5xy^2 + 4y^2}{x^2 + x - 2}$ given that $y(0) = -\frac{1}{2}$.

10 Find the general solution to the differential equation $\frac{dy}{dx} = \frac{x^2y + y}{x^2 - 1}$.

Example 10

Self Tutor

A salmon farm has an initial population of 80. The population grows according to the differential equation $\frac{dP}{dt} = \frac{1}{5}P$, where t is the time in years.

a Write an expression for P in terms of t.

b Find the salmon population after 10 years.

a

$$\frac{dP}{dt} = \frac{1}{5}P$$

$$\therefore \frac{1}{P}\frac{dP}{dt} = \frac{1}{5}$$

$$\therefore \int \frac{1}{P}\frac{dP}{dt}\,dt = \int \frac{1}{5}\,dt$$

$$\therefore \int \frac{1}{P}\,dP = \int \frac{1}{5}\,dt$$

$$\therefore \ln|P| = \frac{1}{5}t + c$$

$$\therefore P = e^{\frac{t}{5}+c} \quad \{\text{since } P \geqslant 0\}$$

$$\therefore P = Ae^{\frac{t}{5}} \quad \{A = e^c\}$$

$\frac{dP}{dt} = kP$ for some constant k indicates an **exponential relationship** between P and t.



When $t = 0$, $P = 80$

$\therefore \quad 80 = Ae^0$ and so $A = 80$

$\therefore \quad P = 80e^{\frac{t}{5}}$

b When $t = 10$, $P = 80e^2$

$\therefore \quad P \approx 591$

After 10 years, there are approximately 591 salmon.



11 A population of rabbits in a field grows according to the differential equation $\frac{dP}{dt} = \frac{1}{2}P$, where t is the time in months. The initial population was 40 rabbits.

a Write an equation for P in terms of t.

b Find the rabbit population after 6 months.

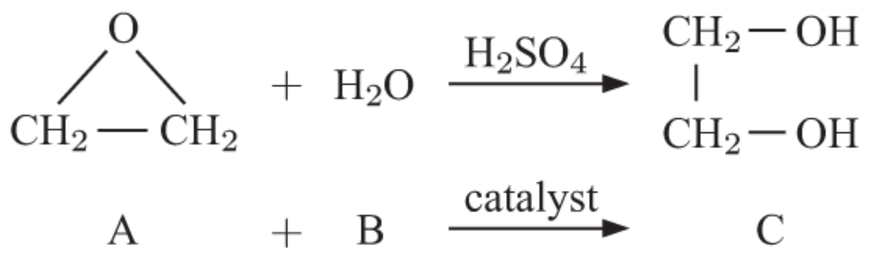
12 When a transistor radio is switched off, the current I (in milliamps) falls away according to the differential equation $\frac{dI}{dt} = -0.4I$ where t is the time in milliseconds. At the instant the radio is switched off, the current is 350 milliamps.

a Write an equation for I in terms of t.

b Find the current after 5 milliseconds.

c How long will it take for the current to fall to 20 milliamps?

13 Ethylene oxide reacts with water in the presence of the catalyst sulphuric acid to form ethylene glycol. Since water is present in excess, the rate of change in concentration of ethylene oxide (A) will be given by $\frac{dC_A}{dt} = -kC_A$ where the reaction rate constant $k = 0.31$ min^{-1}.



The initial concentration of ethylene oxide is 1 mol L^{-1}. Find the time required for 80% of the ethylene oxide to be used up.

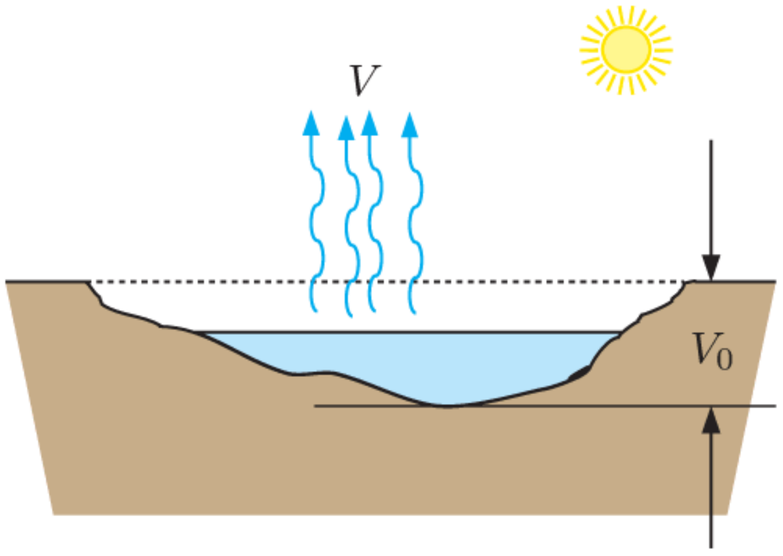
14 In the "inversion" of raw sugar, the rate of change in the weight w kg of raw sugar is directly proportional to the weight w. After 10 hours, 80% of the sugar has been "inverted". What percentage of raw sugar remains after 30 hours?

15 Water evaporates from a lake at a rate proportional to the volume of water remaining.

Suppose V is the total amount of water evaporated after t days, and V_0 is the initial volume of water in the lake.

a Explain why $\frac{dV}{dt} = k(V_0 - V)$.

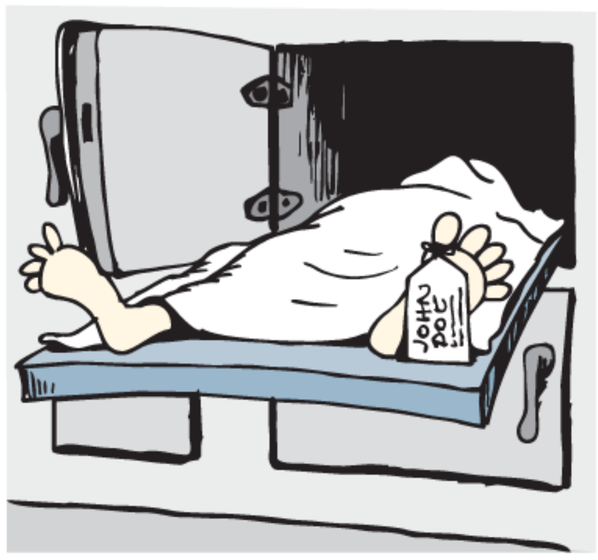
b If 50% of the water evaporates in 20 days, find the percentage of the original water remaining after 50 days without rain.



16 *Newton's law of cooling* states that the rate at which an object changes temperature is proportional to the difference between its temperature T and that of the surrounding medium T_m.

So, $\dfrac{dT}{dt} \propto (T - T_m)$.

Use Newton's law of cooling to solve these problems:

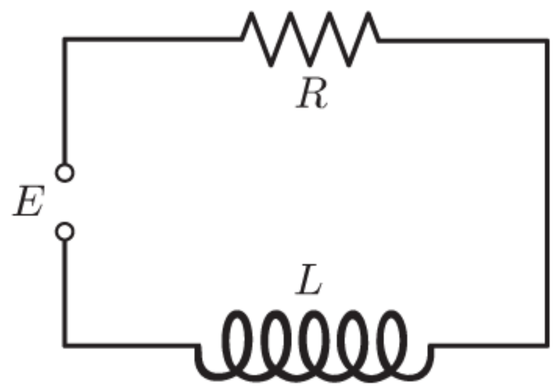


a The temperature inside a refrigerator is maintained at 5°C. An object at 100°C is placed in the refrigerator to cool. After 1 minute its temperature drops to 80°C. How long will it take for the temperature to drop to 10°C?

b At 6 am the temperature of a corpse was 13°C. By 9 am it had fallen to 9°C. Given that the temperature of a living body is 37°C and the temperature of the surroundings is constant at 5°C, estimate the time of death.

17 In an RL-circuit, the current I amps changes according to the differential equation $L\dfrac{dI}{dt} + RI = E$ where

L is the induction in henrys,
R is the resistance in ohms,
E is the voltage drop in volts, and
t is the time in seconds.



Suppose $L = 0.3$, $R = 10$, and $E = 20$.

a Find a general solution for $I(t)$.

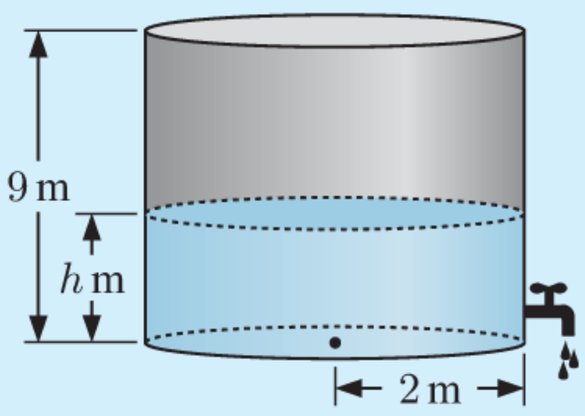
b Find a particular solution for $I(t)$ if $I(0) = 0$ amps.

c By considering I as $t \to \infty$, find the limiting current.

d Find the time required for the current to reach 99% of its limiting value.

Example 11 **Self Tutor**

Water flows out of a tap at the bottom of a large cylindrical tank with base radius 2 m. The rate at which the water flows is proportional to the square root of the depth of the water remaining in the tank. Initially the tank is full to a depth of 9 m. After 15 minutes the water is 4 m deep. How long will it take for the tank to empty?



We are given that $\dfrac{dV}{dt} \propto \sqrt{h}$ where h is the depth of the water.

$$\therefore \quad \frac{dV}{dt} = k\sqrt{h} \quad \text{where } k \text{ is a constant.}$$

$$\therefore \quad \frac{dV}{dh}\frac{dh}{dt} = k\sqrt{h} \qquad \{\text{chain rule}\}$$

$$\therefore \quad 4\pi\frac{dh}{dt} = k\sqrt{h} \qquad \{V = \pi r^2 h = 4\pi h \quad \therefore \quad \frac{dV}{dh} = 4\pi\}$$

$$\therefore \quad \frac{4\pi}{\sqrt{h}}\frac{dh}{dt} = k$$

$$\therefore \quad \int 4\pi h^{-\frac{1}{2}}\frac{dh}{dt}\,dt = \int k\,dt$$

$$\therefore \quad 4\pi \int h^{-\frac{1}{2}}\, dh = \int k\, dt$$

$$\therefore \quad 4\pi \frac{h^{\frac{1}{2}}}{\frac{1}{2}} = kt + c$$

$$\therefore \quad 8\pi\sqrt{h} = kt + c$$

Now when $t = 0$, $h = 9$

$$\therefore \quad 8\pi\sqrt{9} = c$$

$$\therefore \quad c = 24\pi$$

$$\therefore \quad 8\pi\sqrt{h} = kt + 24\pi$$

Also, when $t = 15$, $h = 4$

$$\therefore \quad 8\pi\sqrt{4} = 15k + 24\pi$$

$$\therefore \quad 15k = -8\pi$$

$$\therefore \quad k = -\tfrac{8\pi}{15}$$

So, the equation connecting the depth of the water and the time t is $8\pi\sqrt{h} = -\frac{8\pi}{15}t + 24\pi$.

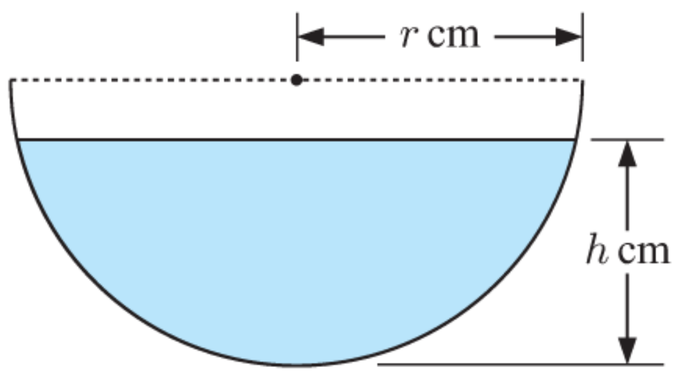
The tank is empty when $h = 0$. This occurs when $\frac{8\pi}{15}t = 24\pi$

$$\therefore \quad t = 45$$

The tank empties in 45 minutes.

18 Water flows out of a tap at the bottom of a cylindrical tank of height 4 m and radius 2 m. The tank is initially full, and the water escapes at a rate proportional to the square root of the depth of the water remaining. After 2 hours, the water is 1 m deep. How long will it take for the tank to empty?

19 Water evaporates from a hemispherical bowl with radius r cm at the rate $\frac{dV}{dt} = -r^2$, where t is the time in hours.



The volume of water of depth h, in a hemispherical bowl of radius r, is given by $V = \frac{1}{3}\pi h^2(3r - h)$.

a Use $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$ to write a differential equation connecting h and t, given that r is a constant.

b Suppose the bowl's radius is 10 cm and that initially the bowl is full of water.

i Show that $t = \frac{\pi}{300}(h^3 - 30h^2 + 2000)$.

ii Find the time taken for the depth of the water to fall to 5 cm.

iii How long will it take for the bowl to empty?

20 Since water and oil are *immiscible*, oil spilt in water will form a cylindrical patch on the surface of the water. The patch spreads at a rate proportional to the thickness of the patch, which is the height of the cylinder.

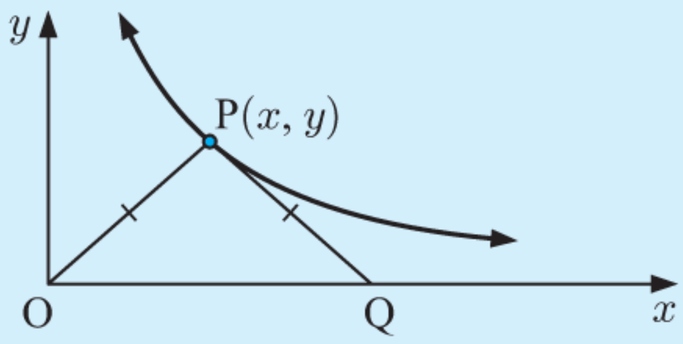
a Suppose V cm^3 of oil is spilt on a still surface. Show that the radius r of the spill after t seconds is given by $r = \sqrt[3]{\frac{3kV}{\pi}t + c}$ cm where c, k are constants.

b Suppose 1 litre of oil is spilt on a still lake. The initial radius of the spill is 20 cm, and after 2 seconds it has increased to 50 cm. How long will it take for the spill radius to reach 5 m?

Example 12 **Self Tutor**

Consider a curve in the first quadrant. The tangent at any point P cuts the x-axis at Q such that OP = PQ, where O is the origin. Given that the curve passes through $(1, 4)$, find its equation.

Since OP = PQ, if P is (x, y) then Q is $(2x, 0)$.



Since (PQ) is the tangent to the curve at P, the gradient of the curve at P is the same as the gradient of [PQ].

$$\text{Hence} \quad \frac{dy}{dx} = \frac{-y}{x}$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$$

Integrating both sides with respect to x gives $\displaystyle\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$

$$\therefore \quad \ln|y| = -\ln|x| + c$$

$$\therefore \quad \ln|x| + \ln|y| = c$$

$$\therefore \quad \ln|xy| = c$$

$$\therefore \quad xy = e^c = k \quad \text{where } k \text{ is a constant.}$$

Since the curve passes through $(1, 4)$, $1 \times 4 = k$

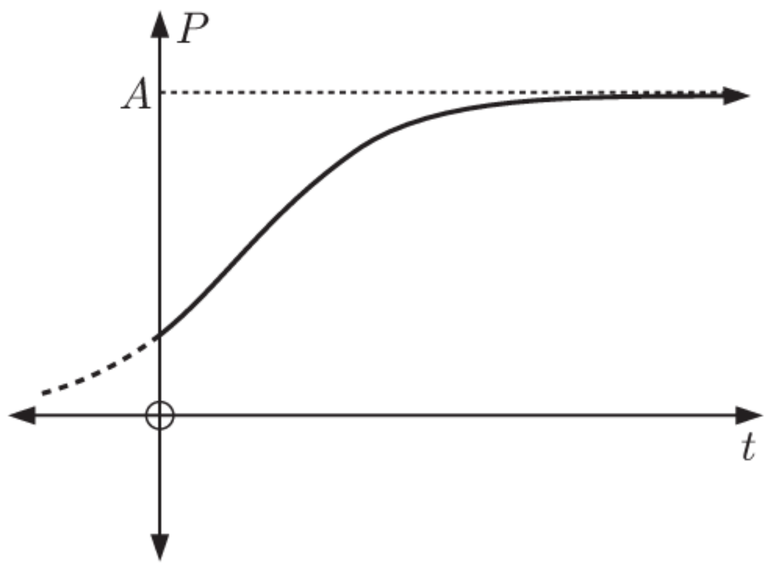
$\therefore$ the equation of the curve is $xy = 4$ or $y = \dfrac{4}{x}$, where $x > 0$.

21 The tangent at any point P on a curve cuts the x-axis at the point Q. $\widehat{OPQ} = 90°$, where O is the origin. Given that the curve passes through $(1, 2)$, find its equation.

22 The tangent at any point P on a curve cuts the x-axis at A and the y-axis at B. Given that AP : PB $= 2 : 1$ and that the curve passes through $(1, 1)$, find the equation of the curve.

E LOGISTIC GROWTH

Many populations grow exponentially at first, but then level off to approach a maximum value A due to limited resources such as land or food. This growth model is called **logistic growth**.



The logistic model can also be used to describe:

- the number of people in a town to be infected by a virus
- the number of "active" molecules in a chemical reaction, whereby a reaction between an "active" and an "inactive" molecule is required to produce two "active" molecules.

Logistic growth is governed by the differential equation

$$\frac{dP}{dt} = \underbrace{kP}\left(1 - \frac{P}{A}\right)$$

This factor controls the exponential nature of the growth when P is small.

This factor ensures that the growth approaches 0 as P approaches A.

The differential equation for logistic growth is separable, and we also need to use partial fractions.

Example 13 **Self Tutor**

The population of an island is currently 154. Its expected growth rate is given by $\frac{dP}{dt} = 0.16P\left(1 - \frac{P}{500}\right)$, where t is in years.

a Write P as a function of t. **b** Find the population after 10 years.

c Find the time taken for the population to increase to 480.

d What is the limiting population size?

a
$$\frac{dP}{dt} = 0.16P\left(1 - \frac{P}{500}\right) = 0.16P\left(\frac{500 - P}{500}\right)$$

$$\therefore \quad \frac{500}{P(500 - P)}\,\frac{dP}{dt} = 0.16$$

$$\therefore \quad \int \frac{500}{P(500 - P)}\,\frac{dP}{dt}\,dt = \int 0.16\,dt$$

$$\therefore \quad \int \frac{500}{P(500 - P)}\,dP = \int 0.16\,dt$$

$$\therefore \quad \int \left(\frac{1}{P} + \frac{1}{500 - P}\right) dP = \int 0.16\,dt$$

We use partial fractions to help with the integration.



$$\therefore \quad \ln|P| + \tfrac{1}{-1}\ln|500 - P| = 0.16t + c$$

$$\therefore \quad \ln\left|\frac{P}{500 - P}\right| = 0.16t + c$$

$$\therefore \quad \frac{P}{500 - P} = \pm e^{0.16t + c}$$

$$\therefore \quad \frac{500 - P}{P} = be^{-0.16t} \qquad \left\{\text{letting } \; b = \pm\frac{1}{e^c}\right\}$$

Now when $t = 0$, $P = 154$

$$\therefore \quad \tfrac{346}{154} = b$$

So, we have $\frac{500 - P}{P} = \left(\frac{346}{154}\right)e^{-0.16t}$

$$\therefore \quad \frac{500}{P} - 1 = \left(\frac{346}{154}\right)e^{-0.16t}$$

$$\therefore \quad \frac{500}{P} = 1 + \left(\frac{346}{154}\right)e^{-0.16t}$$

$$\therefore \quad P = \frac{500}{1 + \left(\frac{346}{154}\right)e^{-0.16t}}$$

b When $t = 10$, $P = \dfrac{500}{1 + \left(\frac{346}{154}\right)e^{-0.16\times 10}} \approx 344$

So the population after 10 years is approximately 344 people.

c When $P = 480$, $480 = \dfrac{500}{1 + \left(\frac{346}{154}\right)e^{-0.16t}}$

$$\therefore \quad 1 + \left(\tfrac{346}{154}\right)e^{-0.16t} = \tfrac{500}{480}$$

$$\therefore \quad e^{-0.16t} = \tfrac{20}{480} \times \tfrac{154}{346} = \tfrac{77}{4152}$$

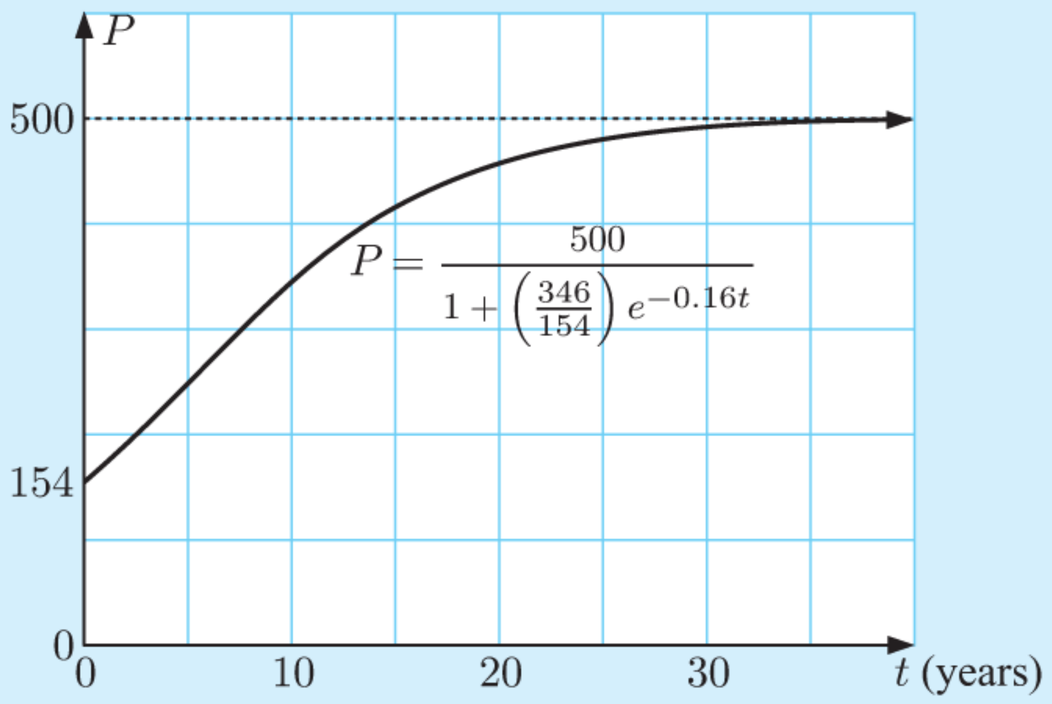
$$\therefore \quad -0.16t = \ln\left(\tfrac{77}{4152}\right)$$

$$\therefore \quad t = \frac{\ln\left(\frac{77}{4152}\right)}{-0.16} \approx 24.9 \text{ years}$$

d as $t \to \infty$, $e^{-0.16t} \to 0$

$$\therefore \quad P \to \frac{500}{1+0} = 500$$

The limiting population is 500.



EXERCISE 25E

1 Consider the logistic differential equation $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{200}\right)$, $P(0) = 20$.

a Write P as a function of t.

b Find the value of P when $t = 10$.

c Discuss the behaviour of P as $t \to \infty$.

d Sketch the graph of P against t.

2 Answer the **Opening Problem** on page **674**.

3 A lily in a pond initially has area 20 cm^2. Its area A grows according to the model $\dfrac{dA}{dt} = 0.1A\left(1 - \dfrac{A}{500}\right)$, where t is the time in days.



a Write A as a function of t.

b Find the area of the lily after 20 days.

c Find the limiting size of the lily.

4 In a small country town, rumours spread very fast. At 8 am on Monday, a rumour begins with 2 people. The number of people N who have heard the rumour grows according to the model $\dfrac{dN}{dt} = 0.8N\left(1 - \dfrac{N}{600}\right)$, where t is the time in hours after 8 am.

a Write N as a function of t.

b How many people have heard the rumour by 11 am?

c How many people do you think live in the town?

d At what time would 500 people have heard the rumour?

5 There are 10^{30} molecules involved in a chemical reaction. Initially, 200 of the molecules are "active", and any reaction between an "active" and an "inactive" molecule produces two "active" molecules. The number of "active" molecules grows according to the differential equation

$\frac{dN}{dt} = kN\left(1 - \frac{N}{10^{30}}\right)$, where t is the time in seconds.

a Solve the differential equation, and hence write N in terms of k and t.

b Given that 1.5×10^7 molecules were "active" after 10^{-5} seconds, find k.

c At what time would you expect the reaction to be 99% complete?

6 14 European foxes were released in Australia in 1845 for sport hunting. Spreading rapidly out of control, the fox is now found throughout the mainland, except in the tropical northern regions.

In 1900 there were 30 000 foxes in Australia, and today their population is steady at around 95 000.



a What features of the growth in the fox population suggest that a logistic model is appropriate?

b Suppose the population of foxes F grows according to the differential equation

$\frac{dF}{dt} = kF\left(1 - \frac{F}{A}\right)$, where t is the number of years since 1845.

i State the value of A.

ii Solve the differential equation, and use the information provided to write F in terms of t.

c Estimate the fox population in 1920.

d Estimate the time at which the fox population was: **i** 15 000 **ii** 65 000.

e Sketch the graph of F against t.

f When was the population growth rate a maximum? How does this appear on the graph of F against t?

7 Before Europeans came to Australia, the numbat population was growing logistically. 220 years ago, at time $t = 0$, the numbat population was 12 000.

Europeans brought predators into the environment, and the numbat population was decimated. 20 years ago, the population of numbats was only 200.

The differential equation governing the population growth over this 200 year period is $\frac{dP}{dt} = \frac{1}{10}P\left(1 - \frac{P}{20\,000}\right)\left(1 - \alpha\sqrt{t}\right)$ where $\alpha > 0$ is a constant which takes into account the effect of introduced predators.

a Solve this differential equation to give a general solution involving α for the population at time t, $0 \leqslant t \leqslant 200$.

b Use $P(0)$ and $P(200)$ to solve for all unknowns, including α.

c Sketch the graph of P against t.

d At what time was the numbat population a maximum? What was the maximum population?

e In recent times, a concentrated effort has been made to ensure the survival of the numbats. If nothing had been done about the predators' threat to the numbat population, would the numbats still exist today?

f What can be said about the effect of the introduced species as time progresses? Do you think this is realistic?

HISTORICAL NOTE — THE LOGISTIC MODEL

The logistic model was first given its name in 1845 by the Belgian mathematician **Pierre François Verhulst**, in his study of population growth. Verhulst originally described the logistic model as $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, where r is a measure of the growth rate, and K is the limiting population.



Pierre François Verhulst

The model was rediscovered in 1920 by the American scientists **Raymond Pearl** and **Lowell Reed**, who promoted its use.

The $\boldsymbol{r/K}$ **selection theory** in ecology derives its name from the symbols in Verhulst's logistic model. The theory hypothesises that species are driven towards either *r-selection* or *K-selection.*

r-selected species are characterised by high growth rates and high birth rates, but with individuals having a relatively low chance of surviving to adulthood. Examples of r-selected species include insects and small rodents.

K-selected species operate in dense groups close to their limiting population. They typically produce fewer offspring, but which have high chance of survival. They are characterised by their ability to compete for limited resources. Examples of K-selected species include elephants, humans, and whales.

F HOMOGENEOUS DIFFERENTIAL EQUATIONS $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

Differential equations of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, where $y = y(x)$, are known as **homogeneous differential equations**.

They can be solved using the substitution $y = vx$ where v is a function of x. The substitution will always reduce the differential equation to a separable differential equation as follows.

If $y = vx$ where v is a function of x, then

$$\frac{dy}{dx} = \frac{dv}{dx}\,x + v \quad \{\text{product rule}\}$$

$$\therefore \quad \frac{dv}{dx}\,x + v = f\left(\frac{vx}{x}\right) = f(v)$$

$$\therefore \quad \frac{dv}{dx} = \frac{f(v) - v}{x}$$

$$\therefore \quad \frac{1}{f(v) - v}\,\frac{dv}{dx} = \frac{1}{x} \qquad \text{which is a separable differential equation.}$$

Example 14 **Self Tutor**

a Show that $\frac{dy}{dx} = \frac{x+2y}{x}$ is homogeneous, and hence solve for y in terms of x.

b If $y = \frac{3}{2}$ when $x = 3$, find the particular solution.

a $\frac{dy}{dx} = \frac{x+2y}{x} = 1 + 2\frac{y}{x}$ so the differential equation is homogeneous.

Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$ {product rule}

Comparing with the differential equation, $v + x\frac{dv}{dx} = 1 + 2v$

$\therefore \quad x\frac{dv}{dx} = 1 + v$

$\therefore \quad \frac{1}{1+v}\frac{dv}{dx} = \frac{1}{x}$

$\therefore \quad \int \frac{1}{v+1}\,dv = \int \frac{1}{x}\,dx$

$\therefore \quad \ln|v+1| = \ln|x| + c$

$\therefore \quad \ln|v+1| = \ln|Ax| \quad$ where $\ln|A| = c$

$\therefore \quad v + 1 = Ax$

$\therefore \quad \frac{y}{x} + 1 = Ax$

$\therefore \quad y = Ax^2 - x, \quad$ where A is a constant.

b Substituting $y = \frac{3}{2}$ and $x = 3$ into the general solution, we find $\frac{3}{2} = A \times 3^2 - 3$

$\therefore \quad 9A = \frac{9}{2}$

$\therefore \quad A = \frac{1}{2}$

$\therefore$ the particular solution is $y = \frac{1}{2}x^2 - x$.

EXERCISE 25F

1 **a** Show that $\frac{dy}{dx} = \frac{x-y}{x}$ is homogeneous, and hence solve for y in terms of x.

b If $y = 2$ when $x = 1$, find the particular solution.

2 Find the particular solution to $\frac{dy}{dx} = \frac{x+2y}{4x}$, $y(1) = 2$.

3 Show that each differential equation is homogeneous. Hence write y in terms of x.

a $\frac{dy}{dx} = \frac{xy+y^2}{x^2}$ **b** $\frac{dy}{dx} = \cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$ **c** $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

4 Suppose $\frac{dy}{dx} = \frac{x+y}{x-y}$, and $y(1) = 1$. Find the exact value of x when $y = 0$.

5 **a** Show that the substitution $y = vx$ (where v is a function of x) will reduce all inhomogeneous differential equations of the form $\frac{dy}{dx} = \frac{y}{x} + f\left(\frac{y}{x}\right) g(x)$ to separable form.

b Solve $x\frac{dy}{dx} = y + e^{\frac{y}{x}}$ using this method.

G THE INTEGRATING FACTOR METHOD

Consider a differential equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$.

Generally, this type of equation is not separable.

However, suppose there is a function $I(x)$, called an **integrating factor**, such that

$$\frac{d}{dx}\left(I(x)\,y\right) = I(x)\left(\frac{dy}{dx} + P(x)\,y\right) \quad \ldots. \ (*)$$
$$= I(x)\,Q(x)$$

If such an integration factor exists, then integrating both sides with respect to x would give

$$I(x)\,y = \int I(x)\,Q(x)\,dx$$

$$\therefore \quad y = \frac{1}{I(x)} \int I(x)\,Q(x)\,dx \quad \text{and we could hence find a solution for } y.$$

Now from $(*)$, $\quad I(x)\,\frac{dy}{dx} + I'(x)\,y = I(x)\,\frac{dy}{dx} + I(x)\,P(x)\,y$

$$\therefore \quad I'(x) = I(x)\,P(x)$$

$$\therefore \quad \frac{I'(x)}{I(x)} = P(x)$$

Integrating both sides with respect to x, $\quad \int \frac{I'(x)}{I(x)}\,dx = \int P(x)\,dx$

$$\therefore \quad \ln|\,I\,| + c = \int P(x)\,dx$$

$$\therefore \quad I(x) = Ae^{\int P(x)\,dx} \quad \text{where} \quad A = e^{-c}.$$

By convention we let $A = 1$, so the **integrating factor** is $\quad \boldsymbol{I(x) = e^{\int P(x)\,dx}}$.

THE INTEGRATING FACTOR METHOD

Suppose $\frac{dy}{dx} + P(x)\,y = Q(x)$ where $y = y(x)$.

1 Calculate the **integrating factor** $\boldsymbol{I(x) = e^{\int P(x)\,dx}}$. You do not need a constant of integration.

2 Multiply both sides of the differential equation by $I(x)$.

3 Simplify the LHS and hence obtain $I(x)\,y = \int I(x)\,Q(x)\,dx + c$, where c is a constant.

4 Integrate to obtain the general solution.

Example 15 **Self Tutor**

Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

The integrating factor is $I(x) = e^{\int 3x^2\,dx} = e^{x^3}$.

Multiplying both sides of the differential equation by e^{x^3} gives

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

$$\therefore \quad \frac{d}{dx}\left(ye^{x^3}\right) = 6x^2e^{x^3}$$

$$\therefore \quad ye^{x^3} = \int 6x^2e^{x^3}\,dx$$

$$\therefore \quad ye^{x^3} = 2e^{x^3} + c$$

$$\therefore \quad y = 2 + ce^{-x^3}$$

EXERCISE 25G

1 Solve using the integrating factor method:

a $\frac{dy}{dx} + 4y = 12$ **b** $\frac{dy}{dx} - 3y = e^x, \quad y(0) = 2$

2 Solve $\cos x\,\frac{dy}{dx} = y\sin x + \sin 2x$ given $y(0) = 1$.

3 Solve using an integrating factor:

a $\frac{dy}{dx} + y = x + e^x, \quad y(1) = 1$ **b** $x\,\frac{dy}{dx} + y = x\cos x$

c $(x+1)y + x\,\frac{dy}{dx} = x - x^2$

H MACLAURIN SERIES DEVELOPED FROM A DIFFERENTIAL EQUATION

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $(0, y_0)$.

Depending on the form of $f(x, y)$, it may not be possible to obtain an explicit particular solution $y(x)$.

However, provided the necessary derivatives exist, we can determine the nth degree Maclaurin polynomial approximation to the particular solution $y(x)$ about $x = 0$ as follows:

Step 1: Use $\frac{dy}{dx} = f(x, y)$ to find the derivatives:

$$y' = f(x, y)$$
$$y'' = f'(x, y)$$
$$y''' = f''(x, y)$$
$$\vdots$$
$$y^{(n)} = f^{(n-1)}(x, y)$$

Step 2: Substitute $(0, y_0)$ to find:

$$y'(0) = f(0, y_0)$$
$$y''(0) = f'(0, y_0)$$
$$y'''(0) = f''(0, y_0)$$
$$\vdots$$
$$y^{(n)}(0) = f^{(n-1)}(0, y_0)$$

Example 16 **Self Tutor**

Consider the differential equation $\frac{dy}{dx} = x - 4xy$ where $y(0) = 1$.

a Without solving the system, find the first three non-zero terms of the Maclaurin polynomial expansion for y.

b Solve the system exactly to find the particular solution.

a
$$\frac{dy}{dx} = x - 4xy = x(1 - 4y)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(x(1-4y)\right)$$
$$= 1 - 4y + x\left(-4\,\frac{dy}{dx}\right)$$

$$\therefore \quad \frac{d^3y}{dx^3} = -4\,\frac{dy}{dx} + 1 \times \left(-4\,\frac{dy}{dx}\right) + x\left(-4\,\frac{d^2y}{dx^2}\right)$$
$$= -8\,\frac{dy}{dx} - 4x\,\frac{d^2y}{dx^2}$$

$$\therefore \quad \frac{d^4y}{dx^4} = -8\,\frac{d^2y}{dx^2} - 4\,\frac{d^2y}{dx^2} + x\left(-4\,\frac{d^3y}{dx^3}\right)$$
$$= -12\,\frac{d^2y}{dx^2} - 4x\,\frac{d^3y}{dx^3}$$

Now $y(0) = 1$, so at $(0, 1)$:
$$\frac{dy}{dx} = 0$$
$$\frac{d^2y}{dx^2} = 1 - 4 + 0 = -3$$
$$\frac{d^3y}{dx^3} = 0$$
$$\frac{d^4y}{dx^4} = 36$$

For each derivative, we substitute $x = 0$, $y = 1$, and the values calculated for the previous derivatives.



$\therefore$ y has Maclaurin polynomial $1 - \frac{3x^2}{2!} + \frac{36x^4}{4!} \ldots.$

which is $1 - \frac{3}{2}x^2 + \frac{3}{2}x^4 \ldots.$

b
$$\frac{dy}{dx} = x(1 - 4y)$$
$$\therefore \quad \frac{1}{1-4y}\,\frac{dy}{dx} = x$$
$$\therefore \quad \int \frac{1}{1-4y}\,dy = \int x\,dx$$
$$\therefore \quad -\tfrac{1}{4}\ln|1-4y| = \frac{x^2}{2} + c$$
$$\therefore \quad \ln|1-4y| = -2x^2 + c$$
$$\therefore \quad 1 - 4y = Ae^{-2x^2}$$
$$\therefore \quad y = \tfrac{1}{4} - Be^{-2x^2}$$

But $y(0) = 1$
$$\therefore \quad \tfrac{1}{4} - B = 1$$
$$\therefore \quad B = -\tfrac{3}{4}$$
$$\therefore \quad y = \tfrac{1}{4}\left(1 + \frac{3}{e^{2x^2}}\right)$$

EXERCISE 25H

1 Consider the differential equation $\frac{dy}{dx} = \frac{e^{-y}}{3} - 1$, where $y(0) = 0$.

a Find the first three non-zero terms in the Maclaurin polynomial expansion of y.

b Verify that $y = \ln\left(\frac{1+2e^{-x}}{3}\right)$ is a particular solution.

2 Consider the differential equation $\frac{dy}{dx} = 2x - 2 + \frac{y}{x-1}$, where $\frac{dy}{dx} = 1$ when $x = 0$.

a Find the first three non-zero terms in the Maclaurin polynomial expansion of y.

b Solve the system exactly to find the particular solution.

3 Consider the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$, where $y(0) = 2$.

a Find the first three non-zero terms in the Maclaurin polynomial expansion of y.

b Solve the system exactly to find the particular solution.

c Verify that the Maclaurin expansion of your particular solution in **b** is consistent with your answer to **a**.

Example 17 Self Tutor

a Show that if $y = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is convergent for all $x \in \mathbb{R}$, then $\frac{dy}{dx} = y$.

b Solve the differential equation $\frac{dy}{dx} = y$ to obtain the general solution.

c Hence prove that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

a $y = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$

$\therefore \frac{dy}{dx} = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots.$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$= y$

b Consider $\frac{dy}{dx} = y$

$\therefore \frac{1}{y}\frac{dy}{dx} = 1$

$\therefore \int \frac{1}{y}\,dy = \int 1\,dx$

$\therefore \ln|y| = x + c$

$\therefore y = Ae^x$

where $A = \pm e^c$ is a constant.

c Since $y = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is a solution to the differential equation for all $x \in \mathbb{R}$,

$\sum_{k=0}^{\infty} \frac{x^k}{k!} = Ae^x$ for some constant A.

But when $x = 0$, $y = 1$, so $A = 1$.

Hence $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$.

4 Let $y(x) = \sum_{n=0}^{\infty} \frac{p(p-1)....(p-n+1)x^n}{n!}$ for $|x| < 1$. You may assume the series is on this interval.

a Show that for $|x| < 1$, y satisfies the differential equation $(1+x)\frac{dy}{dx} = py$.

b Find the general solution to this differential equation.

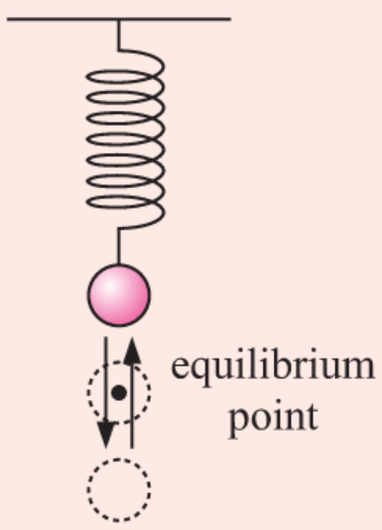
c Hence show that $y(x) = (1+x)^p$ for all $|x| < 1$.

ACTIVITY 1 SIMPLE HARMONIC MOTION

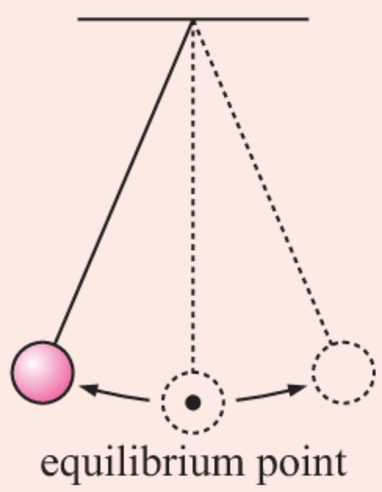
Under **simple harmonic motion**, an object oscillates back and forth about the origin or **equilibrium point**. The object is subject to an acceleration force which is proportional to the object's displacement from the equilibrium point, and in the direction of the equilibrium point.

Simple harmonic motion never exactly occurs in the real world because of friction. However, there are many things that *almost* behave with simple harmonic motion. For example:

- the movement of a mass attached to a spring



- the oscillation of a pendulum.



The acceleration of an object moving with simple harmonic motion is $a = -k^2x$ where x is the displacement from the equilibrium point, and k is a measure of how quickly the object completes a single oscillation.

In the case of a mass on a spring, k is determined by the weight of the object, and the length and stiffness of the spring.

In the case of a pendulum, k is determined by the weight on its end, the length of the pendulum, and its maximum amplitude from the equilibrium point.

From our study of kinematics, we know that the acceleration of the object is $a = \frac{d^2x}{dt^2}$.

We can therefore describe simple harmonic motion by the differential equation $\frac{d^2x}{dt^2} = -k^2x$.

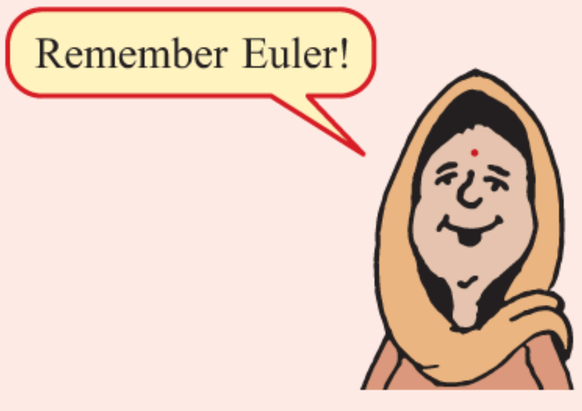
In this Investigation we derive the solution to this differential equation using two different methods.

What to do:

1 Notice that in the differential equation, the second derivative of the function is proportional to the function itself.

a What function types have we studied with this property?

b How are these function types related?



2 **a** Try an exponential solution of the form $x = Ce^{\lambda t}$ where C, λ are constants. Hence show that $x = Ce^{\lambda t}$ is a solution to the differential equation provided $\lambda = \pm ik$.

b Show that $x = C_1 e^{ikt} + C_2 e^{-ikt}$ is a general solution to the differential equation, where C_1, C_2 are constants.

c Use Euler's identity to write this solution in the form $x = B_1 \cos kt + B_2 \sin kt$, where B_1, B_2 are constants.

d By letting $B_1 = A\cos\alpha$ and $B_2 = -A\sin\alpha$, show that the solution can be written in the form $x = A\cos(kt + \alpha)$.

The method presented in **2** requires a guess for the solution, leaving the question open: Is this the *only* solution? We therefore consider an alternative method.

3 **a** Suppose the velocity of the object is $v = \dfrac{dx}{dt}$, so the acceleration is $a = \dfrac{dv}{dt}$.

Use the chain rule to show that $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = a$.

b Using the relationship from **a**, $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = -k^2x$.

Use integration to show that $v^2 = -k^2x^2 + c$.

c Suppose that at the endpoints of its motion, the object reaches maximum displacement $x = \pm A$.

i State the velocity v at these times.

ii Hence show that $v = \pm k\sqrt{A^2 - x^2}$ where the sign is given by the direction of motion.

d Since $v = \dfrac{dx}{dt}$, we can write $\dfrac{dx}{dt} = \pm k\sqrt{A^2 - x^2}$.

i By writing this as a separable differential equation, show that

$$\int \frac{1}{\sqrt{A^2 - x^2}}\,dx = \pm kt + c.$$

ii Perform the integration using the substitution $x = A\cos\phi$.

iii Hence show that the solution can be written in the form $x = A\cos(kt + \alpha)$.

4 Explain why $x = A\sin(kt + \alpha)$ is an equivalent solution to the differential equation.

5 Write the amplitude of the oscillation in terms of A.

6 Write the period of the oscillation in terms of k.

7 A pendulum is swinging with simple harmonic motion about the origin. It is initially at its extreme positive position. The amplitude of the motion is 0.7 m, and the period of the motion is 2 seconds. Use the form $x = A\cos(kt + \alpha)$ to write a formula for the displacement of the pendulum after t seconds.

ACTIVITY 2 LAPLACE TRANSFORMS

Laplace transforms provide a useful link between improper integrals and differential equations. In particular, they are extremely efficient for solving some types of differential equation.



REVIEW SET 25A

1 Verify that $y = 4\ln(x^2 + 3x) + 8$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{8x + 12}{x^2 + 3x}$.

2 Find constants a and b such that $y = ax + b$ is a solution to the differential equation $\dfrac{dy}{dx} = 4x - 2y$.

3 A curve passes through the point $(1, 2)$ and satisfies the differential equation $\dfrac{dy}{dx} = x - 2y$.

Use Euler's method with step size 0.1 to estimate the value of y when $x = 1.6$.

4 Solve the following differential equations:

a $\dfrac{dy}{dx} = \cos 2x - \sin^2 x$ **b** $\dfrac{dy}{dx} = 3 - e^{-2x}$

c $\dfrac{dy}{dx} = \dfrac{1}{2x + 1}$, $y(0) = 2$ **d** $\dfrac{dy}{dt} - t = te^{t^2}$, $y(1) = 2e$

5 The marginal profit for producing x vases per day is given by $P'(x) = 20 - \dfrac{x}{4}$ pounds per vase. If no vases are made then a loss of £250 each day occurs.

a Find the profit function $P(x)$.

b What is the maximum profit, and when does it occur?

c What production levels enable a profit to be made?

6 The rate at which spectators entered a stadium before a football match is given by

$S'(t) = \dfrac{4000e^{-0.05t}}{(1 + 4e^{-0.05t})^2}$ spectators per minute, where t is the time in minutes after noon.



a At what rate were spectators entering the stadium at:

i noon **ii** 12:30 pm?

b Find the derivative of $\dfrac{1}{1 + 4e^{-0.05t}}$, and hence find $\displaystyle\int S'(t)\,dt$.

c Find $\displaystyle\int_0^{60} S'(t)\,dt$, and interpret your answer.

d Given that there were 4000 spectators in the stadium at noon, find the number of spectators in the stadium when the match starts at 1:40 pm.

7 Solve:

a $\dfrac{dy}{dx} = 5x^2y$ **b** $\dfrac{dy}{dx} = 2xy^2 - y^2$ **c** $\dfrac{dx}{dt} = \dfrac{3\sin 2t}{4\sqrt{x}}$

8 A radioactive substance decays at a rate proportional to the mass M remaining at time t. Suppose the initial mass is M_0.

a Construct a differential equation involving M, t, and a constant k to model this situation.

b If the substance is reduced to $\frac{4}{5}$ of its original mass in 30 days, calculate the time required for the substance to decay to half its original mass.

9 Solve the differential equation $\frac{dy}{dx} = -\frac{3}{e^y}$ given that $y = 1$ when $x = 0$.

10 The enzyme urease breaks down urea into ammonia and carbon dioxide in the reaction:

$$NH_2CONH_2 + H_2O \xrightarrow{\text{urease}} 2NH_3 + CO_2$$

The rate of change of concentration C of urea after t seconds is given by

$$\frac{dC}{dt} = \frac{-VC}{K + C}$$

where $V = 2.66 \times 10^{-4}$ mol s^{-1} L^{-1}

and $K = 0.0266$ mol L^{-1}

a Given that the initial concentration of urea is 0.1 mol L^{-1}, write t in terms of C.

b Calculate the time required for 85% of the urea to decompose into ammonia and carbon dioxide.



11 Find the equation of the curve through $(2, 1)$ given that for any point (x, y) on the curve, the y-intercept of the tangent to the curve is $3x^2y^3$.

12 A rare virus is infecting the residents of Smallville. It was introduced initially into the population by a child returning home from boarding school. The number of people N who have been infected after t weeks is given by the differential equation $\frac{dN}{dt} = N\left(1 - \frac{N}{694}\right)$.

a Write N in terms of t.

b Find the number of people who will have had the virus after 4 weeks.

c How long will it take for 500 people to have been infected?

13 Use the substitution $y = vx$ where v is a function of x, to solve the differential equation $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$.

14 By finding a suitable integrating factor, solve:

a $\frac{dy}{dx} - \frac{y}{x} = \sqrt{x}, \quad y(4) = 0$

b $\frac{dy}{dx} = \cos x - y\cot x, \quad y(\frac{\pi}{2}) = 0.$

15 Consider the differential equation $\frac{dy}{dx} = 2xe^{-y}$ where $y(0) = 0$.

a Find the first three non-zero terms in the Maclaurin polynomial expansion of y.

b Solve the system exactly to find the particular solution.

16 Consider the differential equation $x''(t) = -m^2x, \ x(0) = 1, \ x'(0) = 0$.

a Find the complete Maclaurin series for $x(t)$.

b *Hence* solve the differential equation.

c Verify that your exact solution is correct.

REVIEW SET 25B

1 Verify that $y = 3\sin x - 2\cos x$ is a solution to $3\dfrac{dy}{dx} - 2y = 13\cos x$.

2 Consider the differential equation $\dfrac{dy}{dx} = y - 1$.

a Show that $y = ce^x + 1$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0, \pm 1, \pm 2$.

c Find the particular solution which passes through $(0, 4)$.

d Find the equation of the tangent to the particular solution at $(0, 4)$.

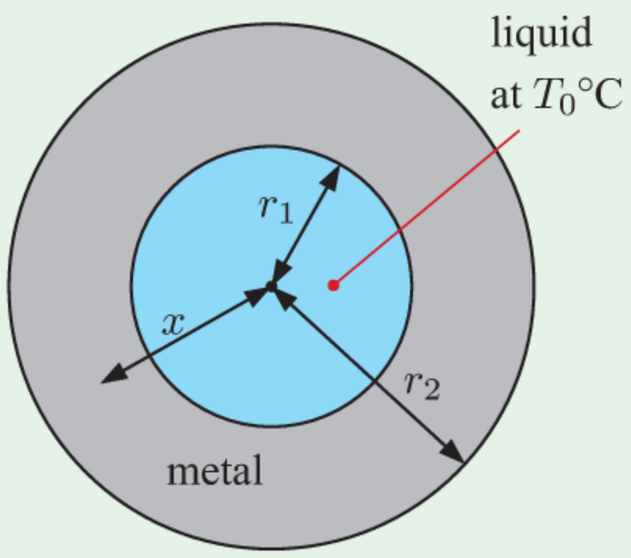
3 Use Euler's method with step size 0.1 to estimate $y(0.5)$ given $\dfrac{dy}{dx} = \sin(x+y)$, $y(0) = 0.5$.

4 Solve the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{e^x}{e^x - 2}$

b $\dfrac{dy}{dx} = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x\right)$, $y\left(\frac{\pi}{2}\right) = 0$

5



A metal tube has an annulus cross-section with radii r_1 and r_2 as shown.

Within the tube, a liquid is maintained at temperature T_0 °C.

Within the metal, the temperature drops from inside to outside according to $\dfrac{dT}{dx} = \dfrac{k}{x}$ where k is a negative constant and x is the distance from the central axis.

Show that the outer surface has temperature $T_0 + k\ln\left(\dfrac{r_2}{r_1}\right)$ °C.

6 Solve:

a $\dfrac{dy}{dx} = 2y^4$

b $(t^2+1)\dfrac{dP}{dt} = Pt$

7 Find the particular solution to:

a $\dfrac{dy}{dx} = \sqrt{y}$ given that $y(0) = 4$

b $\dfrac{dy}{dx} = y\cos x$ given that $y\left(\frac{\pi}{2}\right) = \dfrac{1}{e^2}$.

8 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{xy - 6y}{x^2 - 4}$ given that $y(3) = 1$.

9 Brian has made a cup of tea. The tea's temperature T is initially 85°C, and it cools according to the differential equation $\dfrac{dT}{dt} = -0.1(T - 20)$, where t is the time in minutes.

a Find a formula for T in terms of t.

b Find the temperature of the tea after 4 minutes.

c Describe the behaviour of T as $t \to \infty$.

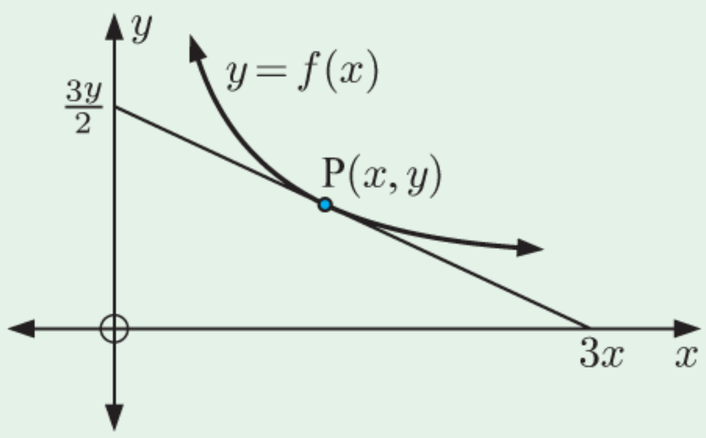
d Sketch the graph of T against t.

e Brian likes to drink his tea at a temperature between 40°C and 65°C.

i How long will he have to wait for his tea to cool down?

ii How much time will he have to drink the tea before it gets too cold?

10



The tangent to a curve at the general point $P(x, y)$ has x-intercept $3x$ and y-intercept $\frac{3y}{2}$.

Given that $x > 0$, find the equation of the curve which passes through the point $(1, 5)$.

11 A water tank of height 1 m has a square base 2 m $\times$ 2 m. When a tap at its base is opened, the water flows out at a rate proportional to the square root of the depth of the water at any given time. Suppose the depth of the water is h m, and V is the volume of water remaining in the tank after t minutes.

a Write down a differential equation involving $\frac{dV}{dt}$ and h to model this situation.

b Explain why $V = 4h$ m^3 at time t, then use the chain rule to write down a differential equation involving $\frac{dh}{dt}$ and h only.

c The tank is initially full. When the tap is opened, the water level drops by 19 cm in 2 minutes. Find the time it takes for the tank to empty.

12 An ecology student is studying the repopulation of wild ostriches in Oudtshoorn after a drought.



The population growth rate is approximately logistic, so $\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$ where $P(t)$ is the population t years after her study begins.

The carrying capacity A is known to be 200 000 ostriches, since this was the population before the drought.

Initially, the student finds that there are 25 000 ostriches. After two years she finds there are 31 200 ostriches.

a Solve the differential equation, and use the population $P(0)$ and $P(2)$ to find the population $P(t)$ after t years.

b Estimate the ostrich population after 5 years.

c Find the time at which the population growth rate is a maximum.

13 Use the substitution $y = vx$ where v is a function of x, to solve the differential equation $xy\,\frac{dy}{dx} = 1 + x + y^2, \quad y(1) = 0.$

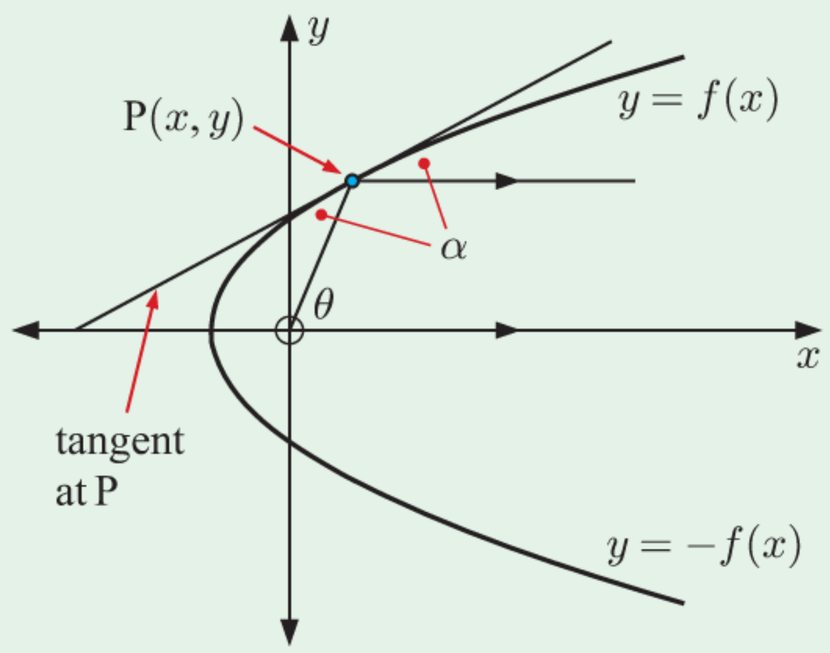
14 By finding a suitable integrating factor, solve $\frac{dy}{dx} + \frac{3y}{x} = 8x^4, \quad y(1) = 0.$

15 Consider the differential equation $\frac{dy}{dx} = y\ln(x + 1)$ where $y(0) = 1$.

a Find the first three non-zero terms in the Maclaurin polynomial expansion of y.

b Find the particular solution to the differential equation exactly.

16 The inside surface of $y = \pm f(x)$ is a mirror. Light is emitted from O$(0, 0)$. All rays that strike the surface of the mirror are reflected so that they emerge parallel to the x-axis.



a Show that $\theta = 2\alpha$.

b Show that the gradient of the tangent at a general point P(x, y) on the mirror is given by $\dfrac{dy}{dx} = \tan\alpha$.

c Use the identity $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ to deduce that $\tan\alpha = \dfrac{\sqrt{x^2+y^2}-x}{y}$.

d Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+y^2}-x}{y}$ by making the substitution $r^2 = x^2 + y^2$.

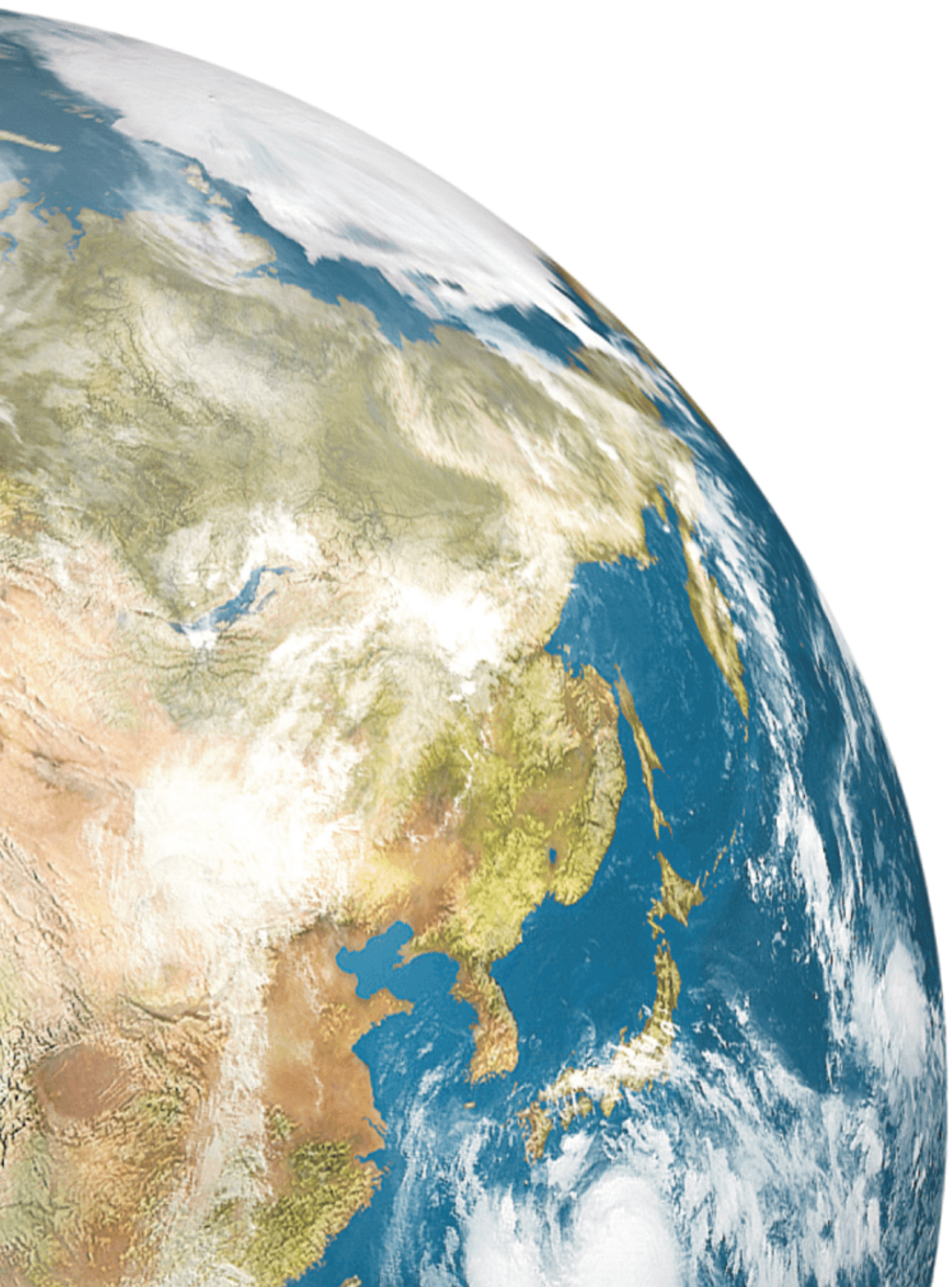
e Describe the shape of the mirror.

Chapter 26

Bivariate statistics

Contents:

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C Line of best fit by eye
D The least squares regression line
E The regression line of x against y



OPENING PROBLEM

At a junior tournament, some young athletes each throw a discus. The *age* and *distance thrown* are recorded for each athlete.

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

Things to think about:

a Do you think the distance an athlete can throw is related to the person's age?

b What happens to the distance thrown as the age of the athlete increases?

c How could you graph the data to more clearly see the relationship between the variables?

d How can we *measure* the relationship between the variables?

In the **Opening Problem**, each athlete has had *two* variables (*age* and *distance thrown*) recorded about them. This type of data is called **bivariate data**. We study it to understand the **relationship** between the two variables.

For example, we expect the *distance thrown* will *depend* on the athlete's *age*, so *age* is the **independent variable** and *disance thrown* is the **dependent variable**.

The **independent** and **dependent** variables are sometimes called the **explanatory** and **response** variables respectively.



In this Chapter we **describe** and **model** relationships between pairs of numerical variables.

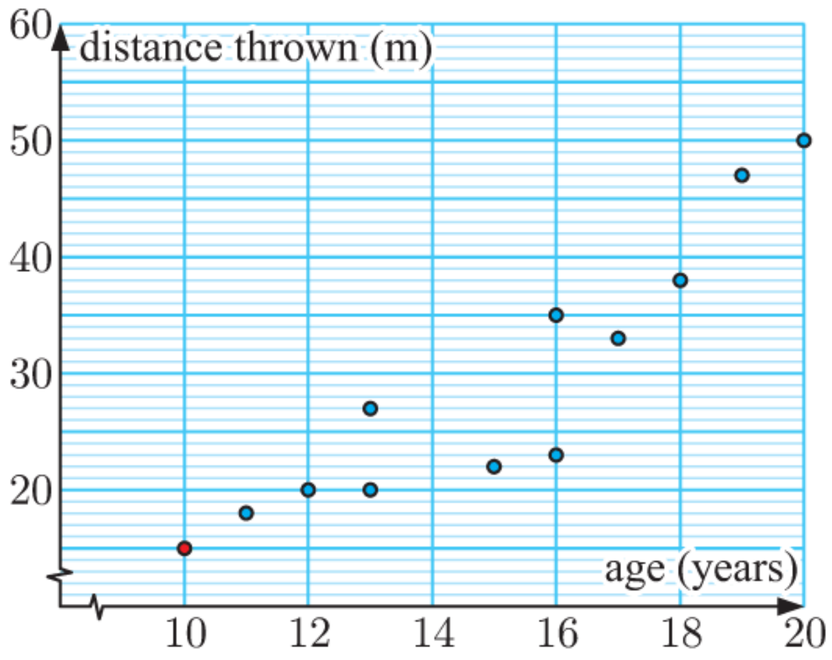
A ASSOCIATION BETWEEN NUMERICAL VARIABLES

We can observe the relationship between two numerical variables using a **scatter diagram**. We usually place the independent variable on the horizontal axis, and the dependent variable on the vertical axis.

In the **Opening Problem**, the independent variable *age* is placed on the horizontal axis, and the dependent variable *distance thrown* is placed on the vertical axis.

We then graph each data value as a point on the scatter diagram. For example, the red point represents athlete H, who is 10 years old and threw the discus 15 metres.

From the general shape formed by the dots, we can see that as the *age* increases, so does the *distance thrown*.

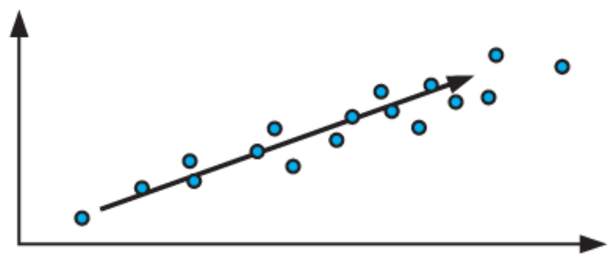


CORRELATION

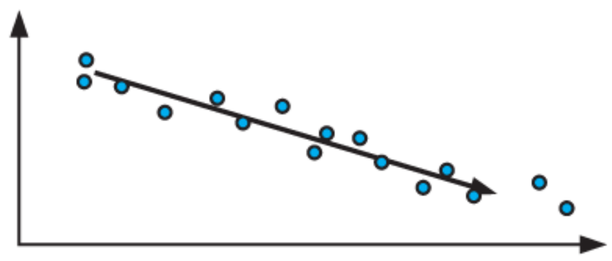
Correlation refers to the relationship or association between two numerical variables.

There are several characteristics we consider when describing the correlation between two variables: direction, linearity, strength, outliers, and causation.

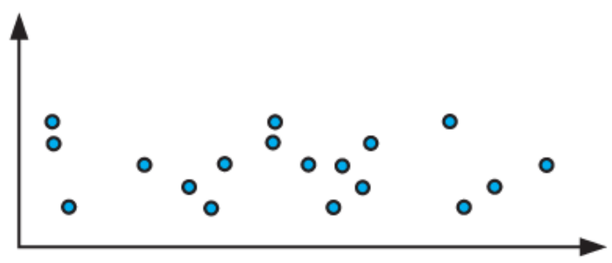
DIRECTION



For a generally *upward* trend, we say that the correlation is **positive**. An increase in the independent variable generally results in an increase in the dependent variable.



For a generally *downward* trend, we say that the correlation is **negative**. An increase in the independent variable generally results in a decrease in the dependent variable.

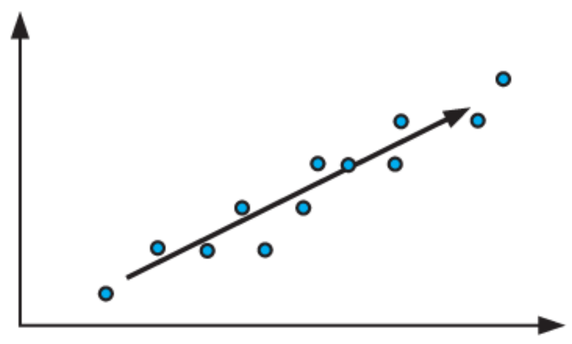


For *randomly scattered* points, with no upward or downward trend, we say there is **no correlation**.

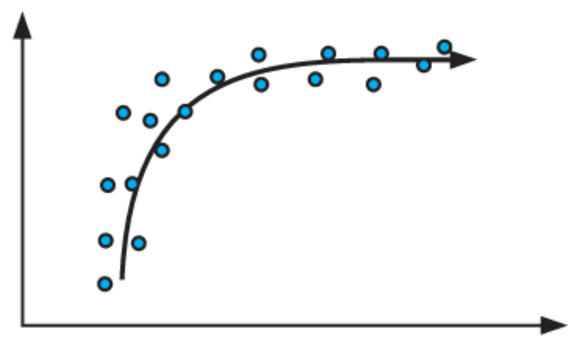
LINEARITY

When a trend exists, if the points approximately form a straight line, we say the trend is **linear**.

These points are roughly linear.

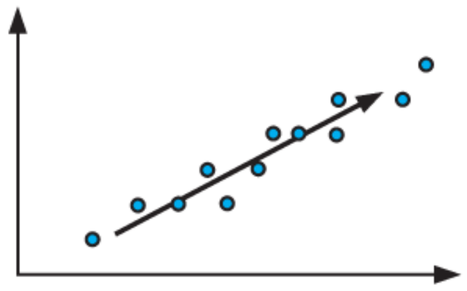
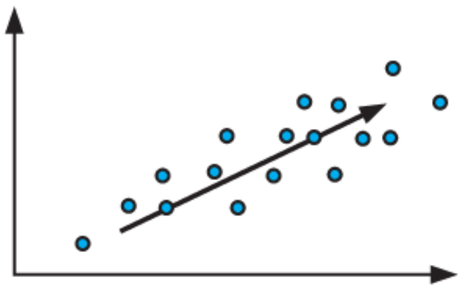
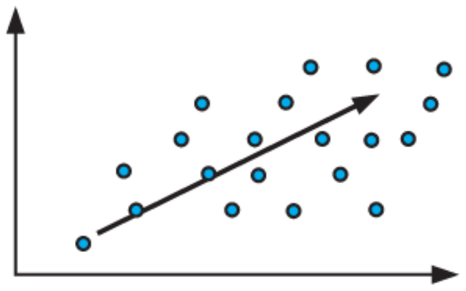
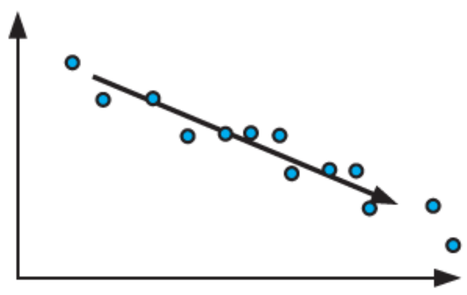
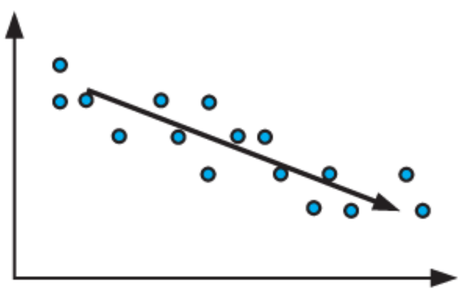
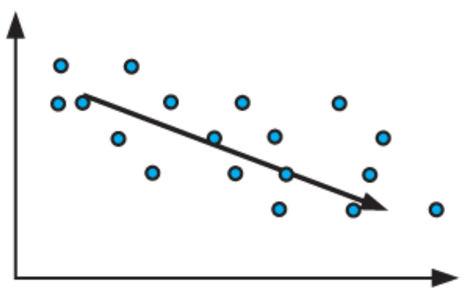


These points do not follow a linear trend.



STRENGTH

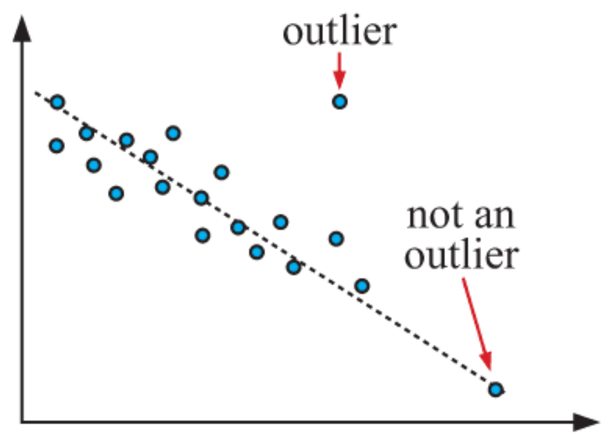
To describe how closely the data follows a pattern or trend, we talk about the **strength** of correlation. It is usually described as either **strong**, **moderate**, or **weak**.

strong	**moderate**	**weak**
 strong positive	 moderate positive	 weak positive
 strong negative	 moderate negative	 weak negative

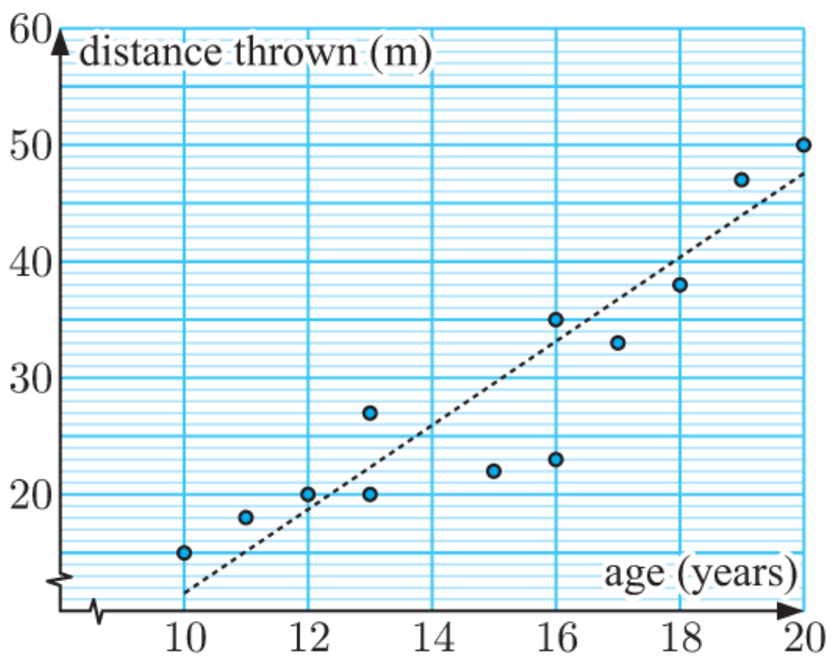
OUTLIERS

Outliers are isolated points which do not follow the trend formed by the main body of data.

If an outlier is the result of a recording or graphing error, it should be discarded. However, if the outlier is a genuine piece of data, it should be kept.



For the scatter diagram of the data in the **Opening Problem**, we can say that there is a strong positive correlation between *age* and *distance thrown*. The relationship appears to be linear, with no outliers.



CAUSALITY

Correlation between two variables does not necessarily mean that one variable *causes* the other.

For example:

- The *arm length* and *running speed* of a sample of young children were measured, and a strong, positive correlation was found between the variables.

  This does *not* mean that short arms cause a reduction in running speed, or that a high running speed causes your arms to grow long.

  Rather, there is a strong, positive correlation between the variables because both *arm length* and *running speed* are closely related to a third variable, *age*. Up to a certain age, both *arm length* and *running speed* increase with *age*.



- The number of television sets sold in London and the number of stray dogs collected in Boston were recorded over several years. A strong, positive correlation was found between the variables.

  Obviously the number of television sets sold in London was not influencing the number of stray dogs collected in Boston. It is coincidental that the variables both increased over this period of time.



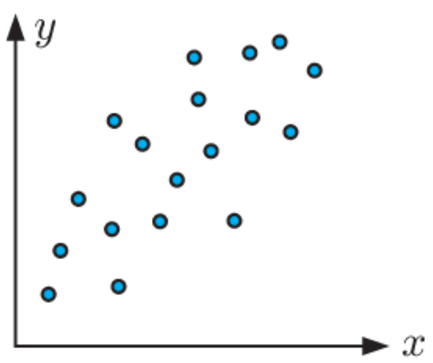
If a change in one variable *causes* a change in the other variable then we say that a **causal relationship** exists between them. In these cases, we can say that the independent variable *explains* the dependent variable. It may be more natural to use the terminology **explanatory variable** and **response variable**.

In cases where a causal relationship is not apparent, we cannot conclude that a causal relationship exists based on high correlation alone.

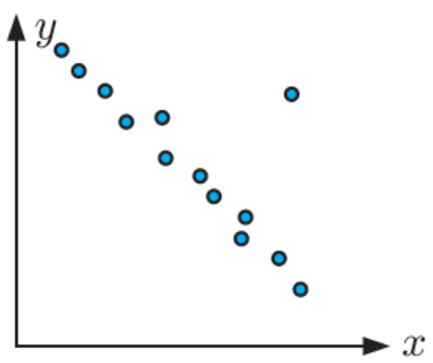
EXERCISE 26A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.

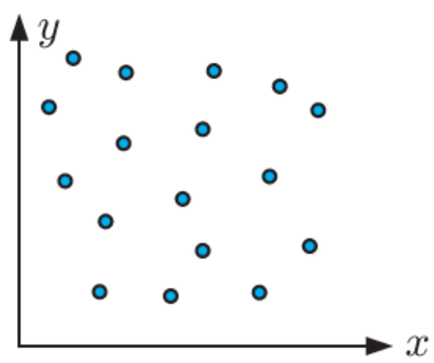
a



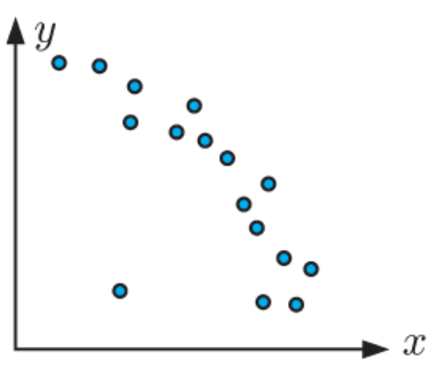
b



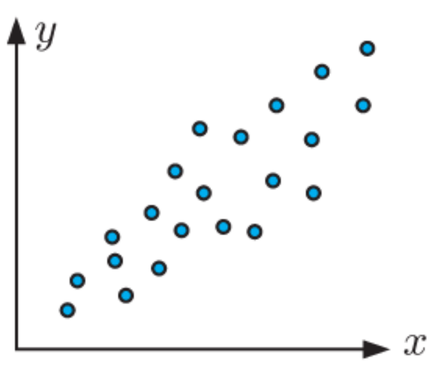
c



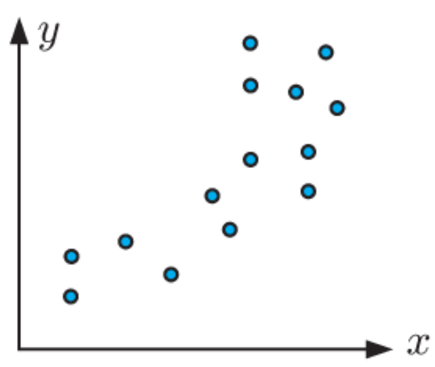
d



e



f



2 Tiffany is a hairdresser. The table below shows the number of hours she worked each day last week, and the number of customers she had.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

a Which is the explanatory variable, and which is the response variable?

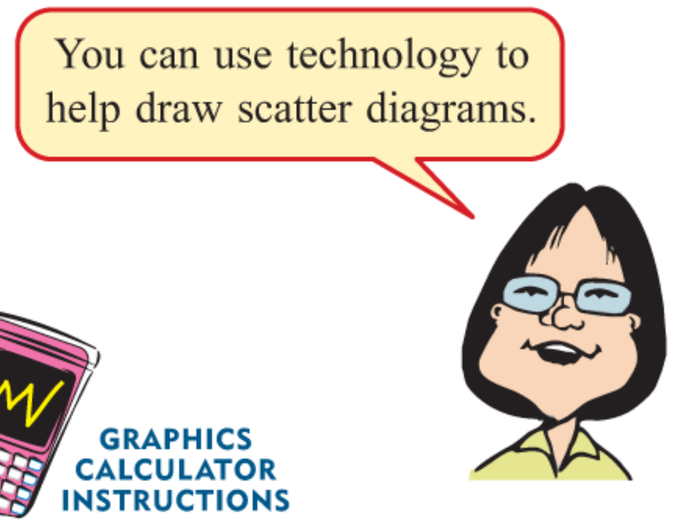
b Draw a scatter diagram of the data.

c On which two days did Tiffany:

i work the same number of hours

ii have the same number of customers?

d Explain why you would expect a positive correlation between the variables.



3 The scores awarded by two judges at an ice skating competition are shown in the table.

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

a Construct a scatter diagram for the data, with Judge A's scores on the horizontal axis and Judge B's scores on the vertical axis.

b Copy and complete the following comments about the scatter diagram:

There appears to be ....., ..., ... correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores .....

c Would it be reasonable to conclude that an increase in Judge A's scores *causes* an increase in Judge B's scores? Explain your answer.

4 Paul owns a company which installs industrial air conditioners. The table below shows the number of workers at the company's last 10 jobs, and the time it took to complete the job.

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

a Which job: **i** took the longest **ii** involved the most workers?

b Draw a scatter diagram to display the data.

c Describe the relationship between the variables *number of workers* and *time*.

5 Choose the scatter diagram which would best illustrate the relationship between the variables x and y.

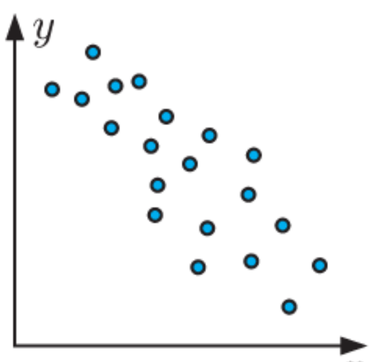
a $x =$ the number of apples bought by customers, $y =$ the total cost of apples bought

b $x =$ the number of pushups a student can perform in one minute,
$y =$ the time taken for the student to run 100 metres

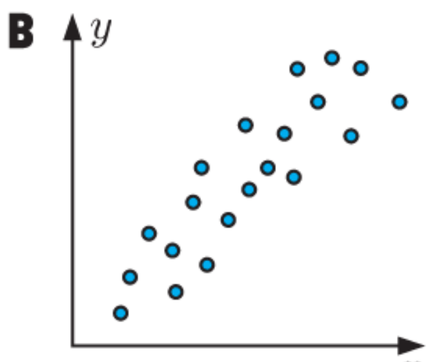
c $x =$ the height of a person, $y =$ the weight of the person

d $x =$ the distance a student travels to school, $y =$ the height of the student's uncle

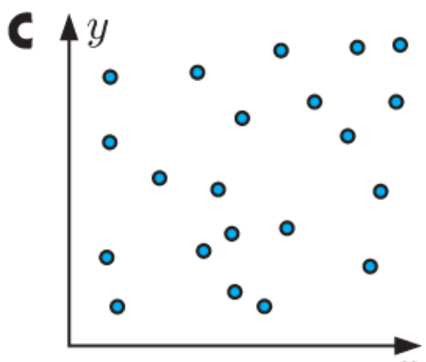
A



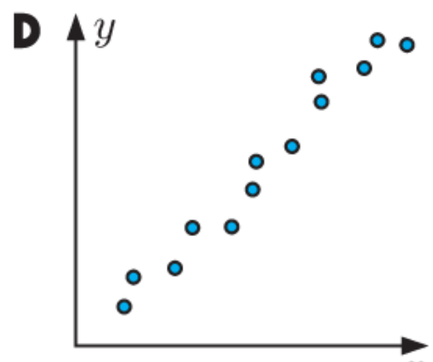
B



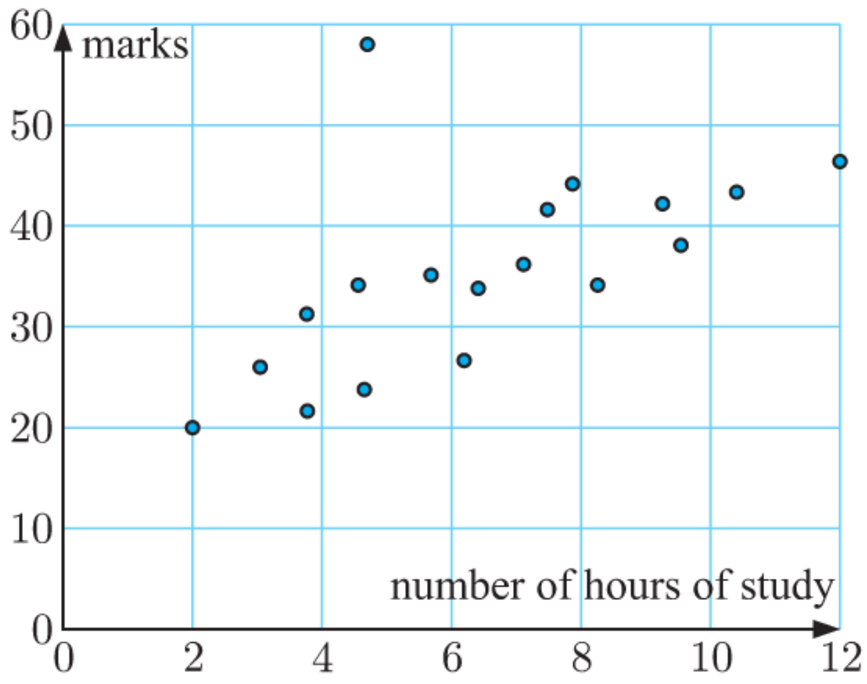
C



D



6 The scatter diagram shows the marks obtained by students in a test out of 50 marks, plotted against the number of hours each student studied for the test.



a Describe the correlation between the variables.

b How should the outlier be treated? Explain your answer.

c Do you think there is a causal relationship between the variables? Explain your answer.

7 When the following pairs of variables were measured, a strong, positive correlation was found between each pair. Discuss whether a causal relationship exists between the variables. If not, suggest a third variable to which they may both be related.

a The lengths of one's left and right feet.

b The damage caused by a fire and the number of firefighters who attend it.

c A company's expenditure on advertising, and the sales they make the following year.

d The heights of parents and the heights of their adult children.



e The numbers of hotels and numbers of service stations in rural towns.

B PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

In the previous Section, we classified the strength of the correlation between two variables as either strong, moderate, or weak. We observed the points on a scatter diagram, and judged how clearly the points formed a linear relationship.

Since this method is *subjective* and relies on the observer's opinion, it is important to get a more precise measure of the strength of linear correlation between the variables. We achieve this using **Pearson's product-moment correlation coefficient** r.

For a set of n data given as ordered pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n),

Pearson's product-moment correlation coefficient is $r = \dfrac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}}$

where $\overline{x}$ and $\overline{y}$ are the means of the x and y data respectively, and $\sum$ means the sum over all the data values.

You are not required to learn this formula, but you should be able to calculate the value of r using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

HISTORICAL NOTE

Karl Pearson (1857 - 1936) was an English statistician who developed the product-moment correlation coefficient together with his academic advisor **Sir Francis Galton**.

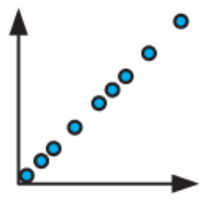
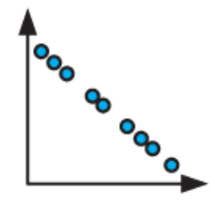
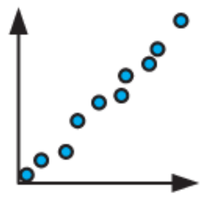
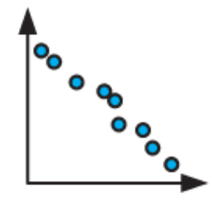
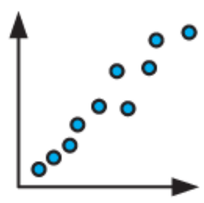
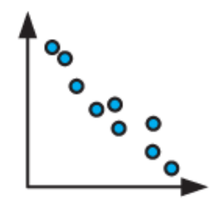
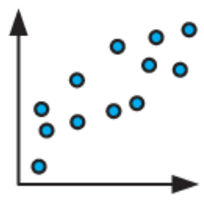
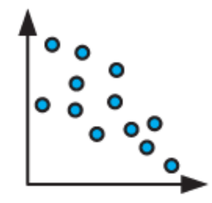
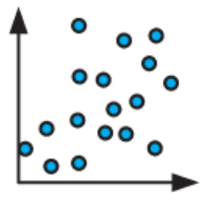
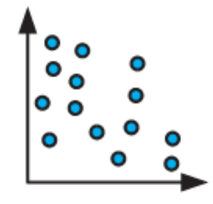
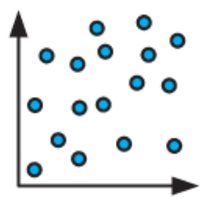
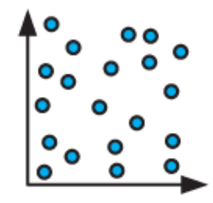
Pearson made many other contributions to statistics including the use of histograms in exploratory data analysis, parameter estimation, and hypothesis testing.

He is considered a key figure in the development of mathematical statistics.

PROPERTIES OF PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

- The values of r range from -1 to $+1$.
- The **sign** of r indicates the **direction** of the correlation.
  - A positive value for r indicates the variables are **positively correlated**.
    An increase in one variable results in an increase in the other.
  - A negative value for r indicates the variables are **negatively correlated**.
    An increase in one variable results in a decrease in the other.
  - If $r = 0$ then there is **no correlation** between the variables.
- The **size** of r indicates the **strength** of the correlation.
  - A value of r close to $+1$ or -1 indicates strong correlation between the variables.
  - A value of r close to zero indicates weak correlation between the variables.

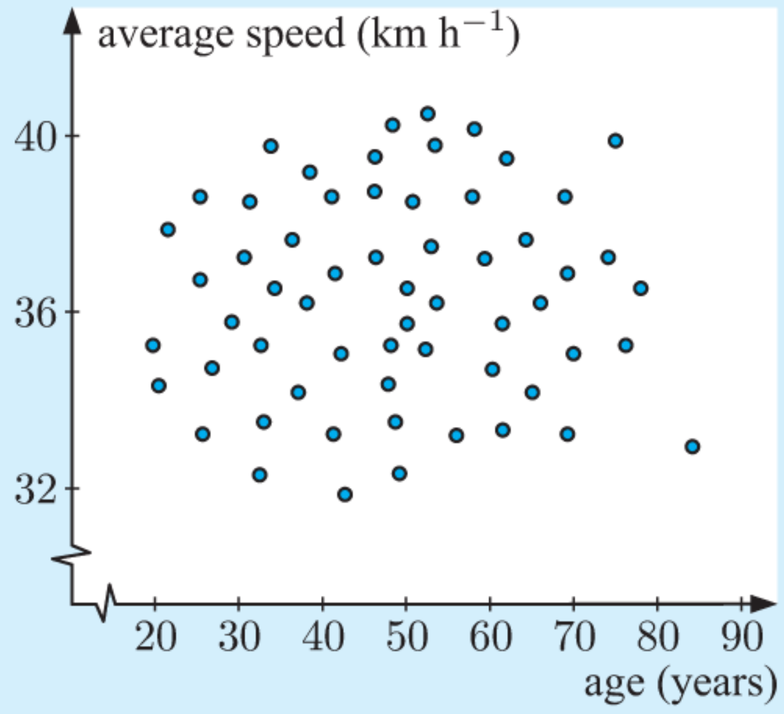
The following table is a guide for describing the strength of linear correlation using r.

Positive correlation		Negative correlation	
$r = 1$	perfect positive correlation 	$r = -1$	perfect negative correlation 
$0.95 \leqslant r < 1$	very strong positive correlation 	$-1 < r \leqslant -0.95$	very strong negative correlation 
$0.87 \leqslant r < 0.95$	strong positive correlation 	$-0.95 < r \leqslant -0.87$	strong negative correlation 
$0.7 \leqslant r < 0.87$	moderate positive correlation 	$-0.87 < r \leqslant -0.7$	moderate negative correlation 
$0.5 \leqslant r < 0.7$	weak positive correlation 	$-0.7 < r \leqslant -0.5$	weak negative correlation 
$0 < r < 0.5$	very weak positive correlation 	$-0.5 < r < 0$	very weak negative correlation 

Example 1 **Self Tutor**

The Department of Road Safety wants to know if there is any association between *average speed* in the metropolitan area and the *age of drivers*. They commission a device to be fitted in the cars of drivers of different ages.

The results are shown in the scatter diagram.
The r-value for this association is $+0.027$.
Describe the association.



Since $0 < r < 0.5$, there is a very weak positive correlation between the two variables.
We observe this in the graph as the points are randomly scattered.

Example 2 **Self Tutor**

The botanical gardens have been trying a new chemical to control the number of beetles infesting their plants. The results of one of their tests are shown in the table.

Sample	*Quantity of chemical* (g)	*Number of surviving beetles*
A	2	11
B	5	6
C	6	4
D	3	6
E	9	3

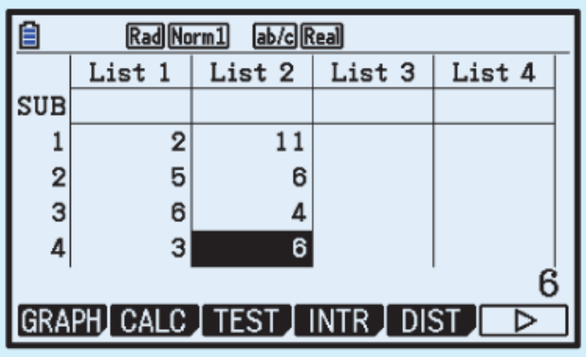
a Draw a scatter diagram for the data.

b Determine the correlation coefficient r.

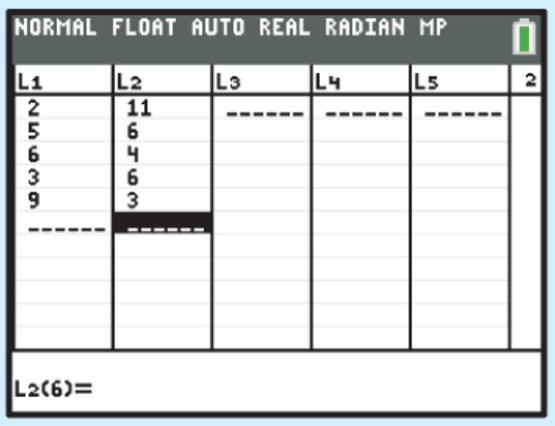
c Describe the correlation between the *quantity of chemical* and the *number of surviving beetles*.

We first enter the data into separate lists:

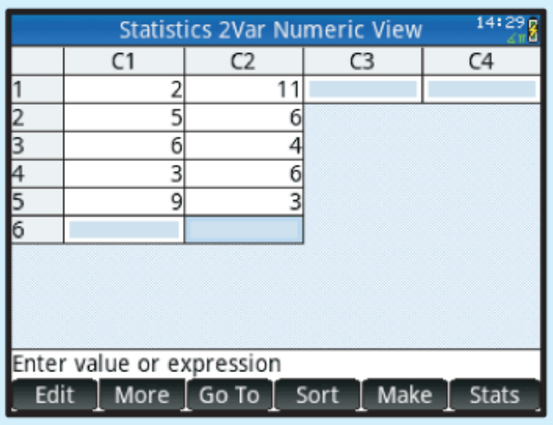
Casio fx-CG50



TI-84 Plus CE

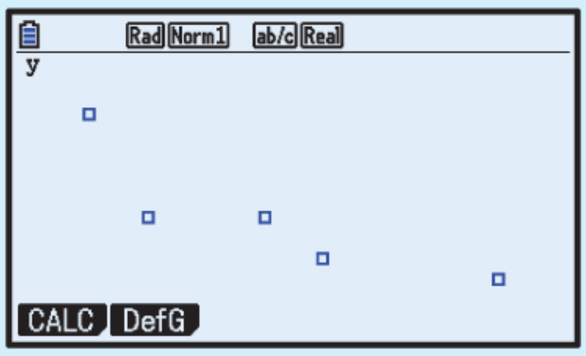


HP Prime

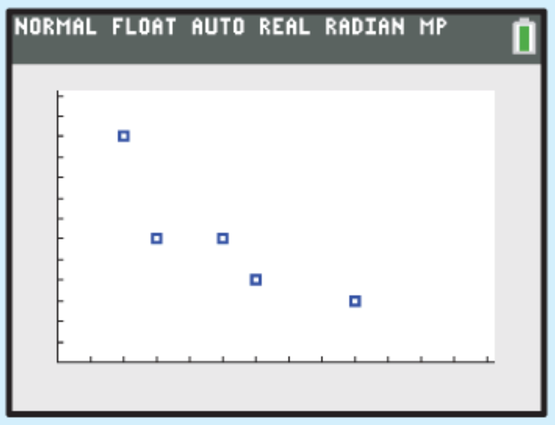


a

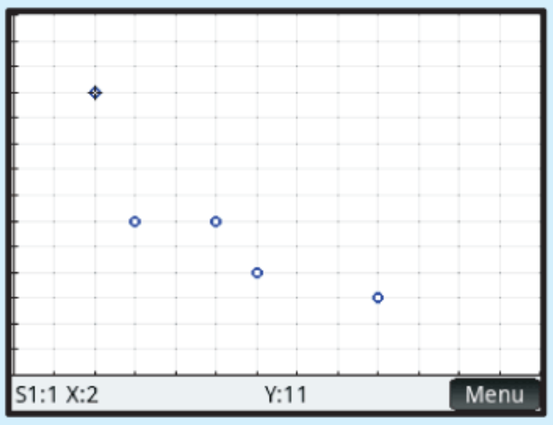
Casio fx-CG50



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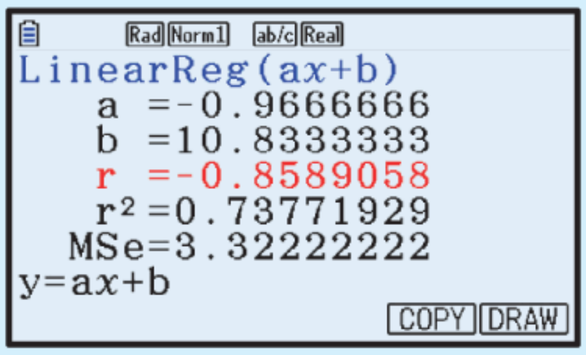


HP Prime

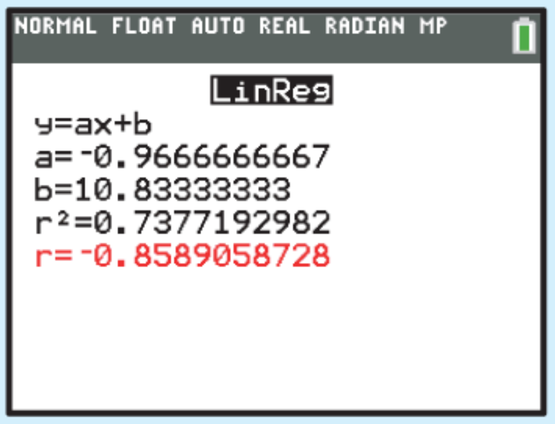


b

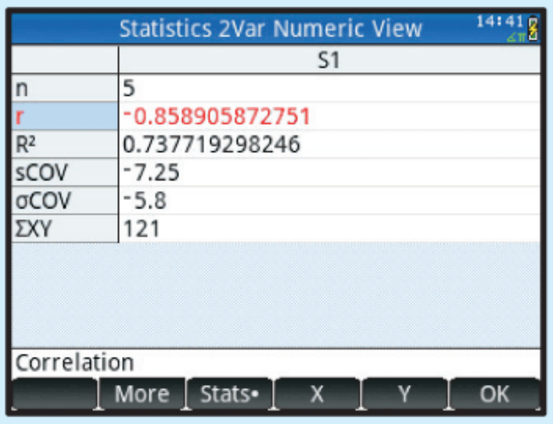
Casio fx-CG50



TI-84 Plus CE



HP Prime



So, $r \approx -0.859$.

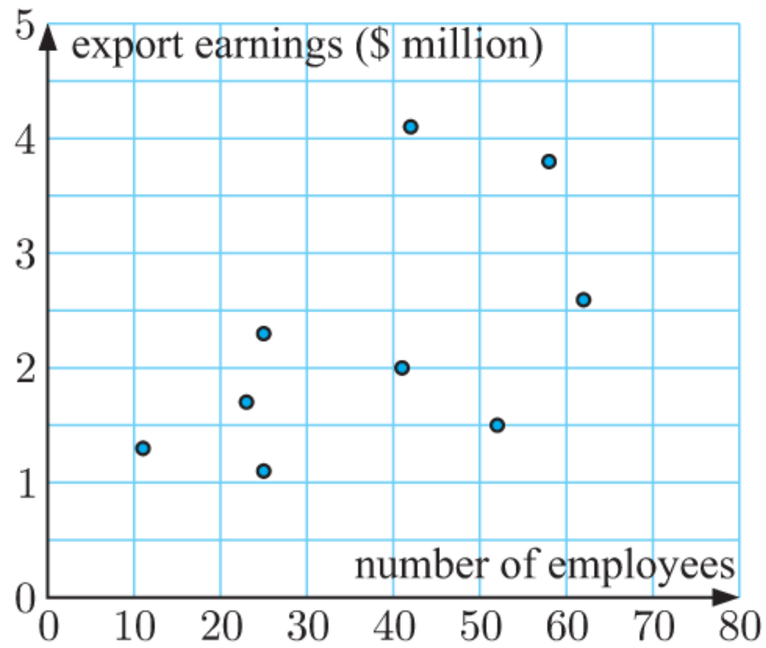
c There is a moderate negative correlation between the *quantity of chemical used* and the *number of surviving beetles*.

In general, the more chemical that is used, the fewer beetles that survive.

EXERCISE 26B

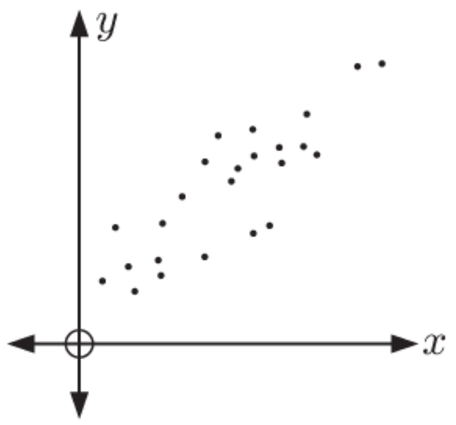
1 In a recent survey, the Department of International Commerce compared the *number of employees of a company* with its *export earnings*. A scatter diagram of their data is shown alongside. The corresponding value of r is 0.556.

Describe the association between the variables.

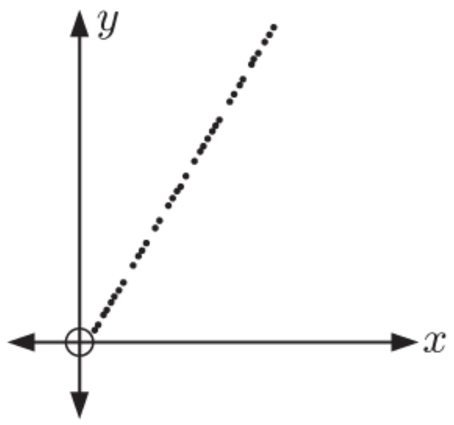


2 Match each scatter diagram with the correct value of r.

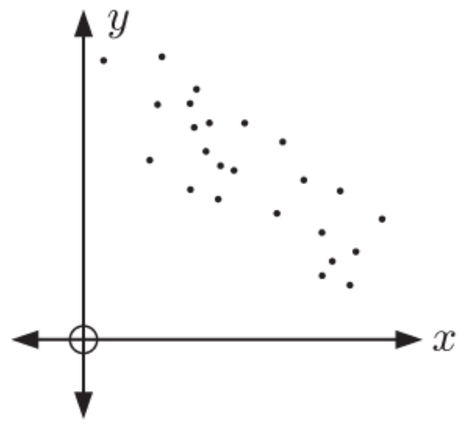
a



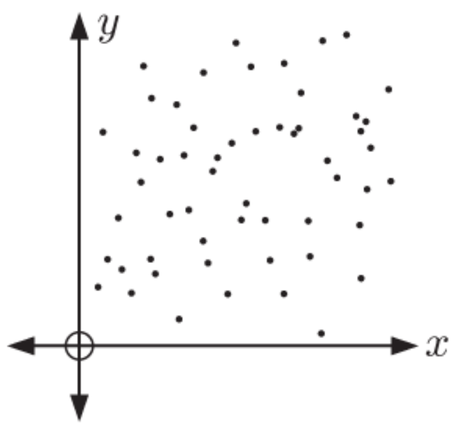
b



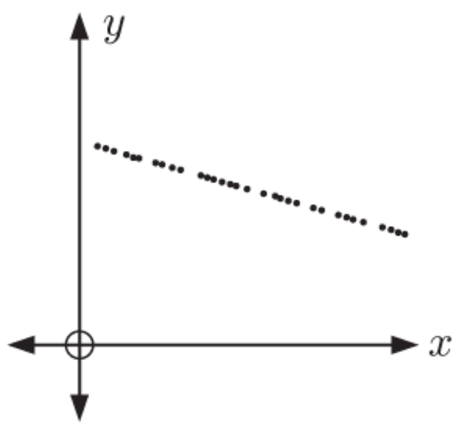
c



d



e



A $r = 1$ **B** $r = 0.6$ **C** $r = 0$ **D** $r = -0.7$ **E** $r = -1$

3 For each of the following data sets:

i Draw a scatter diagram for the data.

ii Calculate Pearson's product-moment correlation coefficient r.

iii Describe the linear correlation between x and y.

a

x	1	2	3	4	5	6
y	3	2	5	5	9	6

b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

4 A selection of students was asked how many phone calls and text messages they received the previous day. The results are shown alongside.

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the linear correlation between *phone calls received* and *text messages received*.

d Give a reason why this correlation may occur.

5 Consider the **Opening Problem** on page **708**.

a Calculate r for the data.

b Hence describe the association between the variables.

6 Jill does her washing every Saturday and hangs her clothes out to dry. She notices that the clothes dry faster some days than others. She investigates the relationship between the temperature and the time her clothes take to dry:

Temperature (x °C)	25	32	27	39	35	24	30	36	29	35
Drying time (y minutes)	100	70	95	25	38	105	70	35	75	40

a Draw a scatter diagram for the data.

b Calculate r.

c Describe the correlation between *temperature* and *drying time*.

7 This table shows the number of supermarkets in 10 towns, and the number of car accidents that have occurred in these towns in the last month.

Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37

a Draw a scatter diagram for the data.

b Calculate r.

c Identify the outlier in the data.

d It was found that the outlier was due to an error in the data collection process.

i Recalculate r with the outlier removed.

ii Describe the relationship between the variables.

iii Discuss the effect of removing the outlier on the value of r.

e Do you think there is a causal relationship between the variables? Explain your answer.

8 A health researcher notices that the incidence of Multiple Sclerosis (MS) is higher in some parts of the world than in others.

To investigate further, she records the *latitude* and *incidence of MS per 100 000 people* of 20 countries.

Latitude (degrees)	55	25	41	22	47	37	56	14	34	25
MS incidence per 100 000	165	95	75	20	180	140	230	15	45	65

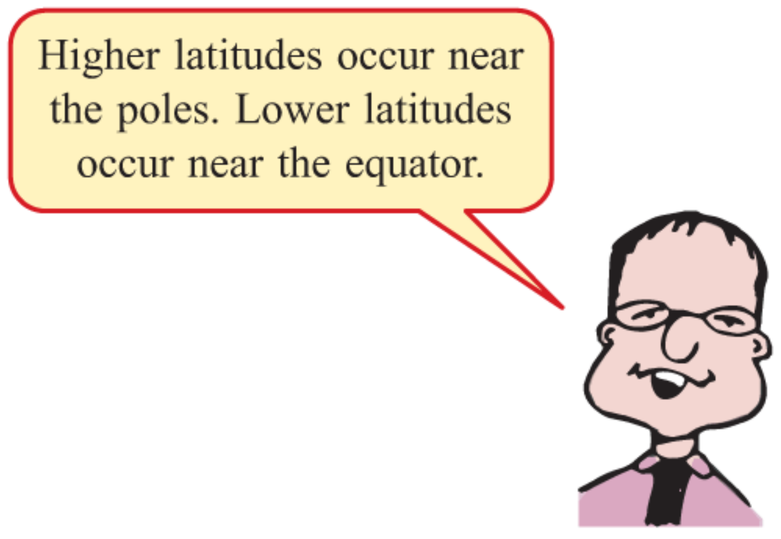
Latitude (degrees)	27	65	10	24	4	56	46	8	50	40
MS incidence per 100 000	30	140	5	15	2	290	95	8	160	105

a Draw a scatter diagram for the data.

b Calculate the value of r.

c Describe the relationship between the variables.

d Is the incidence of MS higher near the equator, or near the poles?



ACTIVITY 1 COMPARING HEIGHT AND FOOT LENGTH

In this Activity, you will explore the relationship between the *height* and *foot length* of the students in your class.

You will need: ruler, tape measure

What to do:

1 Predict whether there will be positive correlation, no correlation, or negative correlation between the *height* and *foot length* of the students in your class.

2 Measure the height and foot length of each student in your class. Record your measurements in a table like the one below:

Student	*Height* (cm)	*Foot length* (cm)



3 Use technology to draw a scatter diagram for the data.

4 Calculate Pearson's product-moment correlation coefficient r for the data.

5 Describe the relationship between *height* and *foot length*. Was your prediction correct?

6 Do you think that a high value of r indicates a causal relationship in this case?

C LINE OF BEST FIT BY EYE

If there is a sufficiently strong linear correlation between two variables, we can draw a line of best fit to illustrate their relationship. In general, it is only worth drawing a line of best fit if the correlation between the variables is strong. There is no fixed rule, but we suggest that a line of best fit is not appropriate if $|r| < 0.85$.

If we draw the line just by observing the points, we call it a **line of best fit by eye**. This line will vary from person to person.

We draw a line of best fit connecting variables x and y as follows:

Step 1: Calculate the mean of the x values $\overline{x}$, and the mean of the y values $\overline{y}$.

Step 2: Mark the **mean point** $(\overline{x}, \overline{y})$ on the scatter diagram.

Step 3: Draw a line through the mean point which fits the trend of the data, and so that about the same number of data points are above the line as below it.

Consider again the data from the **Opening Problem**:

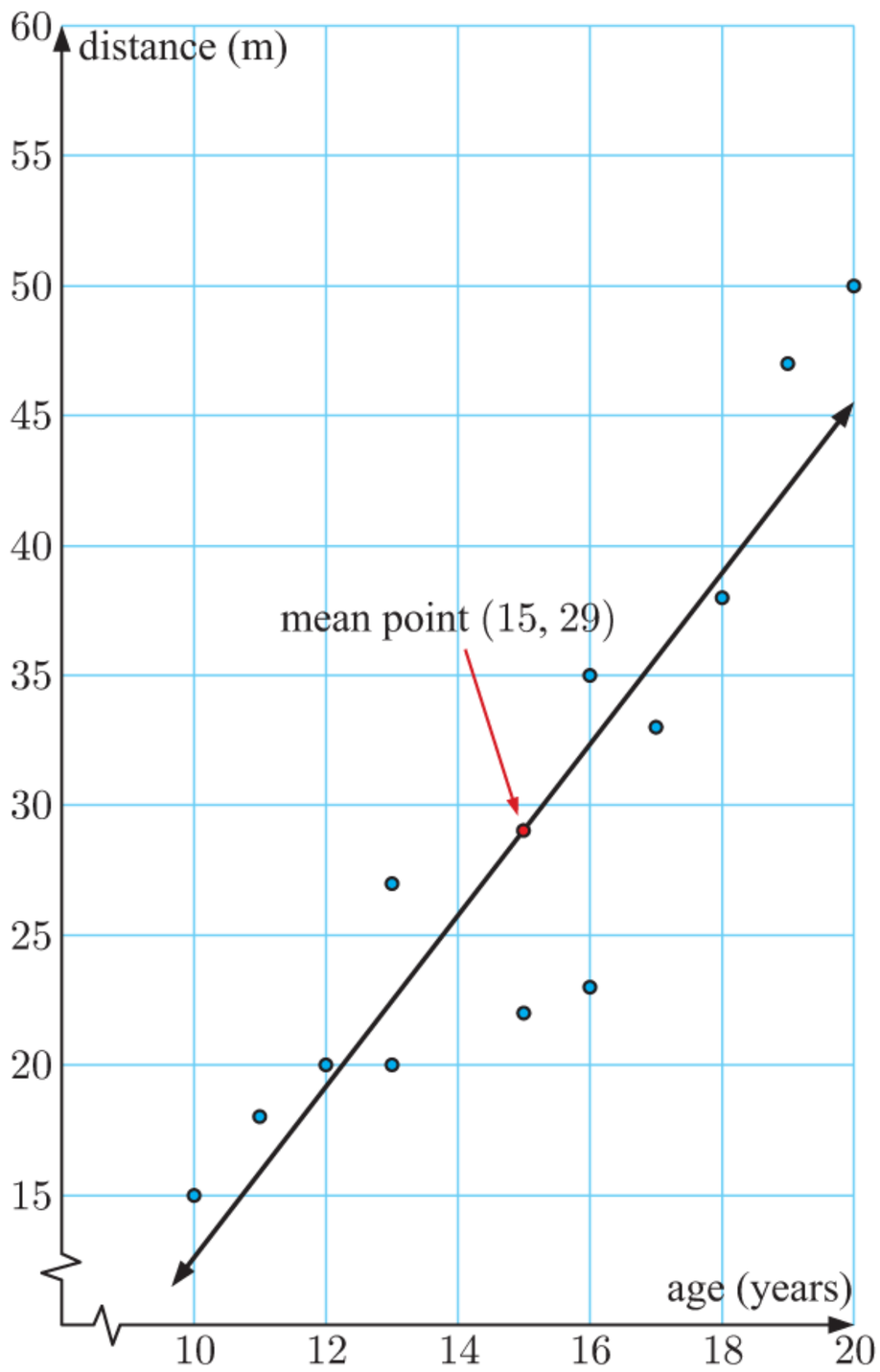
Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

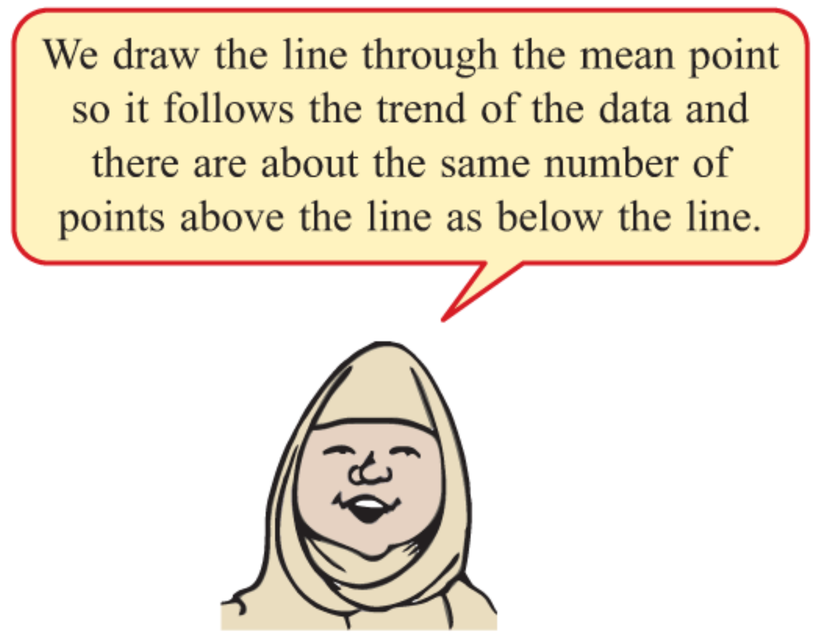
We have seen that there is a strong positive linear correlation between *age* and *distance thrown*.

We can therefore model the data using a line of best fit.

The mean age is 15 years and the mean distance thrown is 29 m. We therefore draw our line of best fit through the mean point (15, 29).

We can use the line of best fit to estimate the value of y for any given value of x, and vice versa.



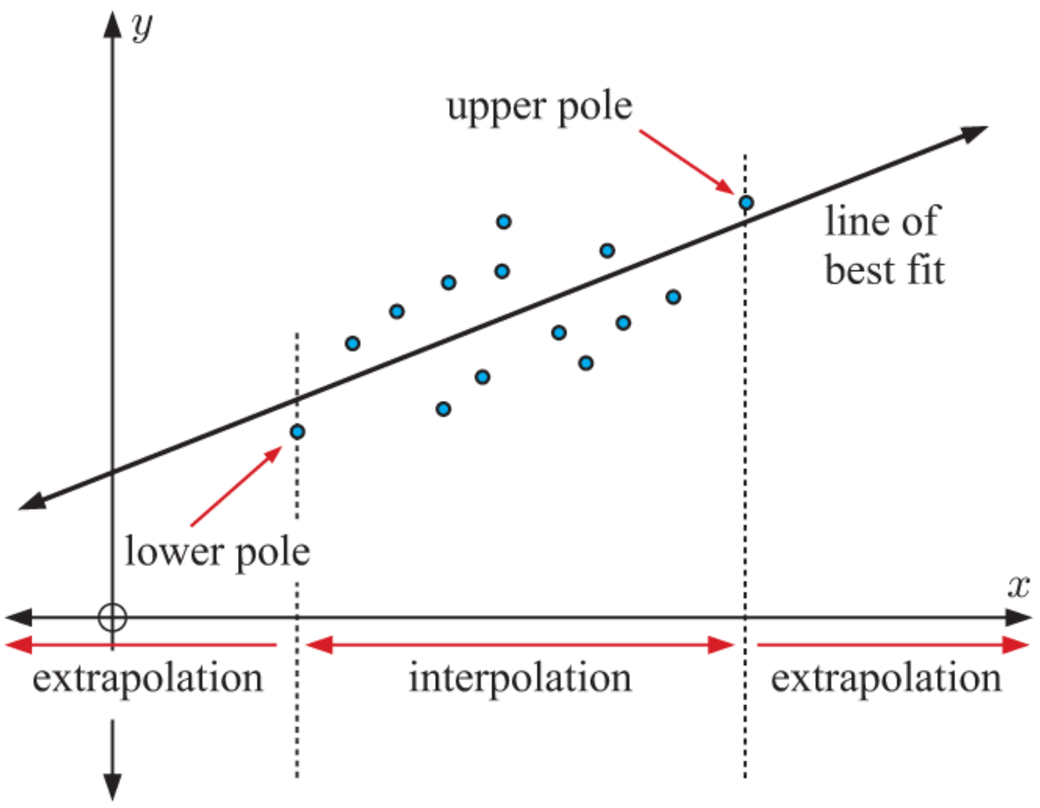


INTERPOLATION AND EXTRAPOLATION

Consider the data in the scatter diagram alongside. The data with the highest and lowest values are called the **poles**.

The line of best fit for the data is also drawn on the scatter diagram. We can use this line to predict the value of one variable for a given value of the other.

- If we predict a y value for an x value **in between** the poles, we say we are **interpolating** in between the poles.
- If we predict a y value for an x value **outside** the poles, we say we are **extrapolating** outside the poles.



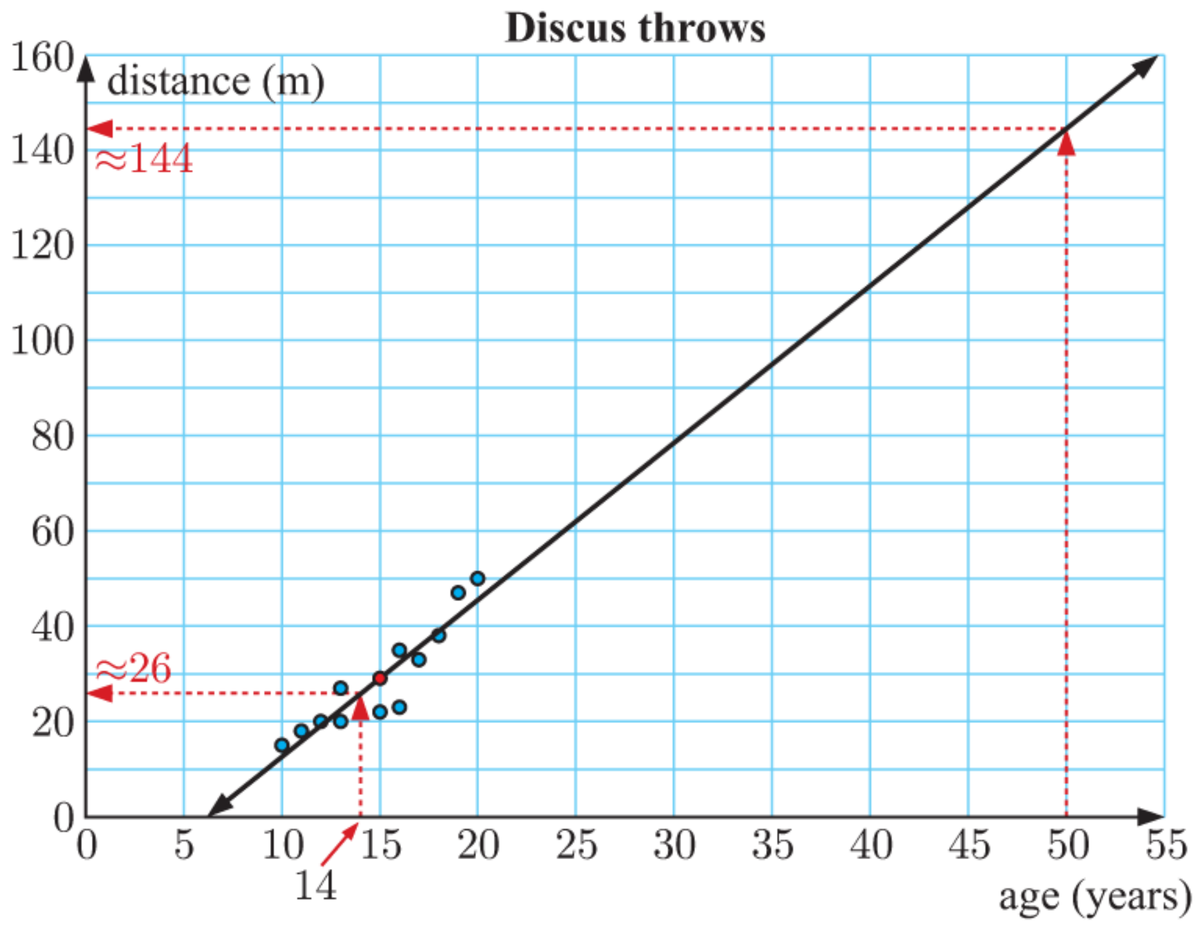
The accuracy of an interpolation depends on how well the linear model fits the data. This can be gauged by the correlation coefficient and by ensuring that the data is randomly scattered around the line of best fit.

The accuracy of an extrapolation depends not only on how well the model fits, but also on the assumption that the linear trend will continue past the poles. The validity of this assumption depends greatly on the situation we are looking at.

For example, consider the line of best fit for the data in the **Opening Problem**. It can be used to predict the distance a discus will be thrown by an athlete of a particular age.

The age 14 is within the range of ages in the original data, so it is reasonable to predict that a 14 year old will be able to throw the discus 26 m.

However, it is unlikely that the linear trend shown in the data will continue far beyond the poles. For example, according to the model, a 50 year old might throw the discus 144 m. This is almost twice the current world record of 76.8 m, so it would clearly be an unreasonable prediction.



Example 3

Self Tutor

On a hot day, six cars were left in the sun in a car park. The length of time each car was left in the sun was recorded, as well as the temperature inside the car at the end of the period.

Car	A	B	C	D	E	F
Time (x minutes)	50	5	25	40	15	45
Temperature (y °C)	47	28	36	42	34	41

a Calculate $\overline{x}$ and $\overline{y}$.

b Draw a scatter diagram for the data.

c Locate the mean point $(\overline{x}, \overline{y})$ on the scatter diagram, then draw a line of best fit through this point.

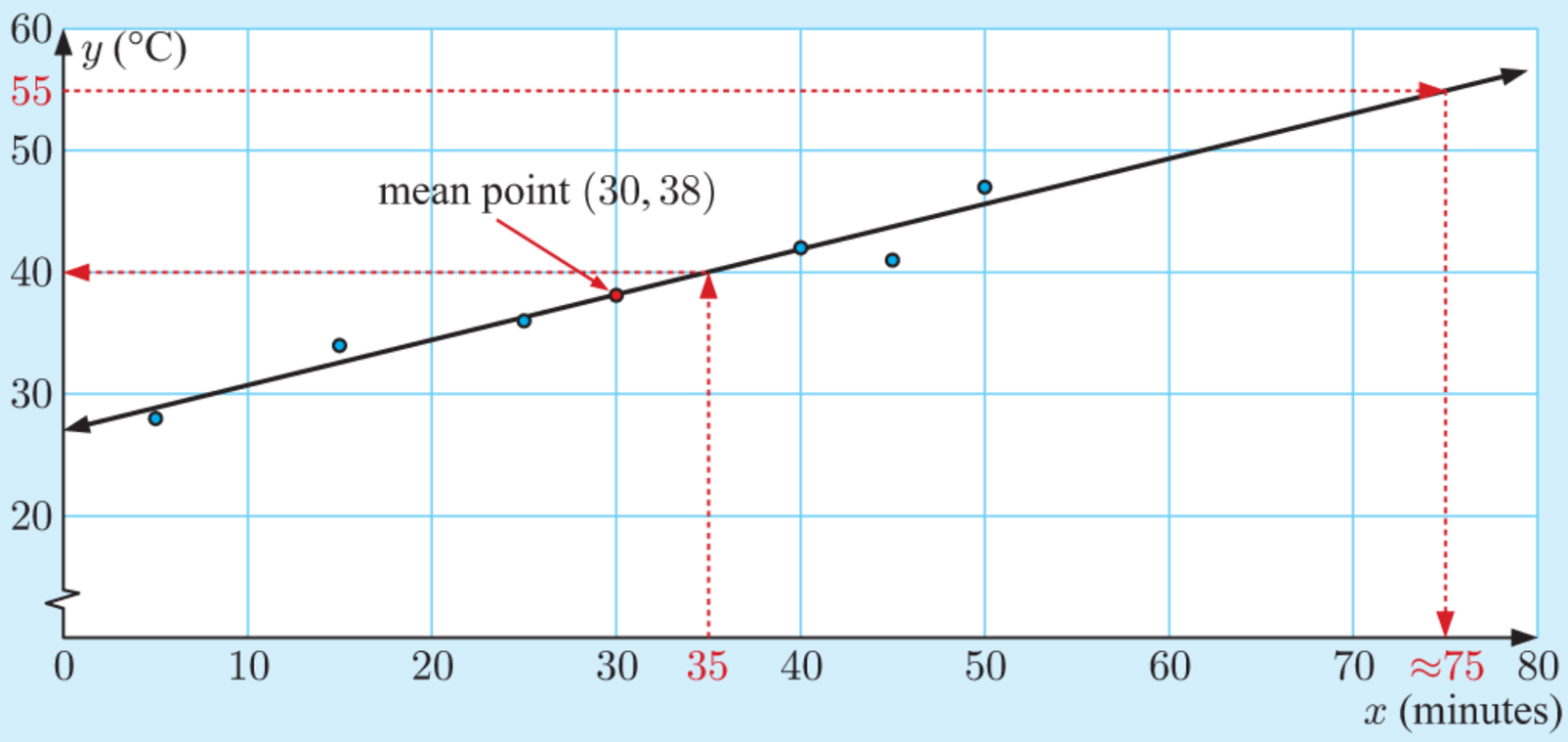
d Predict the temperature of a car which has been left in the sun for 35 minutes.

e Predict how long it would take for a car's temperature to reach 55°C.

f Comment on the reliability of your predictions in **d** and **e**.

a $\overline{x} = \dfrac{50+5+25+40+15+45}{6} = 30, \qquad \overline{y} = \dfrac{47+28+36+42+34+41}{6} = 38$

b, c



d When $x = 35$, $y \approx 40$.
The temperature of a car left in the sun for 35 minutes will be approximately $40°$C.

e When $y = 55$, $x \approx 75$.
It would take approximately 75 minutes for a car's temperature to reach $55°$C.

f The prediction in **d** is reliable, as the data appears linear, and this is an interpolation.
The prediction in **e** may be unreliable, as it is an extrapolation and the linear trend displayed by the data may not continue beyond the 50 minute mark.

EXERCISE 26C

1 Consider the data set:

x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10

a Draw a scatter diagram for the data.
b Does the data appear to be positively or negatively correlated?
c Calculate r for the data.
d Describe the strength of the relationship between x and y.
e Calculate the mean point $(\overline{x}, \overline{y})$.
f Locate the mean point, then use it in drawing a line of best fit.
g Estimate the value of y when $x = 22$.

2 Fifteen students were weighed and their pulse rates were measured:

Weight (x kg)	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
Pulse rate (y beats per min)	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58

a Draw a scatter diagram for the data. **b** Calculate r.
c Describe the relationship between *weight* and *pulse rate*.
d Calculate the mean point $(\overline{x}, \overline{y})$.
e Locate the mean point on the scatter diagram, then use it in drawing a line of best fit.
f Estimate the pulse rate of a 50 kg student. Comment on the reliability of your estimate.

3 The trunk widths and heights of the trees in a garden are given below:

Trunk width (x cm)	35	47	72	40	15	87	20	66	57	24	32
Height (y m)	11	18	24	12	3	30	22	21	17	5	10

a Draw a scatter diagram for the data.
b Which of the points is an outlier?
c How would you describe the tree represented by the outlier?
d Calculate the mean point $(\overline{x}, \overline{y})$.
e Locate the mean point on the scatter diagram, then draw a line of best fit through the mean point.
f Predict the height of a tree with trunk width 120 cm. Comment on the reliability of your prediction.
g Predict the trunk width of a tree with height 10 m. Comment on the reliability of your prediction.

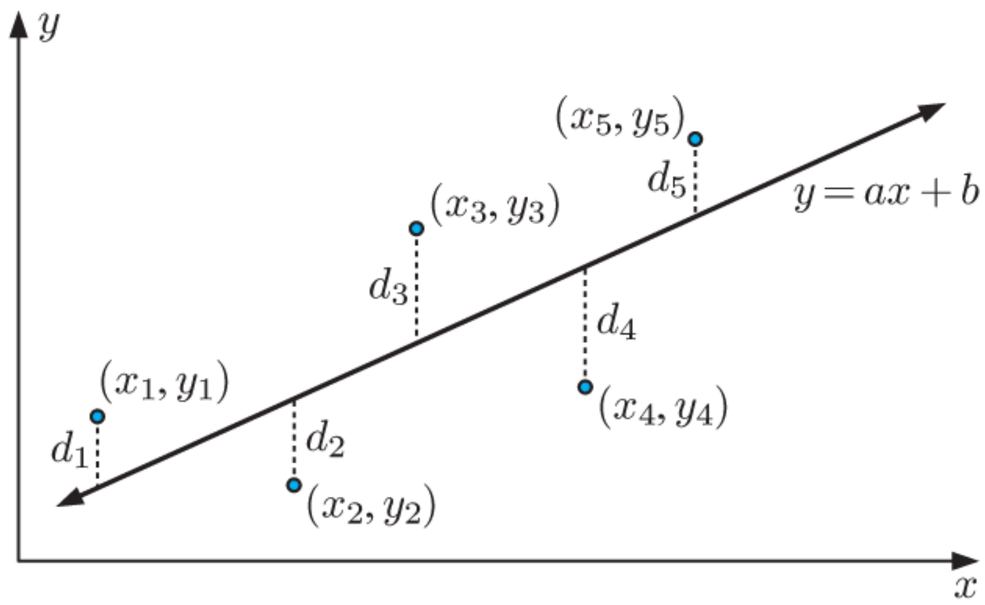


D THE LEAST SQUARES REGRESSION LINE

The problem with drawing a line of best fit by eye is that the line drawn will vary from one person to another. For consistency, we use a method known as **linear regression** to find the equation of the line which best fits the data. The most common method is the method of "**least squares**".

In least squares linear regression, we minimise the sum of the squares of the *vertical* distances between each data point and the **regression line**.

In other words, we need to find the straight line $y = ax+b$ where a and b are chosen to minimise $D = \sum_{i=1}^{n} d_i^2$.



The required values of a and b are $a = \dfrac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2}$ and $b = \overline{y} - a\overline{x}$.

Click on the icon to see how these formulae are derived.

LEAST SQUARES REGRESSION

In this course you will not be required to find the equation of the least squares regression line by hand.

Instead, you can use your **graphics calculator** or the **statistics package**.

STATISTICS PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

Example 4 Self Tutor

The annual income and average weekly grocery bill for a selection of families is shown below:

Income (x thousand pounds)	55	36	25	47	60	64	42	50
Grocery bill (y pounds)	120	90	60	160	190	250	110	150

a Construct a scatter diagram to illustrate the data.

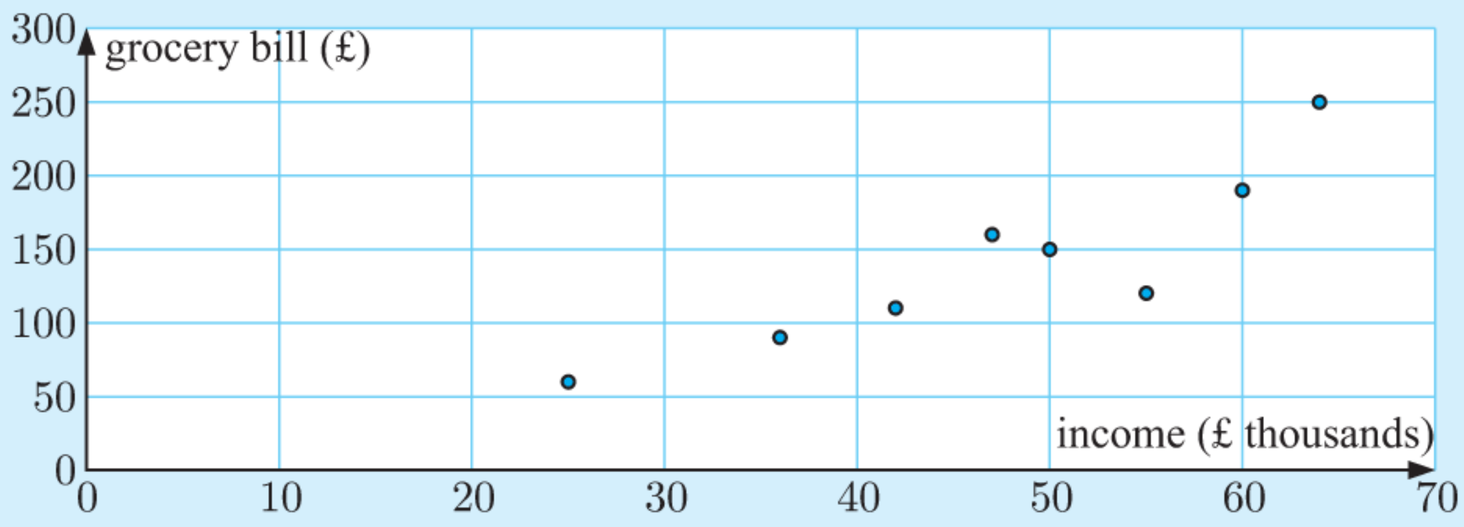
b Find the equation of the regression line. State and interpret its gradient.

c Estimate the weekly grocery bill for a family with an annual income of £95 000.

d Estimate the annual income of a family whose weekly grocery bill is £100.

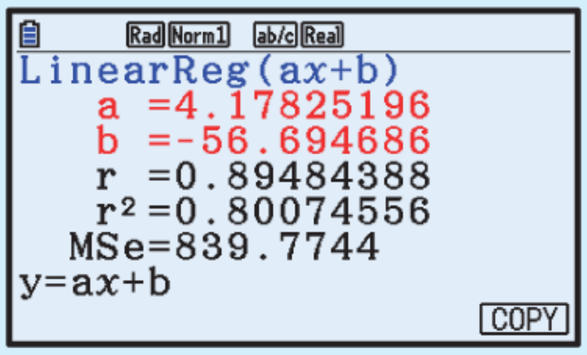
e Comment on whether the estimates in **c** and **d** are likely to be reliable.

a

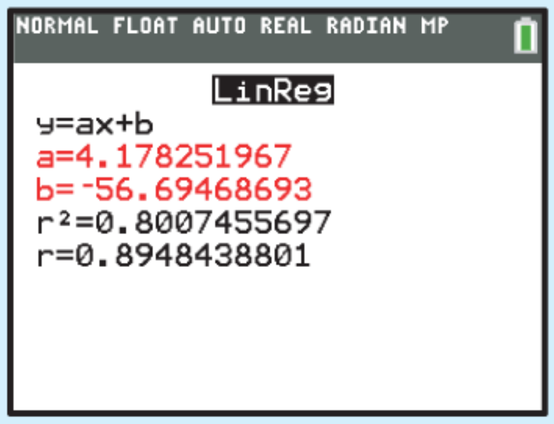


b

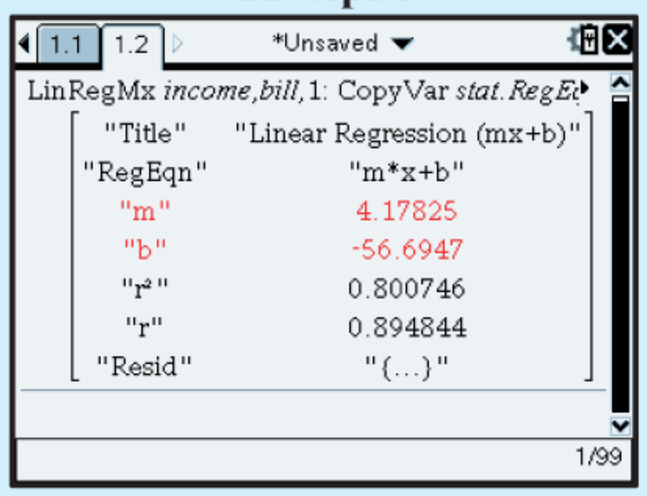
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



Using technology, the regression line is $y \approx 4.18x - 56.7$

The gradient of the regression line ≈ 4.18. This means that for every additional £1000 of income, a family's weekly grocery bill will increase by an average of £4.18.

c When $x = 95$, $y \approx 4.18(95) - 56.7 \approx 340$

So, we expect a family with an income of £95 000 to have a weekly grocery bill of approximately £340.

d When $y = 100$, $100 \approx 4.18x - 56.7$

$\therefore \ 156.7 \approx 4.18x$ {adding 56.7 to both sides}

$\therefore \ x \approx 37.5$ {dividing both sides by 4.18}

So, we expect a family with a weekly grocery bill of £100 to have an annual income of approximately £37 500.

e The estimate in **c** is an extrapolation, so the estimate may not be reliable.

The estimate in **d** is an interpolation and there is strong linear correlation between the variables. We therefore expect this estimate to be reliable.

EXERCISE 26D

1 Consider the data set below.

x	10	4	6	8	9	5	7	1	2	3
y	20	6	8	13	20	12	13	4	2	7



GRAPHICS CALCULATOR INSTRUCTIONS

a Draw a scatter diagram for the data.

b Use technology to find the equation of the regression line, and plot the line on your calculator.

c Use **b** to draw the regression line on your scatter diagram.

2 Steve wanted to see whether there was any relationship between the temperature when he leaves for work in the morning, and the time it takes for him to get to work.

He collected data over a 14 day period:

Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the relationship between the variables.

d Is it reasonable to fit a linear model to this data? Explain your answer.

3 The prices of petrol and the number of customers per hour for sixteen petrol stations are:

Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
Number of customers (y)	45	42	25	48	43	15	19	10

Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
Number of customers (y)	30	23	42	50	12	24	32	17

a Calculate Pearson's product-moment correlation coefficient for the data.

b Describe the relationship between the *petrol price* and the *number of customers*.

c Use technology to find the equation of the regression line.

d State and interpret the gradient of the regression line.

e Estimate the number of customers per hour for a petrol station which sells petrol at 115.9 cents per litre.

f Estimate the petrol price at a petrol station which has 40 customers per hour.

g Comment on the reliability of your estimates in **e** and **f**.

4 To investigate whether speed cameras have an impact on road safety, data was collected from several cities. The number of speed cameras in operation was recorded for each city, as well as the number of accidents over a 7 day period.

Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32

a Draw a scatter diagram for the data.

b Calculate r.

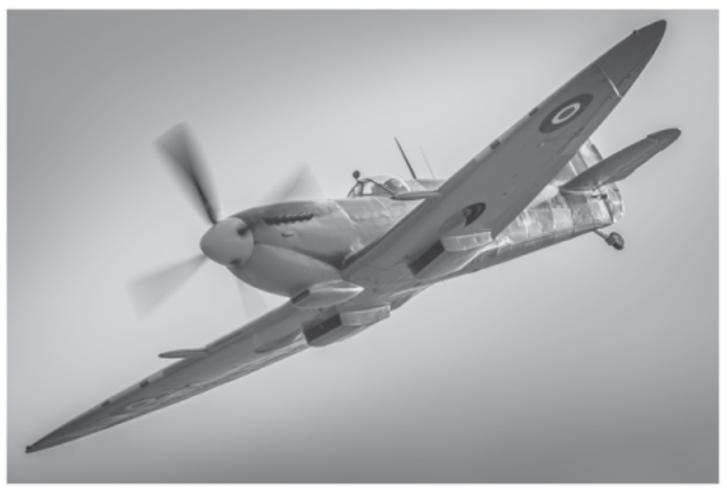
c Describe the relationship between the *number of speed cameras* and the *number of car accidents*.

d Find the equation of the regression line.

e State and interpret the gradient and y-intercept of the regression line.

f Estimate the number of car accidents in a city with 10 speed cameras.

5 The table below contains information about the *maximum speed* and *ceiling* (maximum altitude obtainable) for nineteen World War II fighter planes. The maximum speed is given in $\text{km}\,\text{h}^{-1}$, and the ceiling is given in km.



Maximum speed	*Ceiling*
460	8.84
420	10.06
530	10.97
530	9.906
490	9.448
530	10.36
680	11.73

Maximum speed	*Ceiling*
680	10.66
720	11.27
710	12.64
660	11.12
780	12.80
730	11.88

Maximum speed	*Ceiling*
670	12.49
570	10.66
440	10.51
670	11.58
700	11.73
520	10.36

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the association between *maximum speed* (x) and *ceiling* (y).

d Use technology to find the regression line, and draw the line on your scatter diagram.

e State and interpret the gradient of the regression line.

f Estimate the ceiling for a fighter plane with a maximum speed of 600 $\text{km}\,\text{h}^{-1}$.

g Estimate the maximum speed for a fighter plane with a ceiling of 11 km.

6 A group of children was asked the numbers of hours they spent exercising and watching television each week.

Exercise (x hours per week)	4	1	8	7	10	3	3	2
Television (y hours per week)	12	24	5	9	1	18	11	16

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the correlation between *time exercising* and *time watching television.*

d Find the equation of the regression line, and draw the line on your scatter diagram.

e State and interpret the gradient and y-intercept of the regression line.

f **i** One of the children in the group exercised for 7 hours each week. How much television does this child watch weekly?

ii Use the regression line to predict the amount of television watched each week by a child who exercises for 7 hours each week.

iii Compare your answers to **i** and **ii**.

7 The yield of pumpkins on a farm depends on the quantity of fertiliser used.

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4

a Draw a scatter diagram for the data, and identify the outlier.

b What effect do you think the outlier has on:

i the strength of correlation of the data **ii** the gradient of the regression line?

c Calculate the correlation coefficient:

i with the outlier included **ii** without the outlier.

d Calculate the equation of the regression line:

i with the outlier included **ii** without the outlier.

e If you wish to estimate the yield when 15 g per m^2 of fertiliser is used, which regression line from **d** should be used? Explain your answer.

f Can you explain what may have caused the outlier? Do you think the outlier should be kept when analysing the data?

8 Consider fitting a line through the origin to the bivariate data set $\{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)\}$. The model has the form $y = \beta x$.

a Show that the sum of the squared vertical distances from each point to the line is

$$D = \beta^2 \sum_{i=1}^{n} x_i^2 - 2\beta \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2.$$

b Hence find, in terms of x_i and y_i, the value of β that minimises D.

ACTIVITY 2 ANSCOMBE'S QUARTET

Anscombe's quartet is a collection of four bivariate data sets which have interesting statistical properties.

It was first described in 1973 by the English statistician **Francis Anscombe** (1918 - 2001). At the time, computers were becoming increasingly popular in statistics, as they allowed for more large scale and complex computations to be done within a reasonable amount of time. However, many common statistical packages primarily performed numerical calculations rather than produce graphs. Such output was often limited to those with advanced programming skills.



Francis Anscombe
Photo courtesy of Yale University.

In his 1973 article, Anscombe stressed that:

> "*A computer should make* both *calculations* and *graphs. Both sorts of output should be studied; each will contribute to understanding.*"

The data values for Anscombe's quartet are given in the tables below:

Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

Enter the data into your **graphics calculator** or click on the icon to access the data in the **statistics package**.



What to do:

1 For each data set, use technology to calculate:

a the mean of each variable **b** the population variance of each variable.

Comment on your answers.

2 Find the regression line for each data set. What do you notice?

3 Construct a scatter diagram for each data set, and plot the corresponding regression line on the same set of axes.

4 How do your calculations in **1** and **2** compare to your graphs in **3**? Is a linear model necessarily appropriate for each data set?

5 Why is it important to consider both graphs *and* descriptive statistics when analysing data?

ACTIVITY 3 RESIDUAL PLOTS

In addition to the *correlation coefficient* and the *linearity* of a scatter diagram, we can use a **residual plot** to decide whether a linear model is appropriate. Click on the icon to explore these graphs.



THEORY OF KNOWLEDGE

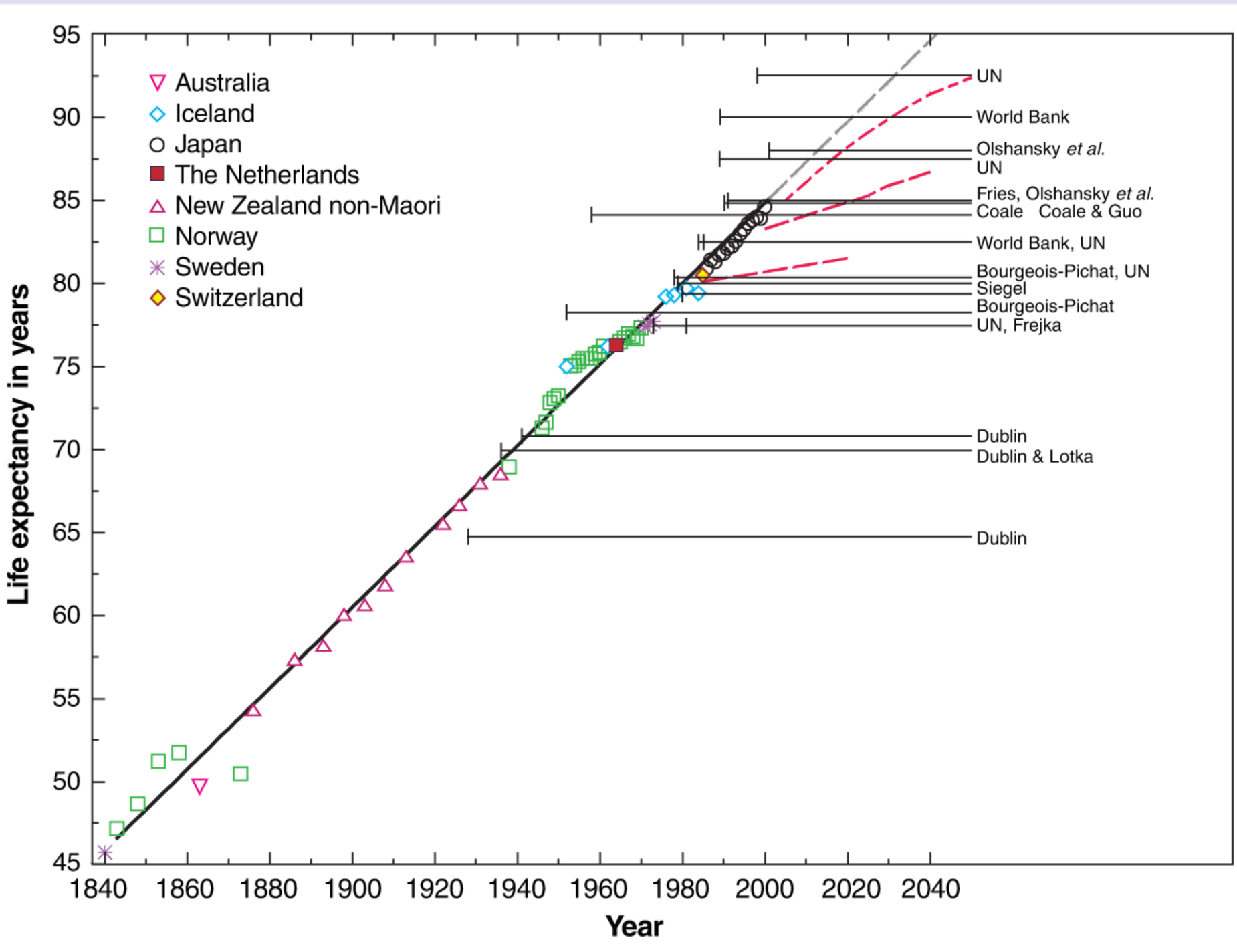
The use of extrapolation for predicting the future leads to debate on many global issues. Even when data shows a strong linear correlation, we need to consider whether it is reasonable for the trend to continue in the long term.

For example, the graph below is based on the article by Oeppen and Vaupel (2002)[1]. It shows female life expectancy from 1840 to the early 2000s, and the country with the highest female life expectancy at each point in time.

Notice that:

- The linear regression trend line is drawn in black, and extrapolated in grey.
- The horizontal black lines show asserted "ceilings" on life expectancy. The vertical line at the left end shows the year of publication.
- The dashed red lines denote projections of female life expectancy in Japan published by the United Nations (UN) in 1986, 1999, and 2001.

Record female life expectancy from 1840 to the present - Oeppen and Vaupel (2002)



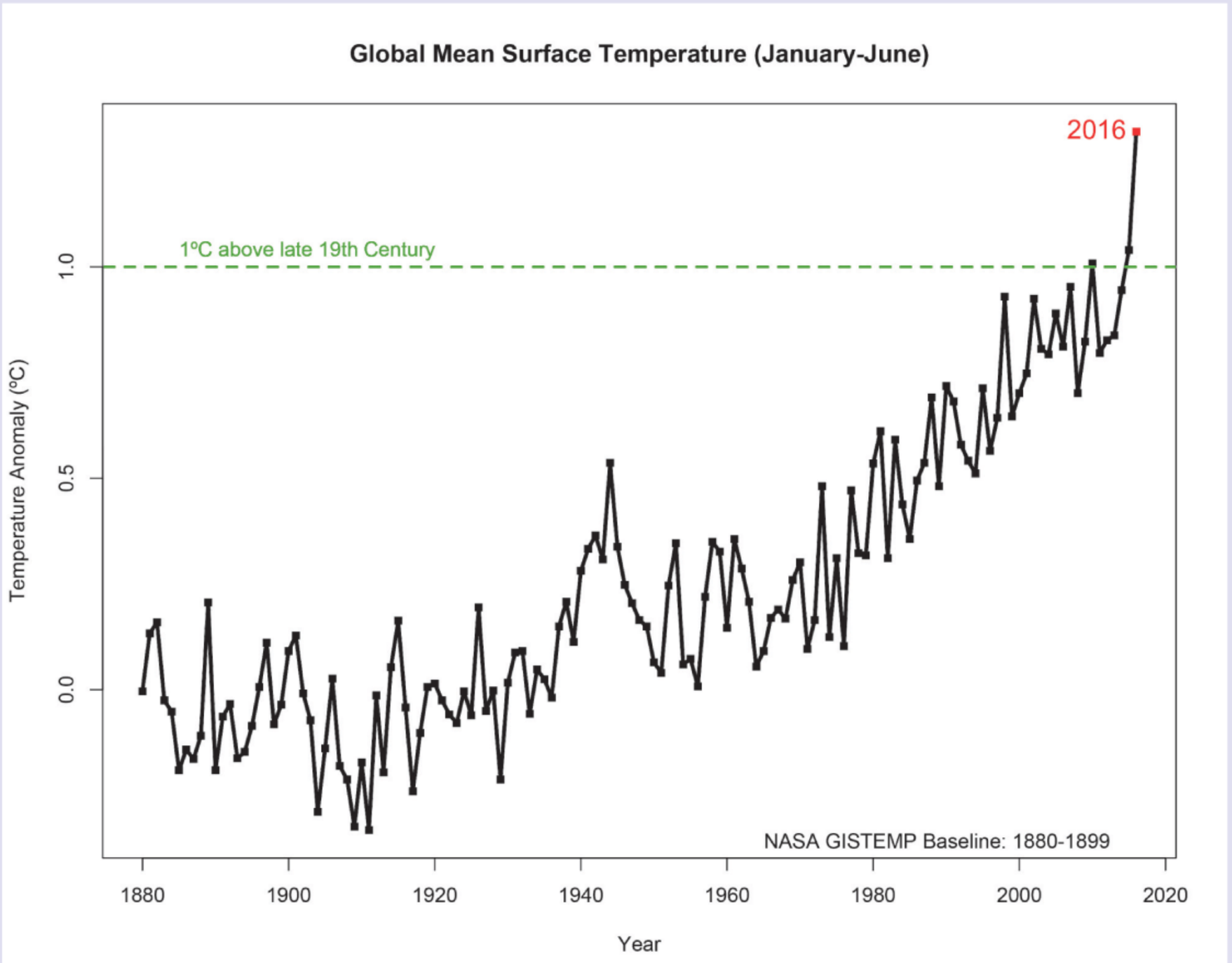
1 Discuss the relationship between the variables.

2 Use the regression line to predict female life expectancy in the year 2100. Do you think this is realistic?

3 Discuss the "ceilings" suggested by publishers over time. Is there evidence to suggest that human life expectancy will approach a limiting "ceiling"?

4 Discuss the accuracy of the UN projections for females in Japan from 1986 to 1999. Is there reason to expect the latest projection will be more reliable?

The graph below shows data from the NASA Goddard Institute for Space Studies[2]. The data for each point is for the first six months of the corresponding year.



5 Discuss the relationship between the variables. Is it reasonable to use a linear model to describe the mean surface temperature of the Earth over time? Is it reasonable to even conclude that the mean surface temperature of the Earth is increasing?

6 How can we predict the mean surface temperature of the Earth in the future?

7 Is mathematical extrapolation valid evidence for dictating environmental policy?

References:

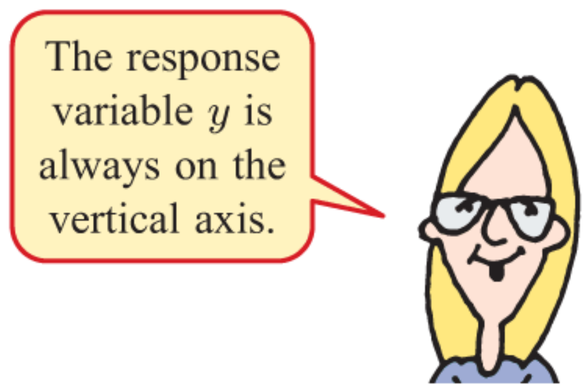
[1] Oeppen and Vaupel, *Broken limits to life expectancy*, Science, **296**, 5570, 1029-1031, 2002.

[2] www.nasa.gov/feature/goddard/2016/climate-trends-continue-to-break-records

E THE REGRESSION LINE OF x AGAINST y

In the previous Section, we saw how linear regression can be used to find a linear model for the response variable y in terms of the explanatory variable x.

In these cases, we generally rely on the x-values being more precise than the y-values. This means that either there is less error involved with their measurement, or that there is naturally less variation associated with the x variable. The distance of each data point from the line is measured in the y-direction, so we are associating all of the "error" with the response variable y. However, in some cases the y-values may be more precisely measured.

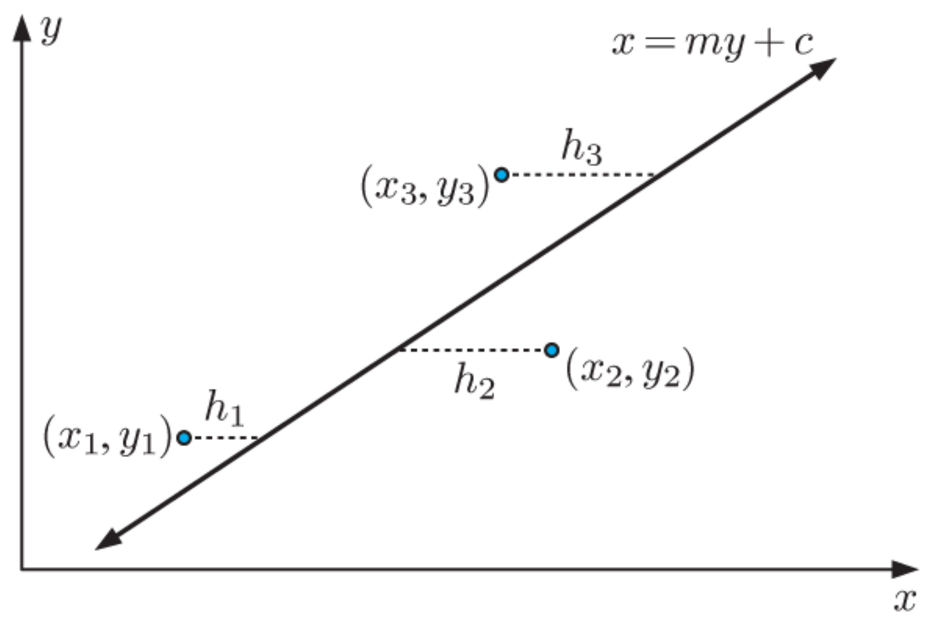


For example:

- When a student studies for a test, their time spent studying x explains their test score y. However, the test score will be more precisely measured than the amount of time spent studying.
- At a breath testing station, police use a breathalyser to estimate the blood alcohol concentration (BAC) of drivers. If the result x is sufficiently high, the driver is required to take a blood test to establish their actual BAC. The blood test result y is a much more precise measurement.

In these scenarios, we consider the regression line of $\boldsymbol{x}$ **against** $\boldsymbol{y}$. This means that we minimise the *horizontal* distances of points from the line, so all of the "error" is associated with the explanatory variable x.

We consider a line of the form $x = my + c$, and choose the constants m and c to minimise $H = \sum_{i=1}^{n} h_i^2$.



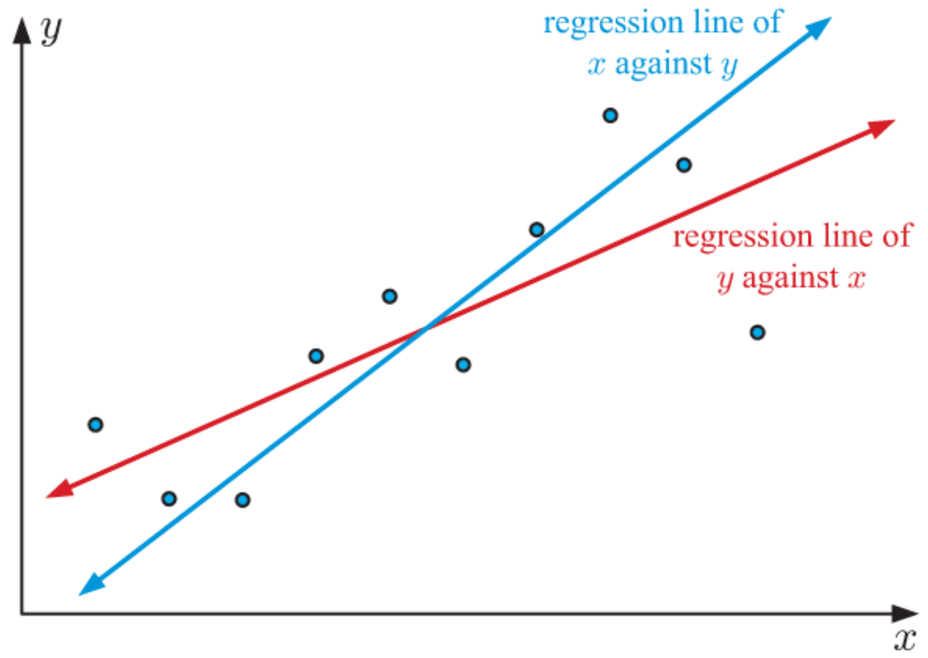
It can be shown that $m = \dfrac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(y_i - \overline{y})^2}$ and $c = \overline{x} - m\overline{y}$.

Rearranging the regression line $x = my + c$ into gradient-intercept form gives $y = \dfrac{1}{m}x - \dfrac{c}{m}$.

The gradient of the rearranged line is $\dfrac{1}{m} = \dfrac{\sum(y_i - \overline{y})^2}{\sum(x_i - \overline{x})(y_i - \overline{y})} \neq a$

and the y-intercept is

$$-\frac{c}{m} = \frac{-(\overline{x} - m\overline{y})}{m}$$
$$= \frac{m\overline{y} - \overline{x}}{m}$$
$$= \overline{y} - \frac{1}{m}\overline{x}$$
$$\neq b \qquad \{\text{since } \frac{1}{m} \neq a\}$$



So, in general, the regression line of x against y is **not** the same as the regression line of y against x.

Example 5

Self Tutor

The data below shows the quantity of feed eaten and the number of eggs laid in a fortnight for 12 Leghorn chickens:

Feed (x kg)	0.57	0.86	0.49	1.37	0.91	0.50
Number of eggs (y)	4	6	3	9	6	3

Feed (x kg)	0.97	1.06	1.00	0.68	1.34	0.94
Number of eggs (y)	6	7	7	4	9	6

a Explain why it would be appropriate to use the regression line of x against y in this case.

b Find the regression line of x against y.

c Use the regression line to estimate:

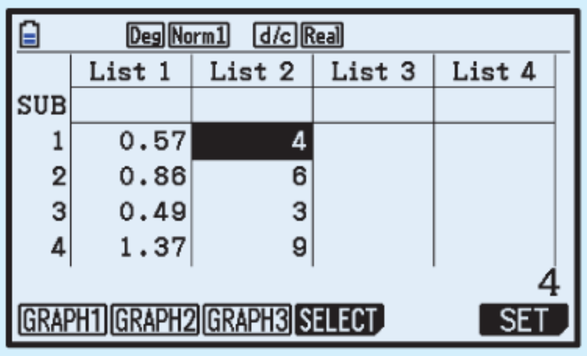
i the quantity of feed required for a Leghorn chicken to lay a dozen eggs

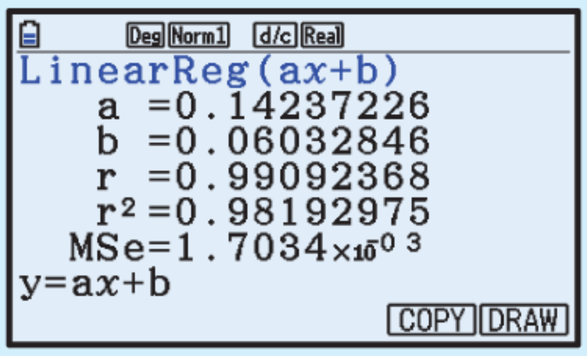
ii the number of eggs laid in a fortnight by a hen that ate 1.2 kg of feed.

a The number of eggs laid can be counted exactly, whereas the quantity of feed usually cannot be measured exactly.

Since the response variable y is more precisely measured than the explanatory variable x, it would be appropriate to use the regression line of x against y in this case.

b





The regression line of x against y is $x \approx 0.142y + 0.0603$ kg.

c **i** When $y = 12$, $x \approx 0.142(12) + 0.0603$

≈ 1.76

We expect a Leghorn chicken to need about 1.76 kg of feed to lay a dozen eggs.

ii When $x = 1.2$, $1.2 \approx 0.142y + 0.0603$

$\therefore \ 1.1397 \approx 0.142y$

$\therefore \ y \approx 8$

We expect a Leghorn chicken eating 1.2 kg of feed to lay about 8 eggs.

EXERCISE 26E

1 The table below shows the amount of time a sample of families spend preparing homemade meals each week, and the amount of money they spend each week on fast food.

Time on homemade meals (x hours)	3.5	6.0	4.0	8.5	7.0	2.5	9.0	7.0	4.0	7.5
Money on fast food (\$$y$)	85	0	60	0	27	100	15	40	59	29

a Explain why it would be appropriate to use the regression line of x against y in this case.

b Find the regression line of x against y.

c Use the regression line to estimate:

i the time spent preparing homemade meals by a family that spends \$45 on fast food

ii the amount of money spent on fast food by a family that spends 5 hours preparing homemade meals.

2 The table below shows how far a group of students live from school, and how long it takes them to travel there each day.

Distance from school (x km)	7.2	4.5	13	1.3	9.9	12.2	19.6	6.1	23.1
Time to travel to school (y min)	17	13	29	2	25	27	41	15	53

a Draw a scatter diagram of the data.

b Which regression line should be used to model the relationship between the variables? Explain your answer.

c Use an appropriate regression line to estimate the travel time of a student who lives 15 km from school.

d Comment on the reliability of your estimate.

3 Eight students swim 200 m breaststroke. Their times y in seconds, and arm lengths x in cm, are shown in the table below:

Length of arm (x cm)	78	73	71	68	76	72	63	69
Breaststroke (y seconds)	123.1	123.7	127.3	132.0	120.8	125.0	140.9	129.0

a Draw a scatter diagram for the data.

b Find the equation of the regression line of:

i y against x **ii** x against y.

c Plot both regression lines on your scatter diagram. What do you notice?

4 Consider the bivariate data set $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$. The equation of the regression line of y against x is $y = ax + b$ and the equation of the regression line of x against y is $x = my + c$.

a Show that $ma = r^2$, where r is Pearson's product-moment correlation coefficient.

b Hence find the condition(s) under which the two regression lines will be the same.

DISCUSSION

Suppose the variables x and y are measured with equally poor precision.

1 Is it necessarily appropriate to choose one regression line over the other in this case?

2 How could we take the poor precision of *both* variables into account when formulating the linear regression model? Consider the "distance" that we minimise when we perform the linear regression.

3 In some situations, the variables depend equally on each other. In these cases we say that the variables are **co-dependent**, and the variables can be placed on either axis of the scatter diagram. Is it sensible to use a regression line to describe the relationship between co-dependent variables?

THEORY OF KNOWLEDGE

Since the 1970s and 1980s, wage discrimination between men and women has been a topic of debate. During that time, Conway and Roberts[1] published a study which used linear regression to show that on average, women with the same qualifications as men were paid less. This would seem to imply that given the same salary, women would be more qualified. However, when the regression was applied the other way, the opposite conclusion was observed.

1 Can you explain why this occurred?

2 Are these two questions necessarily equivalent?
- "Given the same qualifications, do men and women earn the same wage?"
- "Given the same wage, do men and women have the same qualifications?"

3 Is it necessary to consider both regression lines in order to conclude whether discrimination has occurred?

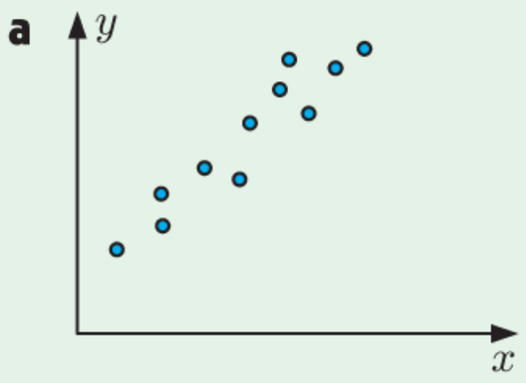
4 Should people be paid according to their qualifications, the job they do, or their capability in doing that job?

[1] Conway, Delores A. and Harry V. Roberts (1983). "Reverse Regression, Fairness, and Employment Discrimination". In: *Journal of Business & Economic Statistics* 1.1, pp. 75-85.

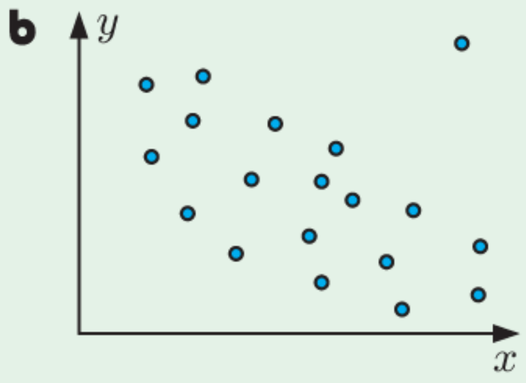
REVIEW SET 26A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.

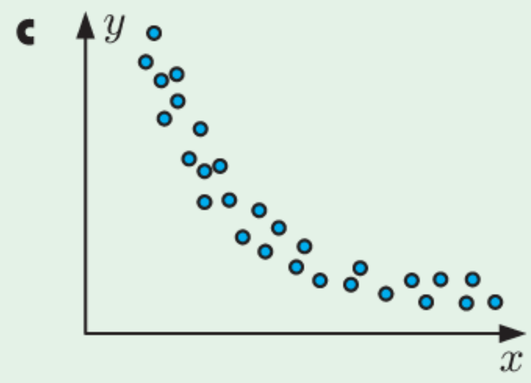
a



b



c



2 Kerry wants to investigate the relationship between the *water bill* and the *electricity bill* for the houses in her neighbourhood.

a Do you think the correlation between the variables is likely to be positive or negative? Explain your answer.

b Is there a causal relationship between the variables? Justify your answer.

3 Consider the data set alongside.

x	2	5	7	10	12	15
y	18	10	13	7	7	5

a Draw a scatter diagram for the data.

b Does the correlation between the variables appear to be positive or negative?

c Calculate Pearson's product-moment correlation coefficient r.

4 The table below shows the ticket and beverage sales for each day of a 12 day music festival:

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient r.

c Describe the correlation between *ticket sales* and *beverage sales*.

5 A clothing store recorded the length of time customers were in the store and the amount they spent.

Time (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
Money (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a Find the mean for each variable.

b Draw a scatter diagram for the data. Plot the mean point, and draw a line of best fit by eye.

c Describe the relationship between *time in the store* and the *money spent*.

6 The ages and heights of children at a playground are given below:

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157

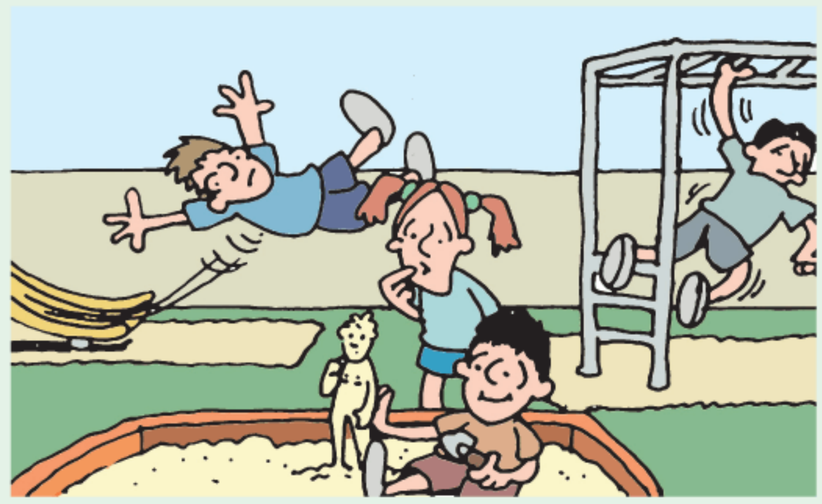
a Draw a scatter diagram for the data.

b Use technology to find the regression line of y against x.

c State and interpret the gradient of the regression line.

d Use the regression line to predict the height of a 5 year old child.

e Based on the given data, at what age would you expect a child to reach 140 cm in height?



7 Tomatoes are sprayed with a pesticide-fertiliser mix. The table below shows the *yield of tomatoes* per bush for various *spray concentrations*.

Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes per bush (y)	67	90	103	120	124	150	82

a Draw a scatter diagram to display the data.

b Determine the value of r and interpret your answer.

c Is there an outlier present that is affecting the correlation?

d The outlier was found to be a recording error. Remove the outlier from the data set, and recalculate r. Is it reasonable to now fit a linear model?

e Determine the equation of the regression line of y against x.

f State and interpret the gradient and y-intercept of the regression line.

g Use your line to estimate:

i the yield if the spray concentration is 7 mL per L

ii the spray concentration if the yield is 200 tomatoes per bush.

h Comment on the reliability of your estimates in **g**.

8 Thomas rode his bicycle for an hour each day for eleven days. He recorded the number of kilometres he rode, along with his estimate of the temperature that day:

Temperature (T °C)	23	24	25	27	28	20	22	21	25	26	24
Distance (d km)	26.5	26.7	24.4	22.8	23.5	32.6	28.7	29.4	24.2	23.2	29.7

a Draw a scatter diagram for the data.

b Explain why it would be appropriate to use the regression line of T against d in this case.

c Find the equation of the regression line of T against d.

d How far would you expect Thomas to ride on a 30°C day?

REVIEW SET 26B

1 For each pair of variables, discuss whether the correlation between the variables is likely to be positive or negative, and whether a causal relationship exists between the variables:

a *price of tickets* and *number of tickets sold*

b *ice cream sales* and *number of shark attacks*.

2 A group of students is comparing their results for a Mathematics test and an Art project:

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66

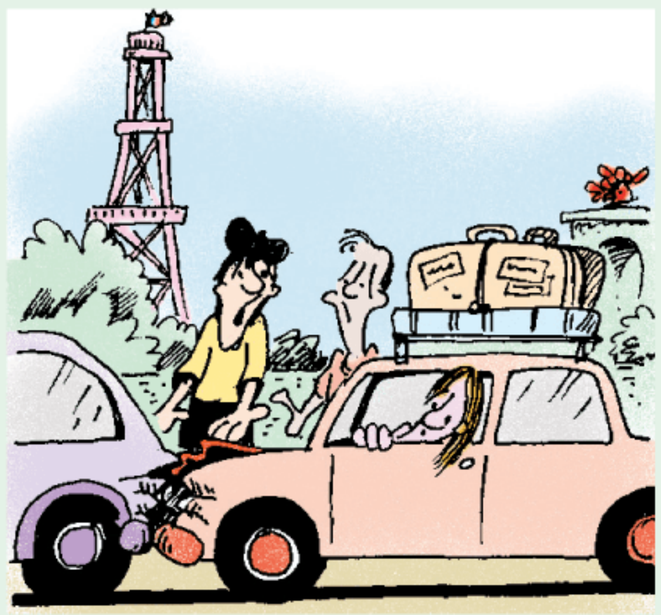
a Construct a scatter diagram for the data.

b Describe the relationship between the Mathematics and Art marks.

c Calculate the correlation coefficient r between the variables.

3 Safety authorities advise drivers to travel three seconds behind the car in front of them. This gives the driver a greater chance of avoiding a collision if the car in front has to brake quickly or is itself involved in an accident.

A test was carried out to find out how long it would take a driver to bring a car to rest from the time a red light was flashed. The following results were recorded for a particular driver in the same car under the same test conditions.



Speed (v km h^{-1})	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a Find the mean point $(\overline{v}, \overline{t})$.

b Draw a scatter diagram of the data. Add the mean point and draw a line of best fit by eye.

c Hence estimate the stopping time for a speed of:

i 55 km h^{-1} **ii** 110 km h^{-1}

d Which of your estimates in **c** is more likely to be reliable?

4 Consider the data set alongside.

x	2	3	6	8	13	16
y	12	17	32	41	50	61

a Calculate the correlation coefficient r.

b Find the regression line of y against x.

c Estimate the value of y when $x = 10$.

5 A craft shop sells canvasses in a variety of sizes. The table below shows the area and price of each canvas type.

Area (x cm^2)	100	225	300	625	850	900
Price (£y)	6	12	13	24	30	35

a Construct a scatter diagram for the data.

b Calculate the correlation coefficient r.

c Describe the correlation between *area* and *price*.

d Find the regression line of y against x, then draw the line on the scatter diagram.

e Estimate the price of a canvas with area 1200 cm^2. Discuss whether your estimate is likely to be reliable.

6 A drinks vendor varies the price of Supa-fizz on a daily basis. He records the number of sales of the drink as shown:

Price ($p)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431

a Produce a scatter diagram for the data.

b Are there any outliers? If so, should they be included in the analysis?

c Calculate the equation of the regression line of s against p.

d State and interpret the gradient of the regression line.

e Do you think the regression line would give a reliable prediction of sales if Supa-fizz was priced at 50 cents? Explain your answer.

7 Eight identical flower beds contain petunias. The different beds were watered different numbers of times each week, and the number of flowers each bed produced was recorded in the table below:

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250

a Draw a scatter diagram for the data, and describe the correlation between the variables.

b Find the equation of the regression line of f against n.

c Is it likely that a causal relationship exists between these two variables? Explain your answer.

d Plot the regression line on the scatter diagram.

e Violet has two beds of petunias. She waters one of the beds 5 times a fortnight and the other 10 times a week.

i How many flowers can she expect from each bed?

ii Discuss which of your estimates is likely to be more reliable.



8 An archer shoots 10 arrows at a target from each of 12 different positions. The table below shows the distance of each position from the target, and how many shots were successful.

Distance from target (x m)	20	25	15	35	40	55	30	45	60	80	65	70
Hits (y)	9	8	8	8	7	6	9	7	4	2	3	3

a Draw a scatter diagram for the data.

b Explain why it would be appropriate to use the regression line of x against y in this case.

c Find the equation of the regression line of x against y.

d Predict the number of hits out of 10 shots fired at a distance of 100 m. Discuss the reliability of your estimate.

Chapter 27

Discrete random variables

Contents:

A Random variables
B Discrete probability distributions
C Expectation
D Variance and standard deviation
E Properties of $aX + b$
F The binomial distribution
G Using technology to find binomial probabilities
H The mean and standard deviation of a binomial distribution



OPENING PROBLEM

In a sideshow game, players have a 50% chance of winning a prize worth \$2, \$5, \$10, or \$20. The probabilities of winning these prizes are given in the table below.

Prize value	\$0	\$2	\$5	\$10	\$20
Probability	0.5	0.35	0.1	0.04	0.01



Things to think about:

a What is the sum of the probabilities in the table? Why must this be the answer?

b What is the most likely outcome from playing the game?

c What is the *average* result from playing the game?

d What is a *fair* price for playing the game?

Many variables in the world around us depend on chance events. Examples of such variables are:

- the number of players in your football team who will score a goal in the next match
- the time it will take you to travel to school tomorrow
- the sum of the numbers when three dice are rolled.

Because of the element of chance in these variables, we cannot predict the exact value they will take when next measured. However, we can often determine the *possible values* the variable can take, and we can assign to each possible value the **probability** of it occurring.

In this Chapter we will extend the ideas of probability we have studied to model the random variation or **distribution** of numerical variables.

A RANDOM VARIABLES

A **random variable** uses numbers to describe the possible outcomes which could result from a random experiment.

A random variable is often represented by a capital letter such as X.

Random variables can be either **discrete** or **continuous**.

A **discrete random variable** X has a set of distinct possible values.

For example, X could be:

- the number of wickets a bowler takes in an innings of cricket, so X could take the values 0, 1, 2, ..., 10
- the number of defective light bulbs in a purchase order of 50, so X could take the values 0, 1, 2, ..., 50.

To determine the value of a discrete random variable, we need to **count**.

A **continuous random variable** X can take any value within some interval on the number line.

For example, X could be:

- the heights of men, which lie in the interval $50 \text{ cm} < X < 250 \text{ cm}$
- the volume of water in a tank, which could lie in the interval $0 \text{ m}^3 < X < 100 \text{ m}^3$.

To determine the value of a continuous random variable, we need to **measure**.

DISCRETE RANDOM VARIABLES

In this Chapter, we will focus on how discrete random variables and their distributions arise.

Example 1 **Self Tutor**

A supermarket has three checkouts A, B, and C. A government inspector checks the weighing scales for accuracy at each checkout. The random variable X is the number of accurate weighing scales at the supermarket.

a List the possible outcomes and the corresponding values of X.

b What value(s) of X correspond to there being:

i one accurate scale ii at least one accurate scale?

a Possible outcomes:

A	B	C	X
✗	✗	✗	0
✓	✗	✗	1
✗	✓	✗	1
✗	✗	✓	1
✗	✓	✓	2
✓	✗	✓	2
✓	✓	✗	2
✓	✓	✓	3

b i $X = 1$

ii $X = 1$, 2, or 3

EXERCISE 27A

1 Classify each random variable as continuous or discrete:

a the quantity of fat in a sausage
b the mark out of 50 for a geography test
c the weight of a Year 12 student
d the volume of water in a cup of coffee
e the number of trout in a lake
f the number of hairs on a cat
g the length of a horse's mane
h the height of a skyscraper.

2 For each scenario:

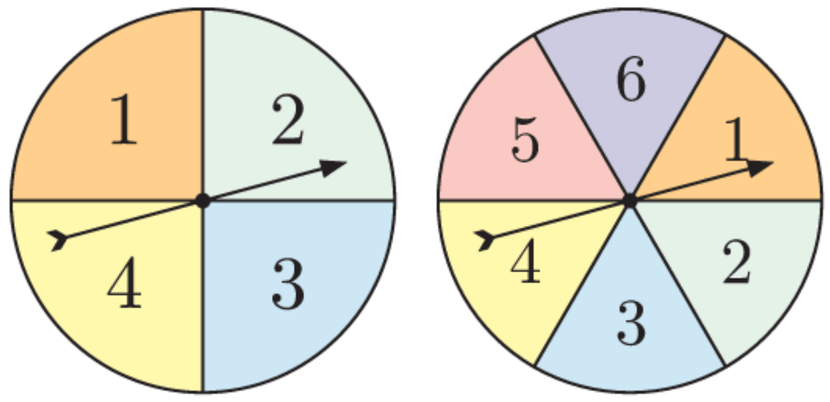
i Identify the random variable being considered.
ii State whether the variable is continuous or discrete.
iii Give possible values for the random variable.

a To measure the rainfall over a 24-hour period in Singapore, water is collected in a rain gauge.

b To investigate the stopping distance for a tyre with a new tread pattern, a braking experiment is carried out.

c To check the reliability of a new type of light switch, switches are repeatedly turned off and on until they fail.

3 Suppose the spinners alongside are spun, and X is the sum of the numbers.



a Explain why X is a discrete random variable.

b State the possible values of X.

4 In the finals series of a baseball championship, the first team to win 4 games wins the championship. Let X represent the number of games played in the finals series.



a State the possible values of X.

b What value(s) of X correspond to the series lasting:

i exactly 5 games **ii** at least 6 games?

5 A supermarket has four checkouts A, B, C, and D. Management checks the weighing devices at each checkout. The random variable X is the number of weighing devices which are accurate.

a What values can X have?

b List the possible outcomes and the corresponding values of X.

c What value(s) of X correspond to:

i exactly two devices being accurate **ii** at least two devices being accurate?

6 Suppose three coins are tossed simultaneously. Let X be the number of heads that result.

a State the possible values of X.

b List the possible outcomes and the corresponding values of X.

c Are the possible values of X equally likely to occur? Explain your answer.

B DISCRETE PROBABILITY DISTRIBUTIONS

For any random variable, there is a corresponding **probability distribution** which describes the probability that the variable will take a particular value.

The probability that the variable X takes value x is denoted $\mathrm{P}(X = x)$.

If X is a random variable with possible values $\{x_1, x_2, x_3, ..., x_n\}$ and corresponding probabilities $\{p_1, p_2, p_3, ..., p_n\}$ such that $\mathrm{P}(X = x_i) = p_i$, $i = 1, ..., n$, then:

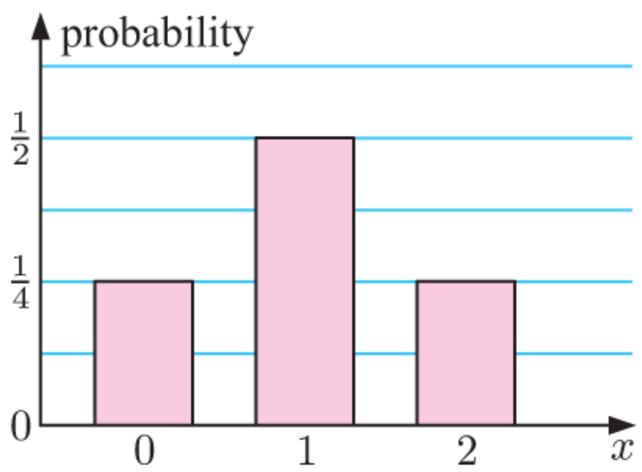
- $0 \leqslant p_i \leqslant 1$ for all $i = 1, ..., n$
- $\sum_{i=1}^{n} p_i = p_1 + p_2 + p_3 + ... + p_n = 1$
- $\{p_1, ..., p_n\}$ describes the **probability distribution** of X.

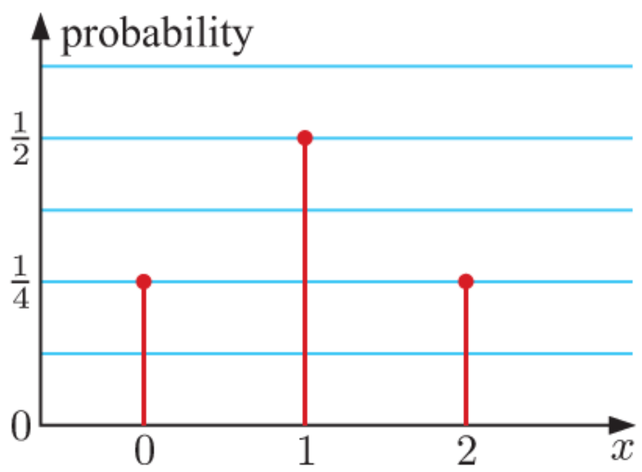
For example, suppose X is the number of heads obtained when 2 coins are tossed. The possible values for X are $\{0, 1, 2\}$ with corresponding probabilities $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$. We see that $0 \leqslant p_i \leqslant 1$ for each value of i, and that the probabilities add up to 1.



We can display this probability distribution in a **table** or a **graph**.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$





UNIFORM DISCRETE RANDOM VARIABLES

If the possible values $x_1, x_2, ..., x_n$ of a discrete random variable X all have the same probability $\frac{1}{n}$ of occurring, then X is a **uniform discrete random variable**.

An example of a uniform discrete random variable is the result X when a die is rolled. The possible values of X are 1, 2, 3, 4, 5, and 6, and each value has probability $\frac{1}{6}$ of occurring.



By contrast, if two dice are rolled, the sum of the resulting numbers Y is *not* a uniform discrete random variable.

THE MODE AND MEDIAN

The **mode** of a discrete probability distribution is the most frequently occurring value of the variable. This is the data value x_i whose probability p_i is the highest.

The **median** of the distribution corresponds to the 50th percentile. If the possible values $\{x_1, x_2, ..., x_n\}$ are listed in ascending order, the median is the value x_j when the cumulative sum $p_1 + p_2 + ... + p_j$ reaches 0.5.

Example 2 **Self Tutor**

A magazine store recorded the number of magazines purchased by its customers in one week. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four, and 5% purchased five. Let X be the number of magazines sold to a randomly selected customer.

a State the possible values of X.

b Construct a probability table for X.

c Graph the probability distribution.

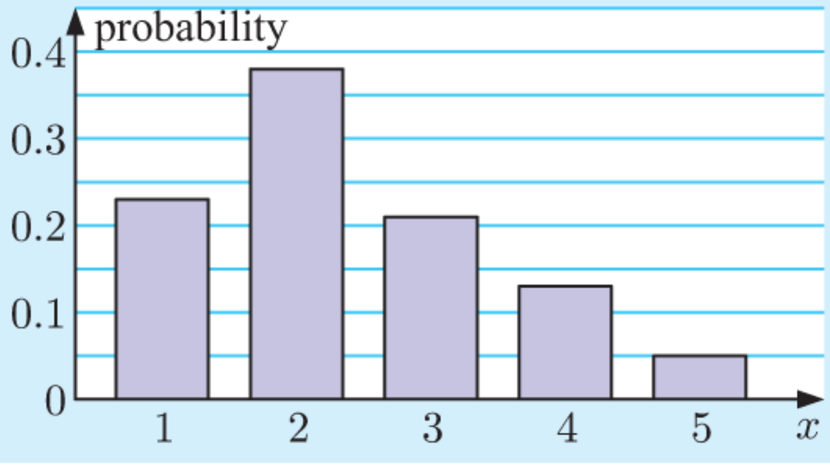
d Find the mode and median of X.

a $X = 1, 2, 3, 4,$ or 5

b

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.23	0.38	0.21	0.13	0.05

c



d Customers are most likely to buy 2 magazines, so this is the mode of X.

Now $p_1 = 0.23$

and $p_1 + p_2 = 0.23 + 0.38 = 0.61$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2 magazines.

We can also describe the probability distribution of a discrete random variable using a **probability mass function** $P(x) = \text{P}(X = x)$. The domain of the probability mass function is the set of possible values of the variable, and the range is the set of values in the probability distribution.

Example 3 **Self Tutor**

Show that $P(x) = \dfrac{x^2 + 1}{34}$, $x = 1, 2, 3, 4$ is a valid probability mass function.

$P(1) = \frac{2}{34}$, $P(2) = \frac{5}{34}$, $P(3) = \frac{10}{34}$, $P(4) = \frac{17}{34}$

All of these values obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum_{i=1}^{n} P(x_i) = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

EXERCISE 27B

1 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3	4
$\text{P}(X = x)$	0.2	0.4	0.15	0.25

ii

x	0	1	2	3
$\text{P}(X = x)$	0.2	0.3	0.4	0.2

iii

x	0	1	2	3	4
$\text{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

iv

x	2	3	4	5
$\text{P}(X = x)$	0.3	0.4	0.5	-0.2

b For which of the probability distributions in **a** is X a uniform random variable?

2 Find k in each of these probability distributions:

a

x	0	1	2
$\text{P}(X = x)$	0.3	k	0.5

b

x	0	1	2	3
$\text{P}(X = x)$	k	$2k$	$3k$	k

3 Consider the probability distribution alongside.

x	0	1	2	3
$\text{P}(X = x)$	0.1	0.25	0.45	a

a Find the value of a.

b Is X a uniform discrete random variable? Explain your answer.

c State the mode of the distribution.

d Find $\text{P}(X \geqslant 2)$.

4 The probability distribution for Jason scoring X home runs in each game during his baseball career is given in the following table:

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a State the value of $P(2)$.

b Find the value of a. Explain what this number means.

c Find the value of $P(1) + P(2) + P(3) + P(4) + P(5)$. Explain what this means.

d Draw a graph of $P(x)$ against x.

e Find the mode and median of the distribution.

5 A policeman inspected the safety of tyres on cars passing through a checkpoint. The number of tyres X which needed replacing on each car followed the probability distribution below.

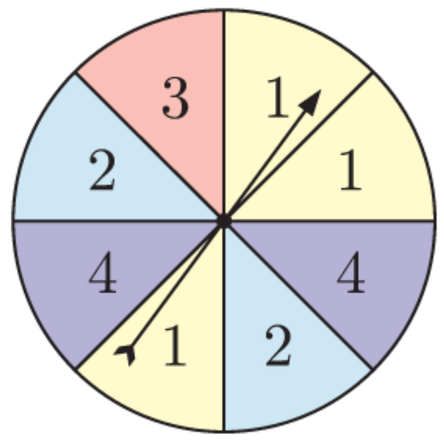
x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.68	0.2	0.06	k	0.02

a Find the value of k.

b Find the mode of the distribution.

c Find $\mathrm{P}(X > 1)$, and interpret this value.

6 Let X be the result when the spinner alongside is spun.



a Display the probability distribution of X in a table.

b Graph the probability distribution.

c Find the mode and median of the distribution.

d Find $\mathrm{P}(X \leqslant 3)$.

7 100 people were surveyed about the number of bedrooms in their house. 24 people had one bedroom, 35 people had two bedrooms, 27 people had three bedrooms, and 14 people had four bedrooms. Let X be the number of bedrooms a randomly selected person has in their house.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

8 A group of 25 basketballers took shots from the free throw line until they scored a goal. 12 of the players only needed one shot, 7 players took two shots, 2 players took three shots, and the rest took four shots. Let X be the number of shots a randomly selected player needs to score a goal.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

9 Show that the following are valid probability mass functions:

a $P(x) = \dfrac{x+1}{10}$ for $x = 0, 1, 2, 3$

b $P(x) = \dfrac{6}{11x}$ for $x = 1, 2, 3$.

10 Find k for the following probability mass functions:

a $P(x) = k(x+2)$ for $x = 1, 2, 3$

b $P(x) = \dfrac{k}{x+1}$ for $x = 0, 1, 2, 3$.

11 A discrete random variable X has the probability mass function $P(x) = \dfrac{4x - x^2}{a}$ for $x = 0, 1, 2, 3$.

a Find the value of a.

b Find $\mathrm{P}(X = 1)$.

c Find the mode of the distribution.

12 The discrete random variable X has probability mass function $P(x) = a\left(\frac{1}{3}\right)^{x-1}$, $x = 1, 2, 3, \ldots$.

a Find the sum of the infinite geometric series $\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1}$.

b Hence find the value of a.

13 A discrete random variable X has probability mass function $P(x) = a\left(\frac{2}{5}\right)^{x}$, $x = 0, 1, 2, 3, \ldots$.
Find the value of a.

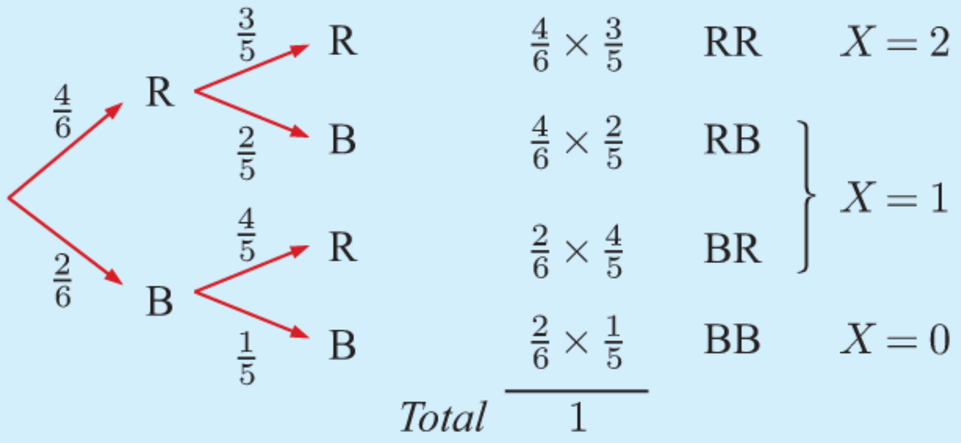
Example 4 — Self Tutor

Two marbles are randomly selected without replacement from a bag containing 4 red and 2 blue marbles. Let X denote the number of red marbles selected.

a Find the probability distribution of X.

b Illustrate the probability distribution using a graph.

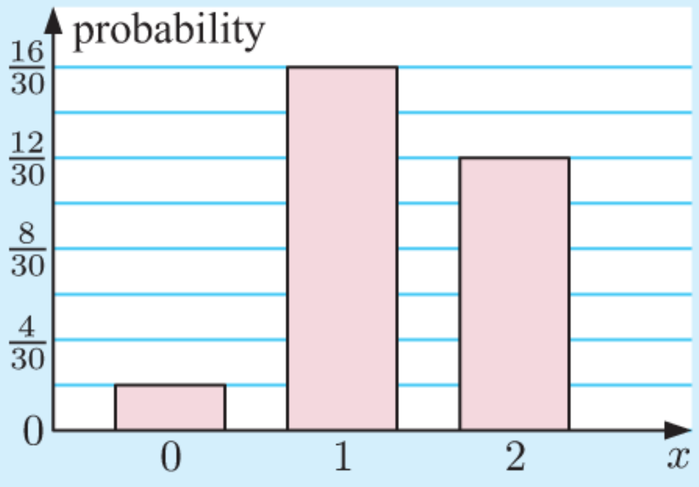
a 1st selection 2nd selection



Total 1

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{2}{30}$	$\frac{16}{30}$	$\frac{12}{30}$

b



14 Two tickets are randomly selected without replacement from a bag containing 5 blue and 3 green tickets. Let X denote the number of blue tickets selected.
Find the probability distribution of X.

15 A hat contains 2 red balls and 2 green balls. Balls are randomly selected without replacement until a green ball is selected. Let X denote the total number of balls selected.

a State the possible values of X.

b Find the probability distribution of X.

16 A pair of dice is rolled. Let S denote the sum of the top faces.

a Display the possible results in a table.

b Find the probability distribution of S.

c Find the mode of the distribution.

d Find $\mathrm{P}(S \geqslant 10)$.

17 **a** We have seen that the Maclaurin series expansion for the exponential function e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Use this expansion to evaluate $\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!}$.

b Suppose X is the number of cars that pass a shop between 3:00 pm and 3:03 pm. The probability distribution for X is given by

$$\mathrm{P}(X = x) = \frac{(0.2)^x e^{-0.2}}{x!} \quad \text{where} \quad x = 0,\ 1,\ 2,\ 3,\ \ldots.$$

i Explain why this is a valid probability distribution.

ii Evaluate $\mathrm{P}(X = 0)$, $\mathrm{P}(X = 1)$, and $\mathrm{P}(X = 2)$.

iii Find the probability that at least three cars will pass the shop in the given period.

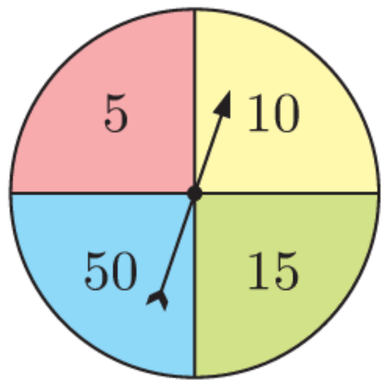
C EXPECTATION

We have already seen how probabilities can be used to predict the number of times we *expect* an event to occur when an experiment is repeated many times.

We can consider the **expected value** or **expectation** of a random variable in a similar way.

EXPECTED VALUE

When the spinner alongside is spun, players are awarded the resulting number of points. On average, how many points can we *expect* to be awarded per spin?



For every 4 spins, we would expect that on average, each score will be spun once. The total score in this case would be $50 + 15 + 10 + 5 = 80$, which is an average of $\frac{80}{4} = 20$ points per spin.

Alternatively, we can write the average score as

$$\begin{aligned} &\tfrac{1}{4}(50 + 15 + 10 + 5) \\ &= \tfrac{1}{4} \times 50 + \tfrac{1}{4} \times 15 + \tfrac{1}{4} \times 10 + \tfrac{1}{4} \times 5 \\ &= 20 \text{ points.} \end{aligned}$$

It is impossible to score 20 points on any given spin, but over many spins we *expect* an average of 20 points per spin.



Notice that each score is multiplied by its probability of occurring.

For a random variable X with possible values $x_1, x_2, x_3, \ldots, x_n$ and associated probabilities $p_1, p_2, \ldots, p_n$, the **expected value** of X is

$$\begin{aligned} \mathbf{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= x_1 p_1 + x_2 p_2 + \ldots + x_n p_n \end{aligned}$$

$\mathrm{E}(X)$ is the **mean** of the probability distribution of X. It is sometimes denoted μ.

Example 5 **Self Tutor**

Consider the magazine store from **Example 2**.

Find the expected number of magazines bought by each customer. Explain what this represents.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

In **Example 2** we found the mode and median for this distribution.



$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05) \\ &= 2.39 \end{aligned}$$

In the long term, the average number of magazines purchased per customer is 2.39 .

EXERCISE 27C.1

1 Find $\mathrm{E}(X)$ for the following probability distributions:

a

x_i	1	2	3
p_i	0.4	0.5	0.1

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

2 Consider the probability distribution alongside.

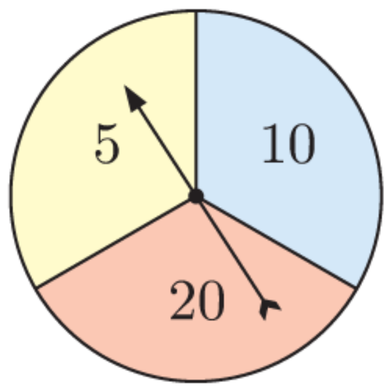
x	1	3	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

a Find the value of a.

b Find the mode of the distribution.

c Find the mean μ of the distribution.

3 When the spinner alongside is spun, players are awarded the resulting number of points. In the long term, how many points can we expect to be awarded per spin?



4 When Ernie goes fishing, he catches 0, 1, 2, or 3 fish, with the probabilities shown.

On average, how many fish would you expect Ernie to catch on a fishing trip?

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

5 Each time Pam visits the library, she borrows either 1, 2, 3, 4, or 5 books, with the probabilities shown.

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

a Find the value of a.

b Find the mode of the distribution.

c On average, how many books does Pam borrow per visit?

6 Lachlan randomly selects a ball from a bag containing 5 red balls, 2 green balls, and 1 white ball. He is then allowed to take a particular number of lollies from a jar according to the colour of the ball.

Find the average number of lollies that Lachlan can expect to receive.

Colour	*Number of lollies*
Red	4
Green	6
White	10

7 When ten-pin bowler Jenna bowls her first bowl of a frame, she always knocks down at least 8 pins. $\frac{1}{3}$ of the time she knocks down 8 pins, and $\frac{2}{5}$ of the time she knocks down 9 pins.

a Find the probability that she knocks down all 10 pins on the first bowl.

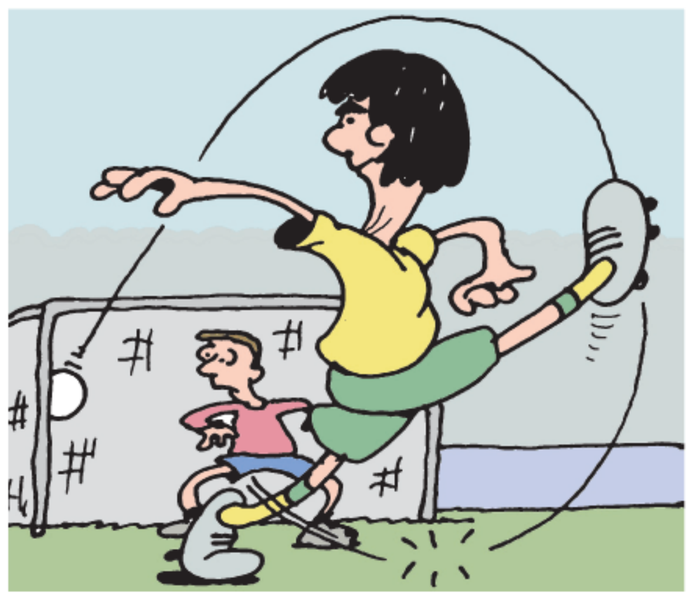
b On average, how many pins does Jenna knock down with her first bowl?

8 Given that $\mathrm{E}(X) = 2.5$, find a and b.

x	1	2	3	4
$\mathrm{P}(X = x)$	0.3	a	b	0.2

9 When Brad's soccer team plays an offensive strategy, they win 30% of the time and lose 55% of the time. When they play a defensive strategy, they win 20% of the time and lose 30% of the time.

On the league table, teams are awarded 3 points for a win, 1 point for a draw, and no points for a loss.



a Find the probability that Brad's team will draw a match under each strategy.

b Calculate the expected number of points per game under each strategy.

c In the long run, is it better for the team to play an offensive or defensive strategy?

d Should the strategy change if teams are awarded 4 points instead of 3 points for a win?

10 Every Thursday, Zoe meets her friends in the city for dinner. There are two car parks nearby, the costs for which are shown below:

Car park A

Time	*Cost*
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	*Cost*
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

Zoe's dinner takes 1 - 2 hours 20% of the time, 2 - 3 hours 70% of the time, and 3 - 4 hours 10% of the time.

a Which car park is cheapest for Zoe if she stays:

i 1 - 2 hours **ii** 2 - 3 hours **iii** 3 - 4 hours?

b When Zoe parks her car, she does not know how long she will stay. Which car park do you recommend for her? Explain your answer.

11 An insurance policy covers a \$20 000 sapphire ring against theft and loss. If the ring is stolen then the insurance company will pay the policy owner in full. If the ring is lost then they will pay the owner \$8000. From past experience, the insurance company knows that the probability of theft is 0.0025, and the probability of loss is 0.03. How much should the company charge to cover the ring in order that their expected return is \$100?



FAIR GAMES

In gambling, the **expected gain** of the player from each game is the expected return or payout from the game, less the amount it cost them to play.

A game is said to be **fair** if the expected gain is zero.

> Suppose X represents the gain of a player from each game.
> The game is **fair** if $\mathrm{E}(X) = 0$.

DISCUSSION

Would you expect a gambling game to be "fair"?

Example 6 **Self Tutor**

In a game of chance, a player spins a square spinner labelled 1, 2, 3, 4. The player wins an amount of money according to the table alongside.

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a Find the expected return for one spin of the spinner.

b Find the expected *gain* of the player if it costs \$5 to play each game.

c Discuss whether you would recommend playing this game.

a Let Y denote the return or payout from each spin.

Each outcome is equally likely, so the probability for each outcome is $\frac{1}{4}$

$\therefore$ the expected return $= \mathrm{E}(Y) = \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4.$

b Let X denote the *gain* of the player from each game.

Since it costs \$5 to play the game, the expected gain $= \mathrm{E}(X) = \mathrm{E}(Y) - \$5$

$= \$4 - \$5$

$= -\$1$

c Since $\mathrm{E}(X) \neq 0$, the game is not fair. In particular, since $\mathrm{E}(X) = -\$1$, we expect the player to lose \$1 on average with each spin. We would not recommend that a person play the game.

EXERCISE 27C.2

1 A dice game costs \$2 to play. If an odd number is rolled, the player receives \$3. If an even number is rolled, the player receives \$1.

Determine whether the game is fair.

2 A man rolls a regular six-sided die. He wins the number of dollars shown on the uppermost face.

a Find the expected return from one roll of the die.

b Find the expected *gain* if it costs \$4 to play the game.

c Would you advise the man to play many games?

3 A roulette wheel has 18 red numbers, 18 black numbers, and 1 green number. Each number has an equal chance of occurring. I place a bet of \$2 on red. If a red is spun, I receive my \$2 back plus another \$2. Otherwise I lose my \$2.



a Calculate the expected gain from this bet.

b If the same bet is made 100 times, what is the expected result?

4 A person pays \$5 to play a game with a pair of coins. If two heads appear then \$10 is won. If a head and a tail appear then \$3 is won. If two tails appear then \$1 is won.
Let X be the gain of the person from each game. Find the expected value of X.

5 In a carnival game, a player randomly selects a ticket from a box of tickets numbered 1 to 20. If the selected number is a multiple of 3, the player wins 5 tokens. If the selected number is a multiple of 10 the player wins 10 tokens.

a Calculate the probability of a player winning: **i** 5 tokens **ii** 10 tokens.

b Let X be the number of tokens won from playing this game. Find the expected value of X.

c If it costs 3 tokens to play the game, would you recommend playing the game many times? Explain your answer.

6 A person selects a disc from a bag containing 10 black discs, 4 blue discs, and 1 gold disc. They win \$1 for a black disc, \$5 for a blue disc, and \$20 for the gold disc. The game costs \$4 to play.

a Calculate the expected gain for this game, and hence show that the game is not fair.

b To make the game fair, the prize money for selecting the gold disc is increased. Find the new prize money for selecting the gold disc.

7 At a charity event there is a money-raising game involving a pair of ordinary dice. The game costs \$$a$ to play. When the two dice are rolled, their sum is described by the variable X. A sum which is less than 4 or between 7 and 9 inclusive gives a return of $\$\frac{a}{3}$. A result between 4 and 6 inclusive gives a return of \$7. A result of 10 or more gives a return of \$21.

a Determine $P(X \leqslant 3)$, $P(4 \leqslant X \leqslant 6)$, $P(7 \leqslant X \leqslant 9)$, and $P(X \geqslant 10)$.

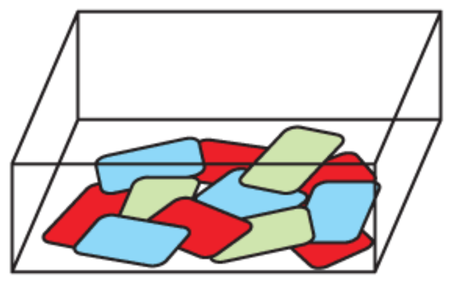
b Show that the expected gain of a player is given by $\frac{1}{6}(35 - 5a)$ dollars.

c What value would a need to have for the game to be "fair"?

d Explain why the organisers would not let a be 4.

e The organisers set $a = 9$ for the event, and the game is played 2406 times. Estimate the amount of money raised by this game.

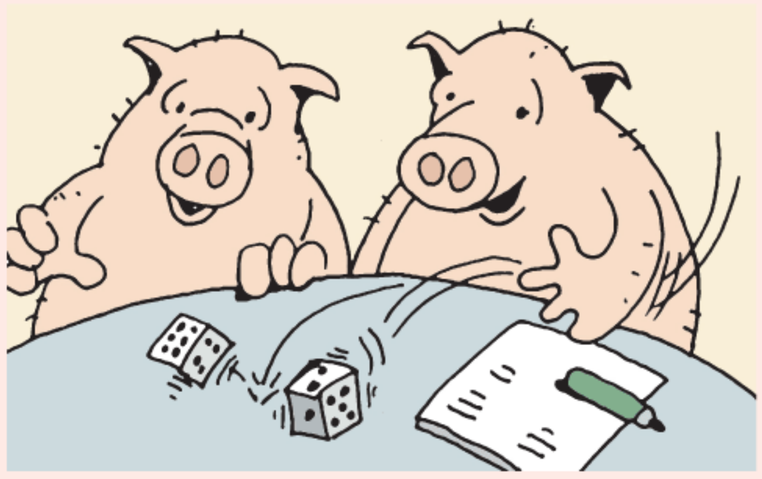
8 In a fundraising game "Lucky 11", a player selects 3 cards without replacement from a box containing 5 red, 4 blue, and 3 green cards.
The player wins \$11 if the cards drawn are all the same colour *or* are one of each colour.
If the organiser of the game wants to make an average of \$1 per game, how much should they charge to play it?



ACTIVITY — GREEDY PIGS

In this Activity, we will play a variant of the dice game **Greedy Pigs**. We will use expected value to find a strategy for playing the game.

In each turn of the game, a player rolls a die a number of times, accumulating points according to the numbers rolled. After each roll, the player can either end their turn and "bank" the points accumulated so far, or continue rolling in an attempt to score more points. However, if the player rolls a 1, the player loses all of the points accumulated on that turn, and their turn is over.



What to do:

1 Play the game in pairs, so that each player has 20 turns.
Which player scored the most points in total? Discuss the strategies you used during the game. Did your strategy change during the game?

2 Expected value can be used to find a strategy that, *on average*, will maximise a player's score.

a Explain why it would not be sensible to:

i stop while you have scored less than 5 points

ii keep going if you have scored over 50 points.

b Suppose you have scored 10 points so far in your turn. Let X be the gain or loss from rolling again.

i Construct a probability distribution for X.

ii Show that $\mathrm{E}(X)$ is positive.

iii What does $\mathrm{E}(X)$ tell us about whether we should roll again at 10 points?

c Find the lowest score at which, on average, it is not beneficial to continue rolling. Hence describe a strategy that will maximise your score in the long term.

d Can you think of situations where this strategy may not be the best strategy for winning the game?

3 How would the strategy for maximising your score change if your turn ended when a 6 was rolled, rather than a 1?

D VARIANCE AND STANDARD DEVIATION

In addition to the mean or expected value, we can also calculate the **variance** and **standard deviation** of a discrete random variable. These values measure the **spread** of the data values we expect to obtain when the experiment is performed many times.

Consider a discrete random variable X with possible values $x_1, x_2, ..., x_k$ and associated probabilities $p_1, p_2, ..., p_k$.

In n trials, we would expect to obtain the frequencies shown in the table.

x_i	*Frequency*
x_1	$f_1 = p_1 \times n$
x_2	$f_2 = p_2 \times n$
$\vdots$	$\vdots$
x_k	$f_k = p_k \times n$

We can calculate the variance of this data using the formula from the **Statistics Chapter** in the **Core Topics HL** book:

$$\begin{aligned}\sigma^2 &= \frac{\sum f_i(x_i - \mu)^2}{n} \\ &= \sum (x_i - \mu)^2 \left(\frac{f_i}{n}\right) \\ &= \sum (x_i - \mu)^2 p_i \\ &= \mathrm{E}[(X - \mu)^2]\end{aligned}$$

For a function $g(x)$, $\mathrm{E}[g(X)] = \sum g(x_i)p_i$.



If a discrete random variable X has k possible values $x_1, x_2, x_3, ..., x_k$
with probabilities $p_1, p_2, p_3, ..., p_k$

then:

- the **expected value** or **mean** is $\mathbf{E(X) = \mu = \sum x_i p_i}$
- the **variance** is $\mathbf{Var(X) = \sigma^2 = E[(X - \mu)^2] = \sum (x_i - \mu)^2 p_i}$
- the **standard deviation** is $\boldsymbol{\sigma(X) = \sqrt{Var(X)} = \sqrt{\sum (x_i - \mu)^2 p_i}}$.

Example 7 **Self Tutor**

In **Examples 2** and **5** we considered the probability distribution for sales in a magazine store:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

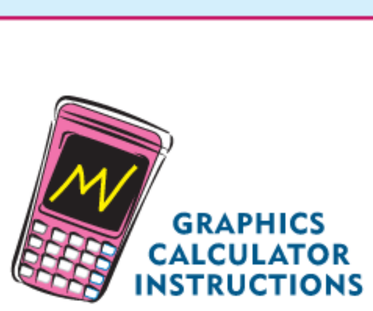
Find the standard deviation for this distribution.

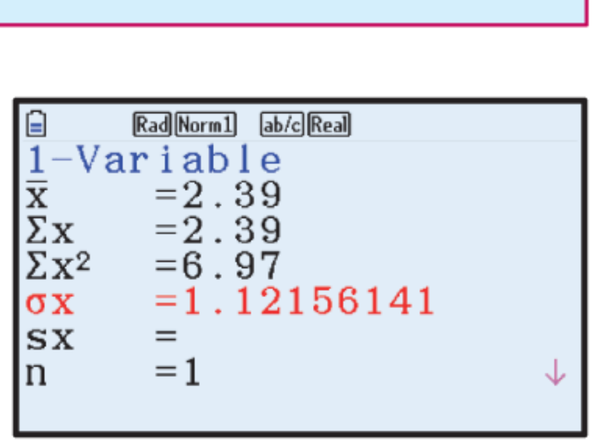
The mean $\mu = \sum x_i p_i = 2.39$ {from **Example 5**}

The standard deviation

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{(1 - 2.39)^2 \times 0.23 + (2 - 2.39)^2 \times 0.38 + ... + (5 - 2.39)^2 \times 0.05} \\ &\approx 1.12\end{aligned}$$

We can also use technology to find the standard deviation of a discrete random variable.





An alternative formula for the variance is $\boldsymbol{\sigma^2 = E(X^2) - \mu^2 = \sum x_i^2 p_i - \mu^2}$.

For example, for the roll of a die:

$$\mu = \sum x_i p_i = 1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + 5(\tfrac{1}{6}) + 6(\tfrac{1}{6}) = 3.5$$

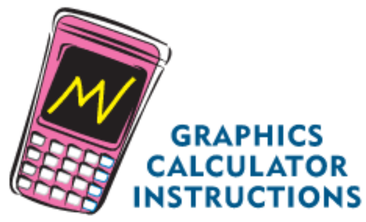
and $\sigma^2 = \sum x_i^2 p_i - \mu^2$

$$= 1^2(\tfrac{1}{6}) + 2^2(\tfrac{1}{6}) + 3^2(\tfrac{1}{6}) + 4^2(\tfrac{1}{6}) + 5^2(\tfrac{1}{6}) + 6^2(\tfrac{1}{6}) - (3.5)^2$$

$$= 2\tfrac{11}{12}$$

$$\therefore \quad \sigma = \sqrt{2\tfrac{11}{12}} \approx 1.708$$

You can check these results using your calculator by generating 800 random digits from 1 to 6. The mean and standard deviation of the results should be a good approximation of the theoretical values above.



EXERCISE 27D

1 For each probability distribution, find the:

i mean μ **ii** variance σ^2 **iii** standard deviation σ.

a

x	1	2	3
$\mathrm{P}(X = x)$	0.3	0.4	0.3

b

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.4	0.1	0.3

Check your answers using technology.

2 For the probability distribution alongside, find the:

x_i	2	4	10	20
p_i	k	0.05	0.35	$3k$

a value of k **b** mode

c mean μ **d** standard deviation σ.

3 The probability distributions below refer to the number of aces served by Michelle and Amanda in each set of tennis they play.

Michelle:

Number of aces	0	1	2	3	4
Probability	0.1	0.15	0.45	0.25	0.05

Amanda:

Number of aces	0	1	2	3	4
Probability	0.2	0.1	0.35	0.2	0.15

a Show that each player is expected to serve an average of 2 aces per set.

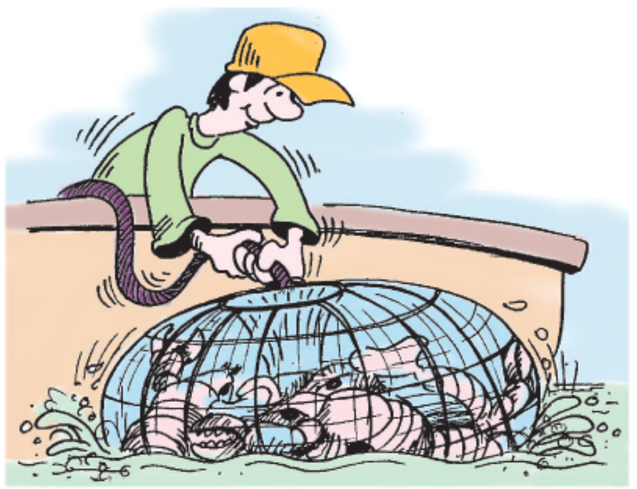
b Calculate the variance and standard deviation of each probability distribution.

c Which player has the greater variation in the number of aces served?

4 A country exports crayfish to overseas markets. The buyers are prepared to pay high prices when the crayfish arrive still alive.

Let X be the number of deaths per dozen crayfish. The probability distribution for X is given by:

x	0	1	2	3	4	5	> 5
$\mathrm{P}(X = x)$	0.54	0.26	0.15	k	0.01	0.01	0.00



a Find k. **b** Over a long period, what is the mean number of deaths per dozen crayfish?

c Find the standard deviation of the distribution.

5 A die is numbered 1, 1, 2, 3, 3, 3. Let X be the result when the die is rolled once.

a Construct the probability distribution for X. **b** Find the mean μ.

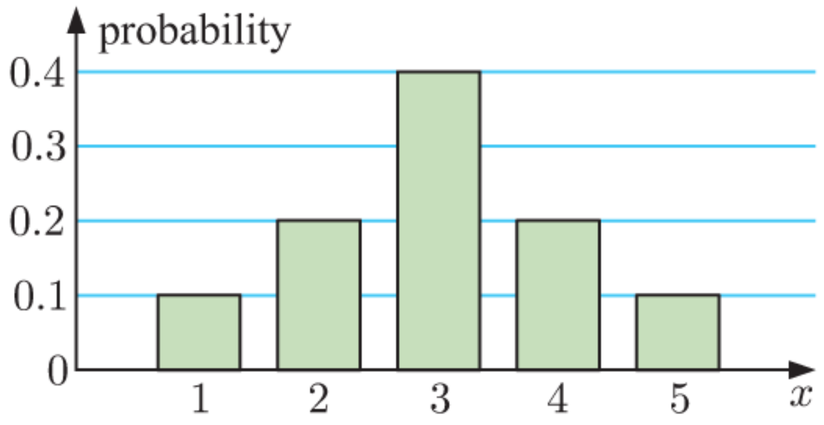
c Find the standard deviation of the distribution.

6 A random variable X has the probability mass function $P(x) = \dfrac{x^2 + x}{20}$, $x = 1, 2, 3$.
For this distribution, calculate the:

a mode **b** median **c** mean μ **d** standard deviation σ.

7 Use $\sigma^2 = \mathrm{E}[(X - \mu)^2]$ to show that $\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$.

8 The probability distribution of a random variable X is shown in the graph.



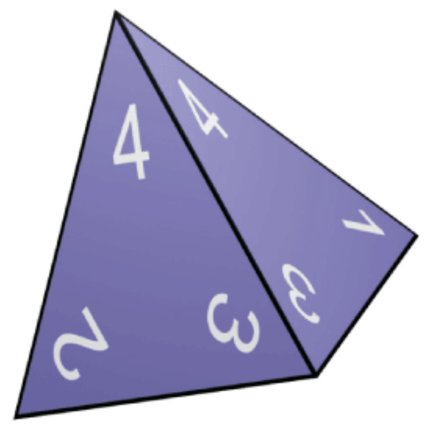
a Copy and complete:

x_i	1	2	3	4	5
p_i					

b Find the mean μ and standard deviation σ of the distribution.

c Find: **i** $\mathrm{P}(\mu - \sigma < x < \mu + \sigma)$ **ii** $\mathrm{P}(\mu - 2\sigma < x < \mu + 2\sigma)$.

9 A tetrahedral die is numbered 1, 2, 3, and 4.
Let X be the number rolled when the die is rolled *once*.
Let Y be the *highest* number rolled when the die is rolled *twice*.



a Would you expect X or Y to have the greater:
i mean **ii** standard deviation?
Explain your answers.

b Construct probability distributions for X and Y.

c Which of the variables is a uniform discrete random variable?

d Calculate the mean and standard deviation of each distribution.

10 Suppose X is a uniform discrete random variable with possible values $X = 1, 2, ..., n$. Show that the variance of X is $\dfrac{n^2 - 1}{12}$.

Hint: $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

E PROPERTIES OF $aX + b$

If we know the expected value, variance, and standard deviation of a random variable X, we can determine these properties for a related random variable $aX + b$, where a and b are constants.

The following properties of $\mathrm{E}(X)$ are useful:

If $\mathrm{E}(X)$ is the expected value of the random variable X, then:

1 $\mathrm{E}(k) = k$ for any constant k

2 $\mathrm{E}(kX) = k\,\mathrm{E}(X)$ for any constant k

3 $\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$ for functions g and h.

Proof:

1 $\mathrm{E}(k) = \sum_{i=1}^{n} kp_i$

$= k\sum_{i=1}^{n} p_i$

$= k(1)$ {probabilities sum to 1}

$= k$

2 $\mathrm{E}(kX) = \sum_{i=1}^{n} (kx_i) \times p_i$

$= k\sum_{i=1}^{n} x_i\, p_i$

$= k\,\mathrm{E}(X)$

3 $\mathrm{E}[g(X) + h(X)]$

$= \sum_{i=1}^{n} [g(x_i) + h(x_i)]\, p_i$

$= \sum_{i=1}^{n} [g(x_i)p_i + h(x_i)p_i]$

$= \sum_{i=1}^{n} [g(x_i)p_i] + \sum_{i=1}^{n} [h(x_i)p_i]$

$= \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$

The expectation of a sum is the sum of the individual expectations.



INVESTIGATION 1 PROPERTIES OF $aX + b$

The purpose of this Investigation is to discover relationships between the properties of X and those of $aX + b$.

What to do:

1 Consider the random variable X with probability distribution:

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

a Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b Consider the random variable $Y = 2X + 3$. The probability distribution of Y is:

y	5	7	9	11	13
$\mathrm{P}(Y = y)$	0.2	0.2	0.2	0.2	0.2

Find $\mathrm{E}(2X + 3)$ and $\mathrm{Var}(2X + 3)$.

c Repeat **b** for:

i $Y = 3X - 2$ **ii** $Y = -2X + 5$ **iii** $Y = \dfrac{X+1}{2}$ **iv** $Y = \dfrac{-X+2}{3}$

2 Make up your own probability distribution for X and repeat **1**.

3 By studying your results, predict the relationship between:

a $\mathrm{E}(X)$ and $\mathrm{E}(aX + b)$ **b** $\mathrm{Var}(X)$ and $\mathrm{Var}(aX + b)$.

4 Using the definition $\sigma(X) = \sqrt{\mathrm{Var}(X)}$ and your result for $\mathrm{Var}(aX + b)$, what can you deduce about $\sigma(aX + b)$?

From the **Investigation** you should have discovered that:

- $\mathbf{E}(aX+b) = a\,\mathbf{E}(X) + b$
- $\mathbf{Var}(aX+b) = a^2\mathbf{Var}(X)$
- $\sigma(aX+b) = |a|\,\sigma(X)$

Example 8 **Self Tutor**

The random variable X has mean 8.1 and standard deviation 2.37. If $Y = 4X - 7$, find the mean and standard deviation of Y.

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(4X-7) \\ &= 4\mathrm{E}(X) - 7 \\ &= 4(8.1) - 7 \\ &= 25.4 \end{aligned} \qquad \begin{aligned} \sigma(Y) &= \sigma(4X-7) \\ &= |4|\,\sigma(X) \\ &= 4(2.37) \\ &= 9.48 \end{aligned}$$

For Y, the mean is 25.4 and the standard deviation is 9.48.

EXERCISE 27E

1 Use the properties of $\mathrm{E}(X)$ to prove that $\mathrm{E}(aX+b) = a\,\mathrm{E}(X) + b$.

2 The mean of X is 3. Find the mean of Y where:

a $Y = 3X + 4$ **b** $Y = -2X + 1$ **c** $Y = \dfrac{4X-2}{3}$

3 X is distributed with mean 6 and standard deviation 2. If $Y = 2X + 5$, find the mean and standard deviation of Y.

4 X is a random variable with mean 5 and standard deviation 2.
Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ for:

a $Y = 2X + 3$ **b** $Y = -5X + 3$ **c** $Y = \dfrac{X-5}{2}$

5 X has the probability distribution:

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

Find:

a $\mathrm{E}(X)$ **b** $\mathrm{Var}(X)$ **c** $\sigma(X)$ **d** $\mathrm{E}(X+1)$

e $\mathrm{Var}(3X+1)$ **f** $\sigma(5-X)$ **g** $\mathrm{E}\left(\dfrac{2X+5}{3}\right)$ **h** $\mathrm{Var}(20-4X)$

6 The score X obtained by rolling a biased pentagonal die has the probability distribution:

x	1	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	a	$\frac{1}{6}$

a Find a.

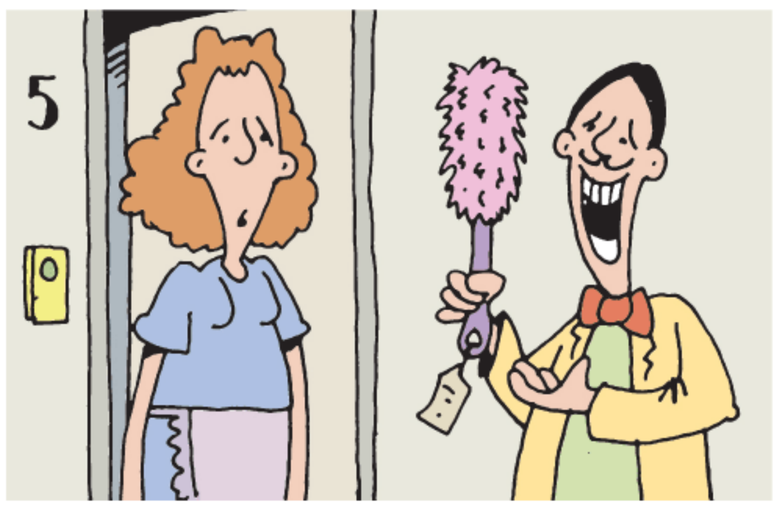
b Find the expected value, variance, and standard deviation of:

i X **ii** $X+4$ **iii** $3X-1$

7 Dominic is a door-to-door salesman. Each day he earns \$100, plus \$25 for each sale made.

From previous experience, Dominic knows that the number of sales X made per day follows the probability distribution below:

x	0	1	2	3	4	5
$\mathrm{P}(X = x)$	0.17	0.25	0.3	0.15	0.1	0.03



a Find:

i $\mathrm{E}(X)$ **ii** $\mathrm{Var}(X)$ **iii** $\sigma(X)$

b Suppose Y is the amount of money Dominic earns, in dollars per day. Write Y in terms of X.

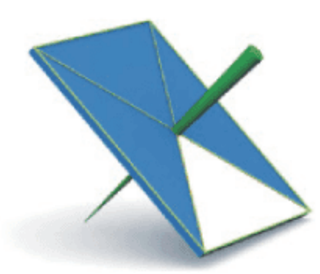
c Hence find:

i $\mathrm{E}(Y)$ **ii** $\mathrm{Var}(Y)$ **iii** $\sigma(Y)$

8 Prove that $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

F THE BINOMIAL DISTRIBUTION

Suppose X = the number of blues which result from spinning this spinner *once*.



The probability distribution of X is:

x	0	1
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{3}{4}$

Now suppose we spin the spinner n times and count the number of blues that result. The probability that we get a blue is the same for each spin, and each spin is independent of every other spin. This is an example of a **binomial experiment**.

In a **binomial experiment**:

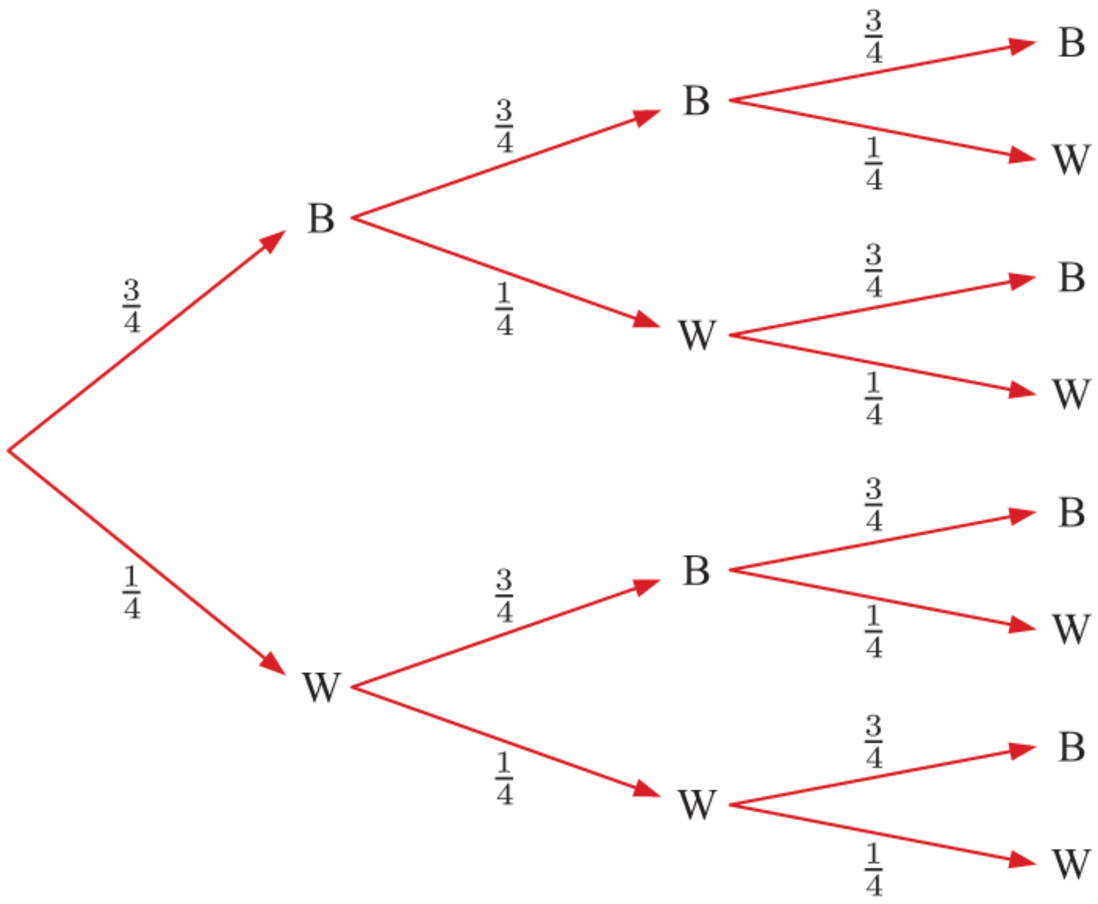
- there are a fixed number of **independent trials**
- there are only *two* possible results for each trial:
  success if some event occurs, or **failure** if the event does not occur
- the probability of success is the same for each trial.

If X is the number of successes in a binomial experiment with n trials, each with probability of success p, then X is a **binomial random variable.**

THE PROBABILITY DISTRIBUTION OF A BINOMIAL RANDOM VARIABLE

Consider the spinner above with 3 blue sectors and 1 white sector. Suppose a "success" is a blue result and let X be the number of "successes" in 3 spins of the spinner. X is a binomial random variable with $n = 3$ and $p = \frac{3}{4}$, and can take the values 0, 1, 2, or 3.

To help determine the probability distribution of X, we first draw a tree diagram and find the probabilities associated with each possible outcome. We let B represent blue and W represent white.



The outcomes have been shaded according to the value of X.

The probabilities associated with each value of X are:

$$\begin{aligned}
\mathrm{P}(X=0) &= \mathrm{P}(\mathrm{WWW}) \\
&= \left(\tfrac{1}{4}\right)^3 && \{\text{outcome shaded green}\} \\
\mathrm{P}(X=1) &= \mathrm{P}(\mathrm{BWW \text{ or } WBW \text{ or } WWB}) \\
&= \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right) \\
&= \textcircled{3} \times \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 && \{\text{outcomes shaded yellow}\} \\
\mathrm{P}(X=2) &= \mathrm{P}(\mathrm{BBW \text{ or } BWB \text{ or } WBB}) \\
&= \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 \\
&= 3 \times \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) && \{\text{outcomes shaded red}\} \\
\mathrm{P}(X=3) &= \mathrm{P}(\mathrm{BBB}) \\
&= \left(\tfrac{3}{4}\right)^3 && \{\text{outcome shaded blue}\}
\end{aligned}$$

Notice that for any particular value of X, each outcome with this property will have the same probability of occurring. The *order* in which blues and whites appear does not matter.

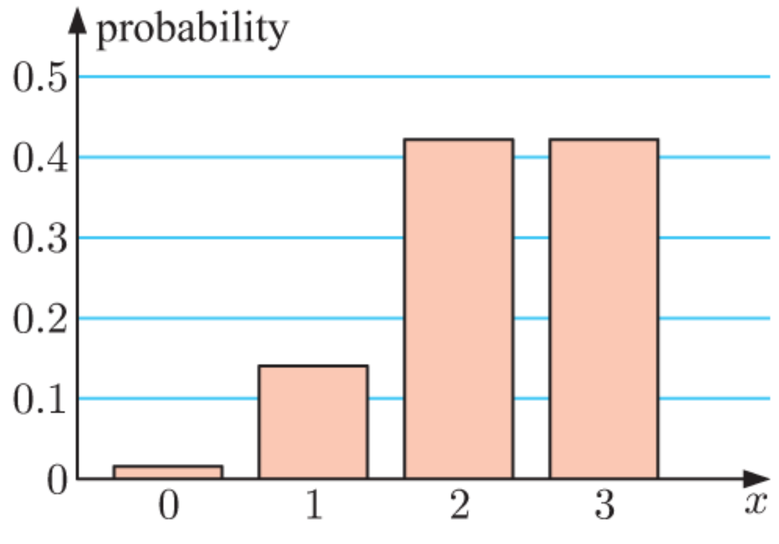
In **Chapter 7** we saw that ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ is the number of ways in which we can obtain r successes from n trials, ignoring order.

For example, the circled factor $\textcircled{3}$ is the number of ways of getting 1 success out of 3 trials, or $\binom{3}{1}$.

We can use this notation to write:

$$\begin{aligned}
\mathrm{P}(X=0) &= \left(\tfrac{1}{4}\right)^3 &&= \binom{3}{0}\left(\tfrac{3}{4}\right)^0\left(\tfrac{1}{4}\right)^3 \approx 0.0156 \\
\mathrm{P}(X=1) &= 3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1 &&= \binom{3}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \approx 0.1406 \\
\mathrm{P}(X=2) &= 3\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 &&= \binom{3}{2}\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)^1 \approx 0.4219 \\
\mathrm{P}(X=3) &= \left(\tfrac{3}{4}\right)^3 &&= \binom{3}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^0 \approx 0.4219
\end{aligned}$$

So, $\mathrm{P}(X=x) = \binom{3}{x}\left(\tfrac{3}{4}\right)^x\left(\tfrac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$.



The sum of the probabilities $\quad \mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$

$$=\left(\tfrac{1}{4}\right)^3+3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)+3\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2+\left(\tfrac{3}{4}\right)^3$$

which is the binomial expansion of $\left(\frac{1}{4}+\frac{3}{4}\right)^3$.

Evaluating, $\left(\frac{1}{4}+\frac{3}{4}\right)^3=1$ which must be the case since all possibilities are covered.

Suppose X is a binomial random variable with n independent trials and probability of success p. The probability mass function of X is:

$$P(x)=\mathrm{P}(X=x)=\binom{n}{x}p^x(1-p)^{n-x} \quad \text{where } x=0,\ 1,\ 2,\ ....,\ n.$$

$\binom{n}{x}$: number of ways x successes can be ordered amongst the n trials

$p^x(1-p)^{n-x}$: probability of obtaining x successes and $n-x$ failures in a particular order

The probability distribution of X is called the **binomial distribution**, and we write $\boldsymbol{X\sim \mathrm{B}(n,\,p)}$.

"$\sim$" reads "is distributed as".



Example 9

Self Tutor

a Expand $\left(\frac{9}{10}+\frac{1}{10}\right)^5$.

b An archer has a 90% chance of hitting a target with each arrow. If 5 arrows are fired, determine the chance of hitting the target:

i twice only **ii** at most 3 times.

a $\left(\frac{9}{10}+\frac{1}{10}\right)^5$

$$=\sum_{k=0}^{5}\binom{5}{k}\left(\tfrac{9}{10}\right)^k\left(\tfrac{1}{10}\right)^{5-k}$$

$$=\left(\tfrac{1}{10}\right)^5+5\left(\tfrac{9}{10}\right)\left(\tfrac{1}{10}\right)^4+10\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^3+10\left(\tfrac{9}{10}\right)^3\left(\tfrac{1}{10}\right)^2+5\left(\tfrac{9}{10}\right)^4\left(\tfrac{1}{10}\right)+\left(\tfrac{9}{10}\right)^5$$

b The probability of success with each arrow is $p=\frac{9}{10}$.

Let X be the number of arrows that hit the target.

The expansion in **a** gives the probability distribution for X.

$\left(\frac{1}{10}\right)^5$	$5\left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^4$	$10\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^3$	$10\left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right)^2$	$5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right)$	$\left(\frac{9}{10}\right)^5$
$\mathrm{P}(X=0)$	$\mathrm{P}(X=1)$	$\mathrm{P}(X=2)$	$\mathrm{P}(X=3)$	$\mathrm{P}(X=4)$	$\mathrm{P}(X=5)$
5 misses	1 hit, 4 misses	2 hits, 3 misses	3 hits, 2 misses	4 hits, 1 miss	5 hits

i $\mathrm{P}(\text{hits twice only})=\mathrm{P}(X=2)$

$$=10\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^3$$

$$=0.0081$$

ii $\mathrm{P}(\text{hits at most 3 times})=\mathrm{P}(X\leqslant 3)$

$$=\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$$

$$=\left(\tfrac{1}{10}\right)^5+5\left(\tfrac{9}{10}\right)\left(\tfrac{1}{10}\right)^4+10\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^3+10\left(\tfrac{9}{10}\right)^3\left(\tfrac{1}{10}\right)^2$$

$$\approx 0.0815$$

EXERCISE 27F

1 For which of these probability experiments does the binomial distribution apply? Explain your answers.

a A coin is thrown 100 times. The variable is the number of heads.

b One hundred coins are each thrown once. The variable is the number of heads.

c A box contains 5 blue and 3 red marbles. I draw out 5 marbles one at a time, replacing each marble before the next is drawn. The variable is the number of red marbles drawn.

d A box contains 5 blue and 3 red marbles. I draw out 5 marbles without replacement. The variable is the number of red marbles drawn.

e A large bin contains ten thousand bolts, 1% of which are faulty. I draw a sample of 10 bolts from the bin. The variable is the number of faulty bolts.

2 **a** Expand $(p+q)^4$.

b If a coin is tossed *four* times, what is the probability of getting:

i 4 heads **ii** 3 heads **iii** 2 heads?

3 **a** Expand $(p+q)^5$.

b If *five* coins are tossed simultaneously, what is the probability of getting:

i 4 heads and 1 tail in any order **ii** 2 heads and 3 tails in any order

iii 4 heads and then 1 tail?

4 **a** Expand $\left(\frac{2}{3}+\frac{1}{3}\right)^4$.

b A box of chocolates contains strawberry creams and almond centres in the ratio 2 : 1. Four chocolates are selected at random, with replacement. Find the probability of getting:

i all strawberry creams **ii** two of each type

iii at least 2 strawberry creams.

5 **a** Expand $\left(\frac{3}{4}+\frac{1}{4}\right)^5$.

b In New Zealand in 1946 there were two different coins of value one florin. These were "normal" kiwis and "flat back" kiwis, in the ratio 3 : 1. From a very large batch of 1946 florins, five were selected at random with replacement. Find the probability that:

i two were "flat backs"

ii at least 3 were "flat backs"

iii at most 3 were "normal" kiwis.



6 The probability mass function for $X \sim \text{B}(n, p)$ is $P(x) = \binom{n}{x} p^x (1-p)^{n-x}$.

a Show that $\sum_{x=0}^{n} P(x) = 1$.

b *Hence* explain why $0 \leqslant P(x) \leqslant 1$ for all $x = 0, 1, ..., n$.

c What do **a** and **b** tell you about $P(x)$?

INVESTIGATION 2 THE GRAPH OF A BINOMIAL DISTRIBUTION

In this Investigation we will explore the graph of a binomial distribution and how its shape varies with changes to n and p.

What to do:



1 Click on the icon to access the demonstration. It shows the graph of the binomial distribution for $X \sim \text{B}(n, p)$. Set $n = 20$ and $p = 0.1$.

a What is the mode of X?

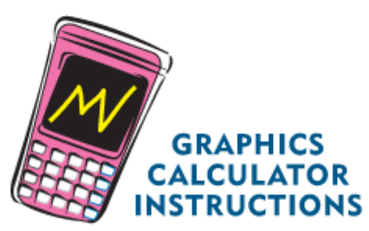
b Describe the shape of the distribution.

2 Use the slider to change the value of p. Describe how the *shape* of the distribution changes as p changes.

3 Reset p to 0.1. Use the slider to change the value of n.
How does this affect the shape of the distribution? What happens to the shape of the binomial distribution as the number of trials n increases?

G USING TECHNOLOGY TO FIND BINOMIAL PROBABILITIES

We can quickly calculate binomial probabilities using a graphics calculator.



For example:

- To find the probability $\text{P}(X = k)$ that the variable takes the value k, we use the **binomial probability function**.
- To find the probability that the variable takes a *range* of values, such as $\text{P}(X \leqslant k)$ or $\text{P}(X \geqslant k)$, we use the **binomial cumulative probability function**.

Some calculator models, such as the **TI-84 Plus CE**, only allow you to calculate $\text{P}(X \leqslant k)$. To find the probability $\text{P}(X \geqslant k)$ for these models, it is often easiest to find the complement $\text{P}(X \leqslant k-1)$ and use $\text{P}(X \geqslant k) = 1 - \text{P}(X \leqslant k-1)$.

Example 10 Self Tutor

72% of union members are in favour of a certain change to their conditions of employment. A random sample of five members is taken. Find the probability that:

a three members are in favour of the change in conditions

b at least three members are in favour of the changed conditions.

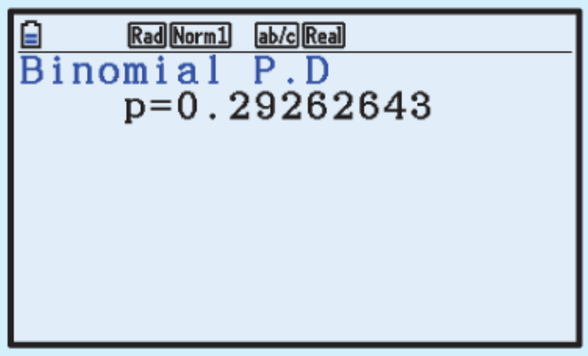
Let X denote the number of members in the sample in favour of the change.

$n = 5$, so $X = 0, 1, 2, 3, 4$, or 5, and $p = 72\% = 0.72$

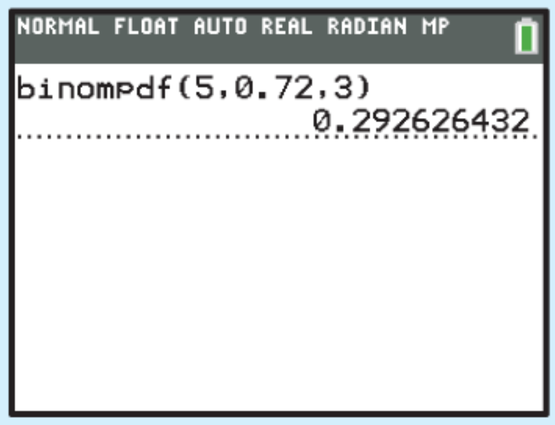
$\therefore \quad X \sim \text{B}(5, 0.72)$.

a $P(X = 3) = \binom{5}{3}(0.72)^3(0.28)^2 \approx 0.293$

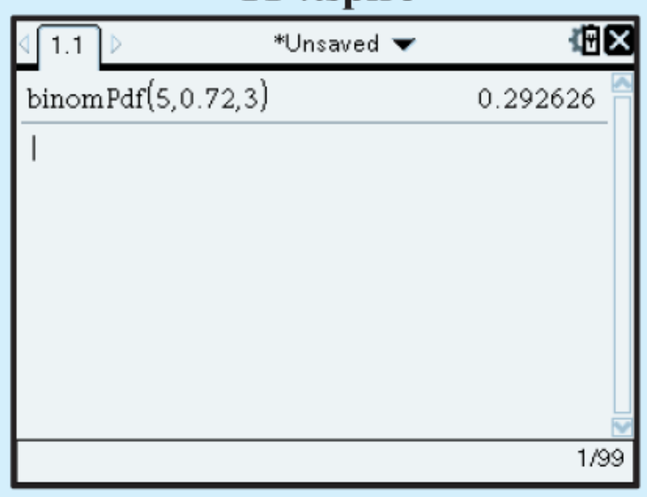
Casio fx-CG50



TI-84 Plus CE

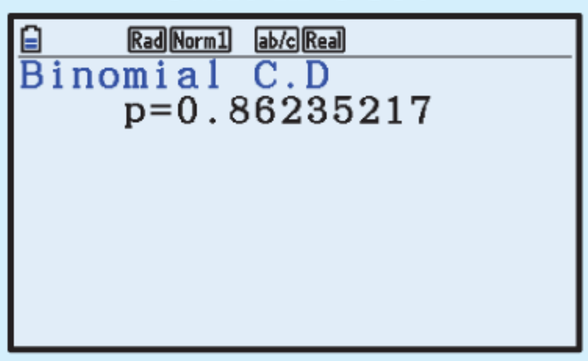


TI-*n*spire

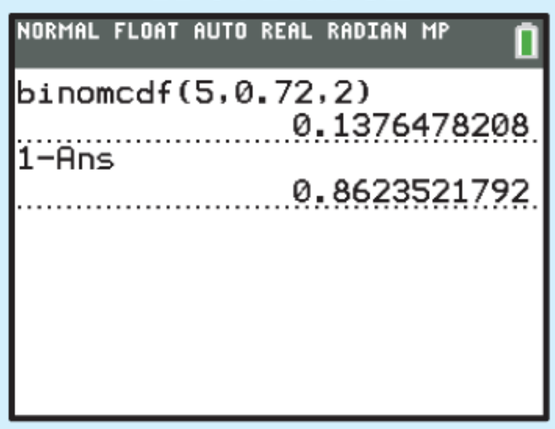


b $P(X \geqslant 3) \approx 0.862$

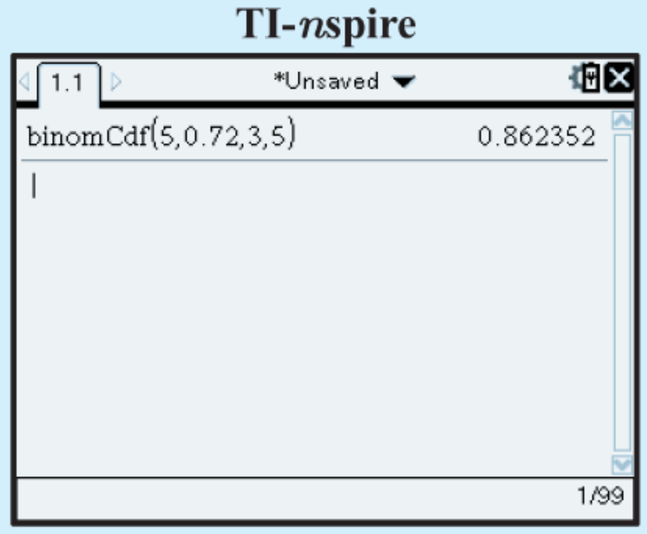
Casio fx-CG50



TI-84 Plus CE



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In examinations, it is unacceptable to write calculator functions such as binompdf(5,0.72,3) ≈ 0.293. You must clearly state the distribution you are using, and the probability that you are calculating.

EXERCISE 27G

1 5% of electric light bulbs are defective at manufacture. 6 bulbs are randomly tested, with each one being replaced before the next is chosen. Determine the probability that:

a two are defective **b** at least one is defective.

2 Records show that 6% of the items assembled on a production line are faulty. A random sample of 12 items is selected with replacement. Find the probability that:

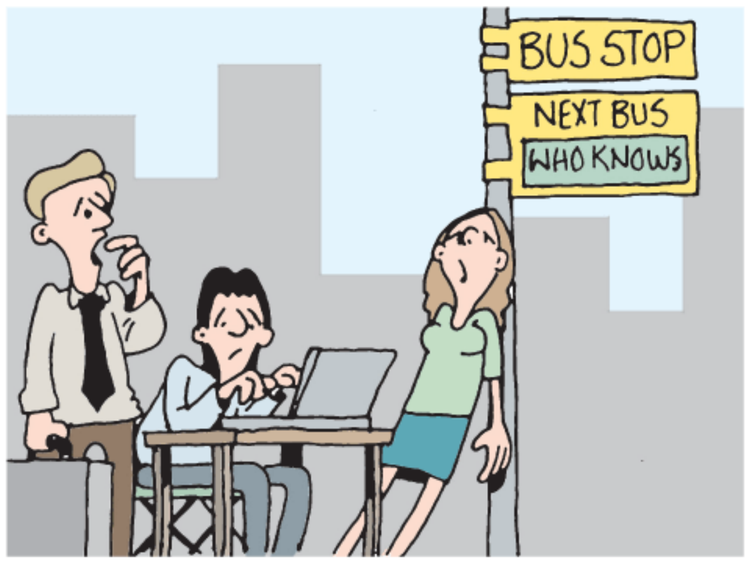
a none will be faulty **b** at most one will be faulty

c at least two will be faulty **d** less than four will be faulty.

3 The local bus service does not have a good reputation. The 8 am bus will run late on average two days out of every five. For any week of the year taken at random, find the probability of the 8 am bus being on time:

a all 7 days **b** only on Monday

c on any 6 days **d** on at least 4 days.



4 In a multiple choice test there are 10 questions. Each question has 5 choices, one of which is correct. Raj knows absolutely nothing about the subject, and guesses each answer at random. Given that the pass mark is 70%, determine the probability that he will pass.

5 An infectious flu virus is spreading through a school. The probability of a randomly selected student having the flu next week is 0.3. Mr C has a class of 25 students.

a Calculate the probability that 2 or more students from Mr C's class will have the flu next week.

b If more than 20% of the students have the flu next week, a class test will have to be cancelled. What is the probability that the test will be cancelled?

6 During a season, a basketball player has an 85% success rate in shooting from the free throw line. In one match the basketballer has 20 shots from the free throw line.
Find the probability that the basketballer is successful with:

a all 20 throws **b** at least 18 throws

c between 14 and 17 (inclusive) throws.

7 Martina beats Jelena in 2 games out of 3 at tennis. What is the probability that Jelena wins a set of tennis 6 games to 4?
Hint: What does the score after 9 games need to be?

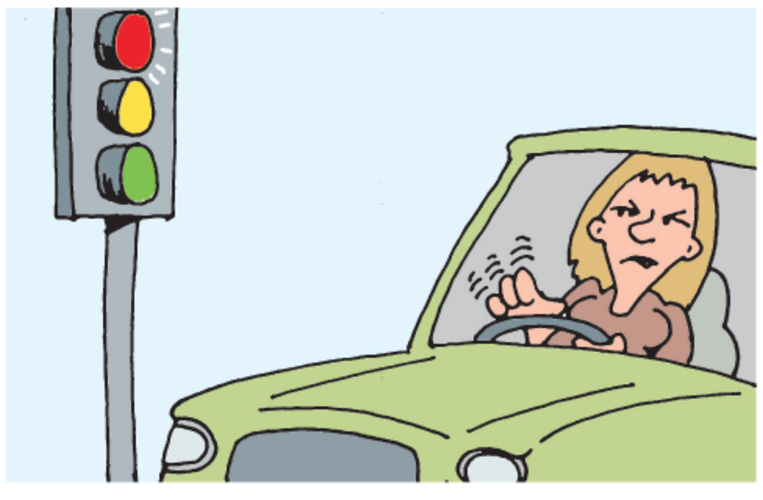
8 A fair coin is tossed 200 times. Find the probability of obtaining:

a between 90 and 110 (inclusive) heads **b** more than 95 but less than 105 heads.

9 **a** Find the probability of rolling double sixes with a pair of dice.

b Suppose a pair of dice is rolled 500 times. Find the probability of rolling between 10 and 20 (inclusive) double sixes.

10 Shelley must pass through 15 traffic lights on her way to work. She has probability 0.6 of being stopped at any given traffic light. If she is stopped at more than 11 traffic lights, she will be late for work.



a Find the probability that Shelley will be late for work on a given day.

b Find the probability that Shelley is on time for work each day of a 5 day week.

c Shelley wants to increase the probability in **b** to at least 80%. She decides to leave home a little earlier, so she must now be stopped at more than 12 traffic lights in order to be late. Has Shelley achieved her goal? Justify your answer.

11 A hot water unit relies on 20 solar components for its power, and will operate provided at least one of its 20 components is working. The probability that an individual solar component will fail in a year is 0.85, and the failure of each individual component is independent of the others.

a Find the probability that the hot water unit will fail within one year.

b Find the smallest number of solar components required to ensure that a hot water service like this one is operating at the end of one year with a probability of at least 0.98.

H THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

INVESTIGATION 3 THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

In this Investigation we will use a calculator to calculate the mean and standard deviation of several binomial distributions. A spreadsheet can also be used to speed up the process.

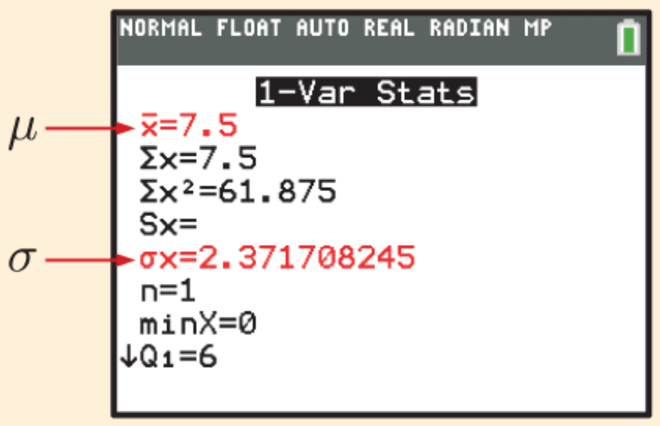


What to do:

1 We will first calculate the mean and standard deviation for the variable $X \sim \text{B}(30, 0.25)$.

a Enter the possible values for X from $x = 0$ to $x = 30$ into **List 1**, and their corresponding binomial probabilities $\text{P}(X = x) = \binom{30}{x} (0.25)^x (0.75)^{30-x}$ into **List 2**.

b Calculate the descriptive statistics for the distribution. You should obtain the results in the screenshot.



2 Copy and complete the following table for distributions with other values of n and p.

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$				
$n = 30$		$\mu = 7.5$ $\sigma \approx 2.3717$		
$n = 50$				

3 Compare your values with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

From this **Investigation** you should have observed the following results:

Suppose X is a binomial random variable with parameters n and p, so $X \sim \text{B}(n, p)$.

- The **mean** of X is $\boldsymbol{\mu = \text{E}(X) = np}$.
- The **variance** of X is $\boldsymbol{\sigma^2 = \text{Var}(X) = np(1-p)}$.
- The **standard deviation** of X is $\boldsymbol{\sigma = \sqrt{\text{Var}(x)} = \sqrt{np(1-p)}}$.

Click on this icon to see a proof for the mean and variance of the binomial distribution.



Example 11 **Self Tutor**

A fair die is rolled twelve times, and X is the number of sixes that result.
Find the mean, variance, and standard deviation of X.

This is a binomial distribution with $n = 12$ and $p = \frac{1}{6}$, so $X \sim \text{B}(12, \frac{1}{6})$.

$$\begin{aligned}\mu &= np \\ &= 12 \times \tfrac{1}{6} \\ &= 2\end{aligned} \qquad \begin{aligned}\sigma^2 &= np(1-p) \\ &= 12 \times \tfrac{1}{6} \times \tfrac{5}{6} \\ &= \tfrac{5}{3}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sigma &= \sqrt{\sigma^2} \\ &= \sqrt{\tfrac{5}{3}} \\ &\approx 1.291\end{aligned}$$

We expect a six to be rolled 2 times, with variance $\frac{5}{3}$ and standard deviation 1.291.

EXERCISE 27H

1 Suppose $X \sim \text{B}(6, p)$. For each of the following cases:

i Find the mean and standard deviation of X.
ii Graph the distribution using a column graph.
iii Comment on the shape of the distribution.

a $p = 0.5$ **b** $p = 0.2$ **c** $p = 0.8$

2 A coin is tossed 10 times and X is the number of heads which occur. Find the mean and variance of X.

3 Bolts produced by a machine vary in quality. The probability that a given bolt is defective is 0.04. Random samples of 30 bolts are taken from the week's production.

a If X is the number of defective bolts in a sample, find the mean and standard deviation of X.
b If Y is the number of non-defective bolts in a sample, find the mean and standard deviation of Y.

4 A city restaurant knows that 13% of reservations are not honoured, which means the group does not arrive. Suppose the restaurant receives 30 reservations. Let the random variable X be the number of groups that do not arrive. Find the mean and standard deviation of X.

5 A new drug has a 75% probability of curing a patient within one week. Suppose 38 patients are treated using this drug. Let X be the number of patients who are cured within a week.

a Find the mean μ and standard deviation σ of X.
b Find $\text{P}(\mu - \sigma < X < \mu + \sigma)$.

6 Let X be the number of heads which occur when a coin is tossed 100 times, and Y be the number of ones which occur when a die is rolled 300 times.

a Show that the mean of both distributions is 50.
b Calculate the standard deviation of each distribution.
c Which variable do you think is more likely to lie between 45 and 55 (inclusive)? Explain your answer.
d Find:

i $\text{P}(45 \leqslant X \leqslant 55)$ **ii** $\text{P}(45 \leqslant Y \leqslant 55)$

REVIEW SET 27A

1 Determine whether the following variables are discrete or continuous:

a the number of attempts to pass a driving test

b the length of time before a phone loses its battery charge

c the number of phone calls made before a salesperson has sold 3 products.

2 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3
$\mathrm{P}(X = x)$	0.6	0.25	0.15

ii

x	0	2	5	10
$\mathrm{P}(X = x)$	0.3	0.5	0.1	0.2

iii

x	0	1	2	3
$\mathrm{P}(X = x)$	0.4	-0.2	0.35	0.45

iv

x	2	3	4	5
$\mathrm{P}(X = x)$	0.25	0.25	0.25	0.25

v

x	2	3
$\mathrm{P}(X = x)$	0.7	0.3

vi

x	0	1
$\mathrm{P}(X = x)$	0.28	0.72

b For which of the probability distributions in **a** is X a uniform discrete random variable?

3 $\mathrm{P}(X = x) = \dfrac{a}{x^2 + 1}$, $x = 0, 1, 2, 3$ is a probability mass function.

a Find the value of a.

b Find $\mathrm{P}(X \geqslant 1)$.

4 A random variable X has the probability mass function $P(x)$ described in the table.

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

a Find k.

b Find $\mathrm{P}(X \geqslant 3)$.

c Find the mode of the distribution.

d Find the expected value $\mathrm{E}(X)$ for the distribution.

5 Three green balls and two yellow balls are placed in a hat. Two balls are randomly drawn without replacement, and X is the number of green balls drawn.

a Explain why X is a discrete random variable.

b State the possible values of X.

c Construct a probability table for X.

d Find the expected number of green balls drawn.

6 The faces of a die are labelled 1, 3, 3, 4, 6, 6. Let X be the result when the die is rolled. Find the expected value of X.

7 Lakshmi rolls a regular six-sided die. She wins twice the number of dollars as the number rolled.

a How much does Lakshmi expect to win from one roll of the die?

b If it costs \$8 to play the game, would you advise Lakshmi to play many games? Explain your answer.

8 For the probability distribution alongside, find the:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.15	0.1	0.35	0.4

a mean μ

b variance σ^2

c standard deviation σ.

9 Suppose X is the number of marsupials entering a park at night. It is suspected that X has a probability mass function $P(x) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, ..., 8$.

a Find the constant a.

b Find the expected number of marsupials entering the park on a given night.

c Find the variance of X.

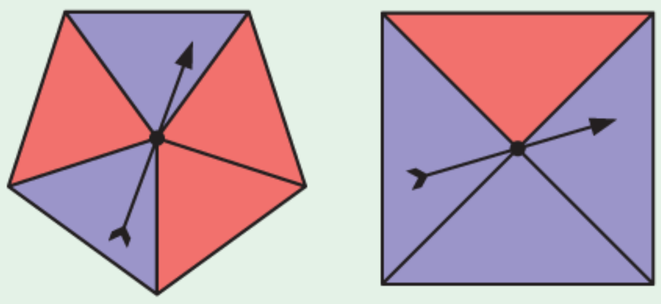
10 X is a discrete random variable with mean 6 and standard deviation 2. If $Y = 4X + 3$, find the mean and standard deviation of Y.

11 **a** Expand $\left(\frac{4}{5} + \frac{1}{5}\right)^5$.

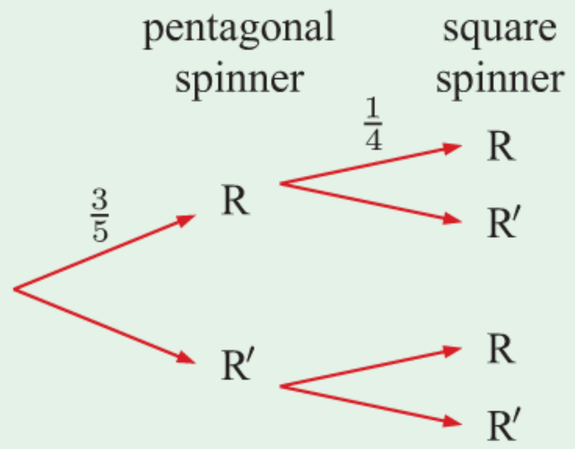
b With every attempt, Jack has an 80% chance of kicking a goal. In one quarter of a match he has 5 kicks for goal. Determine the probability that he scores:

i 3 goals then misses twice **ii** 3 goals and misses twice.

12 Consider the two spinners illustrated:



a Copy and complete the tree diagram which shows all possible results when the two spinners are spun together.



b Calculate the probability that exactly one red will occur.

c The pair of spinners is now spun 10 times. Let X be the number of times that exactly one red occurs.

i State the distribution of X.

ii Write down expressions for $\text{P}(X = 1)$ and $\text{P}(X = 9)$. Hence determine which of these outcomes is more likely.

iii Find the mean and standard deviation of X.

13 A school volleyball team has 9 players, each of whom has a 75% chance of coming to any given game. The team needs at least 6 players to avoid forfeiting the game.

a Find the probability that for a randomly chosen game, the team will:

i have all of its players **ii** have to forfeit the game.

b The team plays 30 games for the season. How many games would you expect the team to forfeit?

REVIEW SET 27B

1 Sally's number of hits in each softball match has the probability distribution shown.

x	0	1	2	3	4	5
$\text{P}(X = x)$	0.07	0.14	k	0.46	0.08	0.02

a State clearly what the random variable represents.

b Find: **i** k **ii** $\text{P}(X \geqslant 2)$ **iii** $\text{P}(1 \leqslant X \leqslant 3)$

c Find the mode and median number of hits.

2 Show that the following are valid probability mass functions:

a $P(x) = \dfrac{e^x}{1+e}$, $x = 0, 1$

b $P(x) = \dfrac{x^2 + x}{40}$, $x = 1, 2, 3, 4$

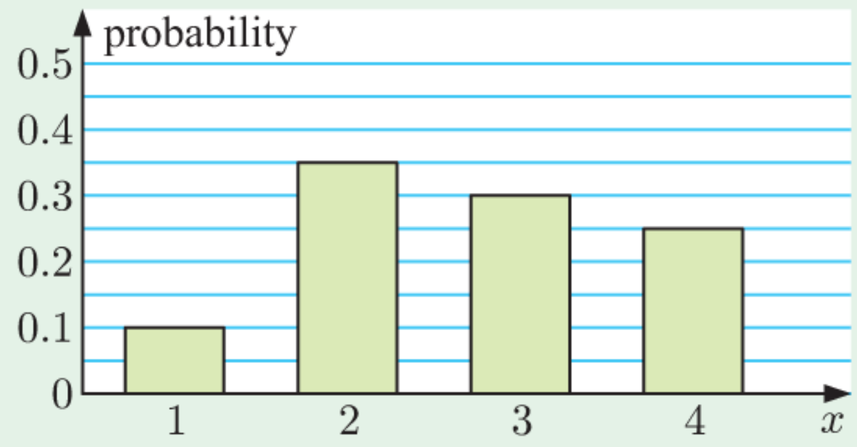
c $P(x) = \log\left(\dfrac{x+1}{x}\right)$, $x = 1, 2, 3, ..., 9$

3 The probability distribution of a random variable X is graphed alongside. Find:

a the mode of X

b the median of X

c the expected value of X.



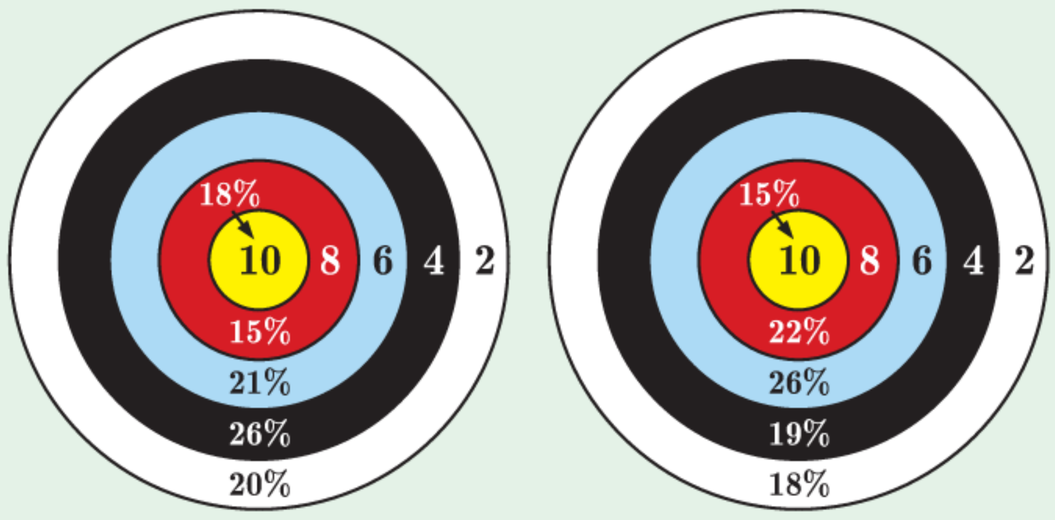
4 The probabilities of Naomi and Rosslyn hitting each section of an archery target are shown alongside.

a On a single shot, who is more likely to score:

i 10 points

ii at least 6 points?

b In the long run, who would you expect to score more points per shot?



5 The numbers from 1 to 20 are written on tickets and placed in a bag. A person draws out a number at random. The person wins \$3 if the number is even, \$6 if the number is a square number, and \$9 if the number is both even and square.

a Calculate the probability that the player wins: **i** \$3 **ii** \$6 **iii** \$9

b How much should be charged to play the game so that it is fair?

6 Suppose X has the probability distribution alongside.

x_i	1	2	3	4
p_i	0.2	a	0.3	b

a Given that $\mathrm{E}(X) = 2.8$, find a and b.

b Find $\mathrm{Var}(X)$.

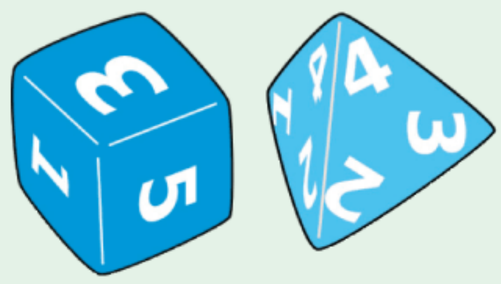
7 When a biased coin is tossed twice, the probability of getting two heads is 0.64.

a What is the probability of tossing a head with a single toss?

b If the coin is tossed 10 times, determine the probability of obtaining:

i exactly 6 heads **ii** at least 6 heads.

8 A 6-sided and 4-sided die are rolled simultaneously. Let X be the number of twos rolled.



a Explain why X is not a binomial random variable.

b Find the probability distribution of X.

c Find:

i the mean of X

ii the standard deviation of X.

9 Caleb is thrown a baseball 4 times. Let X be the number of times Caleb catches the ball. The probability distribution of X is shown alongside.

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

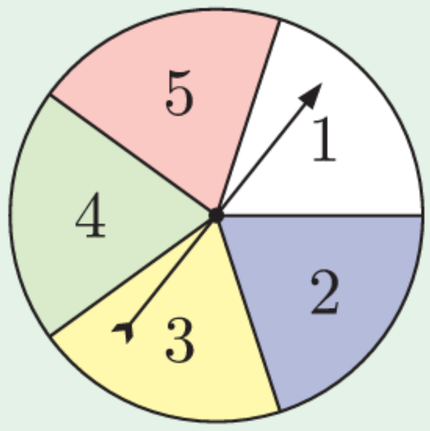
a Find:

i $E(X)$ **ii** $\text{Var}(X)$

b Let Y be the number of times Caleb *drops* the ball.

i Write Y in terms of X. **ii** Hence find $E(Y)$ and $\text{Var}(Y)$.

10 The spinner alongside is spun 20 times. Let X be the number of threes spun.



a Explain why X is a binomial random variable.

b Find the mean and standard deviation of X.

11 24% of visitors to a museum make voluntary donations. On a certain day the museum has 175 visitors.

a Find the expected number of donations.

b Find the probability that:

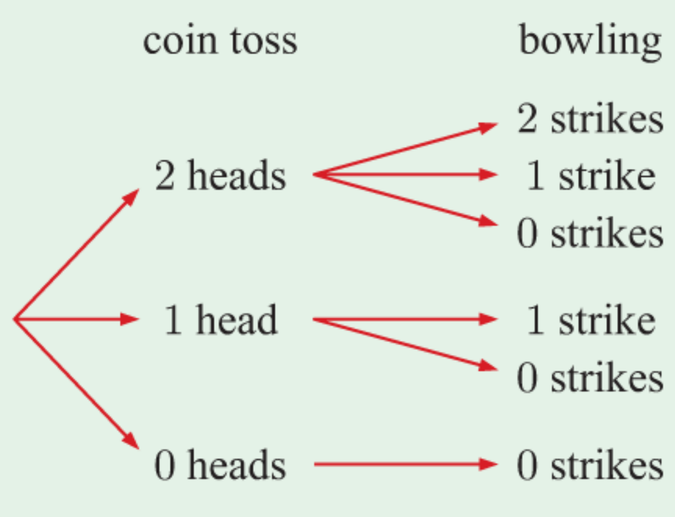
i less than 40 visitors make a donation

ii between 50 and 60 (inclusive) visitors make a donation.

12 Suvi plays a game involving 2 coins and a set of bowling pins. The coins are flipped and the number of heads that result is the number of attempts she gets to knock down the pins. If she knocks all of the pins down on a given attempt, it is called a "strike".

Suvi wins a prize worth \$10 multiplied by the number of strikes she gets. On each attempt, the probability that Suvi gets a strike is $\frac{1}{3}$.

a Copy and complete this tree diagram of possible outcomes.



b Let X be the number of strikes that Suvi gets. Find the probability distribution of X.

c Calculate Suvi's expected return per game.

d Find Suvi's expected gain if the game costs \$5 to play. Would you advise her to play the game many times?

13 It is observed that 3% of all batteries produced by a company are defective.

a For a random sample of 20 batteries, calculate the probability that:

i none are defective **ii** at least one is defective.

b Let X be the number of defectives in a random sample of n batteries.

i Write down an expression for $P(X = 0)$.

ii Calculate the smallest value of n such that $P(X \geqslant 1) \geqslant 0.3$.

Chapter 28

Continuous random variables

Contents:

A Probability density functions
B Measures of centre and spread
C The normal distribution
D Calculating normal probabilities
E The standard normal distribution
F Normal quantiles



OPENING PROBLEM

A salmon breeder is interested in the distribution of the *weight of female adult salmon*, w.

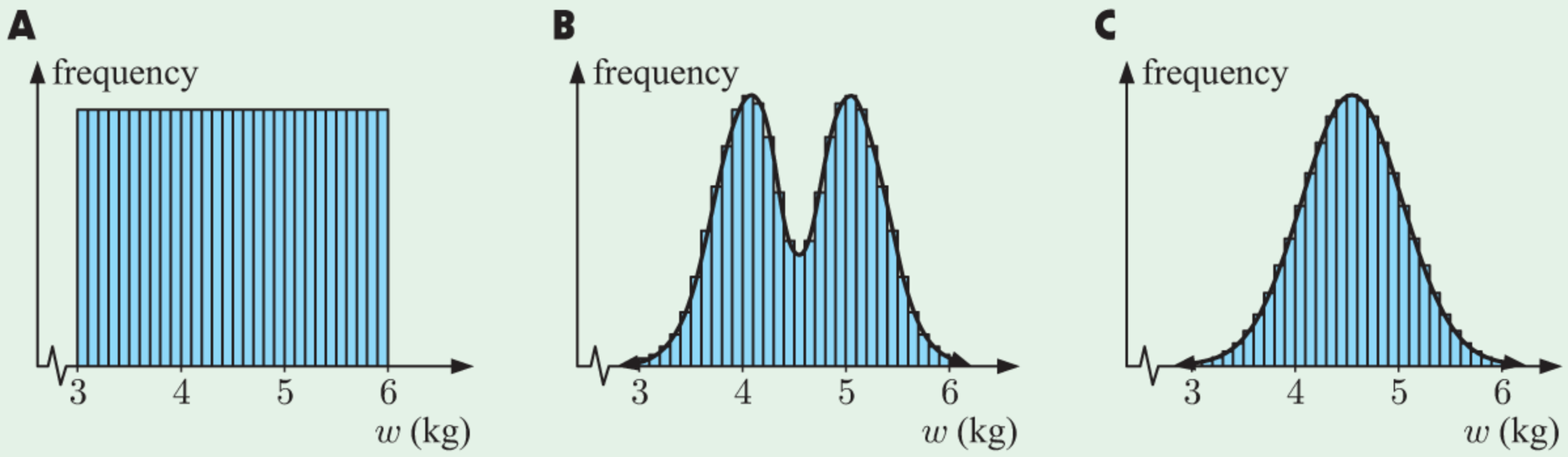
He catches hundreds of female adult fish and records their weights in a frequency table with class intervals $3 \leqslant w < 3.1$ kg, $3.1 \leqslant w < 3.2$ kg, $3.2 \leqslant w < 3.3$ kg, and so on.



The mean weight is 4.73 kg, and the standard deviation is 0.53 kg.

Things to think about:

a Which of these do you think is the most likely distribution for the weights of the female adult salmon?



b How can we use the mean and standard deviation to estimate the proportion of salmon that weigh:

i more than 6 kg **ii** between 4 kg and 6 kg?

c How can we find the weight which:

i 90% of salmon weigh less than **ii** 25% of salmon weigh more than?

In the previous Chapter we looked at discrete random variables and examined binomial probability distributions where the random variable X could take the non-negative integer values $x = 0, 1, 2, 3, 4, ..., n$ for some finite $n \in \mathbb{N}$.

For a **continuous random variable** X, x can take any real value within some reasonable domain. There are infinitely many values X can take, and even if a measuring device enabled us to measure X exactly, the measurements of X from any two members of the population would never be *identical*. This means that the probability that X is exactly equal to any particular value is zero.

For a continuous variable X, $\quad P(X = x) = 0 \quad$ for all x.

For example, the probability that an egg will weigh *exactly* 72.9 g is zero. If you were to weigh an egg on scales that measure to the nearest 0.1 g, a reading of 72.9 g means the weight lies somewhere between 72.85 g and 72.95 g. No matter how accurate your scales are, you can only ever know the weight of an egg within a range.

So, for a continuous variable X, we can only talk about the probability that a measured value lies in an **interval**.

Remembering that $P(X = x) = 0$ for all x,
$P(c \leqslant X \leqslant d) = P(c < X \leqslant d) = P(c \leqslant X < d) = P(c < X < d)$.

A PROBABILITY DENSITY FUNCTIONS

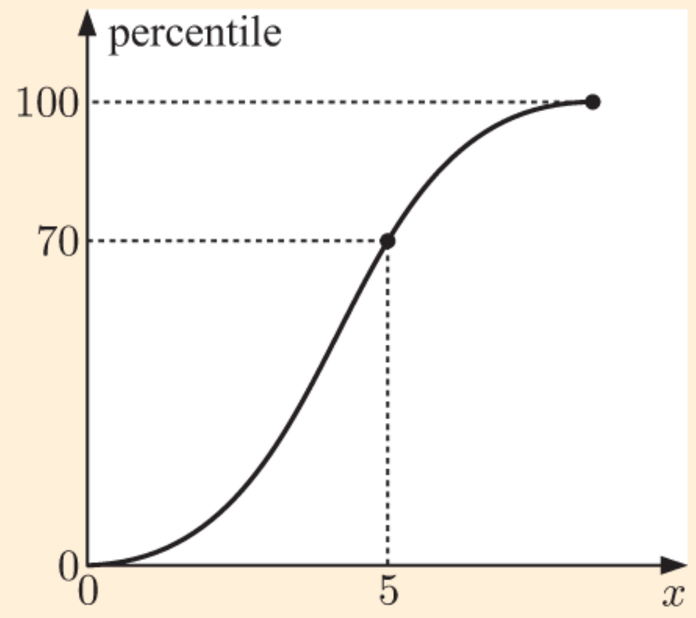
Since the probability that a continuous random variable takes a particular value is always 0, we cannot use a probability mass function to describe its distribution. Instead we use a function called a **probability density function**.

In the following **Investigation** we will see how the concept of a probability density function is motivated by considering the cumulative probability of a continuous random variable.

INVESTIGATION 1 PROBABILITY DENSITY FUNCTIONS

We have previously used a **cumulative frequency graph** to find the proportion of data values below a certain value.

For example, the graph alongside shows the percentile for different values of the variable X. We see that 70% of the data is less than 5.



We can describe the probability of a continuous random variable X more formally using the **cumulative distribution function** defined by $F(x) = \text{P}(X \leqslant x)$.

What to do:

1 Suppose a continuous random variable X has the cumulative distribution function $F(x)$, $a \leqslant x \leqslant b$.

a Explain why $F(x)$ is increasing for all x. What does this mean for $F'(x)$?

b Explain why:

i $F(a) = 0$ and $F(b) = 1$ **ii** $\text{P}(c \leqslant X \leqslant d) = F(d) - F(c)$.

c Find the range of $F(x)$.

2 Suppose a continuous random variable X has the cumulative distribution function $F(x) = \frac{1}{8}x^3$, $0 \leqslant x \leqslant 2$.

a Find:

i $\text{P}(1 \leqslant X \leqslant 2)$ **ii** $\text{P}(1 \leqslant X \leqslant 1.5)$ **iii** $\text{P}(1 \leqslant X \leqslant 1.25)$

iv $\text{P}(1 \leqslant X \leqslant 1.01)$ **v** $\text{P}(1 \leqslant X \leqslant 1.001)$ **vi** $\text{P}(1 \leqslant X \leqslant 1.0001)$

b Divide each probability in **a** by their respective interval widths. What do you notice?

c Differentiate $F(x)$ and find $F'(1)$. Compare $F'(1)$ with your answers to **b**.

d **i** Write an expression for $\text{P}(1 \leqslant X \leqslant 1 + h)$.

ii Explain why $\frac{1}{h}\text{P}(1 \leqslant X \leqslant 1 + h) \approx F'(1)$ for small values of h.

iii Hence explain why the probability that X is *close* to 1 is approximately $hF'(1)$ for small values of h.

e How can we interpret $F'(x)$?

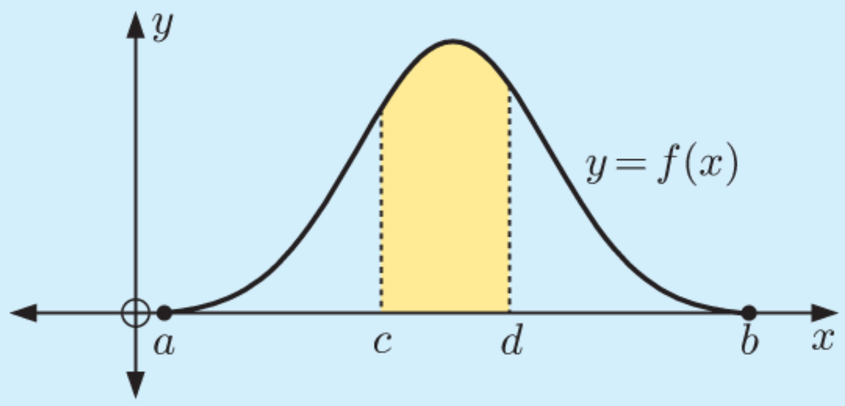
3 The **probability density function** of a continuous random variable is the derivative of the variable's cumulative distribution function.

How can we find probabilities associated with a continuous random variable if we are only given its probability density function?

For a continuous random variable X on the domain $a \leqslant x \leqslant b$, the **probability density function** is a function $f(x)$ such that

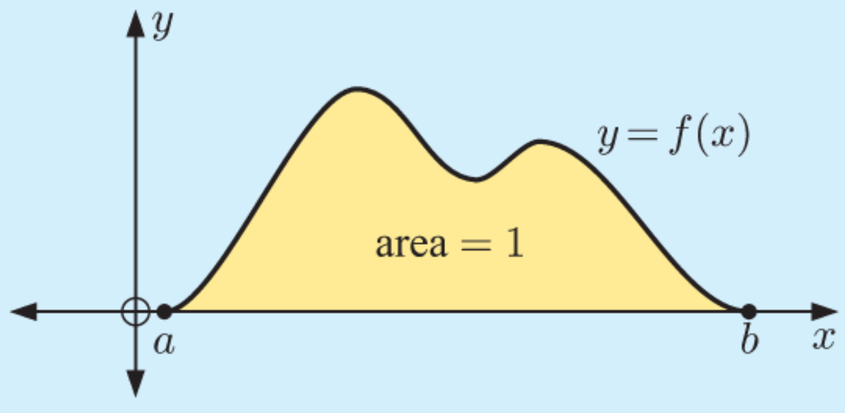
$$\mathbf{P}(c \leqslant X \leqslant d) = \int_c^d f(x)\,dx$$

$=$ the area under $y = f(x)$ between $x = c$ and $x = d$.



For $f(x)$ to be a valid probability density function for X on the domain $a \leqslant x \leqslant b$, it must satisfy:

- $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$
  {as probabilities cannot be negative}
- $\int_a^b f(x)\,dx = 1$
  {as the total probability must be 1}



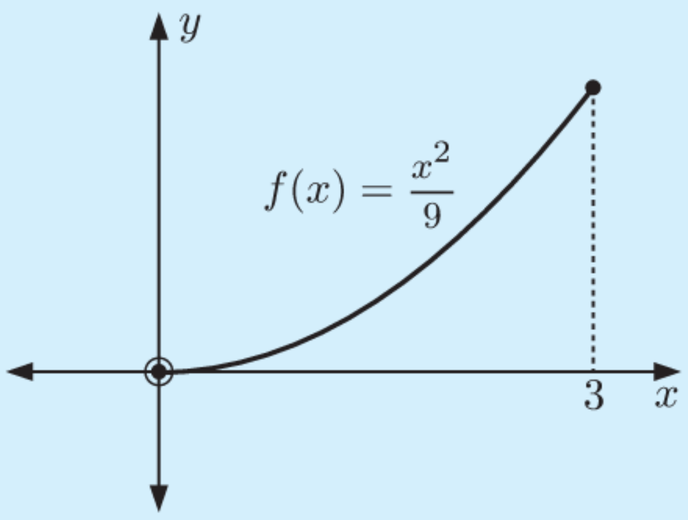
Important: The value of a probability density function is *not* a probability. Rather, it is a measure of the *density* of the probability near that point.

Example 1 **Self Tutor**

Consider the probability density function $f(x) = \frac{x^2}{9}$, $0 \leqslant x \leqslant 3$.

a Check that $f(x)$ is a valid probability density function.

b Find $\mathrm{P}(1 \leqslant X \leqslant 2)$.



a

- From the graph, $f(x) \geqslant 0$ for all $0 \leqslant x \leqslant 3$ ✓
- $\int_0^3 f(x)\,dx = \int_0^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^3 = \frac{27}{27} - 0 = 1$ ✓

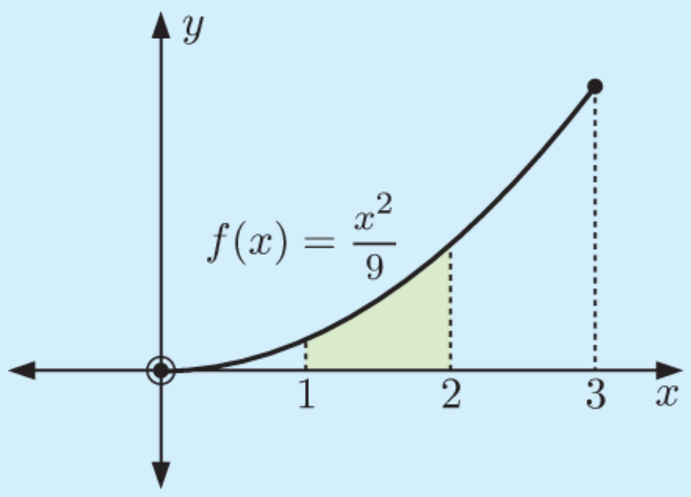
So, $f(x)$ is a valid probability density function.

b $\mathrm{P}(1 \leqslant X \leqslant 2) = \int_1^2 \frac{x^2}{9}\,dx$

$= \left[\frac{x^3}{27}\right]_1^2$

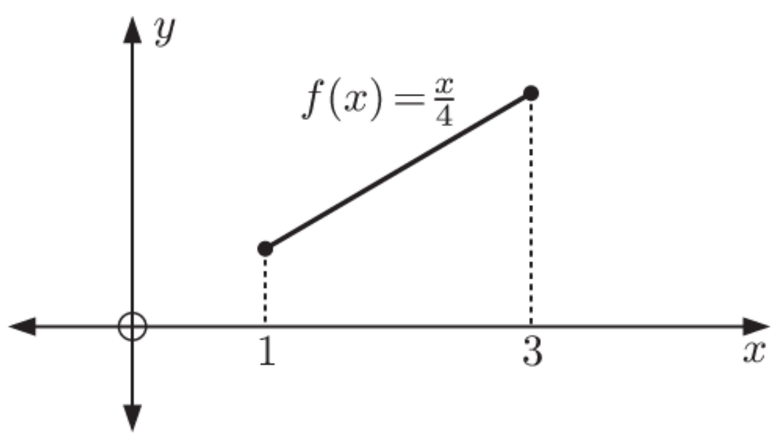
$= \frac{8}{27} - \frac{1}{27}$

$= \frac{7}{27}$



EXERCISE 28A

1 Consider the probability density function $f(x) = \frac{x}{4}$, $1 \leqslant x \leqslant 3$.



a Check that $f(x)$ is a valid probability density function.

b Find $\mathrm{P}(2 \leqslant X \leqslant 3)$.

2 Consider the probability density function $f(x) = \frac{2x}{3} - \frac{2x^2}{9}$, $0 \leqslant x \leqslant 3$.

a Sketch the graph of $y = f(x)$.

b Check that $f(x)$ is a valid probability density function.

c Find $\mathrm{P}(0 \leqslant X \leqslant 1)$.

3 **a** Show that $\int_0^{\frac{3\pi}{2}} \sin 2x \, dx = 1$.

b Explain why $f(x) = \sin 2x$ is *not* a valid probability density function for $0 \leqslant x \leqslant \frac{3\pi}{2}$.

4 A continuous random variable X has probability density function $f(x) = a\sqrt{x}$, $0 \leqslant x \leqslant 4$.

a Find a.

b Hence find $\mathrm{P}(0 \leqslant X \leqslant 1)$.

5 Suppose $f(x) = \frac{1}{x}$.

a Explain why $f(x)$ is not a valid probability density function.

b Find k such that $f(x)$ is a valid probability density function on the restricted domain $1 \leqslant x \leqslant k$.

c Find $\mathrm{P}(1.2 \leqslant X \leqslant 2.4)$ in this case.

6 **a** Find k such that $f(x) = ke^{-x}$ is a probability density function for $0 \leqslant x \leqslant 1$.

b Find $\mathrm{P}(0.1 \leqslant X \leqslant 0.5)$ correct to 3 significant figures.

7 The continuous random variable X has the probability density function $f(x) = ax^4$, $0 \leqslant x \leqslant k$. Given that $\mathrm{P}(X \leqslant \frac{2}{3}) = \frac{1}{243}$, find a and k.

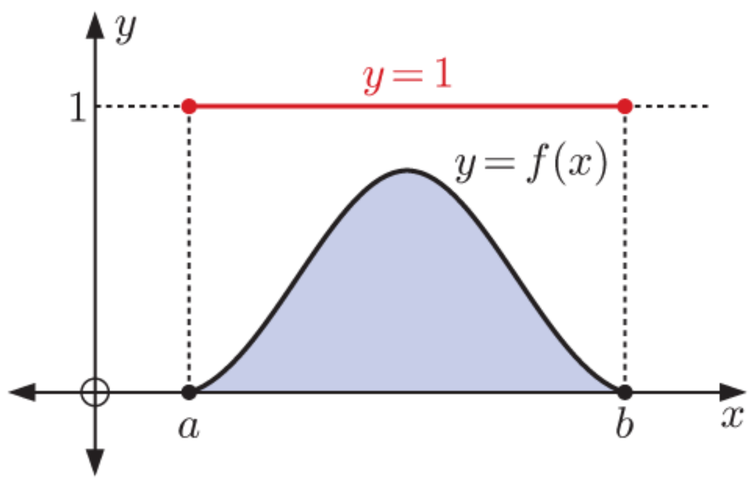
8 Let X be the weight, in kg, of suitcases checked in for plane flights. The probability density function for X is $f(x) = a(-x^2 + 30x - 144)$, $6 \leqslant x \leqslant 24$.

a Find a.

b Find the probability that a randomly selected suitcase:

i weighs between 10 kg and 15 kg **ii** is heavier than 20 kg.

9 Suppose a continuous random variable X has probability density function $f(x)$, $a \leqslant x \leqslant b$.



a Explain why $f(x)$ does not necessarily have to be less than or equal to 1 for all $a \leqslant x \leqslant b$.

b Under what condition(s) is it possible that $f(x) \leqslant 1$ for all $a \leqslant x \leqslant b$?

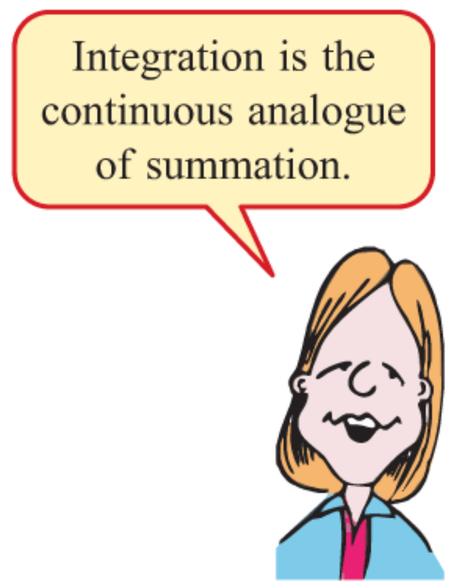
B MEASURES OF CENTRE AND SPREAD

For a continuous random variable X with probability density function $f(x)$ on $a \leqslant x \leqslant b$:

- The **mode** is the value of x which maximises $f(x)$ on $a \leqslant x \leqslant b$.
- The **median** is the value of m such that $\displaystyle\int_a^m f(x)\,dx = \tfrac{1}{2}$.
- The **mean** $\mu = \mathrm{E}(X) = \displaystyle\int_a^b x\,f(x)\,dx$.
- The **variance** $\sigma^2 = \mathrm{Var}(X) = \displaystyle\int_a^b (x-\mu)^2 f(x)\,dx$
$$= \int_a^b x^2 f(x)\,dx - \mu^2.$$
- The **standard deviation** $\sigma = \sqrt{\mathrm{Var}(X)}$.

We can now compare the formulae for expectation and variance of discrete and continuous random variables:

Statistic	*Discrete random variable*	*Continuous random variable*
$\mu = \mathrm{E}(X)$	$\sum x_i p_i$	$\displaystyle\int_a^b x\,f(x)\,dx$
$\sigma^2 = \mathrm{Var}(X)$	$\sum (x_i-\mu)^2 p_i$ $= \sum x_i^2 p_i - \mu^2$	$\displaystyle\int_a^b (x-\mu)^2 f(x)\,dx$ $= \displaystyle\int_a^b x^2 f(x)\,dx - \mu^2$
σ	$\sqrt{\mathrm{Var}(X)}$	

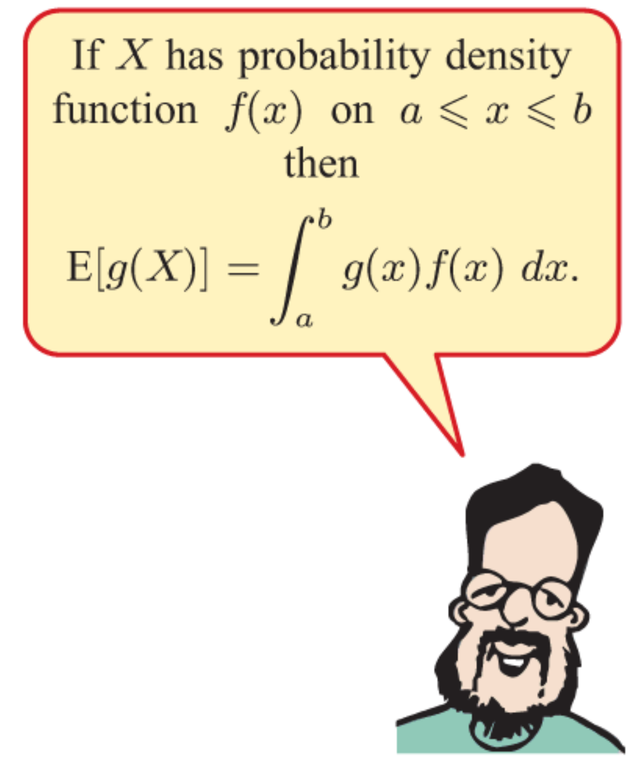


PROPERTIES OF E(X) AND $aX + b$

The properties of $\mathrm{E}(X)$ and $aX+b$ we saw in **Chapter 27** are also true if X is a continuous random variable.

If $\mathrm{E}(X)$ is the expected value of the random variable X, then:

1. $\mathrm{E}(k) = k$ for any constant k
2. $\mathrm{E}(kX) = k\,\mathrm{E}(X)$ for any constant k
3. $\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$ for functions g and h.



For a continuous random variable X and constants a, b:

- $\mathrm{E}(aX+b) = a\,\mathrm{E}(X) + b$
- $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$
- $\sigma(aX+b) = |a|\,\sigma(X)$

Example 2 **Self Tutor**

Consider the probability density function $f(x) = \frac{1}{2}x, \ 0 \leqslant x \leqslant 2$.

a Check that $f(x)$ is a valid probability density function.

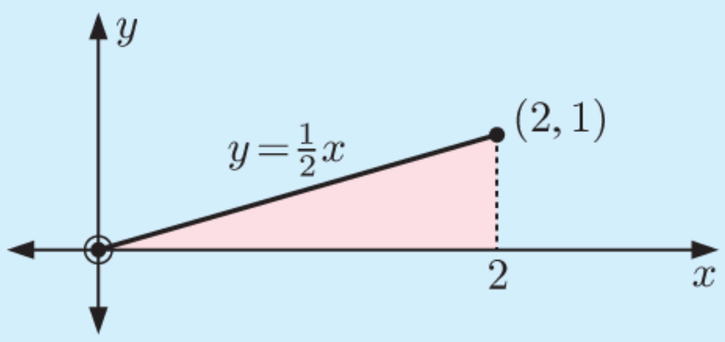
b For this distribution, find the:

i mode ii median iii mean.

c Find $\text{Var}(X)$ and σ.

---

a



- $f(x) \geqslant 0$ for all $0 \leqslant x \leqslant 2$ ✓
- Area $= \frac{1}{2} \times 2 \times 1 = 1$ ✓

  or $\displaystyle\int_0^2 \tfrac{1}{2}x \, dx = \left[\tfrac{1}{4}x^2\right]_0^2 = 1$ ✓

b **i** The maximum value of $f(x)$ is when $x = 2$

$\therefore$ the mode $= 2$

ii The median is the solution of

$$\int_0^m \tfrac{1}{2}x \, dx = \tfrac{1}{2}$$

$$\therefore \ \left[\tfrac{1}{4}x^2\right]_0^m = \tfrac{1}{2}$$

$$\therefore \ \frac{m^2}{4} = \tfrac{1}{2}$$

$$\therefore \ m^2 = 2$$

$$\therefore \ m = \sqrt{2} \quad \{\text{as } \ 0 \leqslant m \leqslant 2\}$$

iii
$$\mu = \int_0^2 x\, f(x)\, dx$$
$$= \int_0^2 \tfrac{1}{2}x^2 \, dx$$
$$= \left[\tfrac{1}{6}x^3\right]_0^2$$
$$= \tfrac{4}{3}$$

c
$$\text{Var}(X) = \int_0^2 x^2 f(x)\, dx - \mu^2$$
$$= \int_0^2 \tfrac{1}{2}x^3 \, dx - \mu^2$$
$$= \left[\tfrac{1}{8}x^4\right]_0^2 - \mu^2$$
$$= 2 - (\tfrac{4}{3})^2$$
$$= \tfrac{2}{9}$$

and $\sigma = \sqrt{\text{Var}(X)}$
$= \frac{\sqrt{2}}{3}$

EXERCISE 28B

1 Consider the probability density function $f(x) = \dfrac{x}{8}, \ 0 \leqslant x \leqslant 4$.

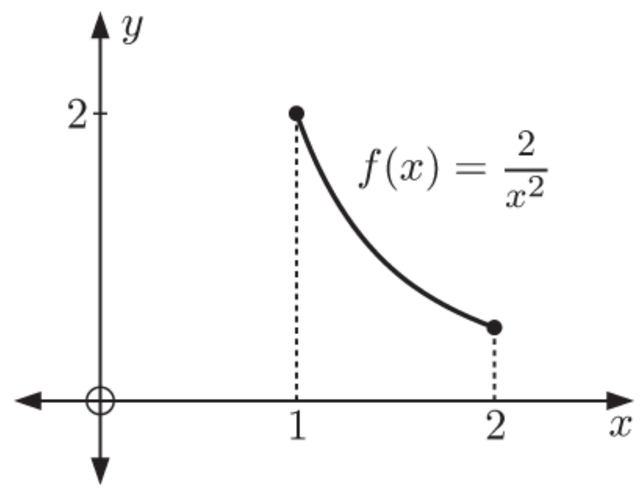
a Check that $f(x)$ is a valid probability density function.

b For this distribution, find the:

i mode ii median iii mean.

c Find $\text{Var}(X)$ and σ.

2 A continuous random variable X has the probability density function $f(x) = \frac{2}{x^2}$, $1 \leqslant x \leqslant 2$.



a Check that $f(x)$ is a valid probability density function.

b Find $\mathrm{P}(1 \leqslant X \leqslant 1.5)$.

c Find:

i $\mathrm{E}(X)$ **ii** $\mathrm{Var}(X)$ **iii** $\sigma(X)$

d Suppose $Y = 2X + 8$. Find:

i $\mathrm{E}(Y)$ **ii** $\mathrm{Var}(Y)$ **iii** $\sigma(Y)$

3 $f(x) = ax(x-4)$, $0 \leqslant x \leqslant 4$ is a probability density function.

a Find a. **b** Sketch the graph of $y = f(x)$.

c For this distribution, find the:

i mean **ii** median **iii** variance **iv** standard deviation.

4 A continuous random variable X has the probability density function $f(x) = \frac{1}{2x}$, $1 \leqslant x \leqslant k$.

a Find k.

b Hence find the exact value of: **i** μ **ii** $\mathrm{Var}(X)$ **iii** σ

c Find the standard deviation of $\left(1 - \frac{1}{3}X\right)$.

5 The probability density function for the random variable X is $f(x) = 5 - 12x$, $0 \leqslant x \leqslant k$.

a Notice that $\int_0^{\frac{1}{2}} (5 - 12x)\,dx = 1$. Explain why $k \neq \frac{1}{2}$ despite this result.

b Without using calculus, find k. **c** Find $\mathrm{P}(0 \leqslant X \leqslant \frac{1}{10})$.

d Find the mean and median value of X.

6 The probability density function for the uniform continuous random variable X is $f(x) = k$, $a \leqslant x \leqslant b$.

a Find k in terms of a and b. **b** Calculate the mean and median of X.

c Calculate $\mathrm{Var}(X)$ and the standard deviation of X.

7 $f(x) = ke^{-x}$, $0 \leqslant x \leqslant 3$ is a probability density function.

a Find k correct to 4 decimal places.

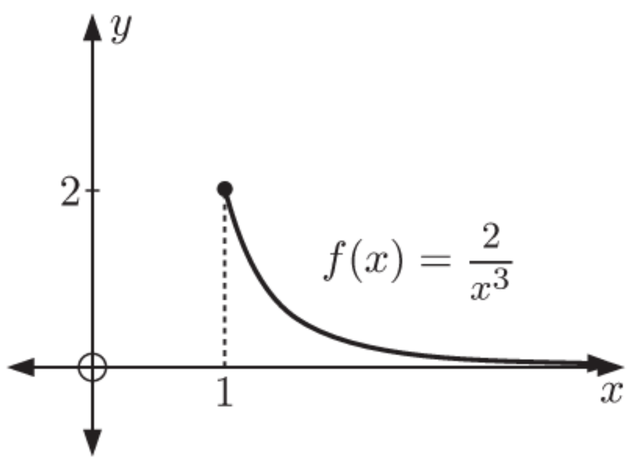
b Use technology to find the mean of the distribution.


GRAPHICS CALCULATOR INSTRUCTIONS

8 $f(x) = kx^2(x-6)$, $0 \leqslant x \leqslant 5$ is a probability density function. Find:

a k **b** the median **c** the mean **d** the variance.

9 The continuous random variable X has the probability density function $f(x) = \frac{2}{x^3}$, $x \geqslant 1$.



a Find $\mathrm{P}(1 \leqslant X \leqslant 3)$. **b** Find the median of X.

c Explain why the mean of X is $\mu = \int_1^{\infty} \frac{2}{x^2}\,dx$.

d Calculate μ using $\mu = \lim_{b \to \infty} \int_1^b \frac{2}{x^2}\,dx$.

10 The time taken X, in hours, to perform a particular task has the probability density function:

$$f(x) = \begin{cases} \frac{125}{18}x^2 & \text{if } 0 \leqslant x < 0.6 \\ \frac{9}{10x^2} & \text{if } 0.6 \leqslant x \leqslant 0.9 \\ 0 & \text{otherwise.} \end{cases}$$

a Sketch the graph of this function.

b Check that $f(x)$ is a valid probability density function.

c Find the mean, median, and mode of X.

d Find the variance and standard deviation of X.

e Find $\text{P}(0.3 < X < 0.7)$ and interpret your answer.

11 The probability distribution function of a continuous random variable X is given by:

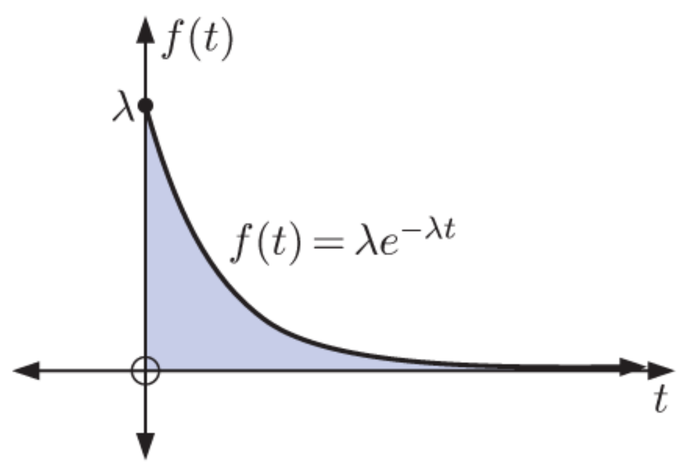
$$f(x) = \begin{cases} \frac{\sin x}{2} & \text{if } 0 \leqslant x \leqslant \frac{\pi}{2} \\ k(x - \frac{\pi}{2}) + 0.5 & \text{if } \frac{\pi}{2} < x \leqslant \frac{\pi}{2} + 2 \\ 0 & \text{otherwise.} \end{cases}$$

a Find $\text{P}(X \leqslant \frac{\pi}{2})$. **b** Find k.

c Write down the median of X.

d Calculate the mean of X.

12 An **exponential random variable** T has probability density function $f(t) = \lambda e^{-\lambda t}$ on the interval $t \geqslant 0$, where $\lambda > 0$ is a constant.



a Explain why $f(t) \geqslant 0$ for all $t \geqslant 0$.

b By considering $\lim\limits_{k \to \infty} \int_0^k f(t)\,dt$, show that $\int_0^\infty f(t)\,dt = 1$.

c **i** Explain why the mean of T is given by $\int_0^\infty \lambda t e^{-\lambda t}\,dt$.

ii Use integration by parts to find an expression for $\int_0^k \lambda t e^{-\lambda t}\,dt$.

iii By considering $\lim\limits_{k \to \infty} \int_0^k \lambda t e^{-\lambda t}\,dt$, show that T has mean $\frac{1}{\lambda}$.

d **i** Explain why the variance of T is given by $\int_0^\infty \lambda t^2 e^{-\lambda t}\,dt - \frac{1}{\lambda^2}$.

ii Use integration by parts twice to find an expression for $\int_0^k \lambda t^2 e^{-\lambda t}\,dt$.

iii By considering $\lim\limits_{k \to \infty} \int_0^k \lambda t^2 e^{-\lambda t}\,dt$, show that T has variance $\frac{1}{\lambda^2}$.

e Let the exponential random variable T be the time in minutes between customers arriving at a shop. The probability density function for T is $f(t) = 0.5e^{-0.5t}$, $t \geqslant 0$.

i Draw the graph of $f(t)$. **ii** Find the mean and variance of T.

iii Find the probability that the next two customers arrive less than 1 minute apart.

iv Find the median time between customers.

13 For a continuous random variable X with probability density function $f(x)$, $x \in \mathcal{D}$, prove that:

a $\mathrm{E}(kX) = k\,\mathrm{E}(X)$ for any constant k

b $\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$ for functions g and h

c $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$.

14 Consider a continuous random variable X with probability function $f(x)$, $x \in \mathcal{D}$. Prove that for any constants a and b:

a $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

b $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

15 Consider a continuous random variable X with probability density function $f_X(x)$ and cumulative distribution function $F_X(x) = \mathrm{P}(X \leqslant x) = \int_a^x f_X(x)\,dx$, $a \leqslant x \leqslant b$.

a Suppose $Y = 2X + 1$.

i Write down the possible values for Y.

ii The cumulative distribution function of Y is defined as $F_Y(y) = \mathrm{P}(Y \leqslant y)$. Write $F_Y(y)$ in terms of F_X.

iii The probability density function of Y is defined as the derivative of its cumulative distribution function, so $f_Y(y) = \dfrac{d}{dy} F_Y(y)$. Write $f_Y(y)$ in terms of f_X.

iv Use $f_Y(y)$ to write $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ in terms of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

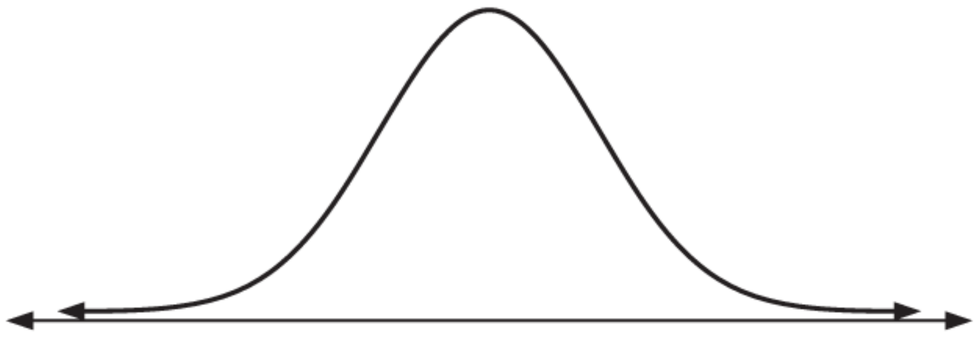
b Suppose we wish to find the distribution of the random variable $Y = g(X)$ where g is an invertible function.

i Show that $F_Y(y) = F_X(g^{-1}(y))$.

ii Show that $f_Y(y) = f_X(g^{-1}(y)) \times \dfrac{d}{dy} g^{-1}(y)$.

C THE NORMAL DISTRIBUTION

For the remainder of the Chapter, we will consider variables with **symmetrical**, **bell-shaped** probability density functions. We call this a **normal distribution**. It is the most important distribution in statistics.



The normal distribution arises in nature when many different factors affect the value of the variable.

For example, consider the apples harvested from an apple orchard. They do not all have the same weight. This variation may be due to genetic factors, the soil, the amount of sunlight reaching the leaves and fruit, weather conditions, and so on.

The result is that most of the fruit will have weights centred about the mean weight, and there will be fewer apples that are much heavier or much lighter than this mean.



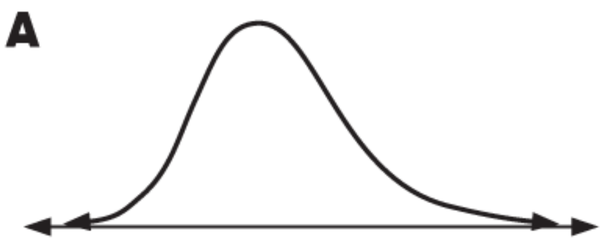
Some examples of quantities that may be normally distributed or approximately normally distributed are:

- the heights of 16 year old boys
- the lengths of adult sharks
- the yields of corn or wheat
- the volumes of liquid in soft drink cans
- the weights of peaches in a harvest
- the life times of batteries

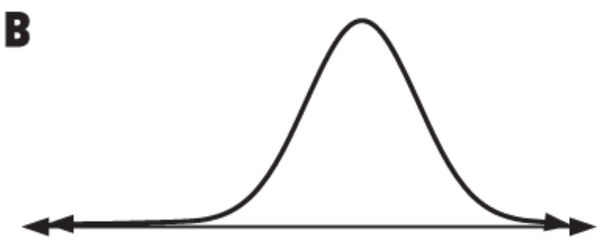
EXERCISE 28C.1

1 Which of the following appear to be normal distribution curves?

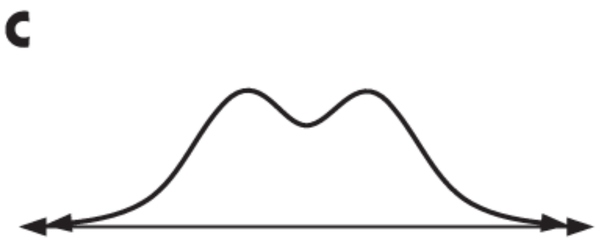
A



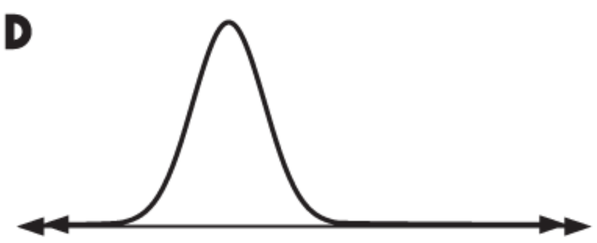
B



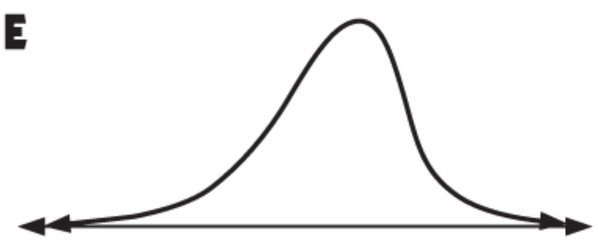
C



D



E



F



2 Explain why it is likely that the following variables will be normally distributed:

a the diameter of wooden rods cut using a lathe

b scores for tests taken by a large population

c the amount of time a student takes to walk to school each day.

3 Discuss whether the following variables are likely to be normally distributed. Sketch a graph to illustrate the possible distribution of each variable.

a the ages of people at a football match

b the distances recorded by a long jumper

c the numbers drawn in a lottery

d the lengths of carrots in a supermarket

e the amounts of time passengers spend waiting in a queue at an airport

f the numbers of brown eggs in a sample of cartons which each contain a dozen eggs

g the numbers of children in families living in Cardiff, Wales

h the heights of buildings in a city.

THE NORMAL PROBABILITY DENSITY FUNCTION

Although all normal distributions have the same general bell-shaped curve, the exact location and shape of the curve is determined by:

- the **mean** μ which measures the **centre** of the distribution
- the **standard deviation** σ which measures the **spread** of the distribution.

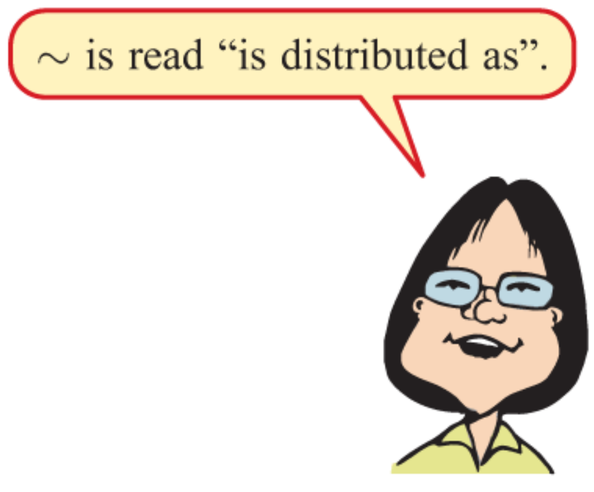
If X is a normally distributed random variable with mean μ and standard deviation σ, we write $\boldsymbol{X \sim N(\mu, \sigma^2)}$.

We say that μ and σ are the **parameters** of the distribution.

The probability density function of X is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{for } x \in \mathbb{R}$$

The graph of this function is called the **normal distribution curve** or just **normal curve**.

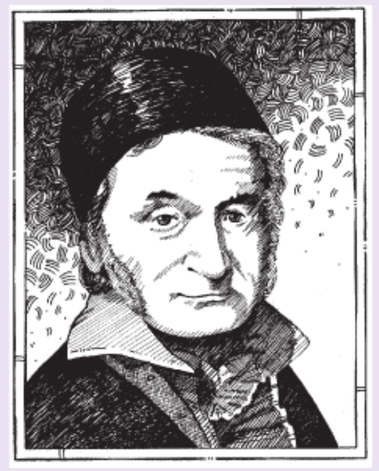


HISTORICAL NOTE

The normal distribution was first characterised by **Carl Friedrich Gauss** in 1809 as a way to rationalise his **method of least squares** for linear regression.

In fact, the normal distribution curve is a special case of the **Gaussian function** $f(x) = ae^{-\frac{1}{2}\left(\frac{x-b}{c}\right)^2}$ where a, b, and c are constants.

Since the normal distribution has such strong ties to Gauss, it is sometimes called the **Gaussian distribution**.



Carl Friedrich Gauss

INVESTIGATION 2 PROPERTIES OF THE NORMAL CURVE

In this Investigation, we will look at some interesting properties of the normal distribution curve $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ with the help of **graphing software** and **calculus**.

What to do:

1 Click on the icon to explore the normal distribution curve and how it changes when μ and σ are altered.

DEMO

a What effects do variations in μ and σ have on the curve? How do these relate to what μ and σ represent?

b Does the curve have a line of symmetry? If so, what is it?

c Is the function ever negative? Why is this important?

d Discuss the behaviour of the normal curve as $x \to \pm\infty$.

e What do you think happens to the *area* under the curve as you change μ and σ?

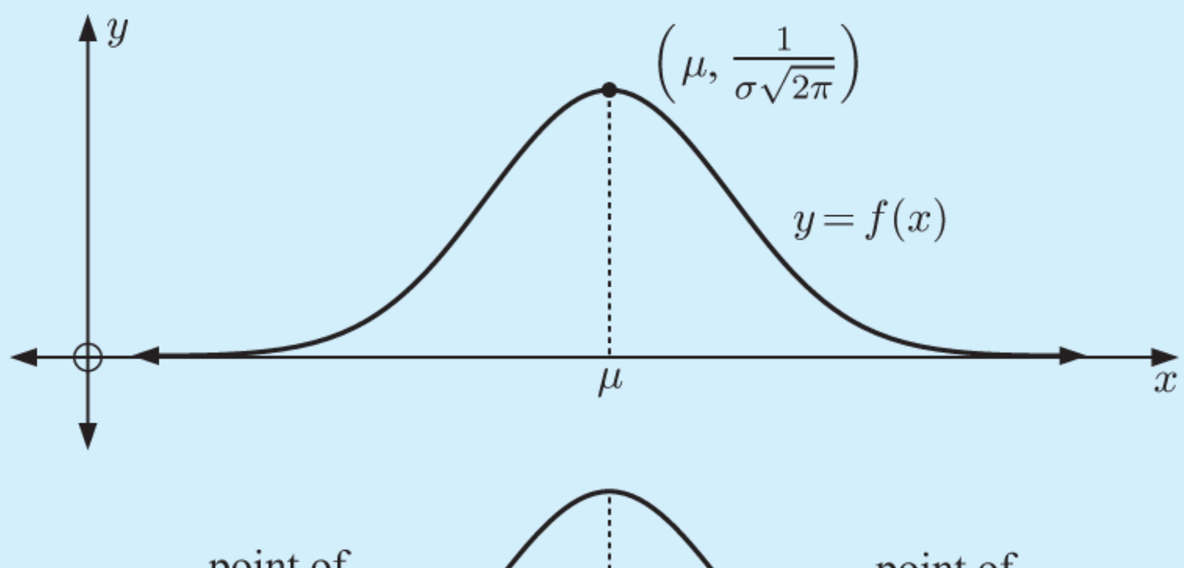
2 For a general normal curve, use calculus to find and classify the:

a stationary point **b** inflection points.

3 Sketch a normal distribution curve to illustrate your answers to **2**.

From the **Investigation**, you should have found that:

- The normal curve is symmetrical about the vertical line $x = \mu$.
- $f(x) > 0$ for all x.
- The x-axis is a horizontal asymptote.
- The maximum occurs at $x = \mu$.



- The points of inflection occur at $x = \mu \pm \sigma$, so the standard deviation is the horizontal distance from the line of symmetry $x = \mu$ to a point of inflection.

EXERCISE 28C.2

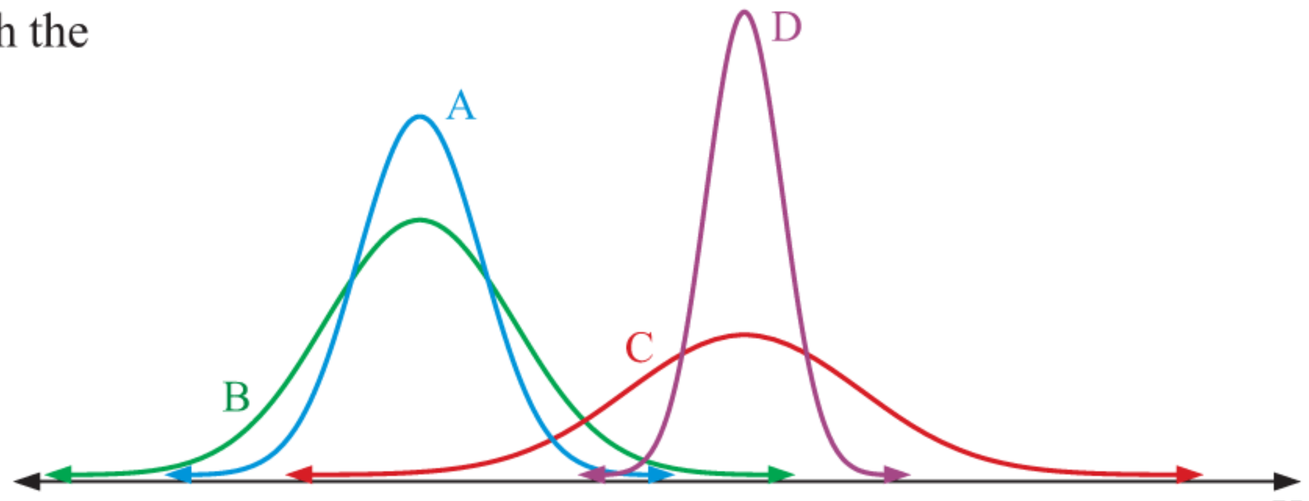
1 Match each pair of parameters with the correct normal distribution curve:

a $\mu = 5,\ \sigma = 2$

b $\mu = 15,\ \sigma = 0.5$

c $\mu = 5,\ \sigma = 1$

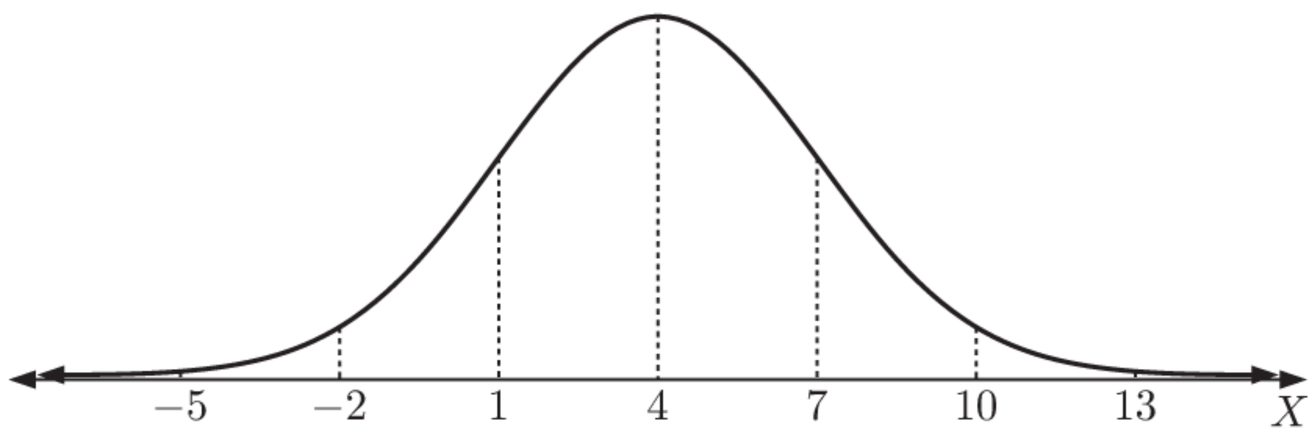
d $\mu = 15,\ \sigma = 3$



2 Sketch the following normal distributions on the same set of axes.

Distribution	*Mean* (mL)	*Standard deviation* (mL)
A	25	5
B	30	2
C	21	10

3 Consider the distribution curve of $X \sim \mathrm{N}(\mu, \sigma^2)$ where $\mu = 4$ and $\sigma = 3$ shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

a $\mathrm{N}(\mu + 2, \sigma^2)$ **b** $\mathrm{N}(\mu, (2\sigma)^2)$ **c** $\mathrm{N}(\mu + 2, (2\sigma)^2)$

d $\mathrm{N}\left(\mu - 1, \left(\frac{\sigma}{3}\right)^2\right)$ **e** $\mathrm{N}\left(3 + \mu, \frac{\sigma^2}{4}\right)$ **f** $\mathrm{N}(2 - \mu, 3\sigma^2)$

PRINTABLE GRAPH

4 **Marquis de Laplace** was the first to calculate the **Gaussian integral** $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Use Laplace's result to show that $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$ for any σ, μ.

Hint: Let $z = \frac{x - \mu}{\sqrt{2}\sigma}$.

5 The nth **moment** of a random variable X is $\mathrm{E}(X^n)$, $n \in \mathbb{Z}^+$. For example, the first moment of X is $\mathrm{E}(X^1) = \mu$, the mean.

The **moment generating function** of X is defined as $M_X(t) = \mathrm{E}(e^{tX})$ where $t \in \mathbb{R}$.

The nth moment of X is the nth derivative of the moment generating function with respect to t, evaluated at 0. In other words, $M_X^{(n)}(0) = \mathrm{E}(X^n)$ for all $n \in \mathbb{Z}^+$.

a Use the Maclaurin series expansion for e^{tX} to show that the moment generating function for X can also be written as $M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{E}(X^k)$.

b Hence show that:

i $M_X'(0) = \mathrm{E}(X)$ **ii** $M_X''(0) = \mathrm{E}(X^2)$

c Prove that $M_X^{(n)}(0) = \mathrm{E}(X^n)$ for all $n \in \mathbb{Z}^+$.

d Consider the normal distribution $X \sim \mathrm{N}(\mu, \sigma^2)$.

i Prove that the moment generating function is $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $t \in \mathbb{R}$.

Hint: If $f(x)$ is the probability density function of X then $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$.

ii Use $M_X(t)$ and the properties of moment generating functions to show that $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

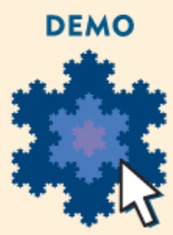
D CALCULATING NORMAL PROBABILITIES

INVESTIGATION 3 PROPORTIONS FROM A NORMAL DISTRIBUTION

In this Investigation we find the proportions of normally distributed data which lie within σ, 2σ, and 3σ of the mean.

What to do:

1 Click on the icon to run a demonstration which randomly generates 1000 data values from a normal distribution with mean μ and standard deviation σ. Set $\mu = 0$ and $\sigma = 1$.



2 Find the endpoints of the interval:

a $\mu - \sigma$ to $\mu + \sigma$ **b** $\mu - 2\sigma$ to $\mu + 2\sigma$ **c** $\mu - 3\sigma$ to $\mu + 3\sigma$

3 Use the frequency table provided to find the proportion of data values which lie between:

a $\mu - \sigma$ and $\mu + \sigma$ **b** $\mu - 2\sigma$ and $\mu + 2\sigma$ **c** $\mu - 3\sigma$ and $\mu + 3\sigma$

4 Repeat **2** and **3** for values of μ and σ of your choosing. Summarise your answers in a table like the one below.

μ	σ	$\mu - \sigma$ to $\mu + \sigma$		$\mu - 2\sigma$ to $\mu + 2\sigma$		$\mu - 3\sigma$ to $\mu + 3\sigma$	
		Interval	*Proportion*	*Interval*	*Proportion*	*Interval*	*Proportion*
0	1	$-1 \leqslant X \leqslant 1$					

5 What do you notice about the proportion of data values in each interval?

6 To more accurately calculate proportions from a normal distribution, we can integrate the normal probability density function. However, we cannot easily write an antiderivative for

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, so we calculate definite integrals numerically using technology.

a Suppose X is normally distributed with $\mu = 0$ and $\sigma = 1$. Use the **graphing package** to help you estimate the following probabilities with an appropriate definite integral:

i $\mathrm{P}(\mu - \sigma \leqslant X \leqslant \mu + \sigma)$ **ii** $\mathrm{P}(\mu - 2\sigma \leqslant X \leqslant \mu + 2\sigma)$

iii $\mathrm{P}(\mu - 3\sigma \leqslant X \leqslant \mu + 3\sigma)$



b Repeat **a** for each of the values of μ and σ that you chose in **4**. How do the definite integrals compare to your proportions from the simulation?

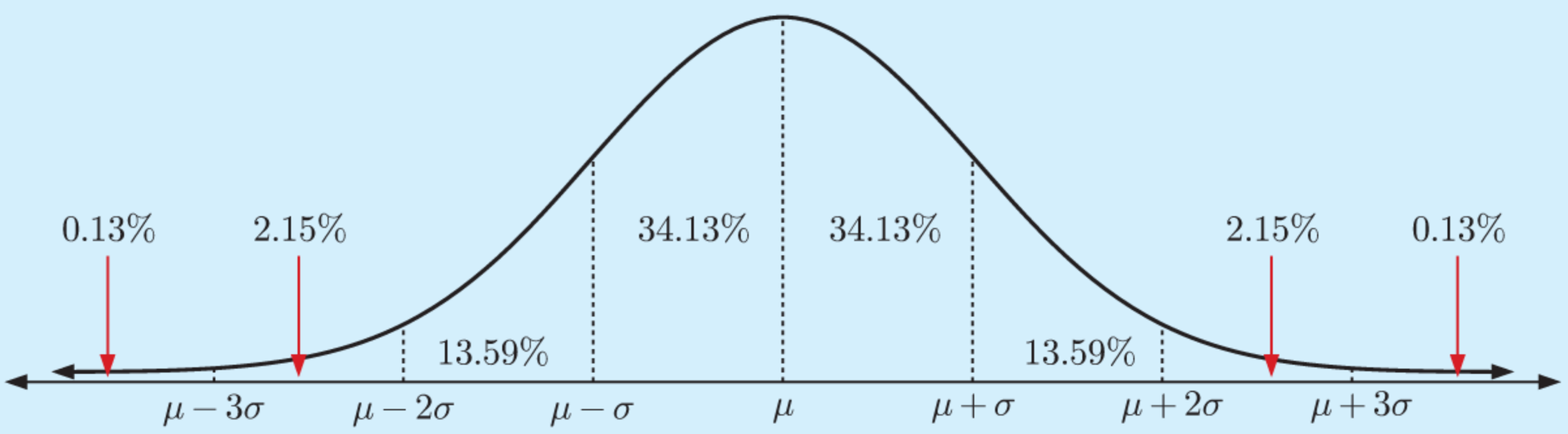
c Does changing the mean and standard deviation of a normal distribution change the proportion of the population that lies within 1, 2, or 3 standard deviations of the mean?

From the **Investigation**, you should have found that:

For any population that is normally distributed with mean μ and standard deviation σ:

- approximately $\mathbf{0.68}$ or $\mathbf{68\%}$ of the population will lie between $\boldsymbol{\mu - \sigma}$ and $\boldsymbol{\mu + \sigma}$
- approximately $\mathbf{0.95}$ or $\mathbf{95\%}$ of the population will lie between $\boldsymbol{\mu - 2\sigma}$ and $\boldsymbol{\mu + 2\sigma}$
- approximately $\mathbf{0.997}$ or $\mathbf{99.7\%}$ of the population will lie between $\boldsymbol{\mu - 3\sigma}$ and $\boldsymbol{\mu + 3\sigma}$.

The proportion of data values that lie within different ranges relative to the mean are:



For any variable that is normally distributed, we can use the mean and standard deviation to estimate the proportion of data that will lie in a given interval. This proportion tells us the probability that a randomly selected member of the population will be in that interval.

Example 3 **Self Tutor**

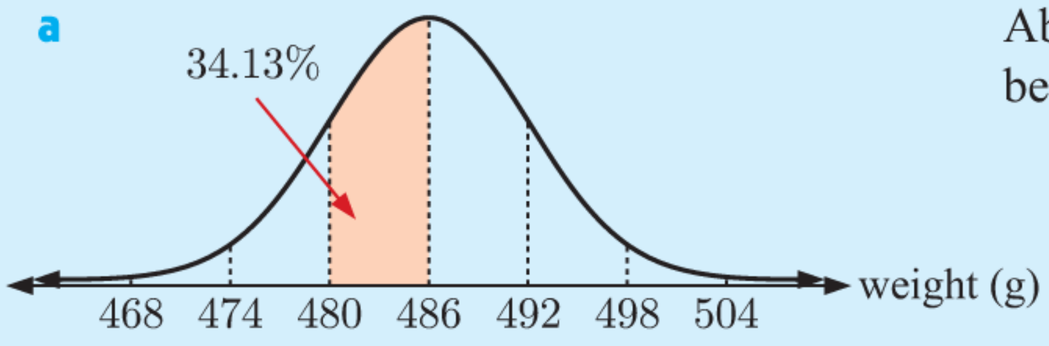
A sample of cans of peaches was taken from a warehouse, and the contents of each can was weighed. The sample mean was 486 g with standard deviation 6 g.

State the proportion of cans that weigh:

a between 480 g and 486 g **b** more than 492 g.

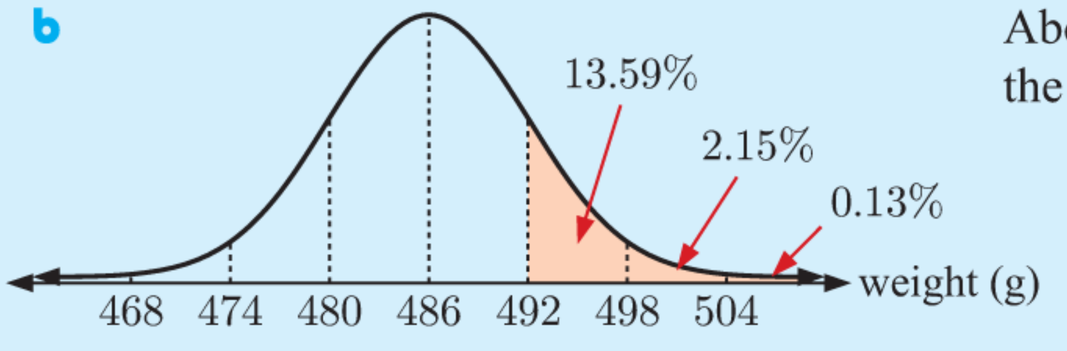
For a manufacturing process such as this, the distribution of weights is approximately normal.

a



About 34.13% of the cans are expected to weigh between 480 g and 486 g.

b



About $13.59\% + 2.15\% + 0.13\% = 15.87\%$ of the cans are expected to weigh more than 492 g.

EXERCISE 28D.1

1 Suppose X is normally distributed with mean 30 and standard deviation 5.

a State the value which is:

i 2 standard deviations above the mean **ii** 1 standard deviation below the mean.

b Describe the following values in terms of the number of standard deviations above or below the mean:

i 35 **ii** 20 **iii** 45

c Draw a curve to illustrate the distribution of X.

d What proportion of values of X are between 25 and 30?

e Find the probability that a randomly selected member of the population will measure between 35 and 40.

2 Suppose the variable X is normally distributed according to the curve shown.

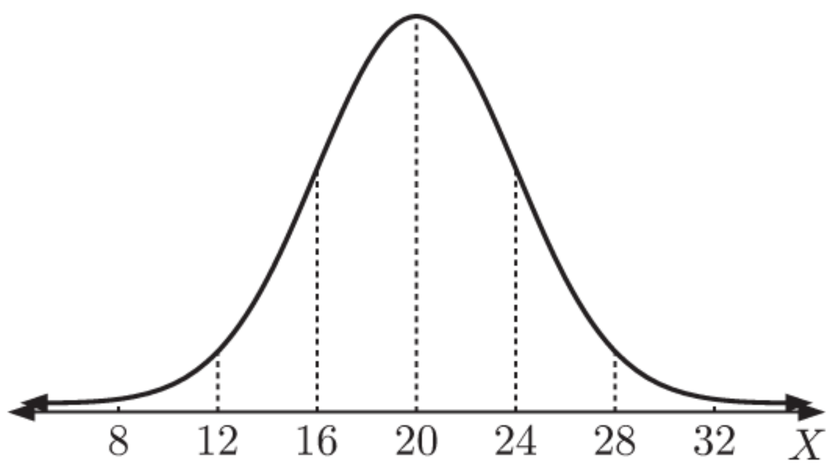
a State the mean and standard deviation of X.

b Find the proportion of values of X which are:

i between 20 and 24

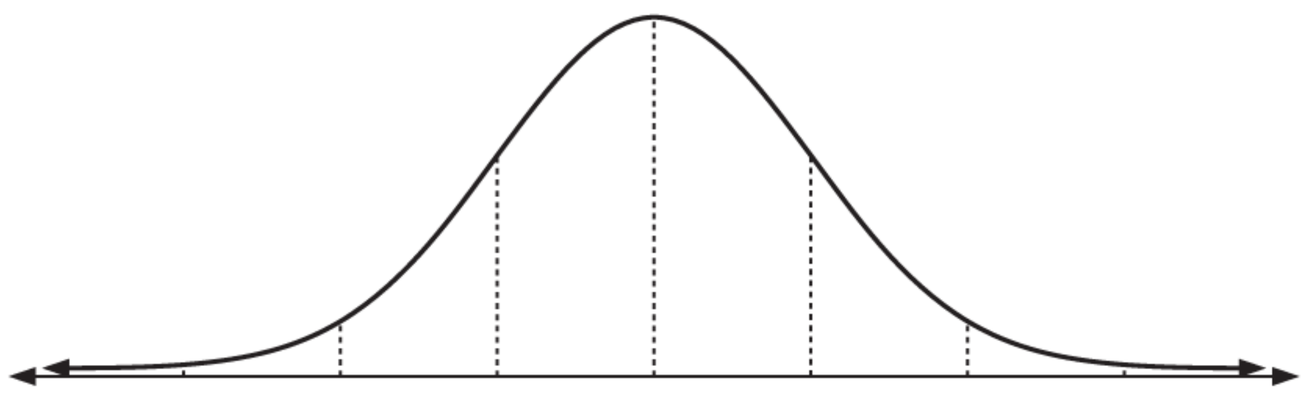
ii between 12 and 16

iii greater than 28.



3 A school's Grade 12 students sat for a Mathematics examination. Their marks were approximately normally distributed with mean 75 and standard deviation 8.

a Copy and complete this bell-shaped curve, assigning scores to the markings on the horizontal axis.



b What proportion of students would you expect to have scored:

i more than 83 **ii** less than 59 **iii** between 67 and 91?

4 State the probability that a randomly selected, normally distributed value:

a lies within one standard deviation either side of the mean

b is more than two standard deviations above the mean.

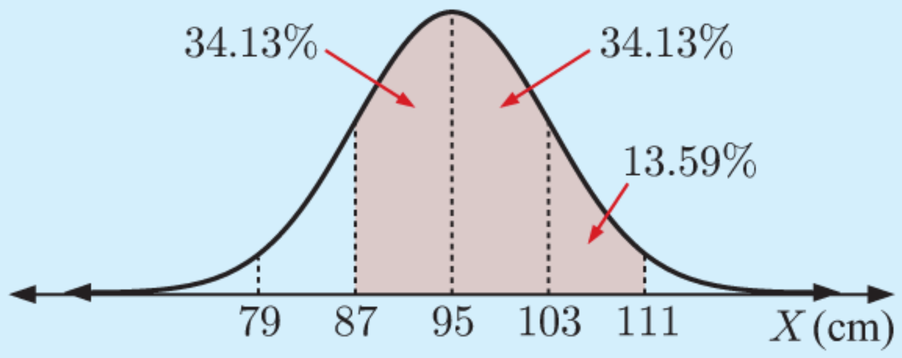
Example 4 **Self Tutor**

The chest measurements of 18 year old male rugby players are normally distributed with mean 95 cm and standard deviation 8 cm.

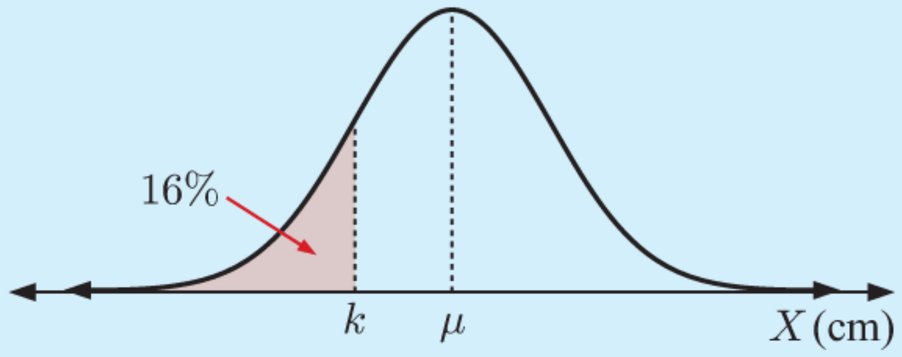
a From a group of 200 18 year old male rugby players, how many would you expect to have a chest measurement between 87 cm and 111 cm?

b Find the value of k such that approximately 16% of chest measurements are below k cm.

a About $34.13\% + 34.13\% + 13.59\% = 81.85\%$ of the rugby players have a chest measurement between 87 cm and 111 cm.
So, we would expect 81.85% of $200 \approx 164$ of the rugby players to have a chest measurement between 87 cm and 111 cm.



b Approximately 16% of data lies more than one standard deviation below the mean.
$\therefore$ k is σ below the mean μ
$\therefore$ $k = 95 - 8$
$= 87$



5 The height of female students at a university is normally distributed with mean 170 cm and standard deviation 8 cm.

a Find the percentage of female students whose height is:
i between 162 cm and 170 cm **ii** between 170 cm and 186 cm.

b Find the probability that a randomly chosen female student has a height:
i less than 154 cm **ii** greater than 162 cm.

c From a group of 500 female university students, how many would you expect to be between 178 cm and 186 cm tall?

d Estimate the value of k such that 16% of the female students are taller than k cm.

6 The weights of the 545 babies born at a maternity hospital last year were normally distributed with mean 3.0 kg and standard deviation 200 grams. Estimate the number that weighed:

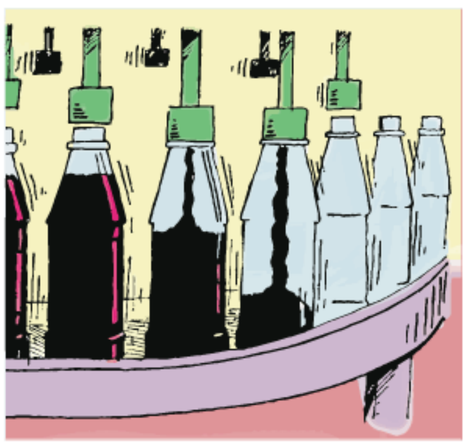
a less than 3.2 kg **b** between 2.8 kg and 3.4 kg.

7 An industrial machine fills an average of 20 000 bottles each day with standard deviation 2000 bottles. Assuming that production is normally distributed and the year comprises 260 working days, estimate the number of working days on which:

a under 18 000 bottles are filled

b over 16 000 bottles are filled

c between 18 000 and 24 000 bottles are filled.



8 Two hundred lifesavers competed in a swimming race. Their times were normally distributed with mean 10 minutes 30 seconds and standard deviation 15 seconds. Estimate the number of competitors who completed the race in a time:

a longer than 11 minutes **b** less than 10 minutes 15 seconds

c between 10 minutes 15 seconds and 10 minutes 45 seconds.

9 The weights of Jason's oranges are normally distributed. 84% of the crop weighs more than 152 grams and 16% weighs more than 200 grams.

a Find μ and σ for the crop.

b What percentage of the oranges weigh between 152 grams and 224 grams?

10 When a particular variety of radish is grown without fertiliser, the weights of the radishes produced are normally distributed with mean 40 g and standard deviation 10 g.
When these radishes are grown in the same conditions but with fertiliser added, their weights are also normally distributed, but with mean 140 g and standard deviation 40 g.

a Determine the proportion of radishes grown:

i without fertiliser which weigh less than 50 grams

ii with fertiliser which weigh less than 60 grams.

b Find the probability that a randomly selected radish weighs between 20 g and 60 g, if it is grown:

i with fertiliser **ii** without fertiliser.

c One radish grown with fertiliser and one radish grown without fertiliser are selected at random. Find the probability that *both* radishes weigh more than 60 g.

USING TECHNOLOGY

When calculating normal distribution probabilities, we have so far only considered numbers that are a whole number of standard deviations from the mean.

To calculate other probabilities, we could use definite integrals of the normal probability density function. However, this function does not have an indefinite integral, so we need to use a numerical approximation. Your **graphics calculator** has built-in functions to calculate these integrals.



Example 5 **Self Tutor**

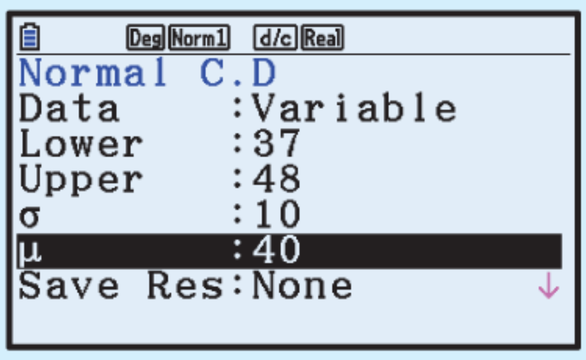
The variable X is normally distributed with mean 40 and standard deviation 10. Find:

a $P(37 < X < 48)$ **b** $P(X > 45)$ **c** $P(X < 26)$

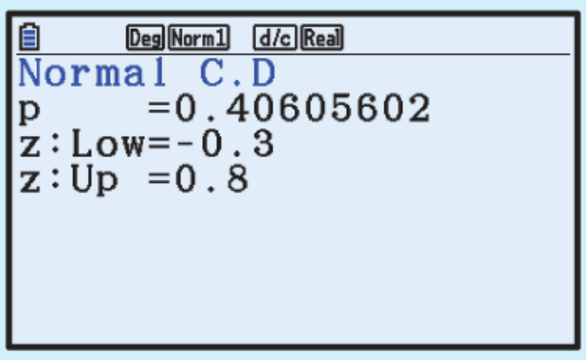
Illustrate your answers.

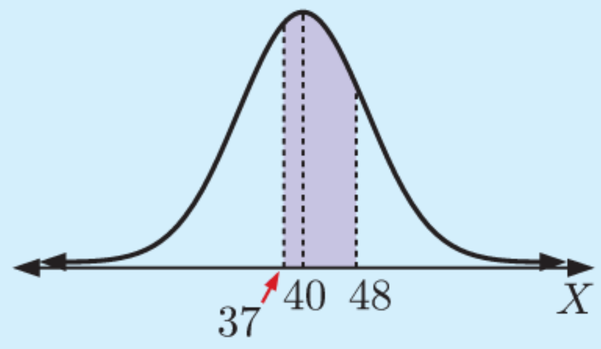
a To find $P(37 < X < 48)$, we set the lower bound to 37 and the upper bound to 48.

Casio fx-CG50



Casio fx-CG50

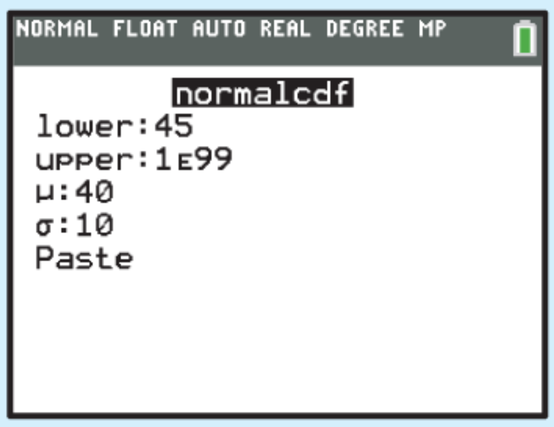




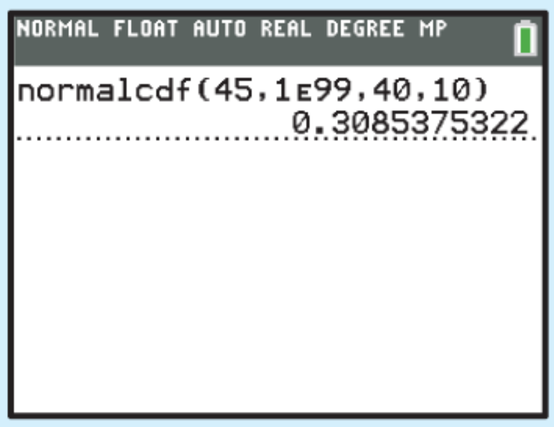
$P(37 < X < 48) \approx 0.406$

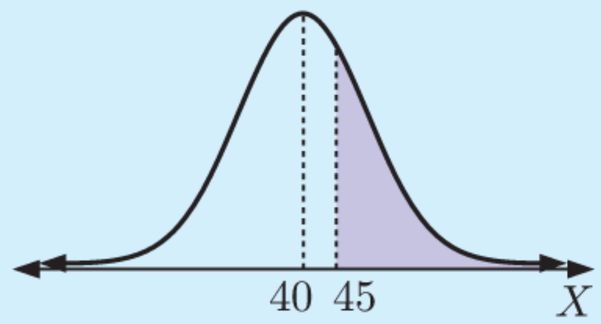
b To find $P(X > 45)$, we use a very high value such as 10^{99} to represent the upper bound.

TI-84 Plus CE



TI-84 Plus CE

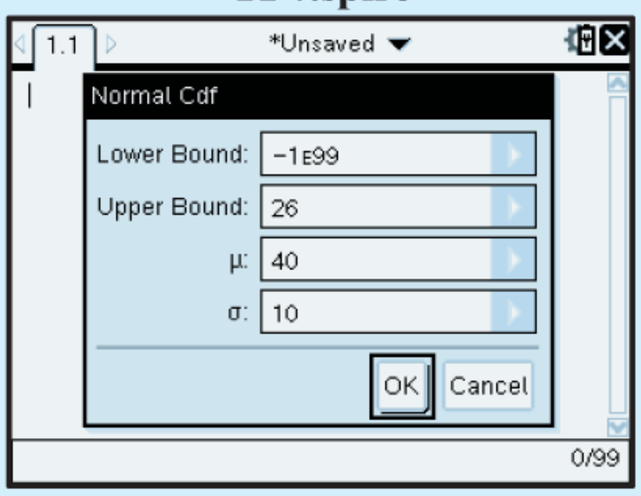




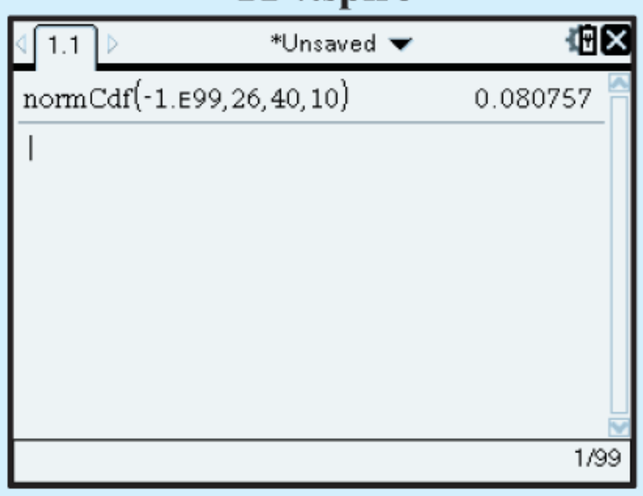
$P(X > 45) \approx 0.309$

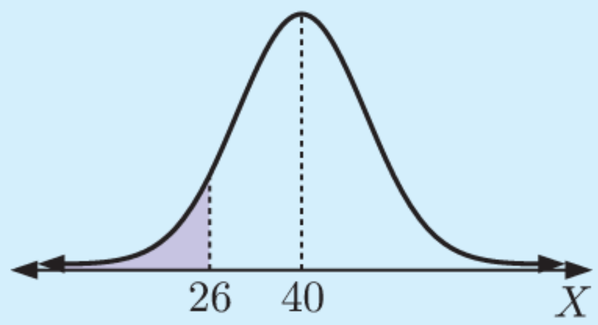
c To find $P(X < 26)$, we use a very low value such as -10^{99} to represent the lower bound.

TI-*n*spire



TI-*n*spire





$P(X < 26) \approx 0.081$

EXERCISE 28D.2

1 Suppose X is normally distributed with mean 60 and standard deviation 5. Find:

a $P(60 \leqslant X \leqslant 65)$ **b** $P(62 \leqslant X \leqslant 67)$

c $P(X \geqslant 64)$ **d** $P(X \leqslant 68)$

e $P(X \leqslant 61)$ **f** $P(57.5 \leqslant X \leqslant 62.5)$

Illustrate your answers.

A continuous random variable X can never be *exactly* 64, so $P(X \geqslant 64) = P(X > 64)$.



2 Suppose X is normally distributed with mean 37 and standard deviation 7.

a Use technology to find $P(X > 40)$.

b Hence find $P(37 \leqslant X \leqslant 40)$ without technology.

3 A machine produces metal bolts. The lengths of these bolts have a normal distribution with mean 19.8 cm and standard deviation 0.3 cm.
If a bolt is selected at random from the machine, find the probability that it will have a length between 19.7 cm and 20 cm.

4 The speed of cars passing a supermarket is normally distributed with mean 46.3 km h^{-1} and standard deviation 7.4 km h^{-1}. Find the probability that a randomly selected car is travelling:

a between 50 and 65 km h^{-1} **b** slower than 60 km h^{-1}

c faster than 50 km h^{-1}.

5 Eels are washed onto a beach after a storm. Their lengths have a normal distribution with mean 41 cm and standard deviation 5.5 cm.

a If an eel is randomly selected, find the probability that it is at least 50 cm long.

b Find the percentage of eels measuring between 40 cm and 50 cm long.

c How many eels from a sample of 200 would you expect to measure at least 45 cm in length?

6 Max's customers put money for charity into a collection box on the front counter of his shop. The weekly collection is approximately normally distributed with mean \$40 and standard deviation \$6.

a On what percentage of weeks would Max expect to collect:

i between \$30 and \$50 **ii** at least \$50?

b How much money would you expect Max to collect in two years?

7 The amount of petrol bought by customers at a petrol station is normally distributed with mean 36 L and standard deviation 7 L.

a What percentage of customers buy:

i less than 28 L of petrol **ii** between 30 L and 40 L of petrol?

b On a particular day, the petrol station has 600 customers.

i How much petrol would you expect the petrol station to sell on this day?

ii How many customers would you expect to buy at least 44 L of petrol?

8 The times Enrique and Damien spend working out at the gym each day are both normally distributed with mean 45 minutes. The standard deviation of Enrique's times is 9 minutes, and the standard deviation of Damien's times is 6 minutes.



a On what percentage of days does:

i Enrique spend between 32 and 40 minutes at the gym

ii Damien spend less than 55 minutes at the gym?

b Tomorrow, who do you think is more likely to spend:

i at least 1 hour at the gym **ii** between 40 minutes and 50 minutes at the gym?

Explain your answers.

c Perform calculations to check your answers to **b**.

Example 6 **Self Tutor**

The times taken by students to complete a puzzle are normally distributed with mean 28.3 minutes and standard deviation 3.6 minutes. Calculate the probability that:

a a randomly selected student took at least 30 minutes to complete the puzzle

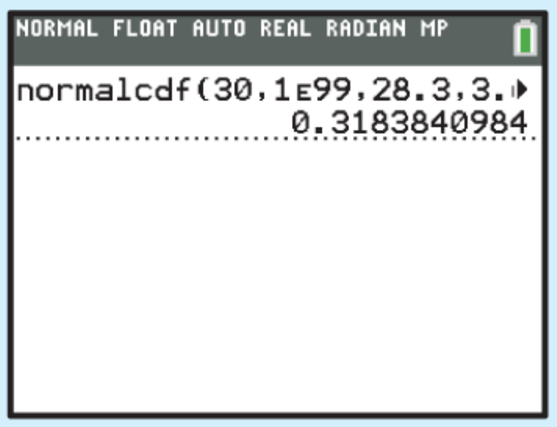
b out of 10 randomly selected students, 5 or fewer of them took at least 30 minutes to complete the puzzle.

a Let X denote the time for a student to complete the puzzle.

$X \sim \text{N}(28.3,\ 3.6^2)$

$\therefore\ \text{P}(X \geqslant 30) \approx 0.318\,38$

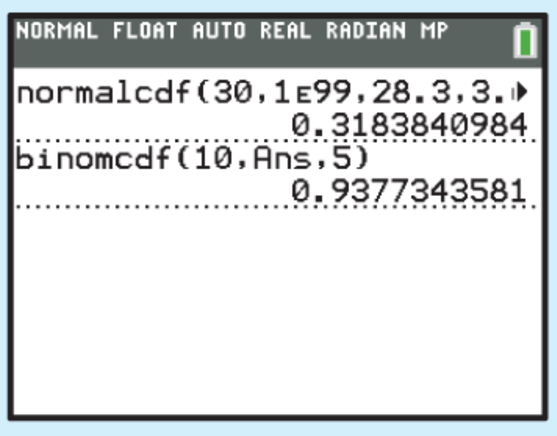
≈ 0.318



b Let Y denote the number of students who took at least 30 minutes to complete the puzzle.

$Y \sim \text{B}(10,\ 0.318\,38)$

$\therefore\ \text{P}(Y \leqslant 5) \approx 0.938$





B$(n,\ p)$ is the binomial distribution with n independent trials, each with probability of success p.

9 Apples from a grower's crop were normally distributed with mean 173 grams and standard deviation 34 grams. Apples weighing less than 130 grams were too small to sell.

a Find the percentage of apples from this crop which were too small to sell.

b Find the probability that in a picker's basket of 100 apples, more than 10 apples were too small to sell.

10 People found to have high blood pressure are prescribed a course of tablets. They have their blood pressure checked at the end of 4 weeks. The drop in blood pressure over the period is normally distributed with mean 5.9 units and standard deviation 1.9 units.

a Find the proportion of people who show a drop of more than 4 units.

b Eight people taking the course of tablets are selected at random. Find the probability that at least six of them will show a drop in blood pressure of more than 4 units.

E THE STANDARD NORMAL DISTRIBUTION

Suppose a random variable X is normally distributed with mean μ and standard deviation σ.

For each value of x we can calculate a **z-score** using the algebraic transformation $z = \dfrac{x - \mu}{\sigma}$.

This algebraic transformation is known as the **Z-transformation**.

INVESTIGATION 4 z-SCORES

In this Investigation we consider how the z-scores for a distribution are themselves distributed.

What to do:

1 Consider the x-values: 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7.

a Draw a histogram of the x-values to check that the distribution is approximately normal.

b Find the mean μ and standard deviation σ of the x-values.

c Calculate the z-score for each x-value.

d Find the mean and standard deviation of the z-scores.

2 Click on the icon to access a demo which randomly generates data values from a normal distribution with given mean and standard deviation. The z-score of each data value is calculated, and histograms of the original data and the z-scores are also shown.



a Generate samples using various values of μ and σ of your choosing.

b Record the mean and standard deviation of the z-scores in a table like the one below.

x-values		*z-scores*	
Mean	*Standard deviation*	*Mean*	*Standard deviation*

c How does the histogram of the z-scores generally compare with the histogram of the x-values?

d What conclusions can you make about the *distribution* of the z-scores?

From the **Investigation**, you should have found that the z-scores were normally distributed with mean 0 and standard deviation 1.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ and $\boldsymbol{Z = \dfrac{X - \mu}{\sigma}}$ then $\boldsymbol{Z \sim \mathrm{N}(0, 1^2)}$.

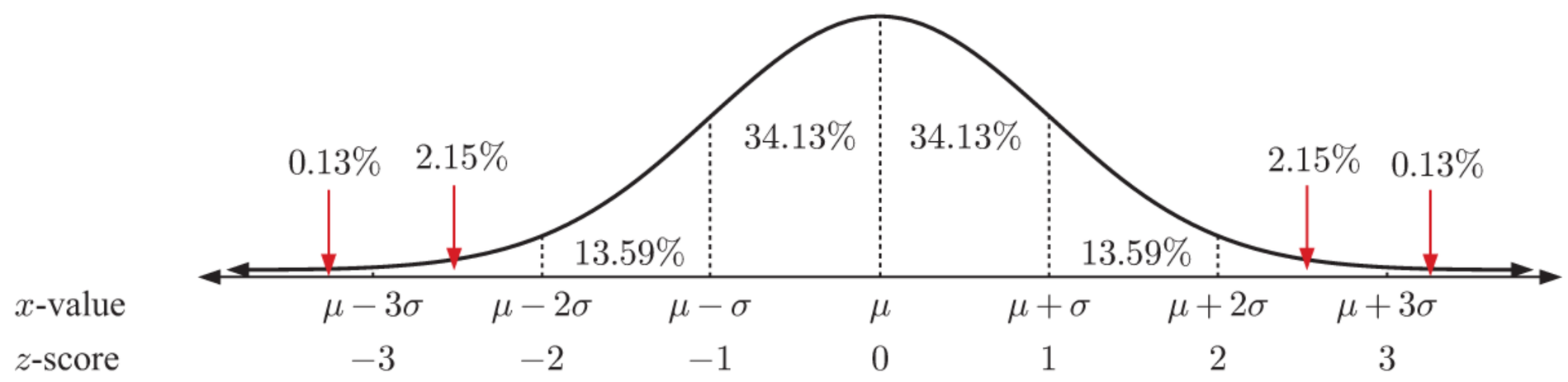
No matter what the parameters μ and σ of the original X-distribution are, we always end up with the same Z-distribution, $Z \sim \mathrm{N}(0, 1^2)$.

Click on the icon to see why the Z-transformation works.



The distribution $\boldsymbol{Z \sim \mathrm{N}(0, 1^2)}$ is called the **standard normal distribution** or **Z-distribution**.

The diagram below shows how z-scores are related to a general normal curve:



Notice that the value of the z-score corresponds to the coefficient of σ in the x-value.

The **z-score** of x is the number of standard deviations x is from the mean.

For example:

- if $z = 1.84$, then x is 1.84 standard deviations to the *right* of the mean
- if $z = -0.273$, then x is 0.273 standard deviations to the *left* of the mean.

z-scores are particularly useful when comparing two populations with different μ and σ. However, these comparisons will only be reasonable if both distributions are approximately normal.

Example 7 **Self Tutor**

Kelly scored 73% in History, where the class mean was 68% and the standard deviation was 10.2%. In Mathematics she scored 66%, where the class mean was 62% and the standard deviation was 6.8%.

In which subject did Kelly perform better compared with the rest of her class?

Assume the scores for both subjects were normally distributed.

Kelly's z-score for History $= \dfrac{73 - 68}{10.2} \approx 0.490$

Kelly's z-score for Mathematics $= \dfrac{66 - 62}{6.8} \approx 0.588$

Kelly's result in Mathematics was 0.588 standard deviations above the mean, whereas her result in History was 0.490 standard deviations above the mean.

$\therefore$ Kelly's result in Mathematics was better compared to her class, even though her percentage was lower.

EXERCISE 28E.1

1 In Emma's classes, the exam results for each subject are normally distributed with the mean μ and standard deviation σ shown in the table.

Subject	*Emma's score*	μ	σ
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Mathematics	84	50	15

a Find the z-score for each of Emma's scores.

b Arrange Emma's subjects from best to worst in terms of the z-scores.

c Explain why the z-scores are a reasonable way to compare Emma's performances with the rest of her class.

2

Subject	*Sergio's score*	μ	σ
Physics	73%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	76%	74%	10.1%
German	91%	86%	9.6%
Biology	58%	62%	5.2%

The table alongside shows Sergio's results in his final examinations, along with the class means and standard deviations.

a Find Sergio's z-score for each subject.

b Hence arrange Sergio's performances in each subject from best to worst.

3 At a swimming competition, Frederick competed in the 50 m freestyle, 100 m backstroke, 200 m breaststroke, and 100 m butterfly events. His times are summarised in the table, along with the event means and standard deviations.

Event	*Time* (seconds)	μ (seconds)	σ (seconds)
50 m freestyle	32.1	27.8	2.2
100 m backstroke	53.5	58.1	4.3
200 m breaststroke	140.0	143.7	6.4
100 m butterfly	59.6	57.7	5.5

a Calculate the z-scores for each of Frederick's times.

b Explain why in this case a lower z-score indicates a better performance.

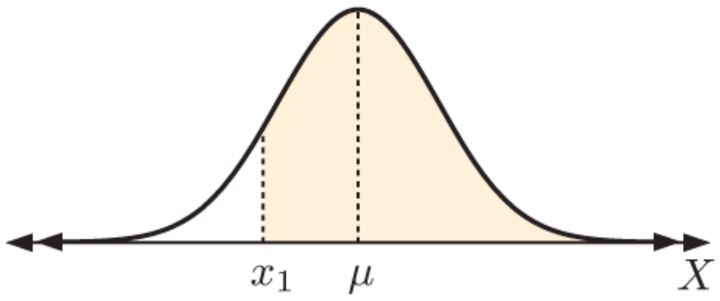
c Hence arrange Frederick's performances in each event from best to worst.

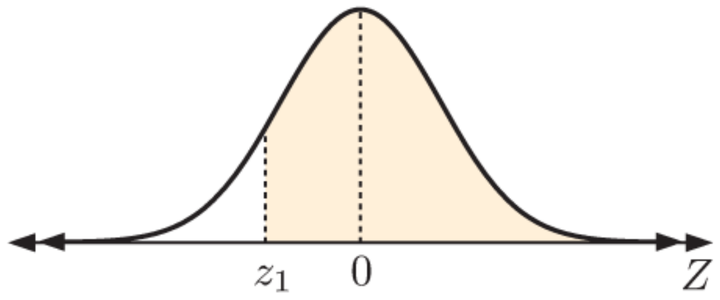
CALCULATING PROBABILITIES USING THE Z-DISTRIBUTION

Consider the variables $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0, 1^2)$.

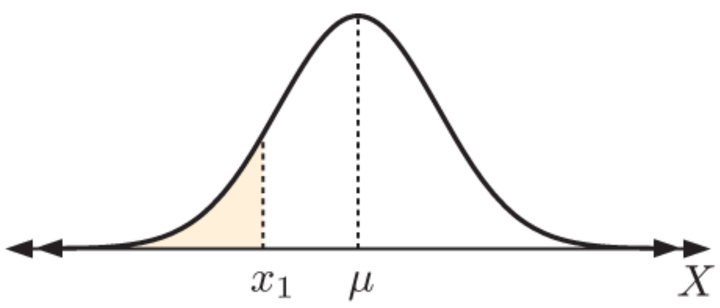
For any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with corresponding z-scores $z_1 = \dfrac{x_1 - \mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$:

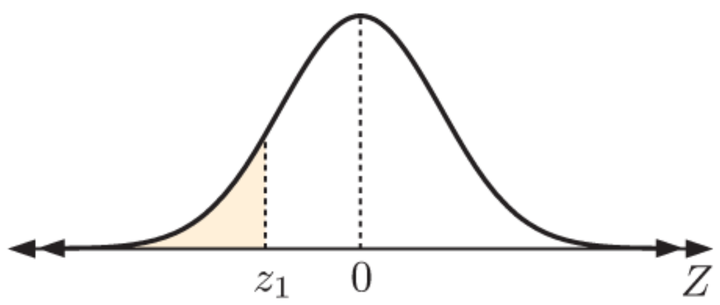
- $\mathrm{P}(X \geqslant x_1) = \mathrm{P}(Z \geqslant z_1)$



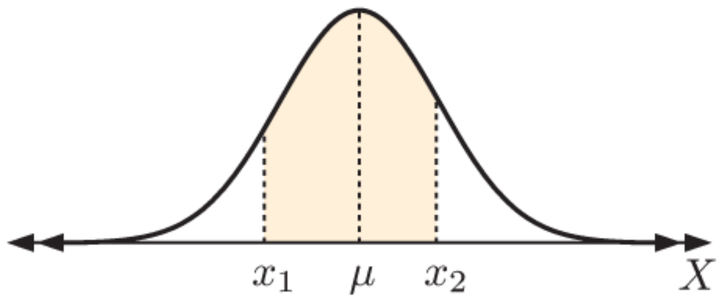


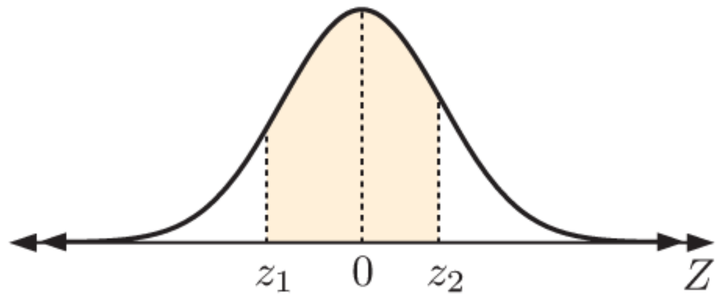
- $\mathrm{P}(X \leqslant x_1) = \mathrm{P}(Z \leqslant z_1)$





- $P(x_1 \leqslant X \leqslant x_2) = P(z_1 \leqslant Z \leqslant z_2)$





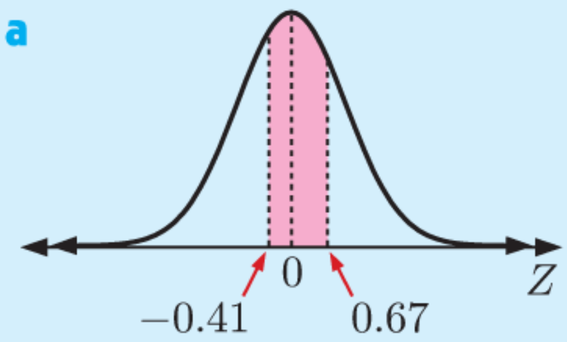
Example 8 **Self Tutor**

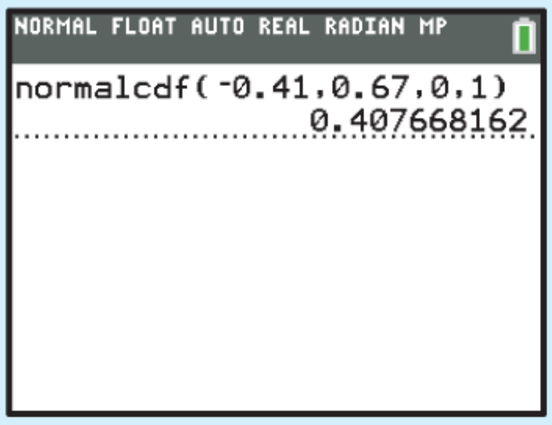
Use technology to illustrate and calculate:

a $P(-0.41 \leqslant Z \leqslant 0.67)$ **b** $P(Z \leqslant 1.5)$ **c** $P(Z > 0.84)$

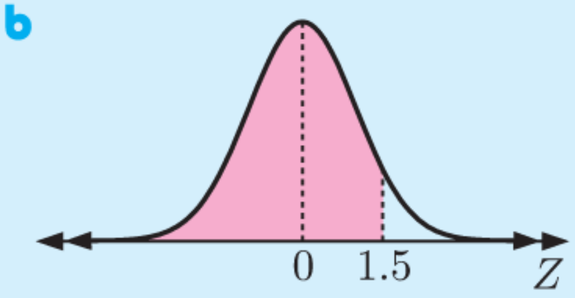
$Z \sim N(0, 1^2)$

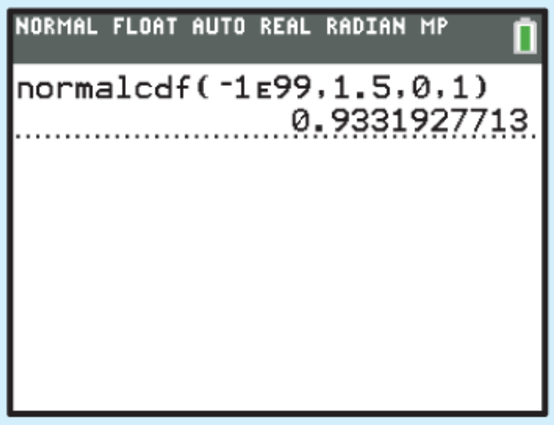
a



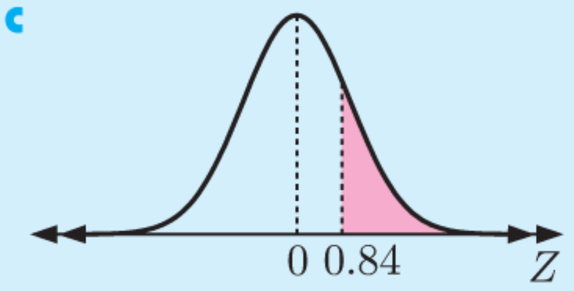


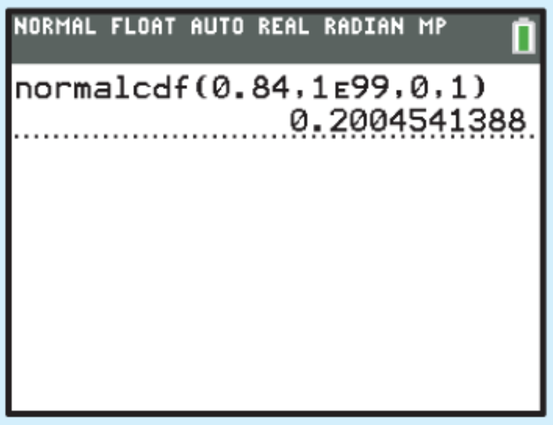
b





c

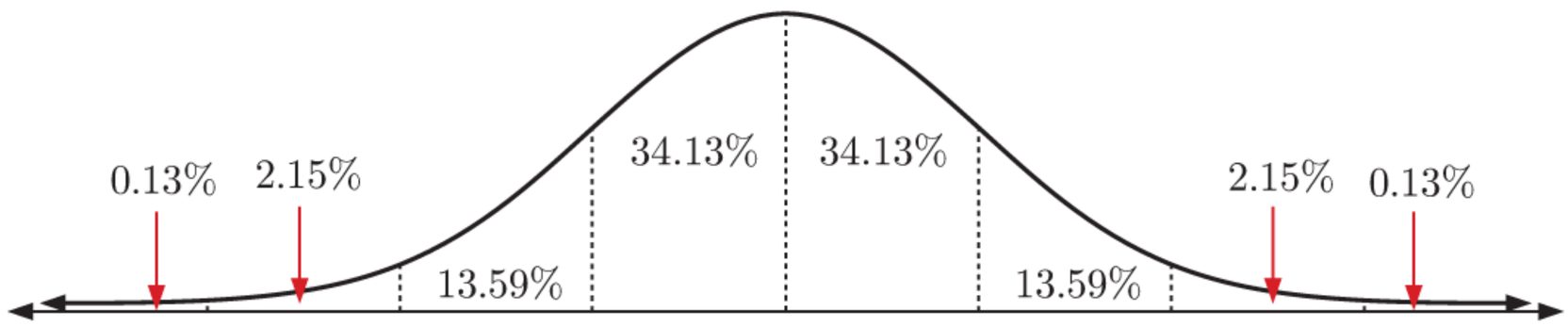




$P(-0.41 \leqslant Z \leqslant 0.67) \approx 0.408$ $P(Z \leqslant 1.5) \approx 0.933$ $P(Z > 0.84) \approx 0.200$

EXERCISE 28E.2

1 Consider the normal distribution curve below.



x-value	$\mu - 3\sigma$	$\mu - 2\sigma$	$\mu - \sigma$	μ	$\mu + \sigma$	$\mu + 2\sigma$	$\mu + 3\sigma$
z-score	-3	-2	-1	0	1	2	3

Use the diagram to calculate the following probabilities. In each case sketch the Z-distribution and shade in the region of interest.

PRINTABLE CURVES

a $P(-1 < Z < 1)$ **b** $P(-1 \leqslant Z \leqslant 3)$ **c** $P(-1 < Z < 0)$

d $P(Z < 2)$ **e** $P(-1 < Z)$ **f** $P(Z \geqslant 1)$

2 For a random variable X, the mean is μ and the standard deviation is σ.

Using properties of $E(X)$ and $\text{Var}(X)$, find: **a** $E\left(\frac{X-\mu}{\sigma}\right)$ **b** $\text{Var}\left(\frac{X-\mu}{\sigma}\right)$.

3 Given $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0, 1^2)$, determine the values of a and b such that:

a $\mathrm{P}(\mu - \sigma < X < \mu + 2\sigma) = \mathrm{P}(a < Z < b)$

b $\mathrm{P}(\mu - 0.5\sigma < X < \mu) = \mathrm{P}(a < Z < b)$

c $\mathrm{P}(0 \leqslant Z \leqslant 3) = \mathrm{P}(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

4 If $Z \sim \mathrm{N}(0, 1^2)$, find the following probabilities using technology. In each case sketch the region under consideration.

a $\mathrm{P}(0.5 \leqslant Z \leqslant 1)$ **b** $\mathrm{P}(-0.86 \leqslant Z \leqslant 0.32)$ **c** $\mathrm{P}(-2.3 \leqslant Z \leqslant 1.5)$

d $\mathrm{P}(Z \leqslant 1.2)$ **e** $\mathrm{P}(Z \leqslant -0.53)$ **f** $\mathrm{P}(Z \geqslant 1.3)$

g $\mathrm{P}(Z \geqslant -1.4)$ **h** $\mathrm{P}(Z > 4)$ **i** $\mathrm{P}(-0.5 < Z < 0.5)$

j $\mathrm{P}(-1.960 \leqslant Z \leqslant 1.960)$ **k** $\mathrm{P}(-1.645 \leqslant Z \leqslant 1.645)$ **l** $\mathrm{P}(|\,Z\,| > 1.645)$

5 **a** Suppose X is normally distributed with mean μ and standard deviation σ.

i Explain why $\mathrm{P}(\mu - 3\sigma < X < \mu + 2\sigma) = \mathrm{P}(-3 < Z < 2)$.

ii Hence find $\mathrm{P}(\mu - 3\sigma < X < \mu + 2\sigma)$.

b For a random variable $X \sim \mathrm{N}(\mu, \sigma^2)$, find:

i $\mathrm{P}(\mu - 2\sigma < X < \mu + 1.5\sigma)$ **ii** $\mathrm{P}(\mu - 2.5\sigma < X < \mu - 0.5\sigma)$

6 Suppose X is normally distributed with mean $\mu = 58.3$ and standard deviation $\sigma = 8.96$.

a Let the z-score of $x_1 = 50.6$ be z_1 and the z-score of $x_2 = 68.9$ be z_2.

i Calculate z_1 and z_2. **ii** Find $\mathrm{P}(z_1 \leqslant Z \leqslant z_2)$.

b Check your answer by calculating $\mathrm{P}(50.6 \leqslant X \leqslant 68.9)$ directly using technology.

HISTORICAL NOTE

The normal distribution has two parameters μ and σ, whereas the standard normal distribution has no parameters. This means that a unique table of probabilities can be constructed for the standard normal distribution.

Before graphics calculators and computer packages, it was impossible to calculate probabilities for a general normal distribution $\mathrm{N}(\mu, \sigma^2)$ directly.

Instead, all data was transformed using the Z-transformation, and the standard normal distribution table was consulted for the required probabilities.

F NORMAL QUANTILES

Consider a population of crabs where the length of a shell, X mm, is normally distributed with mean 70 mm and standard deviation 10 mm.

A biologist wants to protect the population by allowing only the largest 5% of crabs to be harvested. He therefore wants to know what length corresponds to the 95th percentile of crabs.

To answer this question we need to find k such that $\mathrm{P}(X \leqslant k) = 0.95$.

The number k is known as a **quantile**. In this case it is the 95% quantile.



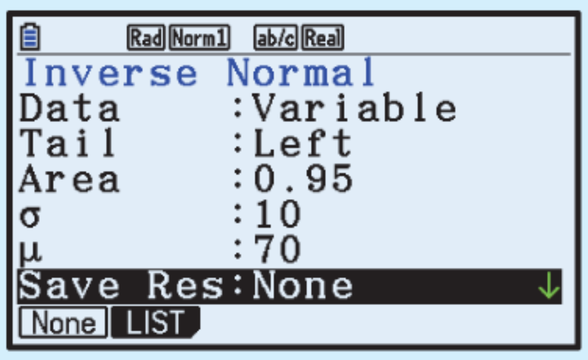
When finding quantiles, we are given a probability and are asked to calculate the corresponding measurement. This is the *inverse* of finding probabilities, so we use the **inverse normal function** on our calculator.

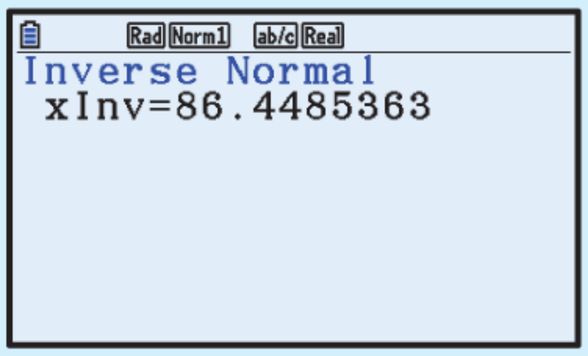


Example 9 **Self Tutor**

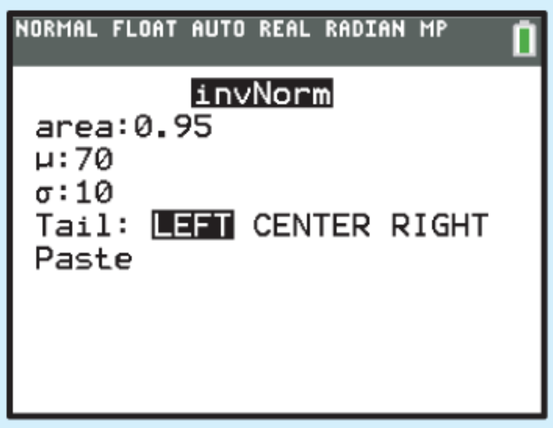
A population of crabs has shell length X mm. X is normally distributed with mean 70 and standard deviation 10. Find k for which $P(X < k) = 0.95$.

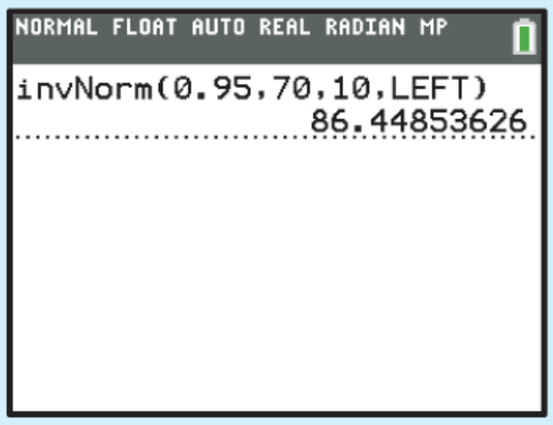
Casio fx-CG50



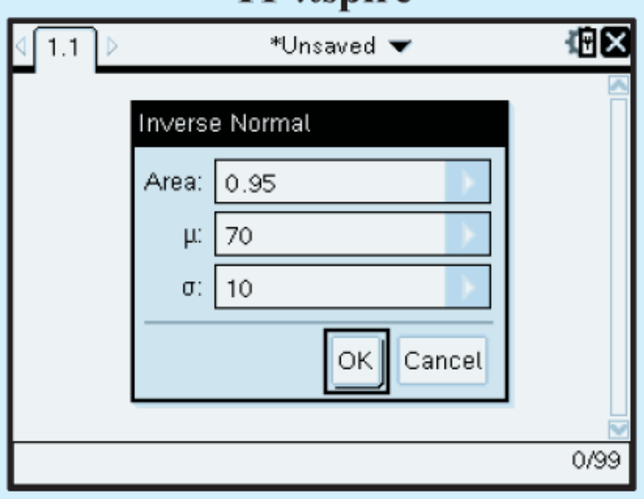


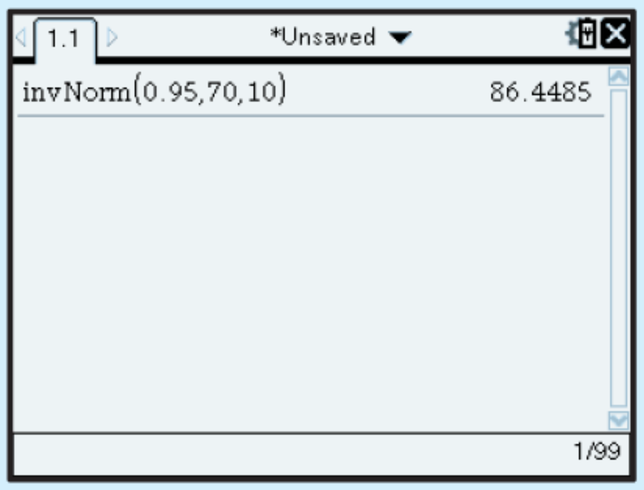
TI-84 Plus CE





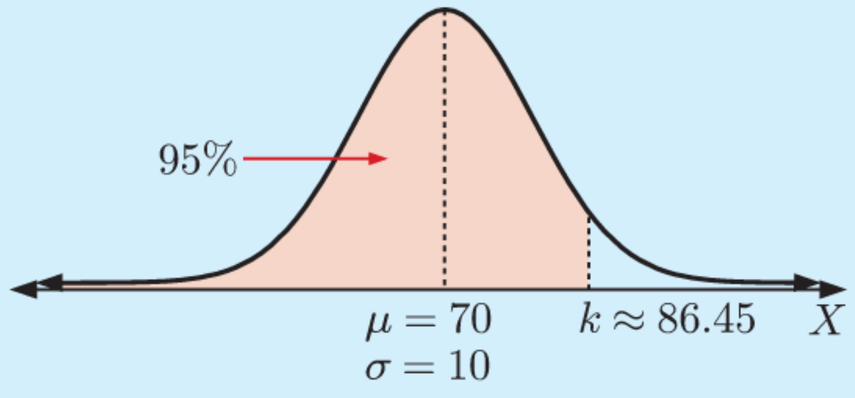
TI-*n*spire



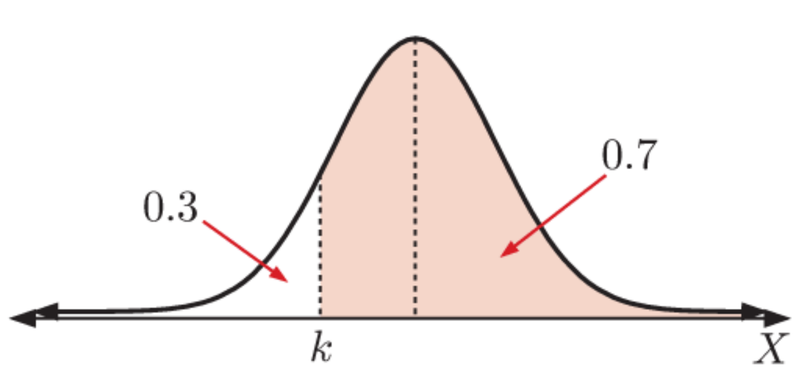


If $P(X < k) = 0.95$ then
$k \approx 86.45$

The 95% quantile corresponds to a shell width of 86.45 mm.



When using the **HP Prime** or **TI-*n*spire** calculators, we must always use the area to the *left* of k. Therefore, to find k such that $P(X > k) = 0.7$, we instead find k such that $P(X < k) = 1 - 0.7 = 0.3$.



EXERCISE 28F.1

1 Suppose X is normally distributed with mean 20 and standard deviation 3. Illustrate with a sketch and find k such that:

a $P(X \leqslant k) = 0.3$ **b** $P(X \leqslant k) = 0.9$ **c** $P(X \leqslant k) = 0.5$

d $P(X > k) = 0.2$ **e** $P(X < k) = 0.62$ **f** $P(X \geqslant k) = 0.13$

2 Suppose Z has the standard normal distribution. Illustrate with a sketch and find k such that:

a $\text{P}(Z \leqslant k) = 0.81$ **b** $\text{P}(Z \leqslant k) = 0.58$ **c** $\text{P}(Z \leqslant k) = 0.17$

d $\text{P}(Z \geqslant k) = 0.95$ **e** $\text{P}(Z \geqslant k) = 0.9$ **f** $\text{P}(Z \geqslant k) = 0.41$

3 Suppose X is normally distributed with mean 30 and standard deviation 5, and $\text{P}(X \leqslant a) = 0.57$.

a Using a diagram, determine whether a is greater or less than 30.

b Use technology to find a.

c Without using technology, find:

i $\text{P}(X \geqslant a)$ **ii** $\text{P}(30 \leqslant X \leqslant a)$

4 Given that X is normally distributed with mean 15 and standard deviation 3, find k such that:

a $\text{P}(X < k) = 0.2$ **b** $\text{P}(X > k) = 0.1$

c $\text{P}(15 - k < X < 15 + k) = 0.9$

5 Suppose X is normally distributed with mean 80 and standard deviation 10.

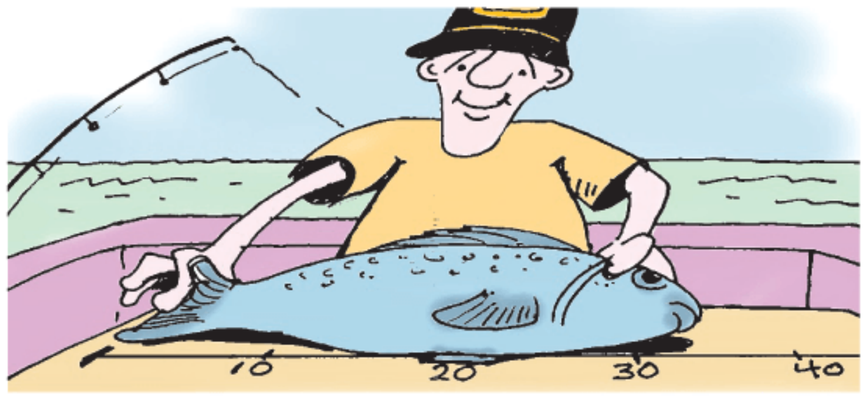
a Find $\text{P}(X \leqslant 72)$.

b Hence find k such that $\text{P}(72 \leqslant X \leqslant k) = 0.1$.

6 Given that $X \sim \text{N}(45, 8^2)$, find a such that:

a $\text{P}(35 \leqslant X \leqslant a) = 0.25$ **b** $\text{P}(a \leqslant X \leqslant 50) = 0.15$ **c** $\text{P}(a \leqslant X \leqslant 54) = 0.6$

7 The lengths of a fish species are normally distributed with mean 35 cm and standard deviation 8 cm. The fisheries department has decided that the smallest 10% of the fish are not to be harvested. What is the size of the smallest fish that can be harvested?



8 The lengths of screws produced by a machine are normally distributed with mean 75 mm and standard deviation 0.1 mm. 1% of the screws are rejected because they are too long. What is the length of the smallest screw to be rejected?

9 The volumes of cool drink in bottles filled by a machine are normally distributed with mean 503 mL and standard deviation 0.5 mL. 1% of the bottles are rejected because they are underfilled, and 2% are rejected because they are overfilled. They are otherwise kept for sale. Find, correct to 1 decimal place, the range of volumes in the bottles that are kept.

10 Abbey goes for a morning walk as long as the temperature is not too cold and not too hot. The morning temperatures are normally distributed with mean 20°C and standard deviation 5°C. Given that the lower limit of Abbey's walking temperatures is 11°C, and that she goes for a walk 95% of the time, find the upper limit of Abbey's walking temperatures.

FINDING AN UNKNOWN MEAN OR STANDARD DEVIATION

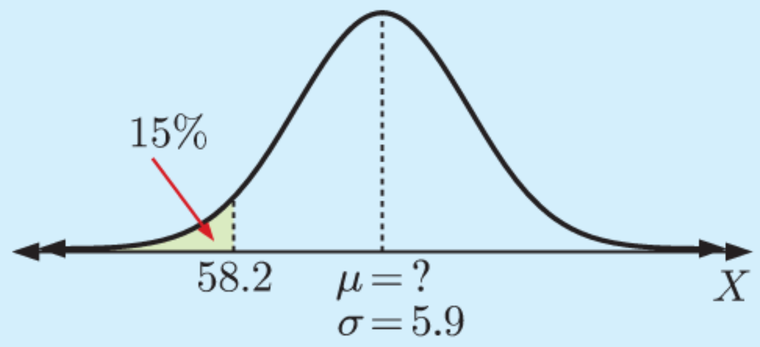
We always need to convert to z-scores if we are trying to find an unknown mean μ or standard deviation σ.

Example 10 **Self Tutor**

The weights of an adult scallop population are known to be normally distributed with a standard deviation of 5.9 g. If 15% of scallops weigh less than 58.2 g, find the mean weight of the population.

Let the mean weight of the population be μ g.

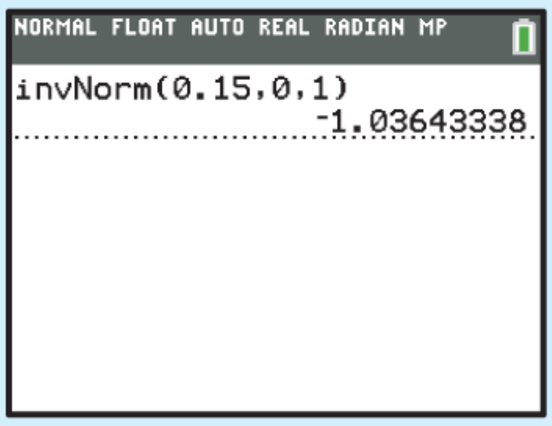
If X g denotes the weight of an adult scallop, then $X \sim \text{N}(\mu, 5.9^2)$.



$$\text{P}(X \leqslant 58.2) = 0.15$$

$$\therefore \text{P}\left(Z \leqslant \frac{58.2 - \mu}{5.9}\right) = 0.15$$

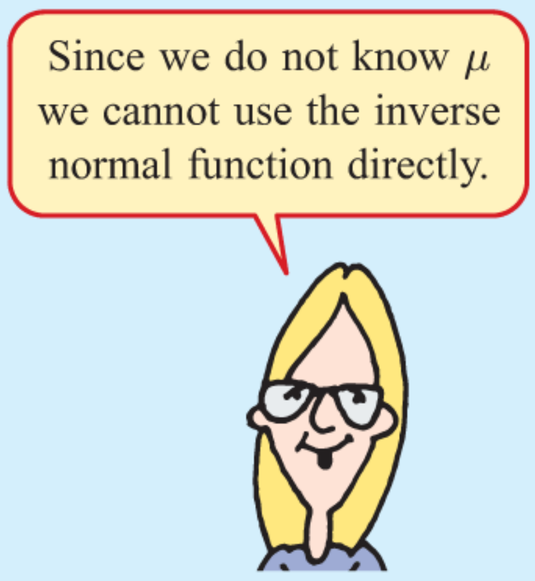
Using the inverse normal function for $\text{N}(0, 1^2)$,



$$\frac{58.2 - \mu}{5.9} \approx -1.0364$$

$$\therefore \quad 58.2 - \mu \approx -6.1$$

$$\mu \approx 64.3$$



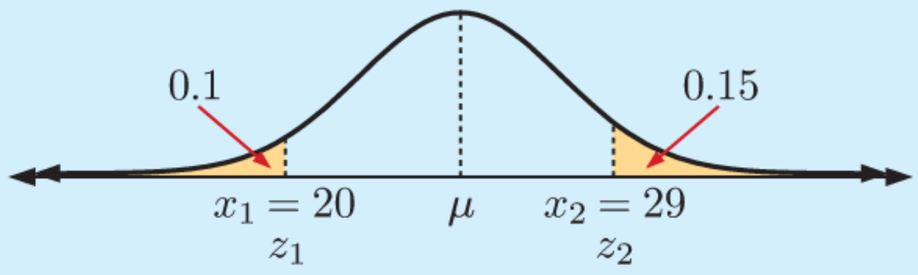
So, the mean weight is approximately 64.3 g.

Example 11 **Self Tutor**

Find the mean and standard deviation of a normally distributed random variable X for which $\text{P}(X \leqslant 20) = 0.1$ and $\text{P}(X \geqslant 29) = 0.15$.

$X \sim \text{N}(\mu, \sigma^2)$ where we have to find μ and σ.

We start by finding z_1 and z_2 which correspond to $x_1 = 20$ and $x_2 = 29$.



Now $\text{P}(X \leqslant x_1) = 0.1$

$$\therefore \text{P}\left(Z \leqslant \frac{20 - \mu}{\sigma}\right) = 0.1$$

$$\therefore \quad z_1 = \frac{20 - \mu}{\sigma} \approx -1.282 \quad \{\text{technology}\}$$

$$\therefore \quad 20 - \mu \approx -1.282\sigma \quad \ldots. \ (1)$$

and $\text{P}(X \leqslant x_2) = 0.85$

$$\therefore \text{P}\left(Z \leqslant \frac{29 - \mu}{\sigma}\right) = 0.85$$

$$\therefore \quad z_2 = \frac{29 - \mu}{\sigma} \approx 1.036 \quad \{\text{technology}\}$$

$$\therefore \quad 29 - \mu \approx 1.036\sigma \quad \ldots. \ (2)$$

Solving (1) and (2) simultaneously we get $\mu \approx 25.0$ and $\sigma \approx 3.88$.

EXERCISE 28F.2

1 Suppose X is normally distributed with standard deviation 6, and that $\text{P}(X < 40) = 0.2$.

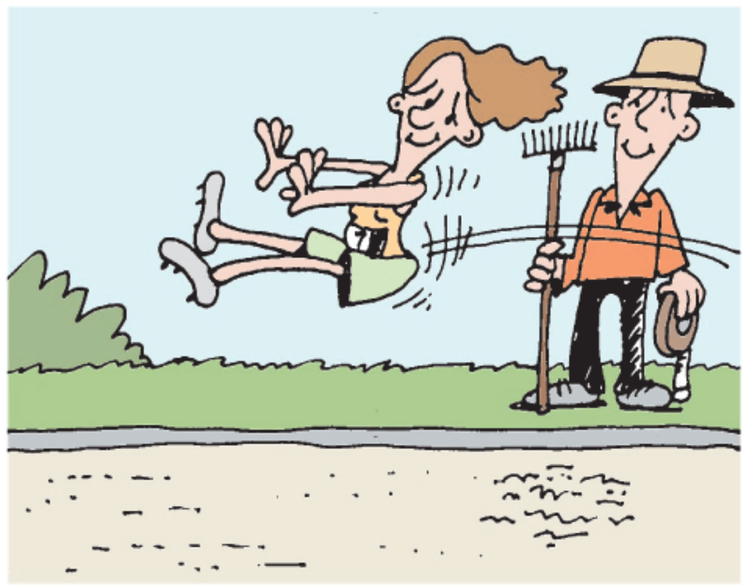
a Would you expect the mean of X to be greater or less than 40? Explain your answer.

b Find the mean of X.

2 Suppose X is normally distributed with mean 15 and $P(X > 20) = 0.1$. Find the standard deviation of X.

3 The IQs of students at a school are normally distributed with a standard deviation of 15. If 20% of the students have an IQ higher than 125, find the mean IQ of students at the school.

4 The distances an athlete jumps are normally distributed with mean 5.2 m. If 15% of the athlete's jumps are less than 5 m, what is the standard deviation?



5 The weekly income of a bakery is normally distributed with mean €6100. If the weekly income exceeds €6000 85% of the time, what is the standard deviation?

6 The arrival times of buses at a depot are normally distributed with standard deviation 5 minutes. If 10% of the buses arrive before 3:55 pm, find the mean arrival time of buses at the depot.

7 Find the mean and standard deviation of a normally distributed random variable X for which $P(X \geqslant 35) = 0.32$ and $P(X \leqslant 8) = 0.26$.

8 **a** Find the mean and standard deviation of a normally distributed random variable X for which $P(X \geqslant 80) = 0.1$ and $P(X \leqslant 30) = 0.15$.

b In a Mathematics examination it was found that 10% of the students scored at least 80, and 15% scored 30 or less. Assuming the scores are normally distributed, what percentage of students scored more than 50?

9 The diameters of pistons manufactured by a company are normally distributed. Only those pistons whose diameters lie between 3.994 cm and 4.006 cm are acceptable.

a Find the mean and the standard deviation of the distribution if 4% of the pistons are rejected as being too small, and 5% are rejected as being too large.

b Determine the probability that the diameter of a randomly chosen piston measures between 3.997 cm and 4.003 cm.

10 Circular metal tokens are used to operate a washing machine in a laundromat. The diameters of the tokens are normally distributed, and only tokens with diameters between 1.94 and 2.06 cm will operate the machine. In the manufacturing process, 2% of the tokens were made too small, and 3% were made too large.

a Find the mean and standard deviation of the distribution.

b Find the probability that at most one token out of a randomly selected sample of 20 will not operate the machine.

c Jake has 3 tokens. Find the probability that they all have diameter greater than 2 cm.

GAME

Click on the icon to play a card game for the normal distribution.

CARD GAME

INVESTIGATION 5 THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

In the previous Chapter, we saw how the **binomial distribution** arises from considering the number of "successes" in a fixed number of independent trials of an experiment.

Suppose $X \sim \text{B}(n, p)$ is the number of successes in n independent trials, each with probability of success p. The probability of getting x successes is:

$$\mathbf{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{where } x = 0, 1, 2, \ldots, n$$

You will have used this formula to calculate probabilities for binomial random variables in cases where the number of trials n is relatively small. As n increases, the probability becomes more difficult to calculate. This is because the binomial coefficient $\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$ becomes very large.

What to do:

1 Click on the icon to access a demonstration which draws the probability distribution of $X \sim \text{B}(n, p)$.



a Set $p = 0.5$ and use the sliders to change the value of n. Describe what happens to the distribution of X as n increases.

b Repeat **a** for p equal to:

i 0.25 **ii** 0.1 **iii** 0.75 **iv** 0.9

Comment on your observations.

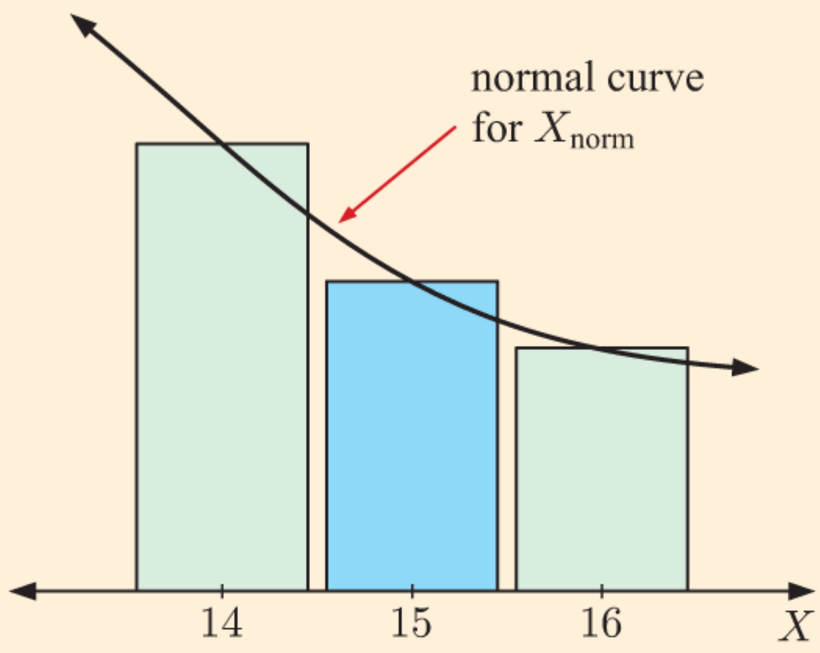
c Do you think that it would be *reasonable* to approximate the binomial distribution with a normal distribution? Explain your answer.

d What should be the mean and standard deviation of this normal distribution?

2 Consider $X \sim \text{B}(50, 0.2)$ and its normal approximation $X_{\text{norm}} \sim \text{N}(\mu, \sigma^2)$.

a Write down expressions for μ and σ.

b Suppose we want to calculate the probability of 15 successes. The diagram alongside shows part of the probability distribution of X with the normal distribution curve of X_{norm} drawn over the top. Explain why $\text{P}(X = 15) \approx \text{P}(14.5 \leqslant X_{\text{norm}} \leqslant 15.5)$.



c Describe how you would estimate the following using the normal approximation:

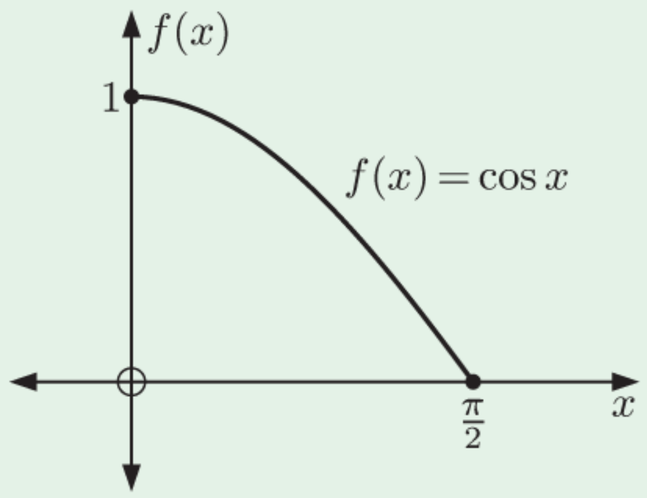
i $\text{P}(X \leqslant 10)$ **ii** $\text{P}(X < 25)$

iii $\text{P}(10 \leqslant X < 25)$

3 It is known that 2% of tyres manufactured by a company are unfit for sale. A quality inspector randomly sampled 500 tyres. Use a normal approximation to estimate the probability that at least 10 tyres in the sample will be unfit for sale.

REVIEW SET 28A

1



Consider the probability density function $f(x) = \cos x$, $0 \leqslant x \leqslant \frac{\pi}{2}$.

a Check that $f(x)$ is a valid probability density function.

b Find $\mathrm{P}(0 \leqslant x \leqslant \frac{\pi}{4})$.

c Find the median of X.

2 A random variable X has probability density function $f(x) = a(x+1)x(x-1)(x-2)$ for $0 \leqslant x \leqslant 1$.

a Show that $a = \frac{30}{11}$.

b Find $\mathrm{P}(0 \leqslant X \leqslant 0.2)$.

c Find the mean of X.

d Hence find the mean of $Y = 5 - \frac{X}{2}$.

3 A continuous random variable X has the probability density function $f(x) = ax(4 - x^2)$, $0 \leqslant x \leqslant 2$.

a Find a.

b Find $\mathrm{P}(0 \leqslant X \leqslant 1)$.

c Find the mean of X.

d Find the standard deviation of X.

4 The average height of 17 year old boys is normally distributed with mean 179 cm and standard deviation 8 cm. Calculate the percentage of 17 year old boys whose heights are:

a more than 195 cm

b between 171 cm and 187 cm

c between 163 cm and 195 cm.

5 The weight of the edible part of a batch of Coffin Bay oysters is normally distributed with mean 38.6 grams and standard deviation 6.3 grams.

a Find the percentage of oysters that weigh between 30 g and 40 g.

b From a sample of 200 oysters, how many would you expect to weigh more than 50 g?

6 The results of a test are normally distributed. The z-score of Harri's test score is -2.

a Interpret this z-score with regard to the mean and standard deviation of the test scores.

b What percentage of students obtained a better score than Harri?

c The mean test score was 61 and Harri's actual score was 47. Find the standard deviation of the test scores.

7 A random variable X is normally distributed with mean 20.5 and standard deviation 4.3. Find:

a $\mathrm{P}(X \geqslant 22)$ **b** $\mathrm{P}(18 \leqslant X \leqslant 22)$ **c** k such that $\mathrm{P}(X \leqslant k) = 0.3$.

8 The distribution of weights in grams of bags of sugar filled by a machine is $X \sim \mathrm{N}(503,\, 2^2)$. Bags less than 500 grams are considered underweight.

a What percentage of bags are underweight?

b If a quality inspector randomly selects 20 bags, what is the probability that 2 or fewer bags are underweight?

9 Suppose X is normally distributed with mean 25 and standard deviation 6. Find k such that:

a $\mathrm{P}(X \leqslant k) = 0.7$ **b** $\mathrm{P}(X \geqslant k) = 0.4$ **c** $\mathrm{P}(20 \leqslant X \leqslant k) = 0.3$

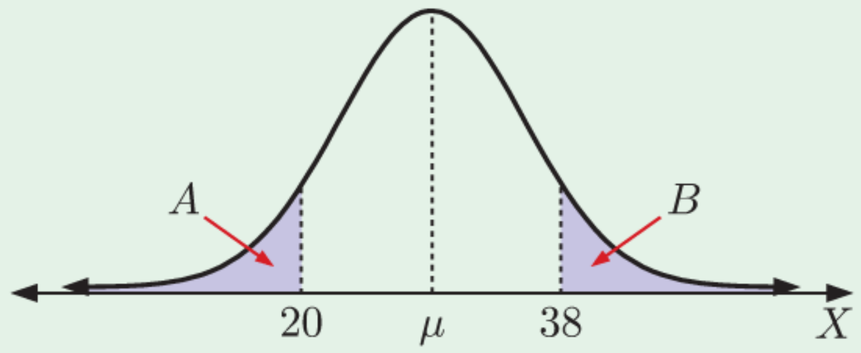
10 The life of a Xenon-brand battery is normally distributed with mean 33.2 weeks and standard deviation 2.8 weeks.

a Find the probability that a randomly selected battery will last at least 35 weeks.

b For how long can the manufacturer expect the batteries to last before 8% of them fail?

11 A random variable X is normally distributed with standard deviation 2.83. Find the probability that a randomly selected score from X will differ from the mean by less than 4.

12 The distribution curve shown corresponds to $X \sim \text{N}(\mu, \sigma^2)$. Area A = Area B = 0.2.



a Find μ and σ.

b Calculate:

i $\text{P}(X \leqslant 35)$ **ii** $\text{P}(23 \leqslant X \leqslant 30)$

13 Machines A and B both produce nails whose lengths are normally distributed. The lengths of nails from machine A have mean 50.2 mm and standard deviation 1.1 mm. The lengths of nails from machine B have mean 50.6 mm and standard deviation 0.8 mm. Nails which are longer than 52 mm or shorter than 48 mm are rejected.

a Find the probability of randomly selecting a nail that has to be rejected from:

i machine A **ii** machine B.

b A quality inspector randomly selects a nail from a randomly chosen machine. Find the probability that the nail was made by machine A *given* that it should be rejected.

14 The random variable T represents the lifetime in years of a component of a solar cell.

The probability density function of T is $f(t) = 0.4e^{-0.4t}$, $t \geqslant 0$.

a Find the probability that a particular component of the solar cell fails within 1 year.

b Each solar cell has 5 of these components which operate independently of each other. The cell will work provided at least 3 of the components continue to work. Find the probability that a solar cell will still operate after 1 year.

REVIEW SET 28B

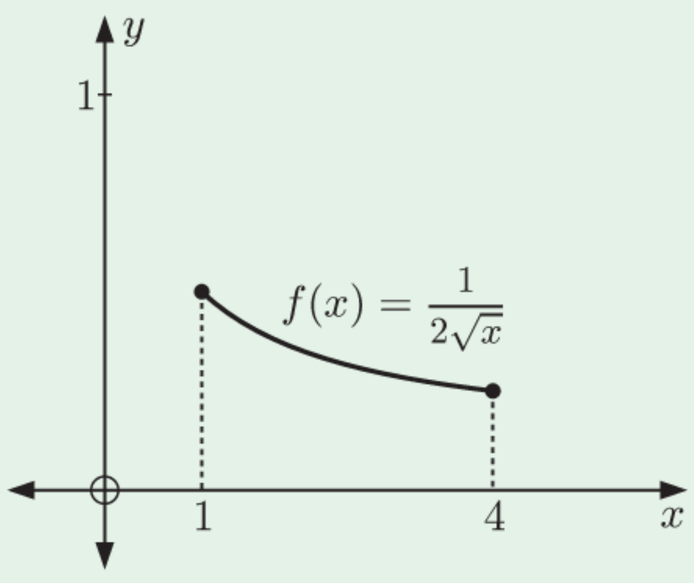
1 The probability density function of a continuous random variable X is $f(x) = \dfrac{1}{2\sqrt{x}}$, $1 \leqslant x \leqslant 4$.

a Check that $f(x)$ is a valid probability density function.

b Find $\text{P}(1 \leqslant X \leqslant 2)$ to 3 decimal places.

c Hence decide whether the median of X is less than 2 or greater than 2.

d Find the median of X.



2 The contents of soft drink cans are normally distributed with mean 327 mL and standard deviation 4.2 mL.

a Find the percentage of cans with contents:

i less than 318.6 mL **ii** between 322.8 mL and 339.6 mL.

b Find the probability that a randomly selected can contains between 327 mL and 331.2 mL.

3 $f(x) = ax(x-3),\ 0 \leqslant x \leqslant 2$ is a probability density function.

a Find a.

b Sketch the graph of $y = f(x)$.

c For this distribution, find the:

i mean **ii** variance **iii** standard deviation.

d Find $\mathrm{P}(1 \leqslant X \leqslant 2)$.

4 The arm lengths of 18 year old females are normally distributed with mean 64 cm and standard deviation 4 cm.

a Find the percentage of 18 year old females whose arm lengths are:

i between 61 cm and 73 cm **ii** greater than 57 cm.

b Find the probability that a randomly chosen 18 year old female has an arm length in the range 50 cm to 65 cm.

c The arm lengths of 70% of the 18 year old females are more than x cm. Find the value of x.

5 The random variable Z has the standard normal distribution $\mathrm{N}(0,\ 1^2)$. Find the value of k for which $\mathrm{P}(-k \leqslant Z \leqslant k) = 0.95$.

6 Suppose X is normally distributed with mean 50 and standard deviation 7. Find:

a $\mathrm{P}(46 \leqslant X \leqslant 55)$ **b** $\mathrm{P}(X \geqslant 60)$ **c** k such that $\mathrm{P}(X > k) = 0.23$.

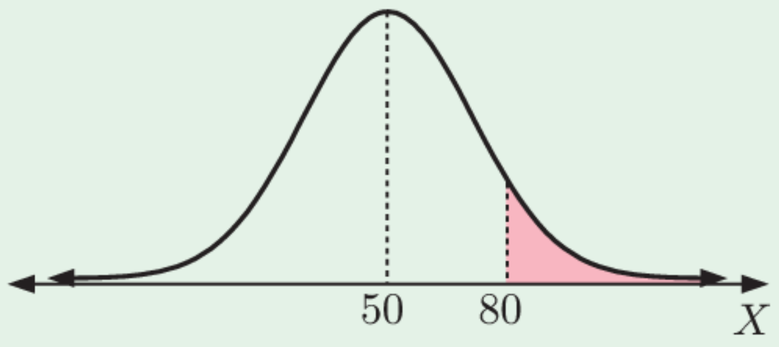
7 In a competition to see who could hold their breath underwater the longest, the times in the preliminary round were normally distributed with mean 150 seconds and standard deviation 12 seconds. If the top 15% of contestants went through to the finals, what time was required to advance?

8 For $X \sim \mathrm{N}(12,\ 2^2)$, find a such that:

a $\mathrm{P}(X < a) = 0.07$ **b** $\mathrm{P}(X > a) = 0.2$ **c** $\mathrm{P}(a \leqslant X \leqslant 11) = 0.1$

9 X is normally distributed with standard deviation 2.1. Let $Z \sim \mathrm{N}(0,\ 1^2)$.
Given that $\mathrm{P}(X > 5.4) = \mathrm{P}(Z > -1.7)$, find the mean of X.

10 A normally distributed random variable X has the distribution curve shown. Its mean is 50, and $\mathrm{P}(X < 90) \approx 0.975$.
Find the shaded area.



11 On an ostrich farm the weights of the birds are found to be normally distributed. The weights of the females have mean 78.6 kg and standard deviation 5.03 kg. The weights of the males have mean 91.3 kg and standard deviation 6.29 kg.

a Find the probability that a randomly selected:

i male will weigh less than 80 kg **ii** female will weigh less than 80 kg

iii female will weigh between 70 and 80 kg.

b 20% of females weigh less than k kg. Find k.

c The middle 90% of the males weigh between a kg and b kg. Find the values of a and b.

d In one field there are 82% females and 18% males. One of these ostriches is selected at random.
Calculate the probability that the ostrich weighs less than 80 kg.

12 The weight of an apple in an apple harvest is normally distributed with mean 300 grams and standard deviation 50 grams. Only apples with weights between 250 grams and 350 grams are considered fit for sale.

a Find the percentage of apples fit for sale.

b In a sample of 100 apples, what is the probability that at least 75 are fit for sale?

13 Giovanni and Beppe are both carrot farmers. The lengths of Giovanni's carrots are normally distributed with mean 22 cm and standard deviation 3.4 cm. The lengths of Beppe's carrots are also normally distributed, with mean 23.5 cm and standard deviation 4.2 cm.

a Find the probability that a carrot is longer than 20 cm, given it comes from:

i Giovanni's farm **ii** Beppe's farm.

b A buyer randomly selects a carrot from each farmer's crop. Calculate the probability that *neither* carrot is longer than 20 cm.

14 The continuous random variable X has probability density function defined by:

$$f(x) = \begin{cases} \frac{x}{5} & \text{for } 0 \leqslant x < 2 \\ \frac{8}{5x^2} & \text{for } 2 \leqslant x \leqslant k \\ 0 & \text{elsewhere.} \end{cases}$$

a Find the value of k.

b Find the exact value of the median of X.

c Find the mean and variance of X.

d Hence find the mean and variance of $Y = 2.4X - 1.2$.

ANSWERS

EXERCISE 1A

1 **a** $\frac{2}{\sqrt{3}}$ **b** $-\frac{1}{\sqrt{3}}$ **c** $-\frac{2}{\sqrt{3}}$ **d** undefined

2 **a** $\operatorname{cosec} x = \frac{5}{3}$, $\sec x = \frac{5}{4}$, $\cot x = \frac{4}{3}$

b $\operatorname{cosec} x = -\frac{3}{\sqrt{5}}$, $\sec x = \frac{3}{2}$, $\cot x = -\frac{2}{\sqrt{5}}$

3 **a** $\sin x = -\frac{\sqrt{7}}{4}$, $\tan x = -\frac{\sqrt{7}}{3}$, $\operatorname{cosec} x = -\frac{4}{\sqrt{7}}$, $\sec x = \frac{4}{3}$, $\cot x = -\frac{3}{\sqrt{7}}$

b $\cos x = -\frac{\sqrt{5}}{3}$, $\tan x = \frac{2}{\sqrt{5}}$, $\operatorname{cosec} x = -\frac{3}{2}$, $\sec x = -\frac{3}{\sqrt{5}}$, $\cot x = \frac{\sqrt{5}}{2}$

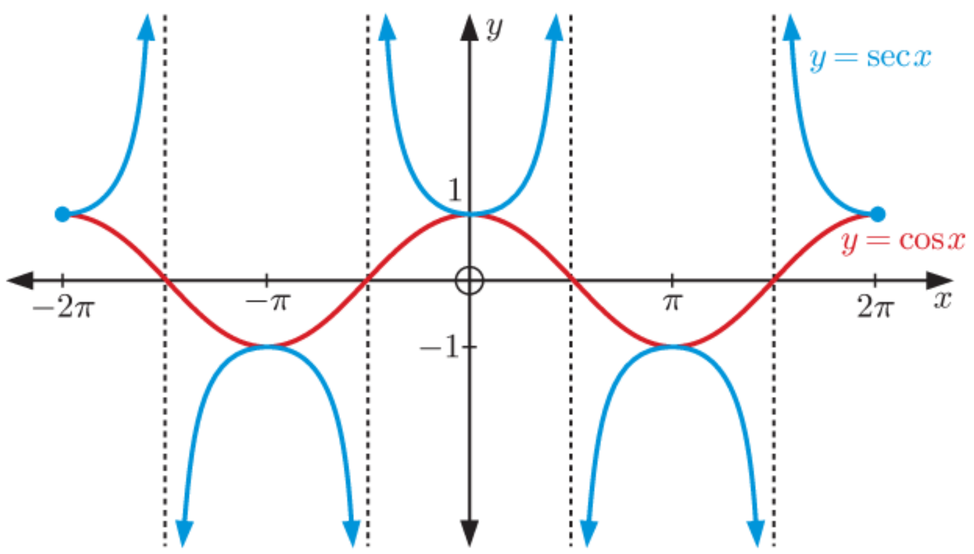
c $\sin x = \frac{\sqrt{21}}{5}$, $\cos x = \frac{2}{5}$, $\tan x = \frac{\sqrt{21}}{2}$, $\operatorname{cosec} x = \frac{5}{\sqrt{21}}$, $\cot x = \frac{2}{\sqrt{21}}$

d $\sin x = \frac{1}{2}$, $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = -\frac{1}{\sqrt{3}}$, $\sec x = -\frac{2}{\sqrt{3}}$, $\cot x = -\sqrt{3}$

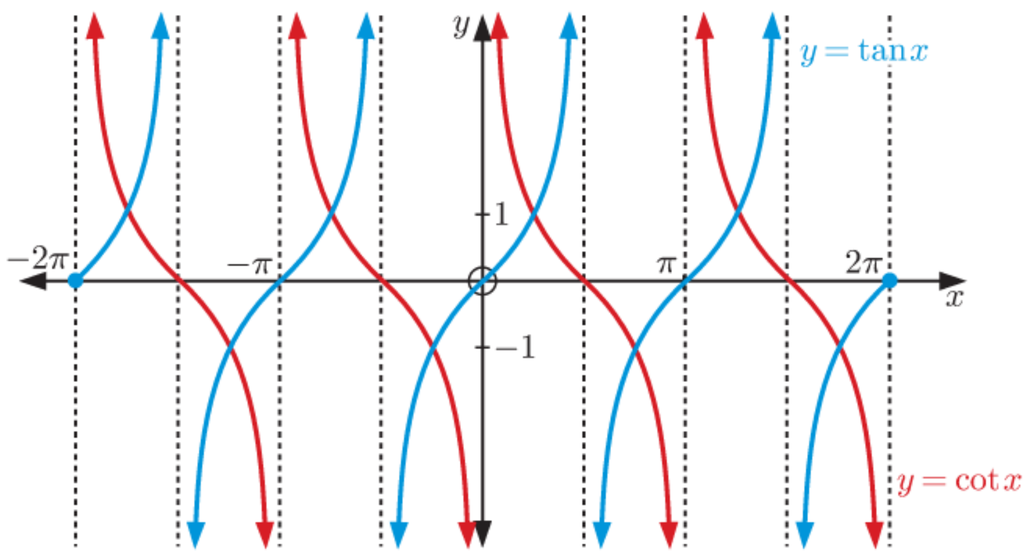
e $\sin \beta = -\frac{1}{\sqrt{5}}$, $\cos \beta = -\frac{2}{\sqrt{5}}$, $\operatorname{cosec} \beta = -\sqrt{5}$, $\sec \beta = -\frac{\sqrt{5}}{2}$, $\cot \beta = 2$

f $\sin \theta = -\frac{3}{5}$, $\cos \theta = -\frac{4}{5}$, $\tan \theta = \frac{3}{4}$, $\operatorname{cosec} \theta = -\frac{5}{3}$, $\sec \theta = -\frac{5}{4}$

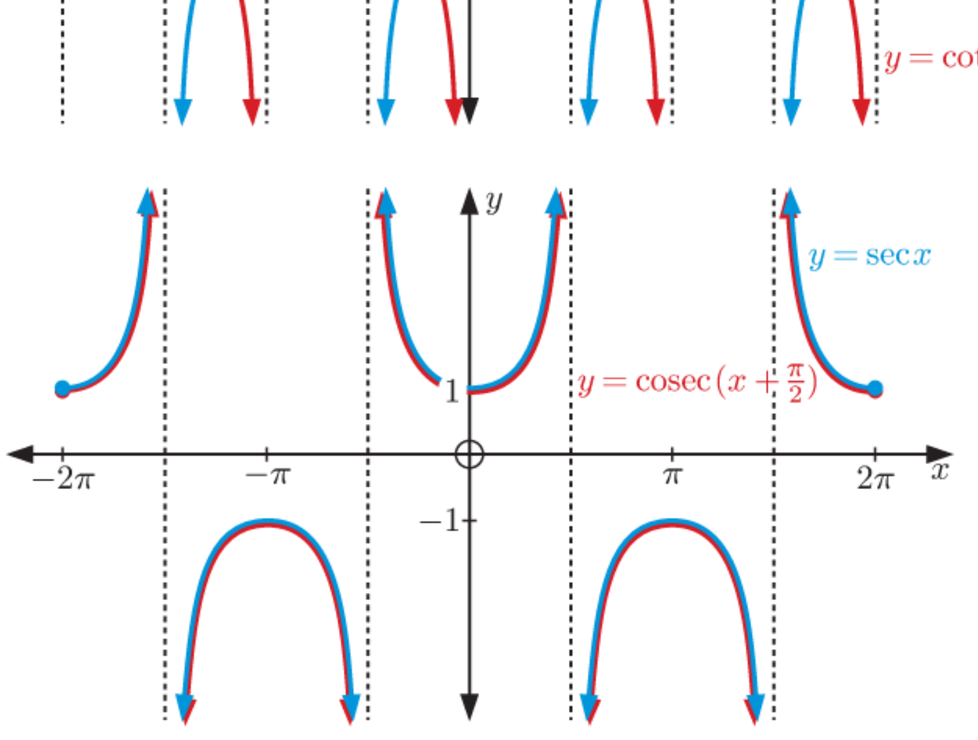
4



5



6



$\sin(x + \frac{\pi}{2}) = \cos x \quad \therefore \quad \frac{1}{\sin(x + \frac{\pi}{2})} = \frac{1}{\cos x}$

$\therefore \quad \operatorname{cosec}(x + \frac{\pi}{2}) = \sec x$

7 **a** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

c $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$, or $\frac{19\pi}{12}$ **d** $x = \frac{13\pi}{12}$ or $\frac{19\pi}{12}$

e $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **f** $x = \frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}$, or $\frac{41\pi}{24}$

8 **a** $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$ **d** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

9 **a** [OD] **b** [AD] **c** [BE] **d** [OB] **e** [OC] **f** [OF]

EXERCISE 1B

1 **a** 0 **b** $-\frac{\pi}{2}$ **c** $\frac{\pi}{4}$ **d** $-\frac{\pi}{4}$ **e** $\frac{\pi}{6}$

f $\frac{5\pi}{6}$ **g** $\frac{\pi}{3}$ **h** $\frac{3\pi}{4}$ **i** $-\frac{\pi}{6}$ **j** ≈ -0.874

k ≈ 1.24 **l** ≈ -1.55

2 **a** (0, 0) **b** (0, 0) **c** (0.739, 0.739)

3 **a** horizontal asymptotes $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$

b No, $y = \sin x$ and $y = \cos x$ do not have horizontal asymptotes.

4 **a** $\arcsin(\sin \frac{\pi}{3}) = \frac{\pi}{3}$ **b** $\arccos(\cos(-\frac{\pi}{6})) = \frac{\pi}{6}$

c $\tan(\arctan(0.3)) = 0.3$ **d** $\cos(\arccos(-\frac{1}{2})) = -\frac{1}{2}$

e $\arctan(\tan \pi) = 0$ **f** $\arcsin(\sin \frac{4\pi}{3}) = -\frac{\pi}{3}$

5 **a** $x = 1$ **b** $x = -\frac{\sqrt{3}}{2}$ **c** $x = -\frac{1}{\sqrt{2}}$

d $x = -\frac{1}{2}$ **e** no solutions **f** $x = 0$

EXERCISE 1C.1

1 **a** $2 \sin \theta$ **b** $3 \cos \theta$ **c** $2 \sin \theta$ **d** $\sin \theta$

e $-2 \tan \theta$ **f** $-3 \cos^2 \theta$

2 **a** $2 \tan x$ **b** $\tan^2 x$ **c** $\sin x$ **d** $\cos x$

e $5 \sin x$ **f** $2 \sec x$

3 **a** 1 **b** 1 **c** $\frac{\cos x}{\sin^2 x}$ **d** $\cos x$ **e** $\cos x$

4 **a** $2 \cos \theta$ **b** $-\tan \theta$ **c** 0 **d** $-\tan \theta$ **e** $\cot \theta$

f $2 \cos \theta$ **g** $\tan \theta$ **h** $\tan \theta$ **i** $2 \tan \theta$

EXERCISE 1C.2

1 **a** 3 **b** -2 **c** -1 **d** $3 \cos^2 \theta$

e $4 \sin^2 \theta$ **f** $\cos \theta$ **g** $-\sin^2 \theta$ **h** $-\cos^2 \theta$

i $-2 \sin^2 \theta$ **j** 1 **k** $\sin \theta$ **l** $\sin \theta$

3 **a** $1 + 2 \sin \theta + \sin^2 \theta$ **b** $\sin^2 \alpha - 4 \sin \alpha + 4$

c $\sec^2 \alpha - 2 \tan \alpha$ **d** $1 + 2 \sin \alpha \cos \alpha$

e $1 - 2 \sin \beta \cos \beta$ **f** $-4 + 4 \cos \alpha - \cos^2 \alpha$

4 **a** $-\tan^2 \beta$ **b** 1 **c** $\sin^2 \alpha$

d $\sin^2 x - \tan^2 x$ **e** 13 **f** $\cos^2 \theta$ **g** 0

EXERCISE 1C.3

1 **a** $(1 + \sin \theta)(1 - \sin \theta)$ **b** $(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)$

c $(\tan \alpha + 1)(\tan \alpha - 1)$ **d** $\sin \beta(2 \sin \beta - 1)$

e $\cos \phi(2 + 3 \cos \phi)$ **f** $3 \sin \theta(\sin \theta - 2)$

g $(\tan \theta + 2)(\tan \theta + 3)$ **h** $(2 \cos \theta + 1)(\cos \theta + 3)$

i $(3 \cos \alpha + 1)(2 \cos \alpha - 1)$ **j** $\tan \alpha(3 \tan \alpha - 2)$

k $(\sec \beta + \operatorname{cosec} \beta)(\sec \beta - \operatorname{cosec} \beta)$

l $(2 \cot x - 1)(\cot x - 1)$

m $(2 \sin x + \cos x)(\sin x + 3 \cos x)$

2 **a** $1+\sin\alpha$ **b** $\tan\beta-1$ **c** $\cos\phi-\sin\phi$
d $\cos\phi+\sin\phi$ **e** $\dfrac{1}{\sin\alpha-\cos\alpha}$ **f** $\dfrac{\cos\theta}{2}$
g $\sin\theta$ **h** $\cos\theta$ **i** $\sec\theta+1$

4 **a** $x=\frac{\pi}{6}$, $\frac{5\pi}{6}$, or $\frac{3\pi}{2}$ **b** no real solutions
c $x=\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **d** $x\approx 0.730,\ 2.41,\ 3.87,$ or 5.55

5 $x=-\frac{11\pi}{12}$, $-\frac{7\pi}{12}$, $\frac{\pi}{12}$, or $\frac{5\pi}{12}$

EXERCISE 1D

2 **a** $\frac{24}{25}$ **b** $-\frac{7}{25}$ **c** $-\frac{24}{7}$ **3** **a** $-\frac{7}{9}$ **b** $\frac{1}{9}$

4 **a** $\cos\alpha=-\frac{\sqrt{5}}{3}$ **b** $\sin 2\alpha=\frac{4\sqrt{5}}{9}$ **c** $\tan 2\alpha=4\sqrt{5}$

5 **a** $\sin\beta=-\frac{\sqrt{21}}{5}$ **b** $\sin 2\beta=-\frac{4\sqrt{21}}{25}$
c $\tan 2\beta=\frac{4\sqrt{21}}{17}$

6 **a** $\frac{1}{3}$ **b** $\frac{2\sqrt{2}}{3}$ **7** **a** $-\frac{1}{\sqrt{3}}$ **b** $\sqrt{\frac{2}{3}}$

8 **a** $\tan A=-\frac{7}{3}$ **b** $\tan A=\frac{3}{2}$ **9** $\tan\frac{\pi}{8}=\sqrt{2}-1$

10 **a** $\sin 2\alpha$ **b** $2\sin 2\alpha$ **c** $\frac{1}{2}\sin 2\alpha$ **d** $\cos 2\beta$
e $-\cos 2\phi$ **f** $\cos 2N$ **g** $-\cos 2M$ **h** $\cos 2\alpha$
i $-\cos 2\alpha$ **j** $\sin 4A$ **k** $\sin 6\alpha$ **l** $\cos 8\theta$
m $-\cos 6\beta$ **n** $\cos 10\alpha$ **o** $-\cos 6D$ **p** $\cos 4A$
q $\cos\alpha$ **r** $-2\cos 6P$

11 $\frac{3}{2}$

13 **a** $x=0$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, or 2π **b** $x=\frac{\pi}{2}$ or $\frac{3\pi}{2}$
c $x=0$, π, or 2π

15 $\theta=-\frac{11\pi}{12}$, $-\frac{7\pi}{12}$, $\frac{\pi}{12}$, or $\frac{5\pi}{12}$

16 **a** $\cos A=\frac{7}{10}$ **b** $\cos A=\frac{3}{4}$

18 **a** $x=0$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, or 2π **b** $x=\frac{\pi}{3}$ or $\frac{5\pi}{3}$
c $x=\frac{\pi}{2}$, $\frac{7\pi}{6}$, or $\frac{11\pi}{6}$
d $x=0$, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, π, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$, $\frac{11\pi}{6}$, or 2π **e** $x=\frac{\pi}{2}$
f $x=\frac{\pi}{4}$ **g** $x=\frac{\pi}{6}$ or $\frac{5\pi}{6}$
h $x=0$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, or 2π

19 **a** C_1 is $y=\cos x$, C_2 is $y=\cos 2x+1$
b A$(\frac{\pi}{3},\frac{1}{2})$, B$(\frac{\pi}{2},0)$, C$(\frac{3\pi}{2},0)$, D$(\frac{5\pi}{3},\frac{1}{2})$

EXERCISE 1E

2 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $\sin\theta$
d $-\cos\alpha$ **e** $-\sin A$ **f** $-\sin\theta$
g $\dfrac{1+\tan\theta}{1-\tan\theta}$ **h** $\dfrac{1+\tan\theta}{1-\tan\theta}$ **i** $\tan\theta$

3 **a** $\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta$ **b** $\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta$
c $-\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta$ **d** $-\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta$

4 **a** $\cos\theta$ **b** $\sin 3A$ **c** $\sin(B-A)$
d $\cos(\alpha-\beta)$ **e** $-\cos(\phi+\theta)$ **f** $2\sin(\alpha-\beta)$
g $\tan\theta$ **h** $\tan 3A$

7 **a** $2+\sqrt{3}$ **b** $-2-\sqrt{3}$ **8** $\frac{7}{17}$ **9** 7

10 $\dfrac{9+5\sqrt{2}}{2}$ **11** $\sqrt{3}$ **12** $\tan A=-\frac{1}{21}$

13 $\tan\alpha=\frac{25}{62}$ **14** $\frac{1}{8}$

15 **a** $\cos 2\alpha$ **b** $-\sin 3\phi$ **c** $\cos\beta$ **d** -1 **e** $\tan 2A$

16 **a** $\dfrac{54-25\sqrt{5}}{22}$ **b** $-4\sqrt{5}$ **17** $\tan A=\pm 1$

18 $\tan(A+B+C)$
$=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$

21 **b** $\theta=-\frac{8\pi}{9}$, $-\frac{4\pi}{9}$, $-\frac{2\pi}{9}$, $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, or $\frac{8\pi}{9}$

22 **a** $\sin 3\theta=-4\sin^3\theta+3\sin\theta$
b $\theta=0$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, 2π, $\frac{9\pi}{4}$, $\frac{11\pi}{4}$, or 3π

23 $k=2,\quad a=\frac{\pi}{6}$

24 **a** $2\cos x+2\sin x=2\sqrt{2}\cos\left(x+\frac{7\pi}{4}\right)$
b $x=\frac{7\pi}{12}$ or $\frac{23\pi}{12}$

25 **a** A vertical stretch with scale factor $\sqrt{10}$, then a translation of ≈ 0.322 units left.
b greatest value $=12$, least value $=2$

27 **a** $2\cos x-5\sin x\approx\sqrt{29}\cos(x+1.19)$
b $x\approx 0.761$ or π
d $x\approx 0.761$ (the solution $x=\pi$ has been lost)

29 **a** 0 **b** $\frac{\pi}{4}$

30 **c** **i** $\frac{1}{2}\sin 4\theta+\frac{1}{2}\sin 2\theta$ **ii** $\frac{1}{2}\sin 7\alpha+\frac{1}{2}\sin 5\alpha$
iii $\sin 6\beta+\sin 4\beta$ **iv** $2\sin 5\theta+2\sin 3\theta$
v $3\sin 7\alpha-3\sin\alpha$ **vi** $\frac{1}{6}\sin 8A-\frac{1}{6}\sin 2A$

31 **c** **i** $\frac{1}{2}\cos 5\theta+\frac{1}{2}\cos 3\theta$ **ii** $\frac{1}{2}\cos 8\alpha+\frac{1}{2}\cos 6\alpha$
iii $\cos 4\beta+\cos 2\beta$ **iv** $3\cos 8x+3\cos 6x$
v $\frac{3}{2}\cos 5P+\frac{3}{2}\cos 3P$ **vi** $\frac{1}{8}\cos 6x+\frac{1}{8}\cos 2x$

32 **c** **i** $\frac{1}{2}\cos 2\theta-\frac{1}{2}\cos 4\theta$ **ii** $\frac{1}{2}\cos 5\alpha-\frac{1}{2}\cos 7\alpha$
iii $\cos 4\beta-\cos 6\beta$ **iv** $2\cos 3\theta-2\cos 5\theta$
v $5\cos 6A-5\cos 10A$ **vi** $\frac{1}{10}\cos 4M-\frac{1}{10}\cos 10M$

33 **a** $\sin A\cos A=\frac{1}{2}\sin 2A$, $\cos^2 A=\frac{1}{2}\cos 2A+\frac{1}{2}$,
$\sin^2 A=\frac{1}{2}-\frac{1}{2}\cos 2A$
c $\cos S+\cos D=2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$
d $\cos D-\cos S=2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

34 **a** $2\sin 3x\cos 2x$ **b** $2\cos 5A\cos 3A$ **c** $-2\sin 2\alpha\sin\alpha$
d $2\cos 4\theta\sin\theta$ **e** $-2\sin 4\alpha\sin 3\alpha$ **f** $2\sin 5\alpha\cos 2\alpha$
g $2\sin 3B\sin B$ **h** $2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$
i $-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

REVIEW SET 1A

1 **a** $\sqrt{2}$ **b** $-\sqrt{3}$ **c** 2

2 **a** $\frac{\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $-\frac{\pi}{6}$

3 **a** $x=\frac{\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x=\frac{2\pi}{3}$ or $\frac{5\pi}{3}$

4 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $\cos\theta$
d $1-\cos\theta$ **e** $\dfrac{1}{\sin\alpha+\cos\alpha}$ **f** $\dfrac{-\cos\alpha}{2}$

5 **a** $-\frac{\sqrt{7}}{4}$ **b** $\frac{3\sqrt{7}}{8}$ **c** $-\frac{1}{8}$
d $-3\sqrt{7}$ **e** $-\sqrt{\frac{4-\sqrt{7}}{8}}$ **f** $\sqrt{\frac{4+\sqrt{7}}{8}}$

7 **a** $\frac{-\sqrt{2}-\sqrt{6}}{4}$ **b** $2-\sqrt{3}$ **8** 60 m

9 $x=0$, $\frac{\pi}{4}$, π, $\frac{5\pi}{4}$, or 2π **11** **c** $x=\frac{16}{3}$ or 3

13 $3\sin x - 5\cos x \approx \sqrt{34}\cos(x + 3.68)$

14 **a** $2\sin x + \sqrt{3}\cos x \approx \sqrt{7}\sin(x + 0.714)$

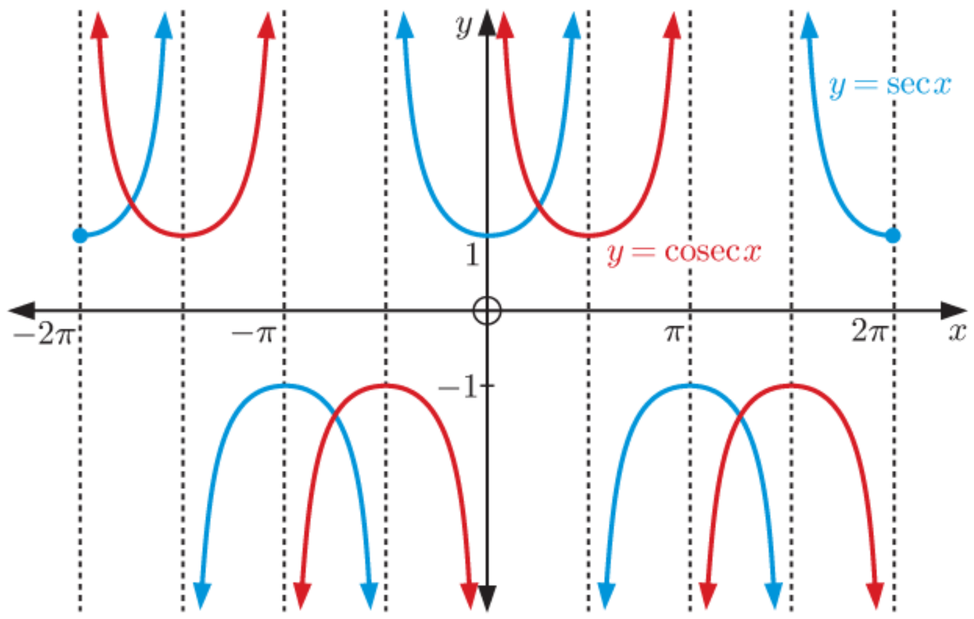
b **i** $A = \sqrt{7}$ **ii** $b \approx 2.43$

15 $\frac{\pi}{4}$

REVIEW SET 1B

1 $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = 2\sqrt{2}$, $\operatorname{cosec} x = -\frac{3}{2\sqrt{2}}$, $\sec x = -3$, $\cot x = \frac{1}{2\sqrt{2}}$

2 **a**



b translation $\frac{\pi}{2}$ units right

3 $x = -\frac{5\pi}{6}$ or $\frac{\pi}{6}$ **4** **a** $x = \frac{\sqrt{3}}{2}$ **b** $x = 2 + \frac{1}{\sqrt{3}}$

5 **a** $\sec x$ **b** $\sin x$ **c** $\cos x$

6 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $5\cos^2\theta$ **d** $-\cos\theta$
e $\operatorname{cosec}\theta$ **f** $\sin 2\theta$

7 **a** $\frac{120}{169}$ **b** $\frac{119}{169}$ **c** $\frac{120}{119}$

10 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}$, or $\frac{\pi}{2}$ **b** $\theta = \frac{\pi}{3}$

11 $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{3\sqrt{3}-\sqrt{7}}{8}$ **12** $\tan\theta = \frac{9}{19}$

13 $3\sin x + 4\cos x \approx 5\sin(x + 0.927)$ **14** 1.5 m

15 **b** $y = 2\sec 2x$ has range $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$

$\therefore \quad \frac{1}{1+\sqrt{2}\sin x} + \frac{1}{1-\sqrt{2}\sin x} = 1$ has no solutions.

EXERCISE 2A

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** $x^{\frac{1}{2}}$ **b** $x^{\frac{3}{2}}$ **c** $x^{-\frac{1}{2}}$ **d** $x^{\frac{5}{2}}$ **e** $x^{-\frac{3}{2}}$

5 **a** ≈ 2.28 **b** ≈ 0.435 **c** ≈ 1.68 **d** ≈ 1.93
e ≈ 0.523

6 **a** $\sqrt[3]{5}$ **b** $\frac{1}{\sqrt{3}}$ **c** $9\sqrt{3}$ **d** $m\sqrt{m}$ **e** $x^3\sqrt{x}$

7 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 2B

1 **a** 1 **b** x **c** $x^{\frac{1}{2}}$ or $\sqrt{x}$

2 **a** $x^5 + 2x^4 + x^2$ **b** $2^{2x} + 2^x$ **c** $x + 1$
d $7^{2x} + 2(7^x)$ **e** $2(3^x) - 1$ **f** $x^2 + 2x + 3$
g $1 + 5(2^{-x})$ **h** $5^x + 1$ **i** $x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$
j $3^{2x} + 5(3^x) + 1$ **k** $2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5$ **l** $2^{3x} - 3(2^{2x}) - 1$

3 **a** $2^{2x} + 2^{x+1} - 3$ **b** $3^{2x} + 7(3^x) + 10$
c $5^{2x} - 6(5^x) + 8$ **d** $2^{2x} + 6(2^x) + 9$
e $3^{2x} - 2(3^x) + 1$ **f** $4^{2x} + 14(4^x) + 49$
g $x - 4$ **h** $4^x - 9$ **i** $x - \frac{1}{x}$ **j** $x^2 + 4 + \frac{4}{x^2}$
k $7^{2x} - 2 + 7^{-2x}$ **l** $25 - 10(2^{-x}) + 2^{-2x}$

4 **a** $5^x(5^x + 1)$ **b** $10(3^n)$ **c** $7^n(1 + 7^{2n})$
d $5(5^n - 1)$ **e** $6(6^{n+1} - 1)$ **f** $16(4^n - 1)$
g $2^n(2^n - 8)$ **h** $\frac{5}{2}(2^n)$ **i** $\frac{9}{2}(2^{2n})$

5 **a** $(3^x + 2)(3^x - 2)$ **b** $(2^x + 5)(2^x - 5)$
c $(4 + 3^x)(4 - 3^x)$ **d** $(5 + 2^x)(5 - 2^x)$
e $(3^x + 2^x)(3^x - 2^x)$ **f** $(2^x + 3)^2$
g $(3^x + 5)^2$ **h** $(2^x - 7)^2$ **i** $(5^x - 2)^2$

6 **a** $(2^x + 1)(2^x - 2)$ **b** $(3^x + 3)(3^x - 2)$
c $(2^x - 3)(2^x - 4)$ **d** $(2^x + 3)(2^x + 6)$
e $(2^x + 4)(2^x - 5)$ **f** $(3^x + 2)(3^x + 7)$
g $(3^x + 5)(3^x - 1)$ **h** $(5^x + 2)(5^x - 1)$
i $(7^x - 4)(7^x - 3)$

7 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x
f $(\frac{3}{4})^a$ **g** $(\frac{8}{3})^k$ **h** 5 **i** 5^n

8 **a** $3^m + 1$ **b** $1 + 6^n$ **c** $4^n + 2^n$ **d** $4^x - 1$
e 6^n **f** 5^n **g** 4 **h** $2^n - 1$ **i** $\frac{1}{2}$

9 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 2C

1 **a** $x = 5$ **b** $x = 2$ **c** $x = 4$ **d** $x = 0$
e $x = -1$ **f** $x = \frac{1}{2}$ **g** $x = -3$ **h** $x = 2$
i $x = -3$ **j** $x = -4$ **k** $x = 2$ **l** $x = \frac{3}{4}$

2 **a** $x = \frac{5}{3}$ **b** $x = -\frac{3}{2}$ **c** $x = -\frac{3}{2}$ **d** $x = -\frac{1}{2}$
e $x = -\frac{2}{3}$ **f** $x = -\frac{5}{4}$ **g** $x = \frac{3}{2}$ **h** $x = \frac{5}{2}$
i $x = \frac{1}{8}$ **j** $x = \frac{9}{2}$ **k** $x = -4$ **l** $x = -\frac{7}{2}$
m $x = 0$ **n** $x = \frac{7}{2}$ **o** $x = -\frac{2}{3}$ **p** $x = -6$

3 **a** $x = \frac{1}{7}$ **b** no solution **c** $x = \frac{5}{2}$
d $x = \frac{1}{3}$ **e** $x = -\frac{1}{4}$ **f** $x = -1$ or 3

4 **a** $x = 3$ **b** $x = 2$ **c** $x = -1$ **d** $x = 2$
e $x = -2$ **f** $x = -2$

5 **a** $x = 1$ or 2 **b** $x = 1$ **c** $x = 1$ or 2
d $x = 1$ **e** $x = 2$ **f** $x = 0$
g $x = 1$ **h** $x = 1$ or -1 **i** $x = 2$
j $x = -2$ or 1 **k** $x = 2$ **l** $x = \frac{1}{2}$

6 $x = \frac{15}{7}$, $y = \frac{10}{7}$

EXERCISE 2D

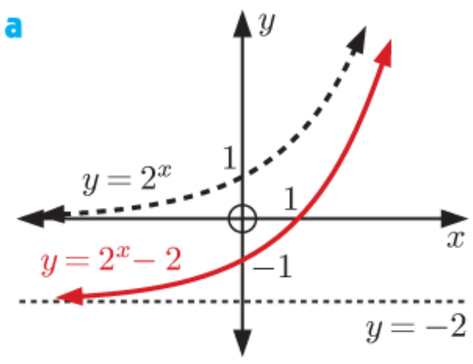
1 **a** **i** ≈ 1.4 **ii** ≈ 1.7 **iii** ≈ 2.8 **iv** ≈ 0.4

b **i** $x \approx 1.6$ **ii** $x \approx -0.7$

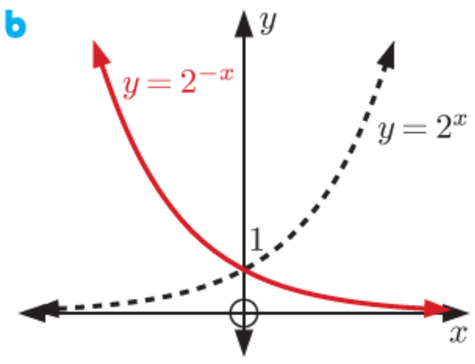
c $y = 2^x$ has a horizontal asymptote of $y = 0$.

2 **a** **C** **b** **B** **c** **E** **d** **A** **e** **D**

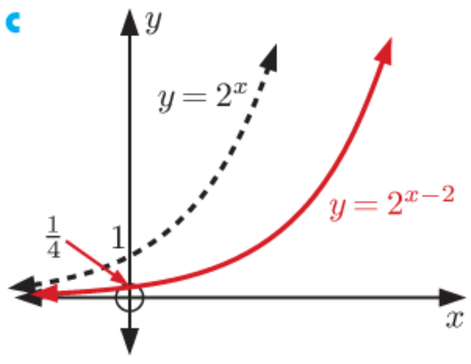
3 **a**



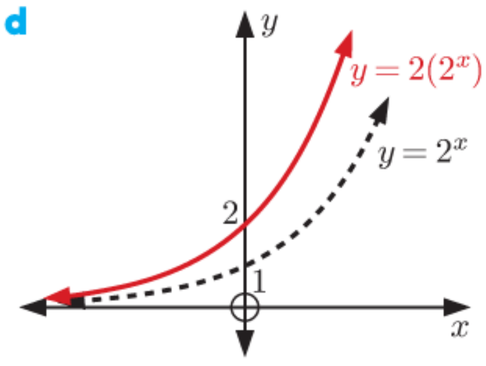
b



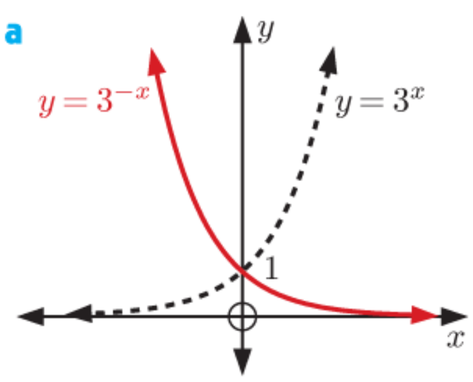
c



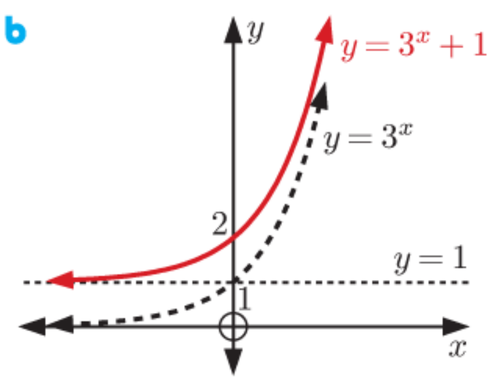
d



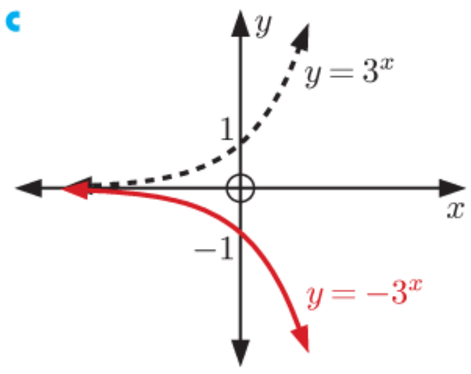
4 **a**



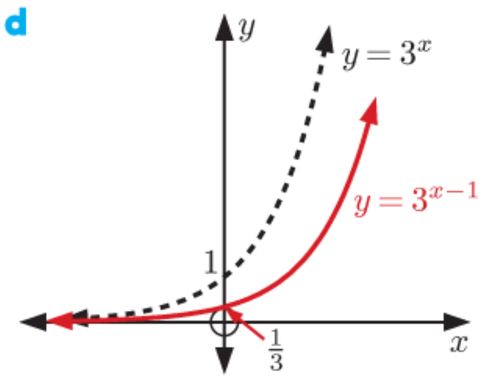
b



c



d

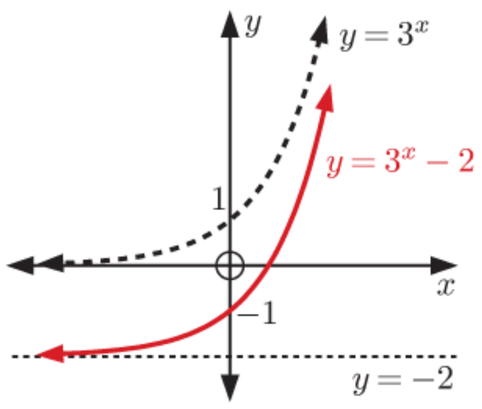


5 **a** $y = 0$ **b** $y = -1$ **c** $y = 3$ **d** $y = 2$
e $y = 0$ **f** $y = -4$

6 **a** **i** -1 **ii** 7
iii $-\frac{17}{9} = -1\frac{8}{9}$

b $y = -2$

c

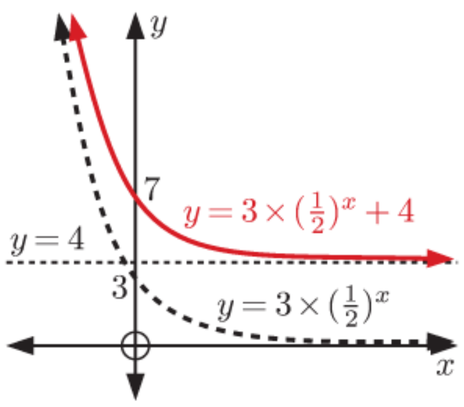


d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > -2\}$

7 **a** **i** 7
ii $\frac{19}{4} = 4\frac{3}{4}$
iii 16

b $y = 4$

c

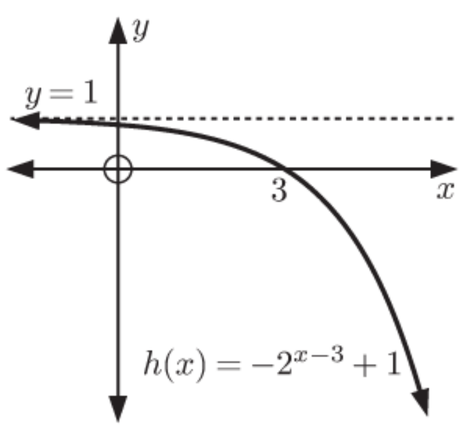


d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 4\}$

8 **a** **i** $\frac{7}{8}$ **ii** 0
iii -7

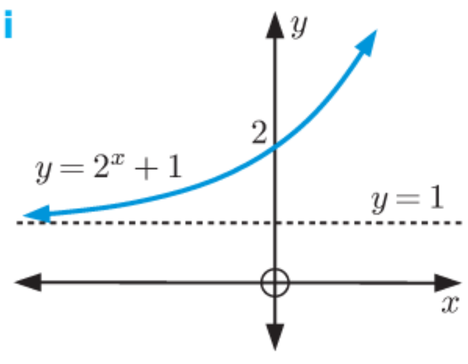
b $y = 1$

c



d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 1\}$

9 **a** **i**



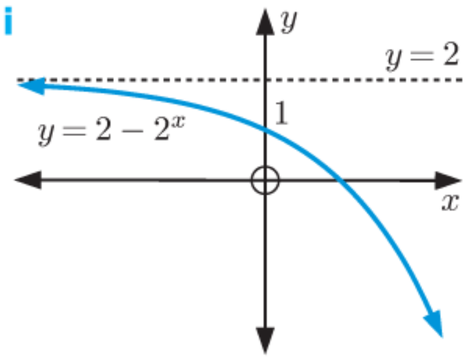
ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 1\}$

iii $y \approx 3.67$

iv as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to 1^+$

v $y = 1$

b **i**



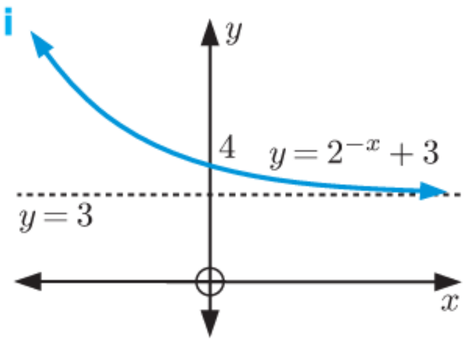
ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 2\}$

iii $y \approx -0.665$

iv as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 2^-$

v $y = 2$

c **i**



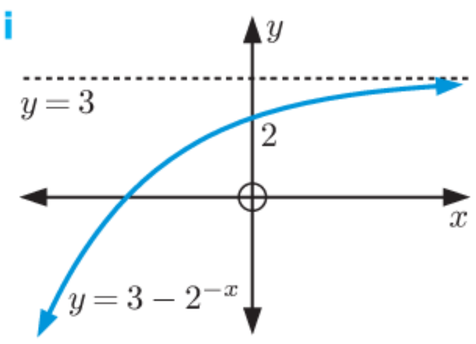
ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 3\}$

iii $y \approx 3.38$

iv as $x \to \infty$, $y \to 3^+$
as $x \to -\infty$, $y \to \infty$

v $y = 3$

d **i**



ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 3\}$

iii $y \approx 2.62$

iv as $x \to \infty$, $y \to 3^-$
as $x \to -\infty$, $y \to -\infty$

v $y = 3$

10 **a** $a = 5,\ b = -10$ **b** $y = 310$

11 **a** P$(0, 2.5)$ **b** $a = 1.5$ **c** $y = 3.5$

12 **a** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant 2\}$
b Domain is $\{x \mid x \neq 0\}$, Range is $\{y \mid y > 0,\ y < -1\}$
c Domain is $\{x \mid x \geqslant 1\}$, Range is $\{y \mid y \geqslant 0\}$

13 **a** $(f \circ g)(x) = 3^{\sqrt{x}} - 9$
Domain is $\{x \mid x \geqslant 0\}$, Range is $\{y \mid y \geqslant -8\}$

b $(g \circ f)(x) = \sqrt{3^x - 9}$
Domain is $\{x \mid x \geqslant 2\}$, Range is $\{y \mid y \geqslant 0\}$

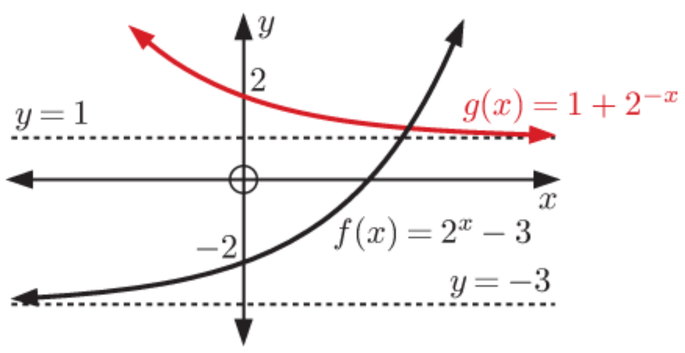
c **i** $x = 4$ **ii** $x = 3$

14 **a** **i** $f(x)$: $y = -3$, $g(x)$: $y = 1$

ii $f(x)$: Range is $\{y \mid y > -3\}$
$g(x)$: Range is $\{y \mid y > 1\}$

iii $f(x)$: y-intercept -2, $g(x)$: y-intercept 2

b

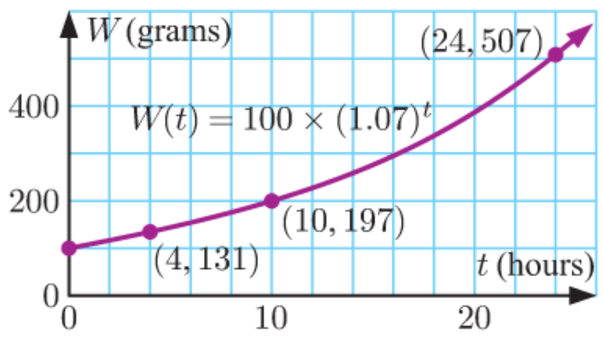


c $-1 + \sqrt{5}$

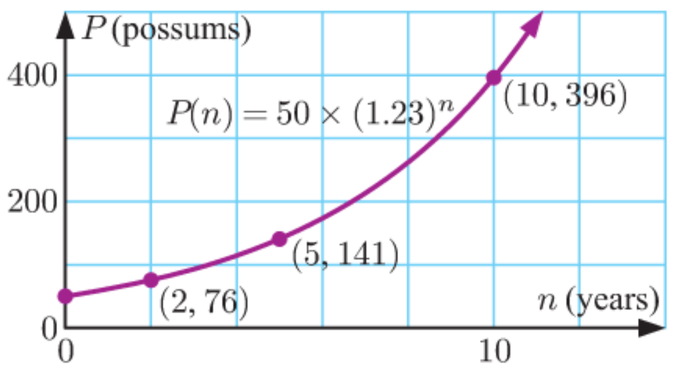
15 **a** $x \approx 3.46$ **b** $x \approx 2.46$ **c** $x \approx 1.16$
d $x \approx -0.738$ **e** $x \approx 1.85$ **f** $x \approx 0.0959$
g $x \approx 6.03$ **h** $x \approx 50.0$ **i** $x \approx 31.0$

EXERCISE 2E.1

1 **a** 100 grams
b **i** ≈ 131 g **ii** ≈ 197 g **iii** ≈ 507 g
c



2 **a** $P_0 = 50$
b **i** ≈ 76 possums **ii** ≈ 141 possums
iii ≈ 396 possums
c



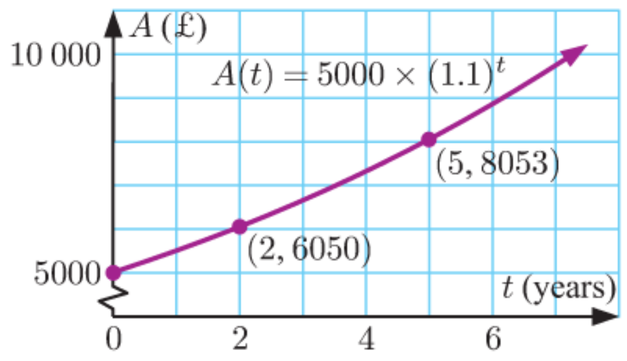
d ≈ 11 years **e** ≈ 11.1 years

3 **a** 4 people **b** ≈ 393 people **c** ≈ 19.9 days

4 **a** $B_0 = 200$
b $a = 1.1$, the bear population is increasing by 10% every year.
c ≈ 1350 bears **d** $\approx 159\%$ increase **e** ≈ 24.2 years

5 **a** **i** V_0 **ii** $2V_0$ **b** 100%
c $\approx 183\%$ increase, it is the percentage increase at 50°C compared with 20°C.

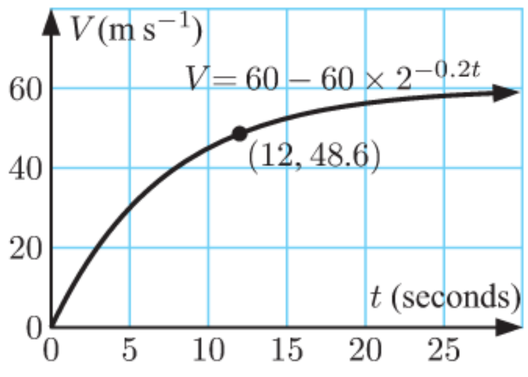
6 **a** $A(t) = 5000 \times (1.1)^t$ **b** **i** £6050 **ii** £8052.55
c



d ≈ 4.93 years

7 **a** $a = 1.08$, the expected value of the house is increasing by 8% per year.
$k = 375\,000$, the original value of the house was \$375 000.
b ≈ 4.98 years

8 **a** When $t = 0$, $V = c - 60 = 0$
$\therefore \quad c = 60$
b $k = -\frac{1}{5} = -0.2$
c $\approx 48.6 \text{ m s}^{-1}$ **d**

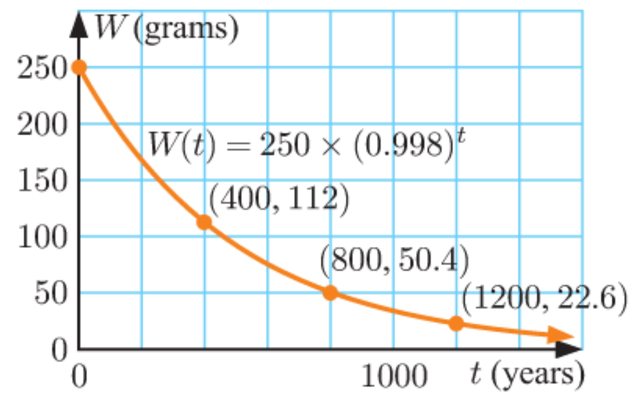


e The parachutist accelerates rapidly until he approaches his terminal velocity of 60 m s^{-1}.

9 ≈ 2.27 hours

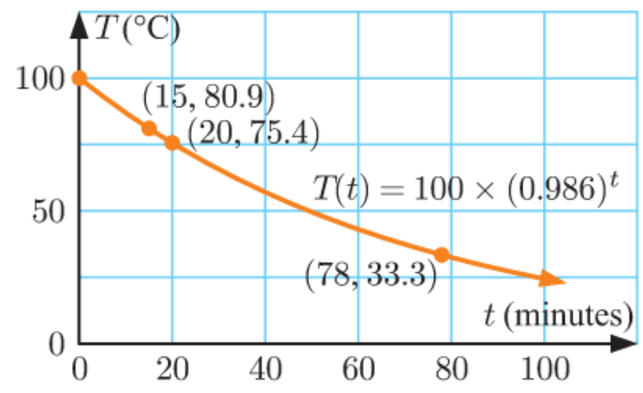
EXERCISE 2E.2

1 **a** 250 g **b** **i** ≈ 112 g **ii** ≈ 50.4 g **iii** ≈ 22.6 g
c

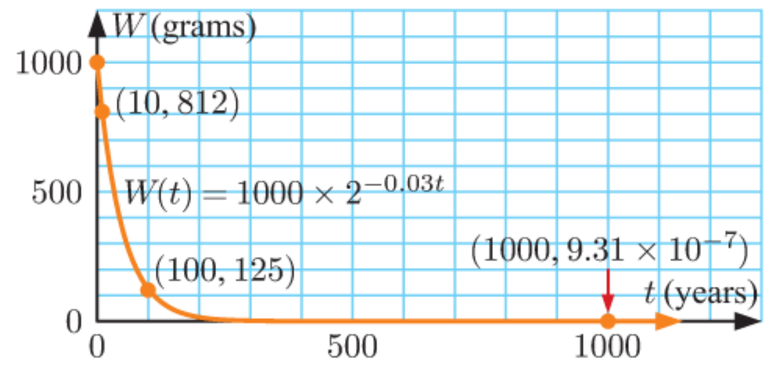


d ≈ 346 years

2 **a** 100°C
b **i** ≈ 80.9°C **ii** ≈ 75.4°C **iii** ≈ 33.3°C
c

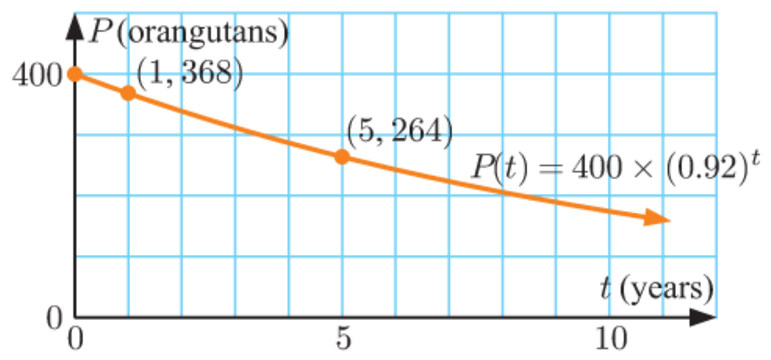


3 **a** 1000 g
b **i** ≈ 812 g **ii** 125 g **iii** $\approx 9.31 \times 10^{-7}$ g
c



d ≈ 221 years **e** $1000(1 - 2^{-0.03t})$ grams

4 **a** $P(t) = 400 \times (0.92)^t$
b **i** 368 orangutans **ii** ≈ 264 orangutans
c

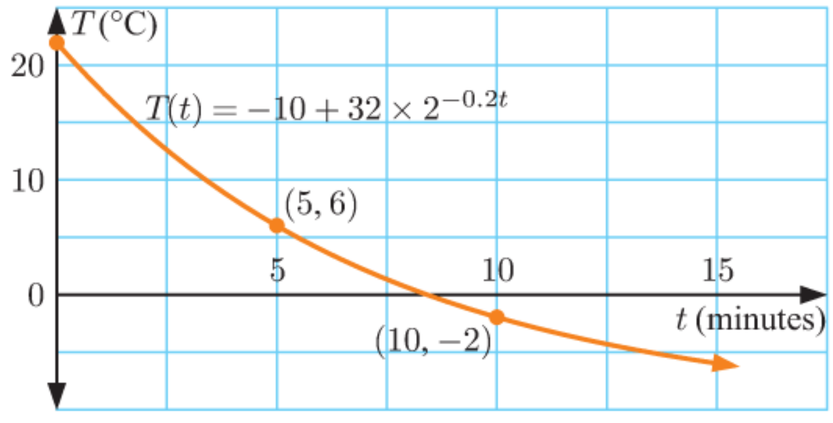


d ≈ 8.31 years, or ≈ 8 years 114 days

5 **a** $L_0 = 10$ units **b** ≈ 2.77 units **c** ≈ 17.9 m
d between ≈ 23.5 m and ≈ 44.9 m

6 **a** \$24 000 **b** $r = 0.85$ **c** 7 years

7 **a** **i** 22°C **ii** 6°C **iii** −2°C
b



c ≈ 8.39 min or ≈ 8 min 23 s
d No, as $32 \times 2^{-0.2t} > 0$ for any value of t.

8 **a** W_0 **b** $\approx 12.9\%$ **c** 45 000 years

9 **a** $A(t) = 150 \times (1.48)^{\frac{t}{3}}$, $B(t) = 400 \times (0.8)^t + 100$
b **i** $t \approx 4.16$ years **ii** $t \approx 3.45$ years **iii** $t \approx 1.69$ years

10 **a** The initial weight of the isotope is 10 mg.

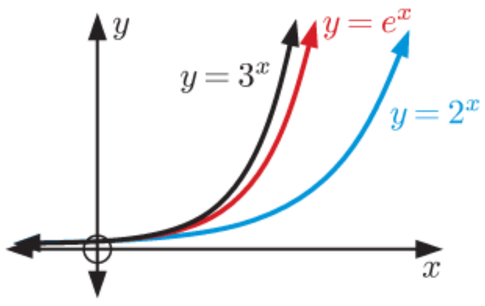
b $a \approx 0.7937$; each day the isotope's weight is decreasing by $\approx 20.63\%$.

c ≈ 6.30 mg **d** **i** ≈ 5.21 days **ii** ≈ 9.00 days

11 ≈ 33.2 minutes or ≈ 33 minutes 13.2 seconds

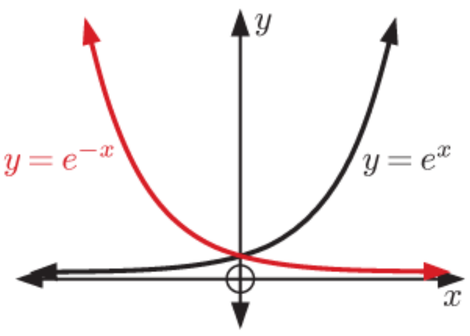
EXERCISE 2F

1



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

2



One is the other reflected in the y-axis.

3 p

4 **a** $e^x > 0$ for all x

b **i** $y \approx 4.12 \times 10^{-9}$ **ii** $y \approx 9.70 \times 10^8$

5 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65 **e** ≈ 0.368

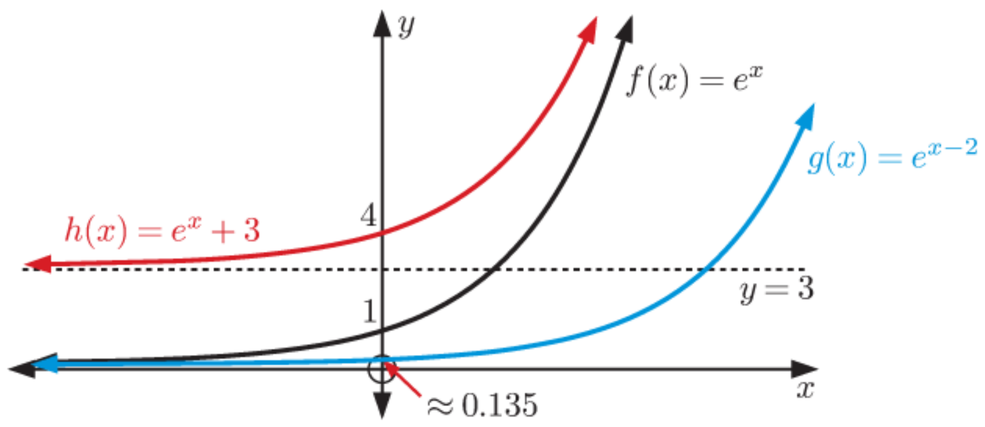
6 **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 **a** ≈ 10.074 **b** $\approx 0.099\,261$ **c** ≈ 125.09
d $\approx 0.007\,994\,5$ **e** ≈ 41.914 **f** ≈ 42.429
g ≈ 3540.3 **h** $\approx 0.006\,342\,4$

8 **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

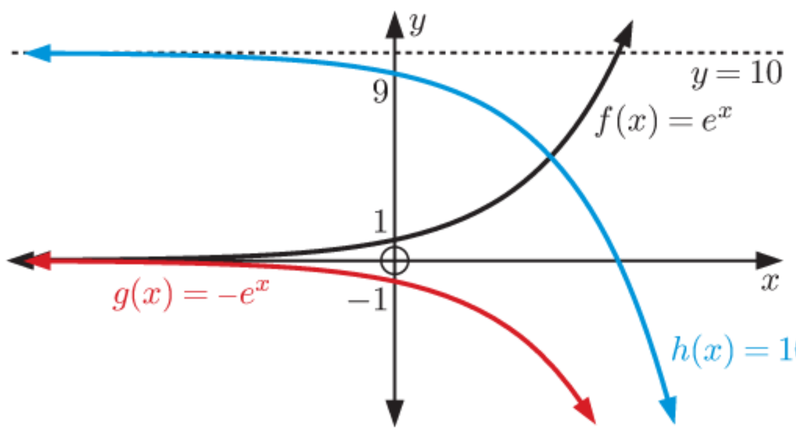
9 **a** $e^x(e^x + 1)$ **b** $(e^x + 4)(e^x - 4)$ **c** $(e^x - 6)(e^x - 2)$

10 **a**



b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y > 0\}$,
Range of h is $\{y \mid y > 3\}$

11 **a**



b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y < 0\}$,
Range of h is $\{y \mid y < 10\}$

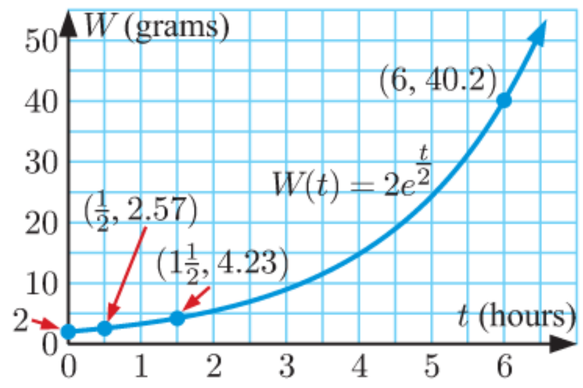
c For f: as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to 0^+$
For g: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 0^-$
For h: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 10^-$

12 **a** $(f \circ g)(x) = e^{\frac{1}{x}} - 1$
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid -1 < y < 0 \text{ or } y > 0\}$

b $(g \circ f)(x) = \dfrac{1}{e^x - 1}$
Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y < -1 \text{ or } y > 0\}$

13 **a** **i** 2 g **ii** ≈ 2.57 g **iii** ≈ 4.23 g **iv** ≈ 40.2 g

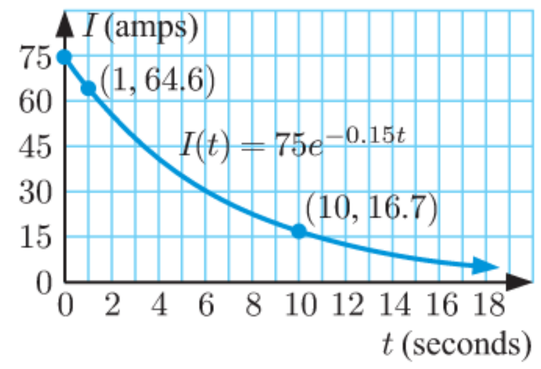
b



14 **a** $x = \frac{1}{2}$ **b** $x = -4$ **c** $x = 0$

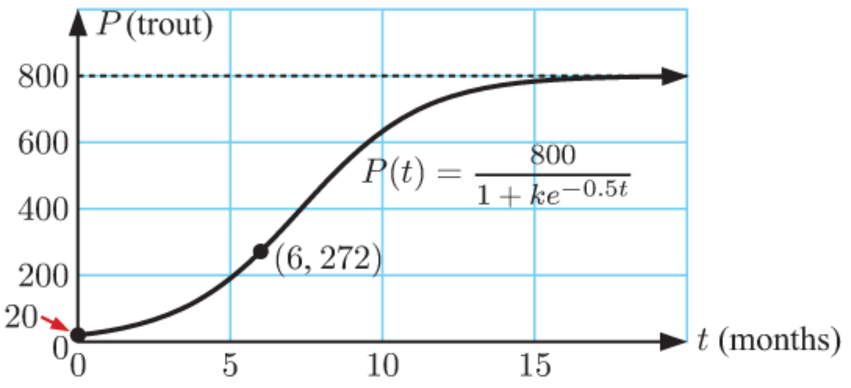
15 **a** **i** ≈ 64.6 amps **ii** ≈ 16.7 amps
c ≈ 28.8 seconds

b



16 **a** $k = 39$ **b** ≈ 272 trout

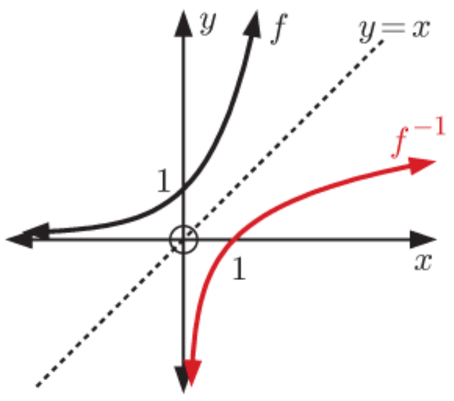
c



d As t increases, the population approaches a limiting value of 800 trout.

e ≈ 9.52 months

17 **a** $f^{-1}(x) = \log_e x$



b Domain of f^{-1} is $\{x \mid x > 0\}$
Range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$

18 $e^1 \approx \sum_{i=0}^{19} \dfrac{1}{k!}\, 1^k \approx 2.718\,281\,828$

REVIEW SET 2A

1 **a** 4 **b** $\frac{1}{9}$ **c** $\frac{1}{3}$

2 **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = -\frac{1}{4}$

3 **a** $1 + e^{2x}$ **b** $2^{2x} + 10(2^x) + 25$ **c** $x - 49$

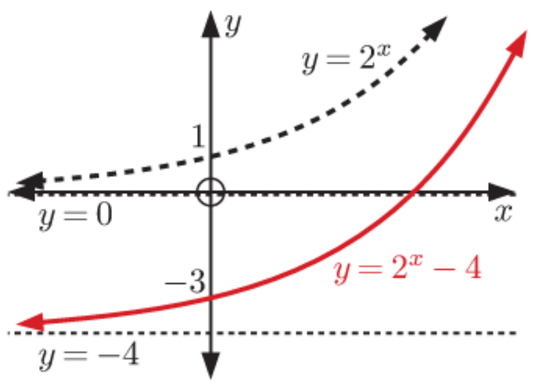
4 **a** $x = 5$ **b** $x = -\frac{5}{2}$ **c** $x = 3$ **5** $k = \frac{3}{2}$

6 **a** **i** ≈ 2.2 **ii** ≈ 0.6

b **i** $x \approx 1.45$ **ii** $x \approx -0.6$ **iii** $x \approx 1.1$

7 **a** 3 **b** 24 **c** $\frac{3}{4}$

8

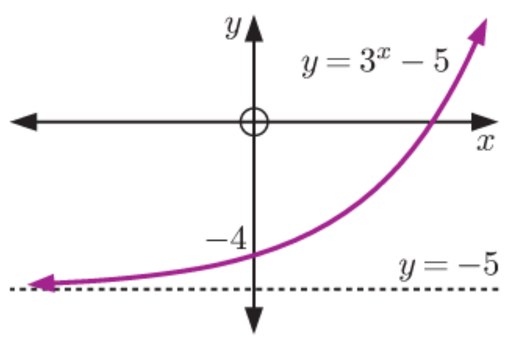


9 **a**

x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

b as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to -5^+$

c



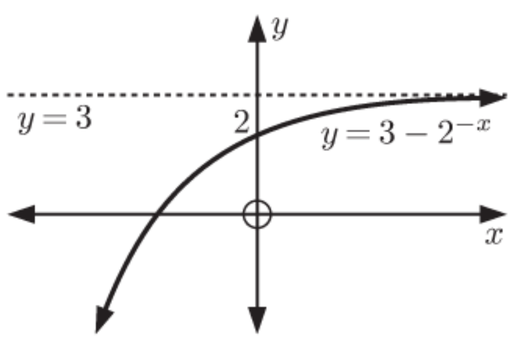
d $y = -5$

10 **a**

x	-2	-1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

b as $x \to \infty$, $y \to 3^-$
as $x \to -\infty$, $y \to -\infty$

c



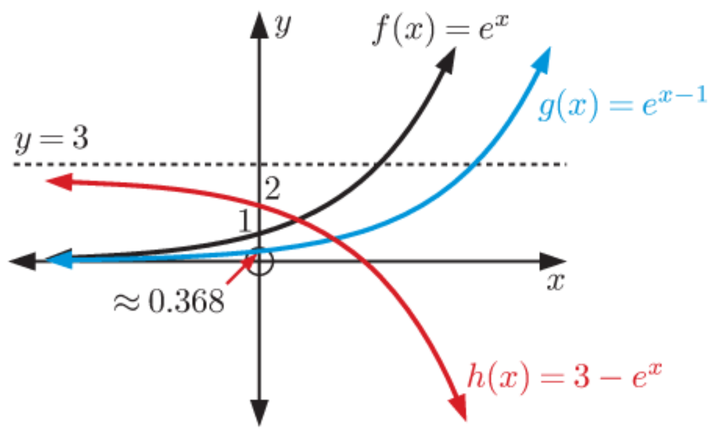
d $y = 3$

11 **a** $(f \circ g)(x) = 2^{3-x^2}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 8\}$

b $(g \circ f)(x) = 3 - 2^{2x} = 3 - 4^x$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

c **i** $x = \pm\sqrt{2}$ **ii** $x = 2$

12 **a**



b For $f(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$
For $g(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$
For $h(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y < 3\}$

c For $f(x)$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0^+$
For $g(x)$: as $x \to \infty$, $g(x) \to \infty$
as $x \to -\infty$, $g(x) \to 0^+$
For $h(x)$: as $x \to \infty$, $h(x) \to -\infty$
as $x \to -\infty$, $h(x) \to 3^-$

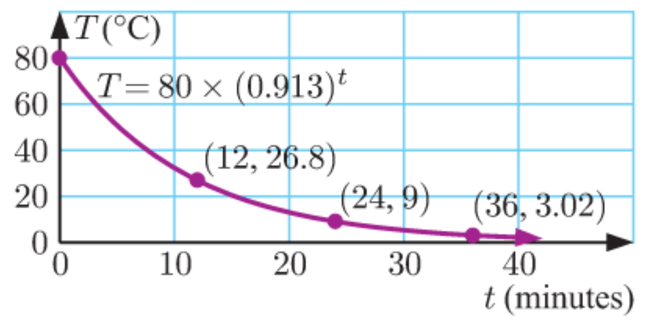
13 about every ≈ 7.92 days

14 **a** 80°C

b **i** ≈ 26.8°C
ii ≈ 9.00°C
iii ≈ 3.02°C

d ≈ 12.8 min

c



REVIEW SET 2B

1 **a** ≈ 3.95 **b** ≈ 0.517 **c** ≈ 3.16

2 **a** $9 - 6e^x + e^{2x}$ **b** $x - 2 - x^{-1}$ **c** $2^x + 1$

3 **a** $8(3^x)$ **b** $(2^x - 4)(2^x + 3)$ **c** $(e^x + 5)(e^x - 3)$

4 **a** $x = -4$ **b** $x = 0$ or 2 **c** $x = -1$ or -2

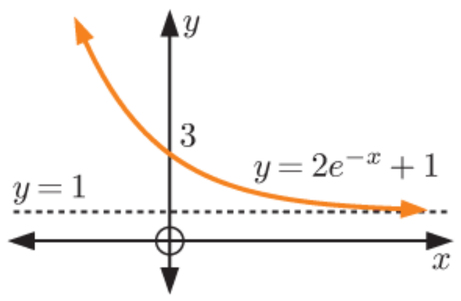
5 **a** $\frac{1}{\sqrt{2}} + 1 \approx 1.71$ **b** $a = -1$

6 **a**

x	-2	-1	0	1	2
y	15.8	6.44	3	1.74	1.27

b as $x \to \infty$, $y \to 1^+$
as $x \to -\infty$, $y \to \infty$

c



d $y = 1$

7 **a** clock: £525, vase: £428

b clock: $V(t) = 500 \times (1.05)^t$
vase: $V(t) = 400 \times (1.07)^t$

c clock $\approx$ £1039.46, vase $\approx$ £1103.61 $\therefore$ the vase

d $500 \times (1.05)^t = 400 \times (1.07)^t$ and solve for t;
$t \approx 11.8$ years

8 Domain is $\{x \mid x \geqslant -1\}$, Range is $\{y \mid y \geqslant 1\}$

9 **a** $f(x)$: y-intercept 2, $g(x)$: y-intercept -1

b $-\frac{5}{2} + \frac{1}{2}\sqrt{5}$ units

10 **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

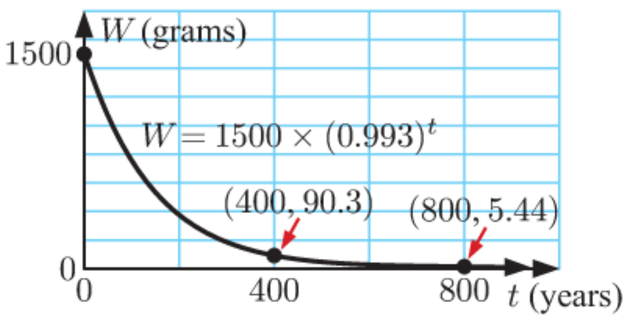
11 **a** y^2 **b** y^{-1} **c** $\frac{1}{\sqrt{y}}$ or $y^{-\frac{1}{2}}$

12 **a** 1500 g

b **i** ≈ 90.3 g
ii ≈ 5.44 g

d ≈ 386 years

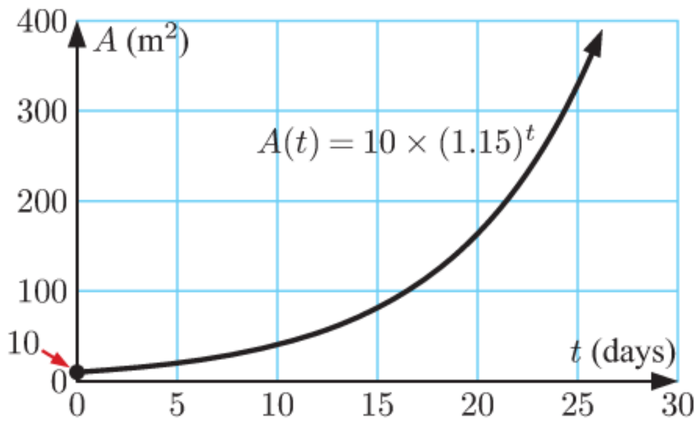
c



13 **a** $A(t) = 10 \times (1.15)^t$

b **i** 13.225 m^2 **ii** ≈ 20.1 m^2

c



d ≈ 24.3 days

EXERCISE 3A

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$
g $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

2 **a** n **b** $a+2$ **c** $1-m$ **d** $a-b$

3 **a** $100 < 237 < 1000$
$\therefore \log 100 < \log 237 < \log 1000$
$\therefore 2 < \log 237 < 3$ **b** ≈ 2.37

4 **a** $-1 < \log(0.6) < 0$ **b** ≈ -0.22

5 **a** ≈ 1.88 **b** ≈ 2.06 **c** ≈ 0.48 **d** ≈ 2.92
e ≈ -0.40 **f** ≈ 3.51 **g** ≈ -2.10 **h** does not exist

6 **a** $x > 1$ **b** $x = 1$ **c** $0 < x < 1$ **d** $x \leqslant 0$

7 **a** $\approx 10^{0.7782}$ **b** $\approx 10^{1.7782}$ **c** $\approx 10^{3.7782}$
d $\approx 10^{-0.2218}$ **e** $\approx 10^{-2.2218}$ **f** $\approx 10^{1.1761}$
g $\approx 10^{3.1761}$ **h** $\approx 10^{0.1761}$ **i** $\approx 10^{-0.8239}$
j $\approx 10^{-3.8239}$

8 **a** **i** ≈ 0.477 **ii** ≈ 2.477
b $\log 300 = \log(3 \times 10^2) = \log(10^{\log 3} \times 10^2) = \ldots.$

9 **a** **i** ≈ 0.699 **ii** ≈ -1.301
b $\log(0.05) = \log(5 \times 10^{-2}) = \log(10^{\log 5} \times 10^{-2}) = \ldots.$

10 **a** $x = 100$ **b** $x = 10$ **c** $x = 1$
d $x = \frac{1}{10}$ **e** $x = \sqrt{10}$ **f** $x = \frac{1}{\sqrt{10}}$
g $x = 10\,000$ **h** $x = 0.000\,01$ **i** $x \approx 6.84$
j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

EXERCISE 3B

1 **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$
d $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$
g $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

2 **a** $\log_4 64 = 3$ **b** $\log_5 25 = 2$
c $\log_7 49 = 2$ **d** $\log_2 64 = 6$
e $\log_2\left(\frac{1}{8}\right) = -3$ **f** $\log_{10}(0.01) = -2$
g $\log_2\left(\frac{1}{2}\right) = -1$ **h** $\log_3\left(\frac{1}{27}\right) = -3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 2 **e** 6 **f** 7
g 2 **h** 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$
m 5 **n** $\frac{1}{3}$ **o** $\frac{1}{3}$ **p** $\frac{3}{2}$ **q** 0 **r** 1
s -1 **t** $\frac{3}{4}$ **u** $-\frac{1}{2}$ **v** $\frac{5}{2}$ **w** $-\frac{3}{2}$ **x** $-\frac{3}{4}$

4 **a** 2 **b** -1 **c** $\frac{1}{2}$ **d** 3 **e** $\frac{1}{4}$ **f** $\frac{3}{2}$
g -2 **h** $-\frac{1}{2}$ **i** $\frac{5}{2}$

5 **a** $x = 8$ **b** $x = 2$ **c** $x = \frac{1}{125}$ **d** $x = 3$
e $x = 14$ **f** $x = \sqrt{3}$

6 $\log_b a = \frac{1}{x}$ **7** $x = \frac{2^{2y}+1}{5}$

EXERCISE 3C

1 **a** $\log 16$ **b** $\log 20$ **c** $\log 8$ **d** $\log\left(\frac{p}{m}\right)$
e 1 **f** $\log 2$ **g** 3 **h** 2
i $\log 24$ **j** 1 **k** 0 **l** $\log 28$

2 **a** $\log 700$ **b** $\log\left(\frac{2}{5}\right)$ **c** $\log_2 6$
d $\log_3\left(\frac{5}{9}\right)$ **e** $\log 200$ **f** $\log(0.005)$
g $\log(10^t \times w)$ **h** $\log_m\left(\frac{40}{m^2}\right)$ **i** $\log_5\left(\frac{5}{2}\right)$

3 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log_3\left(\frac{25}{8}\right)$
e 1 **f** $\log\left(\frac{1}{2}\right)$ **g** $\log 20$ **h** $\log 25$ **i** $\log_n\left(\frac{n^2}{10}\right)$

4 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

5 For example, for **a**, $\log 9 = \log(3^2) = 2\log 3$.

7 **a** $p+q$ **b** $2q+r$ **c** $2p+3q$ **d** $r+\frac{1}{2}q-p$
e $r-5p$ **f** $p-2q$

8 **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$
e $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{2}y-3x$

9 **a** 0.86 **b** 2.15 **c** 1.075 **10** $x = \frac{2}{a^2-1}$

11 **a** $\log 384$ **b** $4+\log_2 45$ **12** $\log\left(\frac{x^{40}}{y^{45}}\right)$

13 $\log \sqrt[3]{3}$ **14** **Hint:** Subtract $2xy$ from both sides.

EXERCISE 3D

1 **a** 2 **b** 4 **c** $\frac{3}{2}$ **d** 0 **e** -1
f $\frac{1}{3}$ **g** -2 **h** $-\frac{1}{2}$

2 **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$ **e** a
f $1+a$ **g** $a+b$ **h** ab

3 **a** ≈ 2.485 **b** ≈ 4.220 **c** ≈ 0.336
d ≈ -0.357 **e** ≈ 6.215

4 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.

5 **a** $\approx e^{1.7918}$ **b** $\approx e^{4.0943}$ **c** $\approx e^{8.6995}$
d $\approx e^{-0.5108}$ **e** $\approx e^{-5.1160}$ **f** $\approx e^{2.7081}$
g $\approx e^{7.3132}$ **h** $\approx e^{0.4055}$ **i** $\approx e^{-1.8971}$
j $\approx e^{-8.8049}$

6 **a** $x \approx 20.1$ **b** $x \approx 2.72$ **c** $x = 1$
d $x \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$
g $x \approx 8.54$ **h** $x \approx 0.0370$

7 **a** **i** x **ii** x **b** They are inverses of each other.

8 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\frac{6}{e}\right)$
i $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\frac{20}{e^2}\right)$ **l** $\ln 1 = 0$

9 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$
e $\ln 6$ **f** $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$
i $\ln 16$ **j** $\ln(16e^2)$ **k** $\ln\left(\frac{3}{e}\right)$ **l** $\ln\left(\frac{\sqrt{e}}{8}\right)$

10 For example, for **a**, $\ln 27 = \ln(3^3) = 3\ln 3$.

11 $x = e^4,\ y = \frac{1}{e^2}$

EXERCISE 3E

1 **a** $\log y = x\log 2$ **b** $\log y = \log 20 + 3\log b$
c $\log M = \log a + 4\log d$ **d** $\log T = \log 5 + \frac{1}{2}\log d$
e $\log R = \log b + \frac{1}{2}\log l$ **f** $\log Q = \log a - n\log b$
g $\log y = \log a + x\log b$ **h** $\log F = \log 20 - \frac{1}{2}\log n$
i $\log L = \log a + \log b - \log c$
j $\log N = \frac{1}{2}\log a - \frac{1}{2}\log b$
k $\log S = \log 200 + t\log 2$ **l** $\log y = m\log a - n\log b$

2 **a** $D = 2e$ **b** $F = \frac{5}{t}$ **c** $P = \sqrt{x}$ **d** $M = b^2c$
e $B = \frac{m^3}{n^2}$ **f** $N = \frac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $Q = \frac{a^2}{x}$

3 **a** $D = ex$ **b** $F = \frac{e^2}{p}$ **c** $P = \sqrt{x}$
d $M = e^3y^2$ **e** $B = \frac{t^3}{e}$ **f** $N = \frac{1}{\sqrt[3]{g}}$
g $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$

4 **a** $\log_2 y = \log_2 3 + x$ **b** $x = \log_2\left(\frac{y}{3}\right)$
c **i** $x = 0$ **ii** $x = 2$ **iii** $x \approx 3.32$

5 **a** $x = 9$ **b** $x = 2$ or 4 **c** $x = 25\sqrt{5}$ **d** $x = 200$
e $x = 5$ **f** $x = 3$ **g** $x = 3$ **h** $x = -2$

6 **a** $x = 2,\ y = 4$ **b** $x = 2,\ y = 8$

7 **a** $2^x = 7$
b $\log 2^x = \log 7$
$\therefore\ x \log 2 = \log 7$
$\therefore\ x = \log_2 7 = \frac{\log 7}{\log 2} \approx 2.81$

8 **a** Taking the logarithm in base a of both sides, $x = \log_a b$.
b $\log a^x = \log b$
c Using **b**, $x \log a = \log b$
$\therefore\ x = \frac{\log b}{\log a}$
and using part **a**, $x = \log_a b = \frac{\log b}{\log a}$

EXERCISE 3F

1 **a** ≈ 1.77 **b** ≈ 5.32 **c** ≈ 3.23 **d** ≈ -10.3
e ≈ -2.46 **f** ≈ 5.42

2 2

4 **a** $x = 16$ **b** $x = \frac{1}{5}$ **c** $x = \sqrt[3]{5}$ **d** $x = 64$
e $x = 2$ or 4

5 $\frac{8}{x}$ **6** $2m + 3$

7 **a** We get $\log_9 2 \neq 0$. **b** $x = 81$ **c** $x = a^4$
d **Hint:** First show that $\log_a x = k^{\frac{k}{k-1}}$.

EXERCISE 3G

1 **a** $3^3 = 27,\ 3^4 = 81,\ \therefore$ if $3^x = 40$, then $3 < x < 4$
b $x = \frac{\log 40}{\log 3}$ **c** $x \approx 3.36$

2 **a** **i** $x = \frac{1}{\log 2}$ **ii** $x \approx 3.32$
b **i** $x = \frac{\log 20}{\log 3}$ **ii** $x \approx 2.73$
c **i** $x = \frac{\log 50}{\log 4}$ **ii** $x \approx 2.82$
d **i** $x = 4$ **ii** $x = 4$
e **i** $x = -\frac{1}{\log\left(\frac{3}{4}\right)}$ **ii** $x \approx 8.00$
f **i** $x = \log(0.000\,015)$ **ii** $x \approx -4.82$

3 **a** $x \approx 2.29$ **b** $x \approx 5.13$ **c** $x \approx 0.194$

4 **a** $x = \ln 10$ **b** $x = \ln 1000$ **c** $x = \ln(0.15)$
d $x = 2 \ln 5$ **e** $x = \frac{1}{2} \ln 18$ **f** $x = 0$

5 **a** $x = \frac{\log 25}{\log 2}$ **b** $x = \frac{\log\left(\frac{20}{7}\right)}{\log 1.5}$ **c** $x = \frac{\log(0.6)}{\log(0.8)}$
d $x = -\frac{\log(0.03)}{\log 2}$ **e** $x = \frac{10 \log\left(\frac{10}{3}\right)}{\log 5}$ **f** $x = 4 \ln 8$

6 **a** $x = \frac{\log 3}{\log 5}$ **b** $x = -\frac{\log 8}{\log 3}$ **c** $x = -1$

7 $x = -\frac{2 \ln 2}{\ln 3}$ **8** $x = \frac{3 \ln 2}{\ln 5}$

9 **a** $x = \ln 2$ **b** $x = 0$ **c** $x = \ln 2$ or $\ln 3$ **d** $x = 0$
e $x = \ln 4$ **f** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

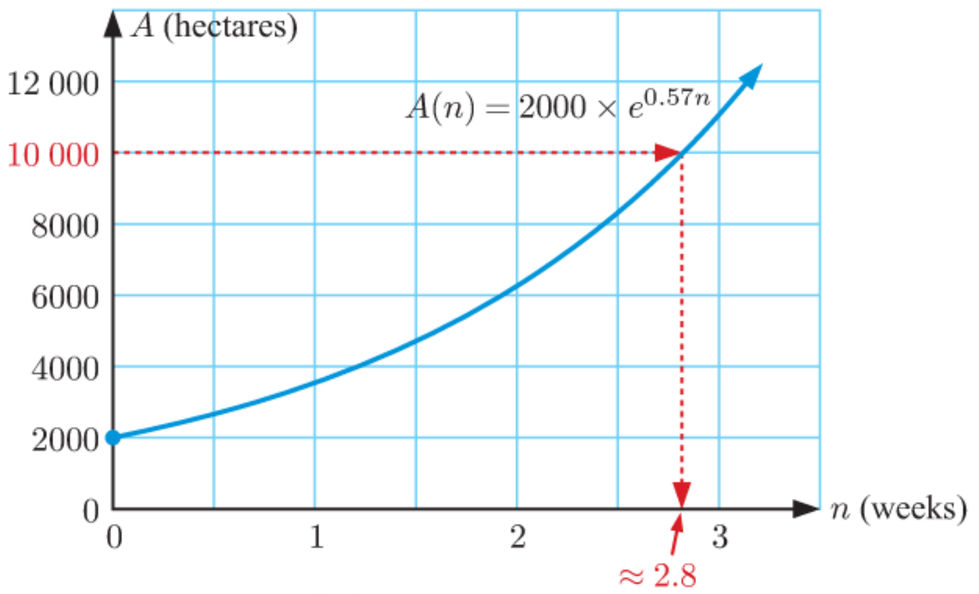
10 **a** $(\ln 3,\ 3)$ **b** $(\ln 2,\ 5)$ **c** $(0,\ 2)$ and $(\ln 5,\ -2)$

11 **a** $(-\ln 2,\ 2)$ **b** $\ln\left(\frac{11}{2}\right)$ units

12 **a** ≈ 2.37 years **b** ≈ 8.36 years

13 **a** ≈ 3.90 hours **b** ≈ 15.5 hours

14 **a**, **b** see graph below



$\therefore$ approximately 2.8 weeks.

15 In ≈ 5.86 years or ≈ 5 years 10 months. **16** 9 years

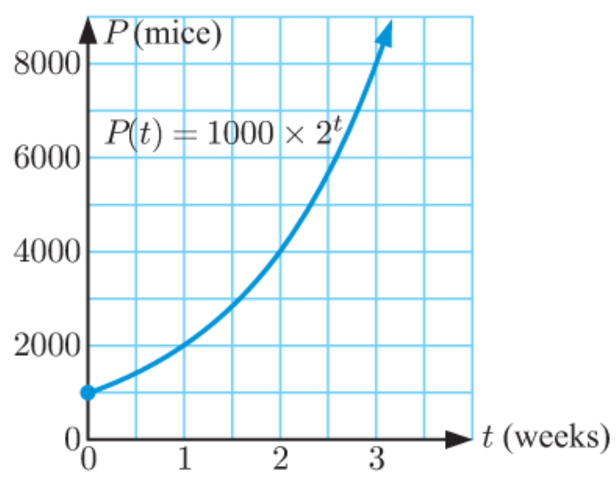
17 **a** $\frac{8.4\%}{12} = 0.7\% = 0.007,\quad r = 1 + 0.007 = 1.007$
b after 74 months

18 **a** ≈ 17.3 years **b** ≈ 92.2 years **c** ≈ 115 years

19 **Hint:** $0.1 \times I_0 = I_0 \times 2^{-0.02t}$
$\therefore\ 0.1 = 2^{-0.02t}$ and solve for t using logarithms.

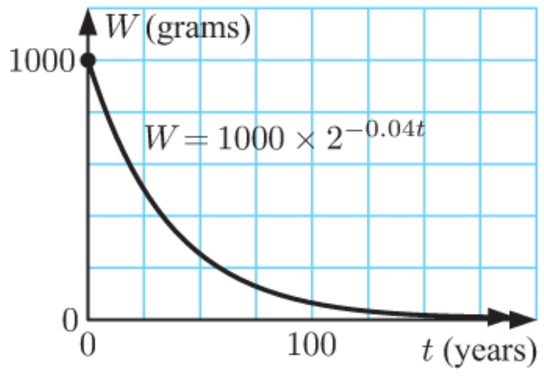
20 **a** **Hint:** Set $V = 40$, solve for t.
b $t = -5 \ln\left(1 - \frac{V}{50}\right)$ s

21 **a**



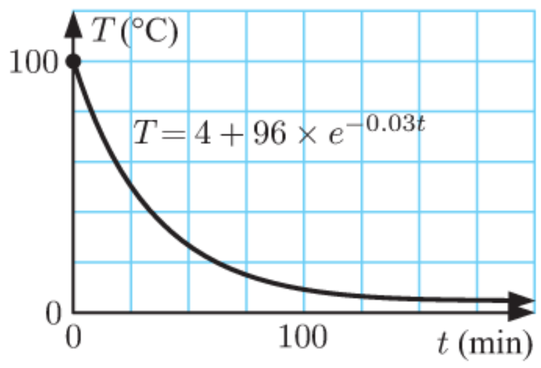
b ≈ 4.32 weeks
c $t = \frac{\log P - 3}{\log 2}$

22 **a**



b $t = \frac{3 - \log W}{0.04 \log 2}$
c **i** $t \approx 141$ years
ii $t \approx 498$ years

23 **a**



b $t = \dfrac{\ln 96 - \ln(T-4)}{0.03}$

c **i** ≈ 50.7 minutes **ii** ≈ 152 minutes

24 **a** decreasing **b** **i** $3900\ \text{m s}^{-1}$ **ii** $\approx 2600\ \text{m s}^{-1}$

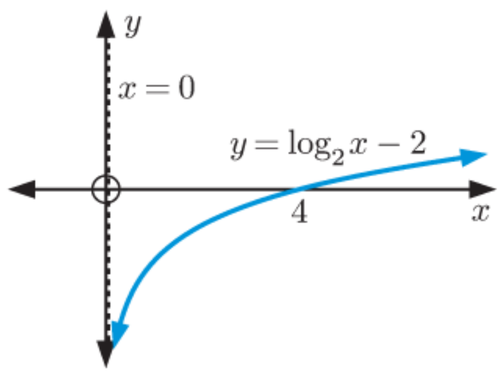
c ≈ 11.8 s

EXERCISE 3H

1 **a** **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept 4, no y-intercept

iii

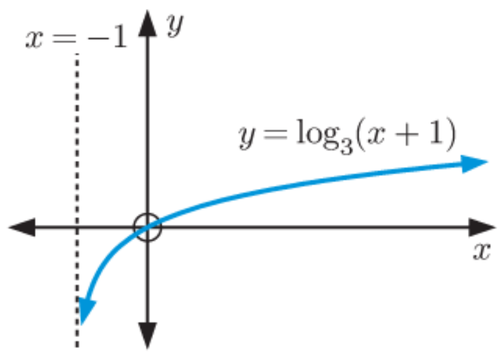


iv $x = 2$ **v** $f^{-1}(x) = 2^{x+2}$

b **i** Domain is $\{x \mid x > -1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -1$, x and y-intercepts 0

iii

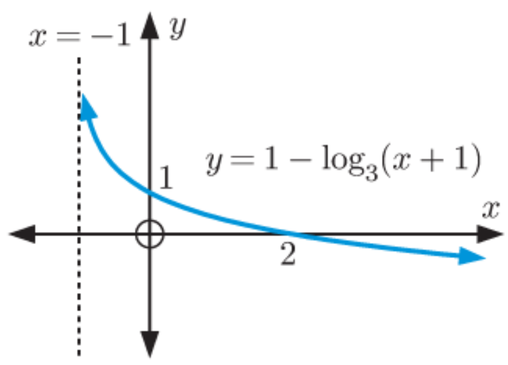


iv $x = -\frac{2}{3}$ **v** $f^{-1}(x) = 3^x - 1$

c **i** Domain is $\{x \mid x > -1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -1$, x-intercept 2, y-intercept 1

iii

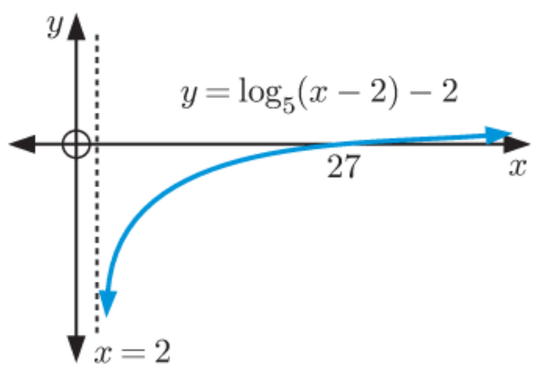


iv $x = 8$ **v** $f^{-1}(x) = 3^{1-x} - 1$

d **i** Domain is $\{x \mid x > 2\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 2$, x-intercept 27, no y-intercept

iii

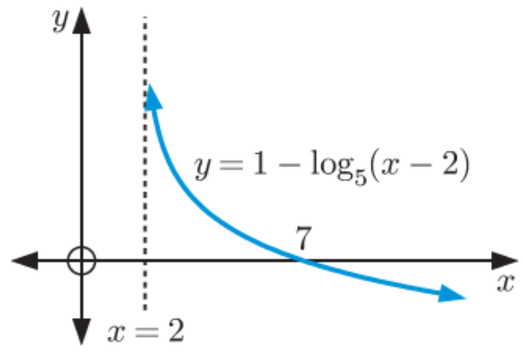


iv $x = 7$ **v** $f^{-1}(x) = 5^{x+2} + 2$

e **i** Domain is $\{x \mid x > 2\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 2$, x-intercept 7, no y-intercept

iii

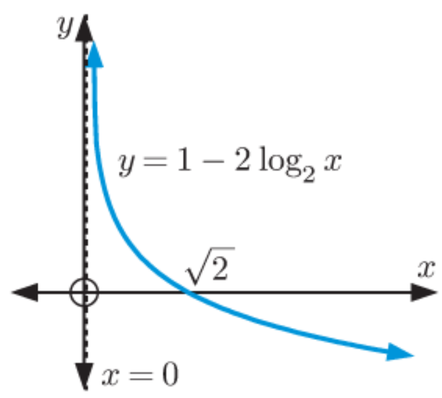


iv $x = 27$ **v** $f^{-1}(x) = 5^{1-x} + 2$

f **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept $\sqrt{2}$, no y-intercept

iii



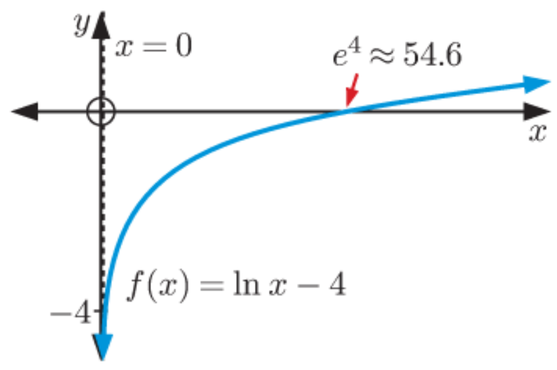
iv $x = 2$ **v** $f^{-1}(x) = 2^{\frac{1-x}{2}}$

2 **a** **i** A translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 0$, x-intercept e^4, no y-intercept

iv



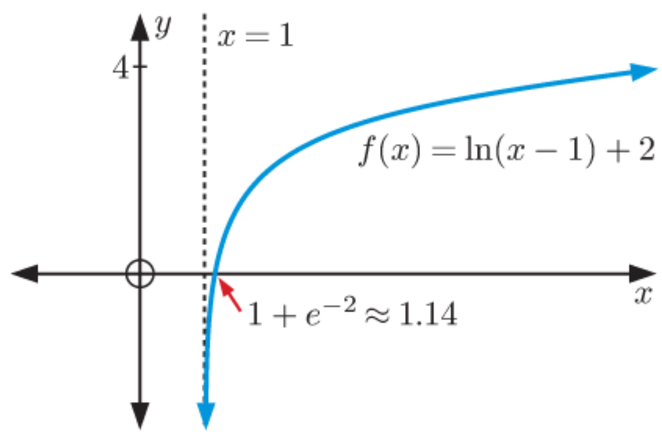
v $f^{-1}(x) = e^{x+4}$

b **i** A translation through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

ii Domain is $\{x \mid x > 1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 1$, x-intercept $1 + e^{-2}$, no y-intercept

iv



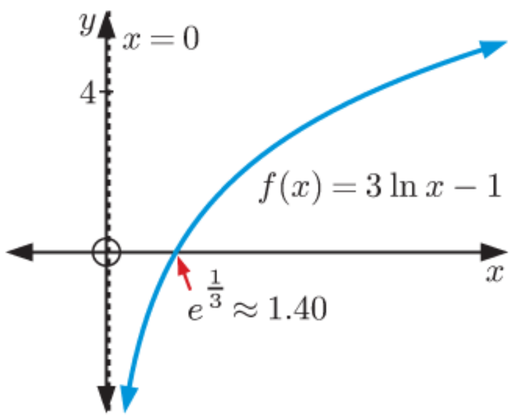
v $f^{-1}(x) = e^{x-2} + 1$

c **i** A vertical stretch with scale factor 3, then a translation through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 0$, x-intercept $e^{\frac{1}{3}}$, no y-intercept

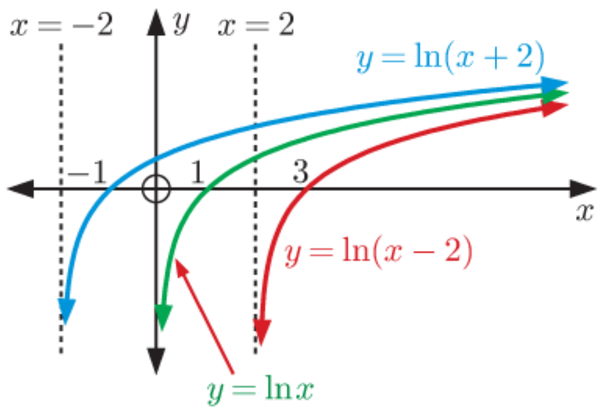
iv



v $f^{-1}(x) = e^{\frac{x+1}{3}}$

3 **a** A is $y = \ln x$ as its x-intercept is 1. B is $y = \ln(x-2)$.

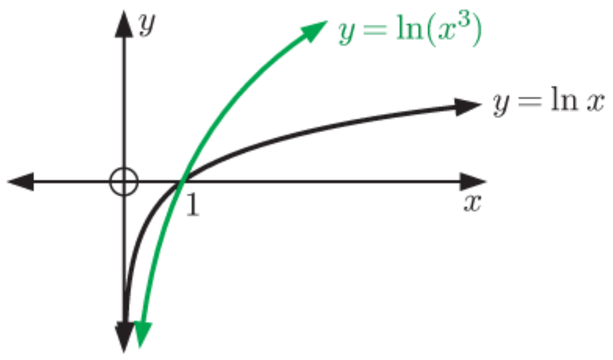
b



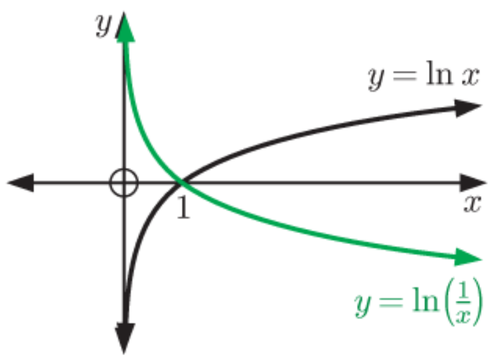
c $y = \ln x$ has vertical asymptote $x = 0$
$y = \ln(x-2)$ has vertical asymptote $x = 2$
$y = \ln(x+2)$ has vertical asymptote $x = -2$

4 $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

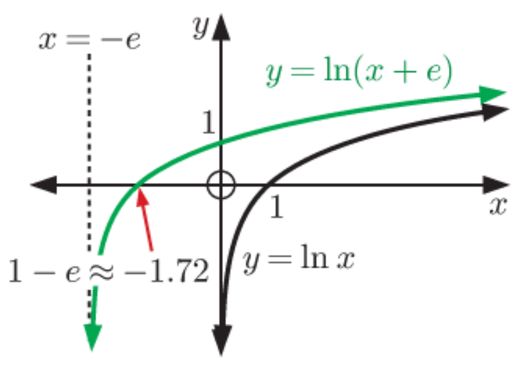
5 **a**



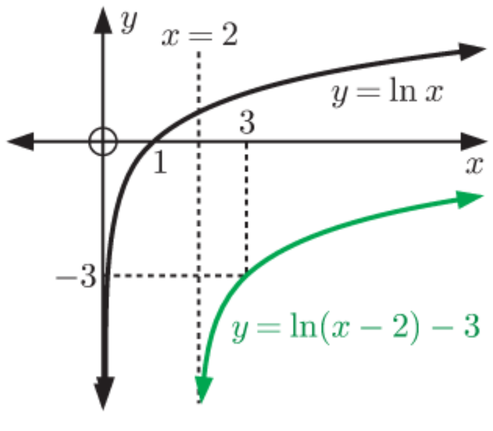
b



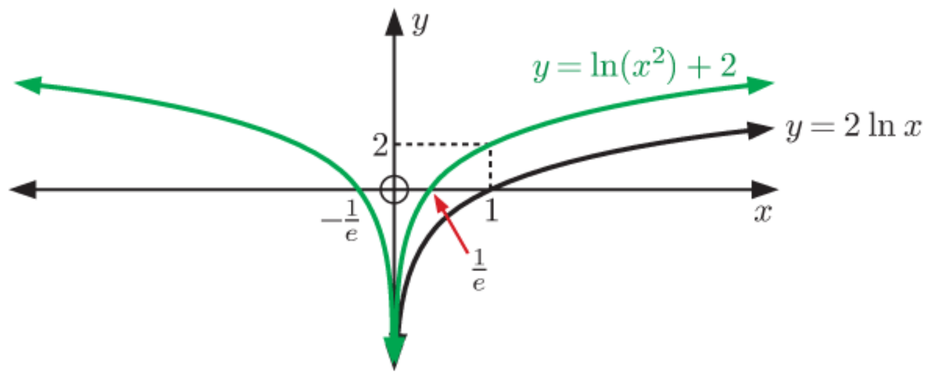
c



d



e



6 A vertical stretch with scale factor $\log_5 2$.

7 **a** **A** is $y = \log_2(x+2)$ since it is increasing.
B is $y = 3 - \log_2(3x+1)$ since it is decreasing.

b **A**: x-intercept -1, y-intercept 1,
vertical asymptote $x = -2$
B: x-intercept $\frac{7}{3}$, y-intercept 3,
vertical asymptote $x = -\frac{1}{3}$

c $(\frac{2}{3},\ 3 - \log_2 3)$

8 **a** $f^{-1}(x) = \log_3 x$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b $f^{-1}(x) = \log_2 x - 1$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

c $f^{-1}(x) = \frac{1}{2}\ln x$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

d $f^{-1}(x) = \log_5(x+3)$
Domain is $\{x \mid x > -3\}$, Range is $\{y \mid y \in \mathbb{R}\}$

9 **a** $b^2 x$ **b** $2\ln b + x$ **c** $x = \dfrac{2\ln b}{b^2 - 1}$

10 **a** $\dfrac{x}{e}$, Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b $x - 1$, Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

11 **a** Domain is $\{x \mid x < \frac{1}{3}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b **i** $x = -\frac{17}{3}$ **ii** $x = \frac{1}{3} - \frac{1}{3\sqrt{2}}$

c $f^{-1}(x) = \dfrac{1 - 2^x}{3}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < \frac{1}{3}\}$

12 $f^{-1}(x) = \frac{1}{2}\ln x$

a $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$

b $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

13 **a** $k = 3$ **b** x-intercepts $1 \pm 2\sqrt{2}$, y-intercept $\log_3 8$

c $g^{-1}(x) = 1 - \sqrt{9 - 3^x}$
Domain is $\{x \mid x \leqslant 2\}$, Range is $\{y \mid -2 < y \leqslant 1\}$

REVIEW SET 3A

1 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a + b + 1$

2 **a** 3 **b** 8 **c** -2 **d** $\frac{1}{2}$ **e** 0
f $\frac{1}{4}$ **g** -1 **h** $\frac{1}{2}$, $k > 0$, $k \neq 1$

3 **a** ≈ 1.431 **b** ≈ -0.237 **c** ≈ 2.602 **d** ≈ 3.689

4 $x = 2 - 3^{2y}$

5 **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\dfrac{25}{e}\right)$ **d** $\ln 3$

6 **a** $\log 144$ **b** $\log_2\left(\frac{16}{9}\right)$ **c** $\log_4 80$

7 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$
d $\frac{1}{2}(A + B)$ **e** $4B - 2A$ **f** $3A - 2B$

8 **a** $\log M = \log a + n \log b$ **b** $\log T = \log 5 - \frac{1}{2}\log l$
c $\log G = 2\log a + \log b - \log c$

9 **a** $x \approx 5.19$ **b** $x \approx 4.29$ **c** $x \approx -0.839$

10 **a** $x = \ln 3$ **b** $x = \ln 3$ or $\ln 4$

11 **a** $P = TQ^{1.5}$ **b** $M = \dfrac{e^{1.2}}{\sqrt{N}}$

12 **a** $x = 0$ or $\ln\left(\frac{2}{3}\right)$ **b** $x = e^2$

13 **a** $x = \frac{1}{8}$ **b** $x \approx 82.7$ **c** $x \approx 0.0316$

14 **a** **i** $x = \dfrac{\log 50}{\log 2}$ **ii** $x \approx 5.64$

b **i** $x = \dfrac{\log 4}{\log 7}$ **ii** $x \approx 0.71$

c **i** $x = \dfrac{-2}{\log(0.6)}$ **ii** $x \approx 9.02$

15 $\log_a\left(\frac{1}{b}\right) = -x$ **16** $4\log_3 5$

17 **Hint:** $2^{4x} - 5 \times 2^{3x} = 0$

18 **a** $x = e^5$ **b** $x = e^{-\frac{2}{3}}$ **c** $x = \ln 400$

d $x = \dfrac{\ln 11 - 1}{2}$ **e** $x = 2\ln 30$

19 **b** $\frac{1}{3}\ln\left(\frac{7}{2}\right)$ **20** $x = 2,\ y = \frac{1}{8}$ or $x = 64,\ y = 4$

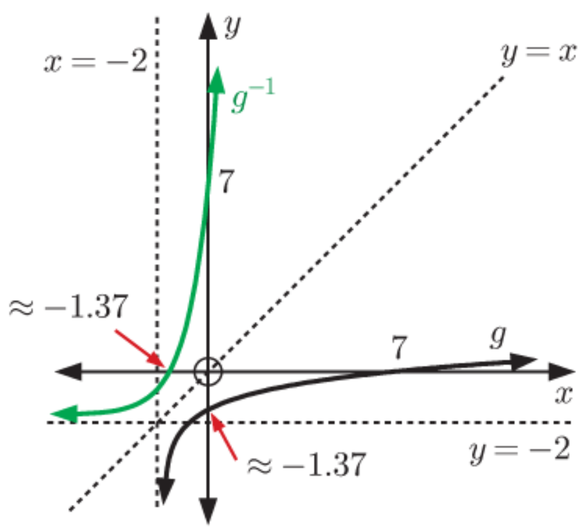
21 **a** A translation through $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

b Domain is $\{x \mid x > -2\}$, Range is $\{y \mid y \in \mathbb{R}\}$

c vertical asymptote is $x = -2$, x-intercept is 7, y-intercept is ≈ -1.37

d $g^{-1}(x) = 3^{x+2} - 2$

e



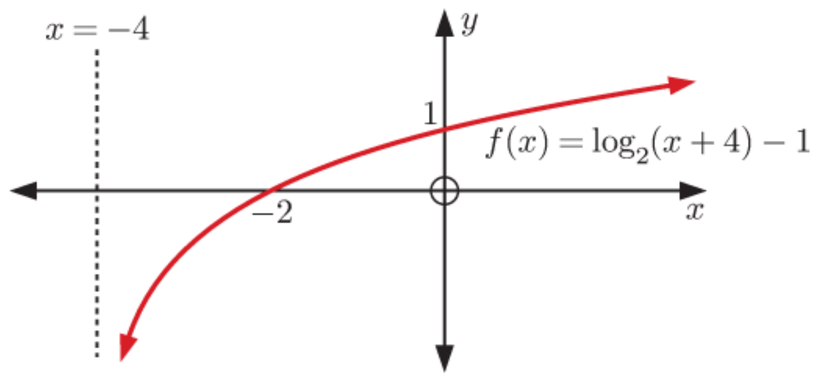
22 **a** ≈ 13.9 weeks **b** ≈ 41.6 weeks **c** ≈ 138 weeks

23 **a** ≈ 4.96 years or ≈ 4 years $11\frac{1}{2}$ months **b** $\approx 74.9\%$

24 **a** **i** Domain is $\{x \mid x > -4\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -4$, x-intercept -2, y-intercept 1

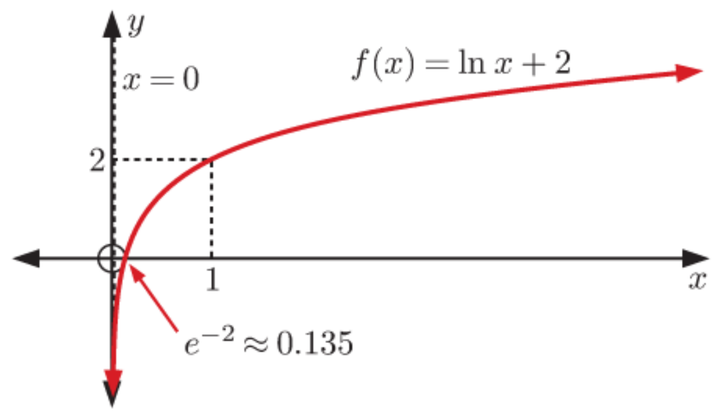
iii



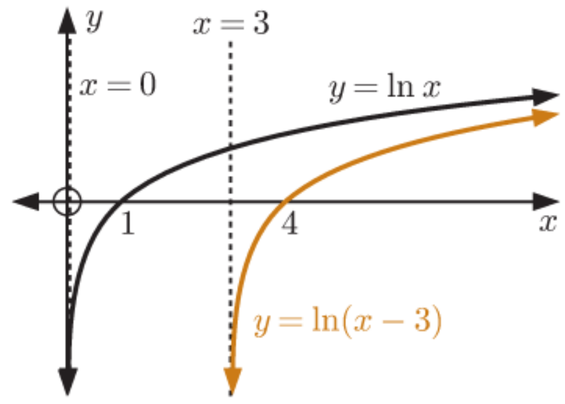
b **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept e^{-2}, no y-intercept

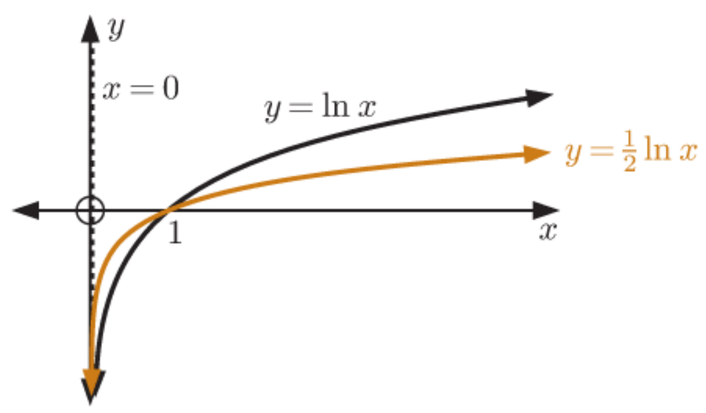
iii



25 **a**



b



26 **a** 9 **b** $\ln 5$

REVIEW SET 3B

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a + b$

2 **a** 7 **b** -3 **c** $-\frac{1}{2}$

3 **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

4 **a** $\frac{3}{2}$ **b** -3 **c** $2x$ **d** $1 - x$

5 **a** $\frac{2}{3}$ **b** $\frac{6}{5}$ **c** 8

6 **a** x^4 **b** 5 **c** $\frac{1}{2}$ **d** $3x$ **e** $-x$ **f** $\log x$

7 **a** $\approx e^{2.9957}$ **b** $\approx e^{8.0064}$ **c** $\approx e^{-2.5903}$

8 **a** **i** $x = \dfrac{\log 7}{\log 5}$ **ii** $x \approx 1.21$

b **i** $x = -\dfrac{1}{\log 2}$ **ii** $x \approx -3.32$

9 **a** $\ln 3$ **b** $\ln 4$ **c** $\ln 125$

10 **a** $x = \dfrac{\ln 70}{2}$ **b** $x = \dfrac{\log\left(\frac{11}{3}\right)}{\log 1.3}$ **c** $x = \dfrac{10\log\left(\frac{16}{5}\right)}{3\log 2}$

11 $x = 1$

13 **a** $\log P = \log 3 + x\log b$ **b** $\log m = 3\log n - 2\log p$

14 **Hint:** Use the change of base rule.

15 **a** $x = 2^{\frac{18}{13}}$ **b** $x = 10^{\frac{3}{11}}$

16 **a** $T = \dfrac{x^2}{y}$ **b** $K = n\sqrt{t}$

17 **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

18

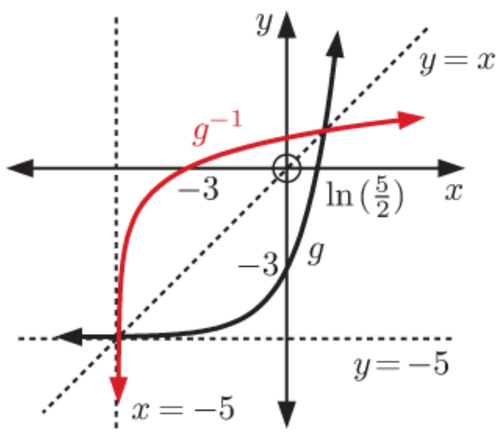
	$y = \log_2 x$	$y = \ln(x + 5)$
Domain	$x > 0$	$x > -5$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

19 **a** $(2^x + 4)(2^x - 5)$ **b** $x = \dfrac{\log 5}{\log 2}$

c **i** $x = \dfrac{1}{p}$ **ii** $x = \dfrac{1}{3p+1}$

20 **a** $g^{-1}(x) = \ln\left(\dfrac{x+5}{2}\right)$

b



c Domain of g is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -5\}$
Domain of g^{-1} is $\{x \mid x > -5\}$, Range is $\{y \mid y \in \mathbb{R}\}$

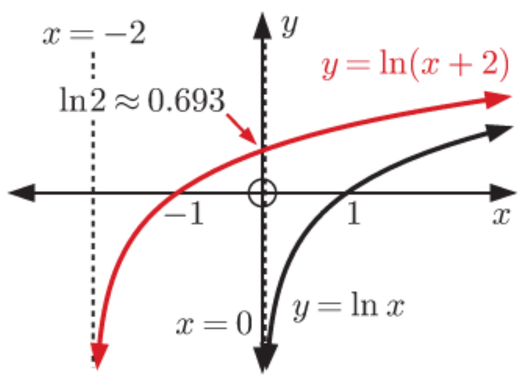
d g has horizontal asymptote $y = -5$,
x-intercept is $\ln\left(\frac{5}{2}\right) \approx 0.916$, y-intercept is -3
g^{-1} has vertical asymptote $x = -5$,
x-intercept is -3, y-intercept is ≈ 0.916

21 **Hint:** Set $T = 40$, and solve for t.

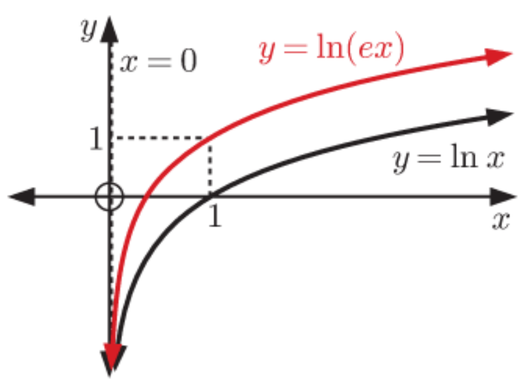
22 **a** 2500 g **b** $\approx$ 3288 years **c** $\approx 42.3\%$

23 **a** $x = \dfrac{2\log 9}{\log 5}$ **b** $x = \ln 30$ **c** $x = \dfrac{1 - \ln 2}{3}$

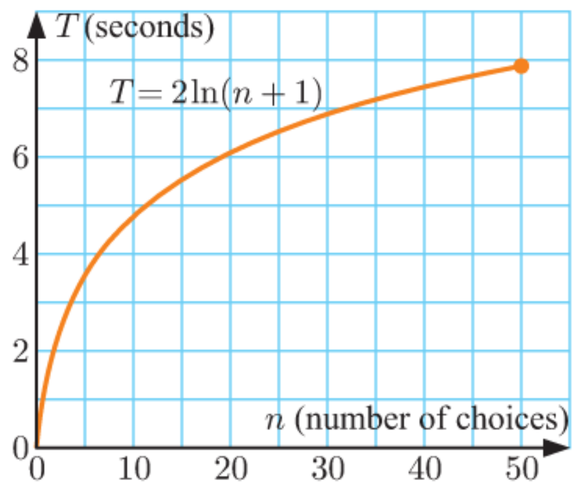
24 **a**



b



25 **a**



b **i** $\approx$ 3.58 seconds **ii** $\approx$ 5.55 seconds

c $\approx$ 1.34 seconds longer

26 **a** **i** $f^{-1}(x) = 3^{x+2}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 0\}$

ii $(f \circ g)(x) = \log_3(3 - \sqrt{x}) - 2$
Domain is $\{x \mid 0 \leqslant x < 9\}$, Range is $\{y \mid y \leqslant -1\}$

iii $(g \circ f)(x) = 3 - \sqrt{\log_3 x - 2}$
Domain is $\{x \mid x \geqslant 9\}$, Range is $\{y \mid y \leqslant 3\}$

b **i** $x = 4$ **ii** $x = 3^{11} = 177\,147$

c $(g \circ f)^{-1}(x) = 3^{x^2 - 6x + 11}$, $x \leqslant 3$
Domain is $\{x \mid x \leqslant 3\}$, Range is $\{y \mid y \geqslant 9\}$

EXERCISE 4A

1

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3 + 2i$	3	2	$3 - 2i$
$5 - i$	5	-1	$5 + i$
3	3	0	3
0	0	0	0
$-3 + 4i$	-3	4	$-3 - 4i$
$-7 - 2i$	-7	-2	$-7 + 2i$
$-11i$	0	-11	$11i$
$i\sqrt{3}$	0	$\sqrt{3}$	$-i\sqrt{3}$

2 **a** $3i$ **b** $8i$ **c** $\frac{1}{2}i$ **d** $i\sqrt{5}$ **e** $i\sqrt{8}$

3 **a** $x = \pm 5$ **b** $x = \pm 5i$ **c** $x = \pm\sqrt{5}$
d $x = \pm i\sqrt{5}$ **e** $x = \pm\frac{3}{2}$ **f** $x = \pm\frac{3}{2}i$

4 **a** $x = 5 \pm 2i$ **b** $x = -3 \pm 4i$ **c** $x = -7 \pm i$
d $x = \frac{3}{2} \pm \frac{1}{2}i$ **e** $x = \sqrt{3} \pm i$ **f** $x = \frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

EXERCISE 4B

1 **a** $(x+3)(x-3)$ **b** $(x+3i)(x-3i)$
c $(x+\sqrt{7})(x-\sqrt{7})$ **d** $(x+i\sqrt{7})(x-i\sqrt{7})$
e $(2x+1)(2x-1)$ **f** $(2x+i)(2x-i)$
g $(\sqrt{2}x+3)(\sqrt{2}x-3)$ **h** $(\sqrt{2}x+3i)(\sqrt{2}x-3i)$

2 $(a+bi)(a-bi) = a^2 - b^2i^2 = a^2 + b^2$ {since $i^2 = -1$}

3 **a** $x = \pm 4i$ **b** $x = \pm 6i$ **c** $x = \pm i\sqrt{5}$
d $x = \pm\frac{1}{3}i$ **e** $x = \pm\frac{5}{2}i$ **f** $x = \pm i\frac{\sqrt{2}}{\sqrt{3}}$

4 **a** $x(x+1)(x-1)$ **b** $x(x+i)(x-i)$
c $(x+1)(x-1)(x+i)(x-i)$
d $(x+2)(x-2)(x+2i)(x-2i)$

5 **a** $x = 0$ or ± 2 **b** $x = 0$ or $\pm 2i$
c $x = 0$ or $\pm\sqrt{3}$ **d** $x = 0$ or $\pm i\sqrt{3}$
e $x = \pm 1$ or $\pm i$ **f** $x = \pm 3$ or $\pm 3i$

6 **a** $x = \pm i\sqrt{3}$ or ± 1 **b** $x = \pm\sqrt{3}$ or $\pm i\sqrt{2}$
c $x = \pm 3i$ or ± 2 **d** $x = \pm i\sqrt{7}$ or $\pm i\sqrt{2}$
e $x = \pm 1$ **f** $x = \pm i$

EXERCISE 4C

1 **a** $7 - i$ **b** $10 - 4i$ **c** $-1 + 2i$ **d** $3 - 3i$
e $4 - 7i$ **f** $12 + i$ **g** $3 + 4i$ **h** $21 - 20i$

2 **a** $-3 + 7i$ **b** $2i$ **c** $-2 + 2i$ **d** $-1 + i$
e $-5 - 12i$ **f** $-5 + i$ **g** $-6 - 4i$ **h** $-1 - 5i$

3 $(a + \cancel{bi}) + (a - \cancel{bi}) = 2a$ which is real

4 **a** $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$,
$i^6 = -1$, $i^7 = -i$, $i^8 = 1$, $i^9 = i$, $i^{-1} = -i$,
$i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$, $i^{-5} = -i$

b $i^{4n+3} = -i$

5 $(1+i)^4 = -4$, $(1+i)^{101} = -2^{50}(1+i)$

6 **a** -6 **b** No, $\sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{36}$.

7 **a** $-\frac{1}{10} - \frac{7}{10}i$ **b** $-\frac{1}{5} + \frac{2}{5}i$ **c** $\frac{7}{5} + \frac{1}{5}i$ **d** $\frac{3}{25} + \frac{4}{25}i$

8 **a** $-\frac{2}{5} + \frac{1}{5}i$ **b** $-\frac{1}{13} + \frac{8}{13}i$ **c** $-\frac{2}{5} + \frac{3}{5}i$
d $\dfrac{2}{x^2+1}$

9 **a** -2 **b** -4 **c** 3 **d** 0

10 $z = 65 - 72i$ **11** $z = i\sqrt{2}$

12 **a** $w = \dfrac{a^2 - b^2 - 1}{(a+1)^2 + b^2} + \dfrac{2ab}{(a+1)^2 + b^2}\, i$

b purely imaginary if $a^2 - b^2 = 1$ and $a \neq 0,\ b \neq 0$

13 $z = 1 + 2i$ or $1 - i$

EXERCISE 4D

1 **a** $x = 0,\ y = -2$ **b** $x = -2$ **c** $x = 3,\ y = 2$

d $x = -\frac{2}{13},\ y = -\frac{3}{13}$

2 **a** $x = 0,\ y = 0$

b $x = 3,\ y = -2$ or $x = 4,\ y = -\frac{3}{2}$

c $x = 2,\ y = -5$ or $x = -\frac{5}{3},\ y = 6$

d $x = -1,\ y = 0$

3 $z = 5 - 4i$ **4** $m = -\frac{1}{11},\ n = \frac{8}{11}$

5 $a = 3,\ b = -5$ or $a = -3,\ b = 5$

6 $z = \frac{4}{3} + \frac{2}{3}i$ or $z = -\frac{4}{3} - \frac{2}{3}i$

EXERCISE 4E

2 z^* **3** **a** $\left(\dfrac{ac+bd}{c^2+d^2}\right) + \left(\dfrac{bc-ad}{c^2+d^2}\right) i$

7 **a** $z^2 = (a^2 - b^2) + (2ab)i$

b $z^3 = (a^3 - 3ab^2) + (3a^2b - b^3)i$

8 **a** $a = 0$ or $b = 0,\ a \neq -1$

b $a^2 - b^2 = 1$ and $a \neq 0,\ b \neq 0$

REVIEW SET 4A

1 **a** $x = 3 \pm i\sqrt{2}$ **b** $x = \sqrt{2} \pm i$

2 **a** $x = \pm\frac{4}{\sqrt{3}}i$ **b** $x = \pm\sqrt{3}$ or $\pm i\sqrt{5}$

3 **a** $12 + 5i$ **b** $-1 + i$ **c** $18 + 26i$

4 $\mathcal{Re}(z) = \frac{7\sqrt{3}}{4},\ \mathcal{Im}(z) = -\frac{3}{4}$ **5** $z = 2 - 2i$

6 **a** $a = 4,\ b = 0$ **b** $a = 3,\ b = -4$

c $a = 3,\ b = -7$ or $a = 14,\ b = -\frac{3}{2}$

7 $z = \frac{3}{5} - \frac{1}{5}i$ **8** $a = 0,\ b = -1$ **10** $x = \sqrt{5}$ $(y = 0)$

REVIEW SET 4B

1 **a** $(x - 11i)(x + 11i)$ **b** $(\sqrt{3}x - i\sqrt{7})(\sqrt{3}x + i\sqrt{7})$

2 $x = 0,\ \pm\frac{1}{2}i$ **3** 7 **4** $-1 + 2i$

5 $z = -\frac{3737}{169} + \frac{4416}{169}i$

6 $w = \dfrac{(a+1)^2 - b^2}{(a+1)^2 + b^2} + \left(\dfrac{2(a+1)b}{(a+1)^2 + b^2}\right) i$

w is purely imaginary if $b = \pm(a+1),\ a \neq -1,\ b \neq 0$

7 **a** $x = 0,\ y = 0$ **b** $x = 5,\ y = -7$

c $x = 0,\ y = 0,$ or $x = 1,\ y = 0,$

or $x = -\frac{1}{2},\ y = \frac{\sqrt{3}}{2},$ or $x = -\frac{1}{2},\ y = -\frac{\sqrt{3}}{2}$

d $x = -1,\ y = 2$

8 $z = 3 - 2i$ or $-3 + 2i$

EXERCISE 5A

1 **a** **i** 3 **ii** 2 **iii** 5 **b** **i** 4 **ii** 1 **iii** -2

c **i** 5 **ii** -3 **iii** 1 **d** **i** 3 **ii** 5 **iii** 0

e **i** 4 **ii** -1 **iii** 7 **f** **i** 3 **ii** -4 **iii** 7

2 **a** 7 **b** -4 **c** 0 **d** $\frac{1}{4}$

EXERCISE 5B

1 **a** $3x^2 + 6x + 9$ **b** $5x^2 + 7x + 9$ **c** $-7x^2 - 8x - 9$

d $4x^4 + 13x^3 + 28x^2 + 27x + 18$

2 **a** $x^3 + x^2 - 4x + 7$ **b** $x^3 - x^2 - 2x + 3$

c $3x^3 + 2x^2 - 11x + 19$ **d** $2x^3 - x^2 - x + 5$

e $x^5 - x^4 - x^3 + 8x^2 - 11x + 10$

f $x^4 - 2x^3 + 5x^2 - 4x + 4$

3 **a** $2x^3 - 3x^2 + 4x + 3$ **b** $x^4 + x^3 - 7x^2 + 7x - 2$

c $x^3 + 6x^2 + 12x + 8$ **d** $4x^4 - 4x^3 + 13x^2 - 6x + 9$

e $16x^4 - 32x^3 + 24x^2 - 8x + 1$

f $18x^4 - 87x^3 + 56x^2 + 20x - 16$

4 **a** $6x^3 - 11x^2 + 18x - 5$ **b** $8x^3 + 18x^2 - x + 10$

c $-2x^3 + 7x^2 + 13x + 10$ **d** $2x^3 - 7x^2 + 4x + 4$

e $2x^4 - 2x^3 - 9x^2 + 11x - 2$

f $15x^4 + x^3 - x^2 + 7x - 6$ **g** $x^4 - 2x^3 + 7x^2 - 6x + 9$

h $4x^4 + 4x^3 - 15x^2 - 8x + 16$

i $8x^3 + 60x^2 + 150x + 125$

j $x^6 + 2x^5 + x^4 - 4x^3 - 4x^2 + 4$

5 degree of $Q(x)$ is 4 $\{(x-1) \times x^3 = x^4 \ldots.\}$

6 degree of $Q(x)$ is 3

7 degree is 6, leading coefficient is -15, constant is -10

8 **a** If $p \neq q$, the degree of $P(x) + Q(x)$ is the maximum of p and q since the highest power of x is the maximum of p and q.

If $p = q$, the degree of $P(x) + Q(x) \leqslant p$, since the coefficients of the higher powers of x may cancel.

b $kP(x)$ has degree p since any non-zero scalar multiple of a polynomial results in a polynomial of the same degree.

c $P(x)\,Q(x)$ has degree $p + q$ since the highest power of x is $x^p \times x^q = x^{p+q}$.

d $(P(x))^2$ has degree $2p$ since the highest power of x is $x^p \times x^p = x^{2p}$.

EXERCISE 5C

1 **a** yes **b** no **c** no **d** yes

2 **a** $-1,\ \frac{5}{3}$ **b** $-3 \pm i$ **c** $3 \pm \sqrt{3}$ **d** $0,\ \pm 2$

e $0,\ \pm i\sqrt{2}$ **f** $\pm 1,\ \pm i\sqrt{5}$

3 **a** $1,\ -\frac{2}{5}$ **b** $-\frac{1}{2},\ \pm i\sqrt{3}$ **c** $0,\ 1 \pm i$

d $0,\ \pm\sqrt{5}$ **e** $0,\ \pm i\sqrt{5}$ **f** $\pm i\sqrt{2},\ \pm\sqrt{5}$

5 **a** $(2x + 3)(x - 5)$ **b** $(z - 3 + i\sqrt{7})(z - 3 - i\sqrt{7})$

c $x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$ **d** $z(3z - 2)(2z + 1)$

e $(z + 1)(z - 1)(z + \sqrt{5})(z - \sqrt{5})$

f $(z + i)(z - i)(z + \sqrt{2})(z - \sqrt{2})$

6 **a** $(x - 8)(x + 2)$; zeros are 8 and -2

b $(2x + 3)(x - 5)$; zeros are $-\frac{3}{2}$ and 5

c no real linear factors; no real zeros

d $x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$; zeros are $0,\ -1 \pm \sqrt{5}$

e $z(3z - 2)(2z + 1)$; zeros are $0,\ \frac{2}{3}$, or $-\frac{1}{2}$

f real linear factors $(z + \sqrt{3})(z - \sqrt{3})$; real zeros are $\pm\sqrt{3}$

7 $P(\alpha) = P(\beta) = P(\gamma) = 0$

8 **a** $P(\alpha) = 0,\ P(\beta) = 0,\ P(\gamma) = 0$

b If $P(x)$ has a factor of $(x - a)$, then $P(a) = 0$ which implies that $P(x)$ has x-intercept a which is distinct from the x-intercepts of α, β, and γ.

But $P(x)$ has only three x-intercepts and since $a \neq \alpha$, β, or γ, a cannot be one of them.

$\therefore$ $P(x)$ cannot have a factor of $(x - a)$.

9 **a** $P(x) = ax(3x + 2),\ a \neq 0$

b $P(x) = a(x^2 - 4x + 1),\ a \neq 0$
c $P(x) = a(x+1)(x-1+\sqrt{2}),\ a \neq 0$
d $P(x) = a(x^2 - 6x + 10),\ a \neq 0$
e $P(x) = a(x^2 - 2x + 10),\ a \neq 0$
f $P(x) = a(x^2 + 4x + 29),\ a \neq 0$
g $P(x) = a(x^2 + 6x + 7),\ a \neq 0$
h $P(x) = a(x^2 + 2),\ a \neq 0$
i $P(x) = a(x^2 - 2\sqrt{2}x + 3),\ a \neq 0$

11 **a** $P(x) = a(x^2 - 4)(x - 3),\ a \neq 0$
b $P(x) = a(x+2)(x^2 + 16),\ a \neq 0$
c $P(x) = a(x-3)(x^2 + 2x + 2),\ a \neq 0$
d $P(x) = a(x+5)(x^2 + 6x + 13),\ a \neq 0$
e $P(x) = a(x+1)(x^2 - 6x + 7),\ a \neq 0$
f $P(x) = a(x - \sqrt{2})(x^2 + 4x + 6),\ a \neq 0$

12 **a** $P(x) = ax(x+2)(x-1)(x-5),\ a \neq 0$
b $P(x) = a(x^2 - 1)(x^2 - 2),\ a \neq 0$
c $P(x) = a(x-2)(x+1)(x^2 + 3),\ a \neq 0$
d $P(x) = ax(x-4)(x^2 - 4x + 1),\ a \neq 0$
e $P(x) = a(x^2 - 3)(x^2 - 2x + 2),\ a \neq 0$
f $P(x) = a(x^2 - 4x - 1)(x^2 + 4x + 13),\ a \neq 0$

13 **a** **Hint:** Show that the sum and product of the zeros must be rational. Then use the fact that the sum of the zeros is rational to derive the form of the second zero.
b $a = -2,\ b = -1$
c $P(x) = a(x^2 - 4x - 1),\quad a \neq 0$

14 **a** **Hint:** Let the quadratic be $\alpha x^2 + \beta x + \gamma$ and use the properties of complex conjugates.
b $a = -6,\ b = 10$
c $P(x) = a(x^2 - 2\sqrt{2}x + 3),\quad a \neq 0$
d $a = -2,\ b = 8$ or $a = 0,\ b = 0$

EXERCISE 5D

1 **a** $a = 2,\ b = 5,\ c = 5$ **b** $a = 4,\ b = 5,\ c = 14$
c $a = 3,\ b = 4,\ c = 3$

2 **a** $a = 2,\ b = -2$ or $a = -2,\ b = 2$
b $a = 2,\ b = -1$ or $a = -1,\ b = 2$
c $a = 3,\ b = -1$

4 $a = -2,\ b = 2,\ x = 1 \pm i$ or $-1 \pm \sqrt{3}$

5 $c = \pm\sqrt{12},\ d = \mp\sqrt{12},\ x = -\sqrt{3} \pm \sqrt{2},\ \sqrt{3} \pm \sqrt{2}$

6 **a** $a = 3$, zeros are $-4,\ 1,\ \frac{3}{2}$
b $a = -4$, zeros are $2,\ -1,\ -\frac{2}{3}$

7 **a** $a = -9,\ b = -1,$ zeros are $-1,\ -\frac{1}{2},\ 2,\ 4$
b $a = 1,\ b = -15,$ zeros are $-3,\ \frac{1}{2},\ 1 \pm \sqrt{2}$

8 $k = 13$

9 $x^4 - 2x^3 - 3x^2 + 4x + 3 = (x^2 - x)^2 - 4(x^2 - x) + 3,$
zeros are $\frac{1}{2} \pm \frac{\sqrt{5}}{2},\ \frac{1}{2} \pm \frac{\sqrt{13}}{2}$

10 **a** If $k = 20,\ P(x) = (x-2)^2(x+5)$
If $k = -\frac{832}{27},\ P(x) = \frac{1}{27}(3x - 13)(3x + 8)^2$
b $P(x) = (x+3)^2(x-3)$ or $P(x) = (x-1)^2(x+5)$
c If $m = -2$, zeros are -1 (repeated) and $\frac{2}{3}$.
If $m = \frac{14}{243}$, zeros are $\frac{1}{9}$ (repeated) and $-\frac{14}{9}$.

EXERCISE 5E.1

1 **a** $Q(x) = x,\ R = -3,\ x^2 + 2x - 3 = x(x+2) - 3$
b $Q(x) = x - 4,\ R = -3,\ x^2 - 5x + 1 = (x-4)(x-1) - 3$
c $Q(x) = 2x^2 + 10x + 16,\ R = 35,$
$2x^3 + 6x^2 - 4x + 3 = (2x^2 + 10x + 16)(x-2) + 35$

2 **a** $x^2 - 3x + 6 = (x+1)(x-4) + 10$
b $x^2 + 4x - 11 = (x+1)(x+3) - 14$
c $2x^2 - 7x + 2 = (2x-3)(x-2) - 4$
d $2x^3 + 3x^2 - 3x - 2 = (x^2 + x - 2)(2x+1)$
e $3x^3 + 11x^2 + 8x + 7 = (x^2 + 4x + 4)(3x - 1) + 11$
f $2x^4 - x^3 - x^2 + 7x + 4 = (x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4})(2x+3) + \frac{19}{4}$

3 **a** $x + 2 + \dfrac{9}{x-2}$ **b** $2x + 1 - \dfrac{1}{x+1}$
c $3x - 4 + \dfrac{3}{x+2}$ **d** $x^2 + 3x - 2$
e $2x^2 - 8x + 31 - \dfrac{124}{x+4}$ **f** $x^2 + 3x + 6 + \dfrac{7}{x-2}$

4 **a** $3x^2 - 8x + 4,\quad (x+1)(3x-2)(x-2)$
b $2x^2 + 11x + 12,\quad (x-2)(2x+3)(x+4)$
c $4x^2 + x - 3,\quad (x+5)(4x-3)(x+1)$

5 **a** $\dfrac{x^3 + a^3}{x+a} = x^2 - ax + a^2$
b **i** $(x^2 - x + 1)(x+1)$ **ii** $(x^2 - 2x + 4)(x+2)$
iii $(x^2 - 5x + 25)(x+5)$

6 **a** **i** $\dfrac{x^2 - a^2}{x-a} = x + a$ **ii** $\dfrac{x^3 - a^3}{x-a} = x^2 + ax + a^2$
iii $\dfrac{x^4 - a^4}{x-a} = x^3 + ax^2 + a^2x + a^3$
iv $\dfrac{x^5 - a^5}{x-a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4$
b $(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1})$
c **i** $(x-2)(x^2 + 2x + 4)$
ii $(x-1)(x^3 + x^2 + x + 1)$
iii $(x-2)(x^3 + 2x^2 + 4x + 8)$
iv $(x-3)(x^4 + 3x^3 + 9x^2 + 27x + 81)$

EXERCISE 5E.2

1 **a** quotient is $x + 1$, remainder is $-x - 4$
b quotient is 3, remainder is $-x + 3$
c quotient is $3x$, remainder is $-2x - 1$
d quotient is 0, remainder is $x - 4$

2 **a** $1 - \dfrac{2x}{x^2 + x + 1},\quad x^2 - x + 1 = 1(x^2 + x + 1) - 2x$
b $x - \dfrac{2x}{x^2 + 2},\quad x^3 = x(x^2 + 2) - 2x$
c $x^2 + x + 3 + \dfrac{3x - 4}{x^2 - x + 1},$
$x^4 + 3x^2 + x - 1 = (x^2 + x + 3)(x^2 - x + 1) + 3x - 4$
d $2x + 4 + \dfrac{5x + 2}{(x-1)^2},$
$2x^3 - x + 6 = (2x + 4)(x-1)^2 + 5x + 2$
e $x^2 - 2x + 3 - \dfrac{4x + 3}{(x+1)^2},$
$x^4 = (x^2 - 2x + 3)(x+1)^2 - 4x - 3$
f $x^2 - 3x + 5 + \dfrac{15 - 10x}{(x-1)(x+2)},$
$x^4 - 2x^3 + x + 5 = (x^2 - 3x + 5)(x-1)(x+2) + 15 - 10x$

3 quotient is $x^2 + 2x + 3$, remainder is 7

4 quotient is $x^2 - 3x + 5$, remainder is $15 - 10x$

EXERCISE 5F

1 **a** $P(x) = Q(x)(x-2) + 7$, $P(x)$ divided by $x-2$ leaves a remainder of 7.

b $P(-3) = -8$, $P(x)$ divided by $x+3$ leaves a remainder of -8.

c $P(5) = 11$, $P(x) = Q(x)(x-5) + 11$

2 **a** 1 **b** -19 **c** 1 **3** **a** $a = 3$ **b** $a = 2$

4 $a = -5$, $b = 6$ **5** $a = -3$, $n = 4$ **6** -7

7 **a** $P(x) = Q(x)(2x-1) + R$

$P(\frac{1}{2}) = Q(\frac{1}{2})(2 \times \frac{1}{2} - 1) + R$

$= Q(\frac{1}{2}) \times 0 + R$

$= R$

b **i** -3 **ii** 7

8 -7 **9** $a = 3$, $b = 10$ **10** **a** -3 **b** 1

11 $R(x) = -5x + 3$ **12** $3z - 5$

13 **Hint:** $P(x) = (x-a)(x-b)\,Q(x) + (Cx + D)$

EXERCISE 5G

1 **a** yes **b** no **c** yes

2 **a** $a = 5$ **b** $a = -\frac{79}{8}$

3 **a** $k = -8$, $P(x) = (x+2)(2x+1)(x-2)$

b $k = 2$, $P(x) = x(x-3)(x+\sqrt{2})(x-\sqrt{2})$

4 **a** $k = 18$ **b** $P(x) = (x-4)(3x^2 - 5x - 2)$

c $x = -\frac{1}{3}$, 2, or 4

5 $a = 7$, $b = -14$

6 $k = -1$ or -2

If $k = -1$, zeros are 0, -2, and 3.

If $k = -2$, zeros are $-1 \pm \sqrt{2}$, and 3.

7 $m = -\frac{10}{7}$

8 **a** $a = 7$, $b = -6$ **b** 60

c $P(x) = (x+3)(2x^2 + 3x - 2)$ **d** -3, -2, $\frac{1}{2}$

9 **a** $a = 7$, $b = 2$ **b** $x = -2 \pm \sqrt{6}$ **11** $a = 2$

EXERCISE 5H

1 $a = -4$, $b = 7$

2 $P(x) = a(2x+1)(x^2 - 2x + 10)$, $a \neq 0$

3 $p(x) = 4x^3 - 20x^2 + 36x - 20$

4 **a** $p = -3$, $q = 52$, other zeros are $2 + 3i$, -4

5 $a = -13$, $b = 34$, other roots are $3 - i$, $-2 \pm \sqrt{3}$

6 $a = -4$, $b = 15$, other roots are $2 - i$, $\pm i\sqrt{3}$

7 $a = 3$, $P(z) = (z+3)(z+i\sqrt{3})(z-i\sqrt{3})$

8 $k = 2$, $P(x) = (x+i\sqrt{5})(x-i\sqrt{5})(3x+2)$

9 **a** $x^2 - 2x + (1 + k^2)$ **b** $k = 0, \pm1, \pm2, \pm3$

c $pq = -1, -2, -5, -10$ **d** $1, -2, 1 \pm 2i$

EXERCISE 5I

1 **a** sum $= \frac{3}{2}$, product $= 2$ **b** sum $= \frac{4}{3}$, product $= \frac{5}{3}$

c sum $= 1$, product $= -4$ **d** sum $= \frac{3}{2}$, product $= 4$

e sum $= 0$, product $= 9$ **f** sum $= 0$, product $= -1$

2 **a** sum $= \frac{20}{3}$, product $= \frac{14}{3}$ **b** -40 **c** -28

3 **a** sum $= 3$, product $= 3$ **b** -2 **c** -2

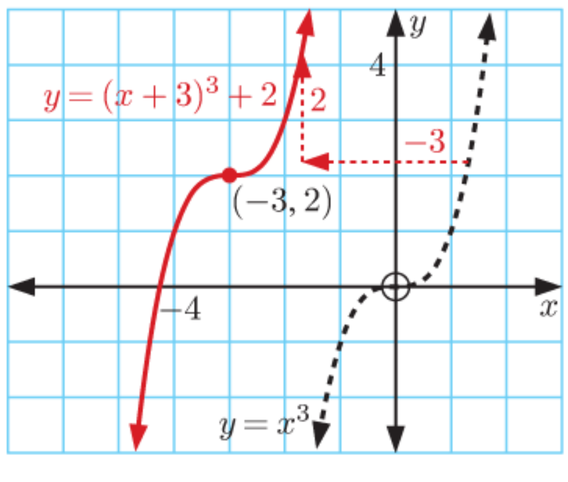
4 **a** $k = 4$ **b** 5 **5** $m = -2$, $n = 1$

6 $a = \pm 2$ **7** **b** $p = -1$, $q = -1$, $r = 1$

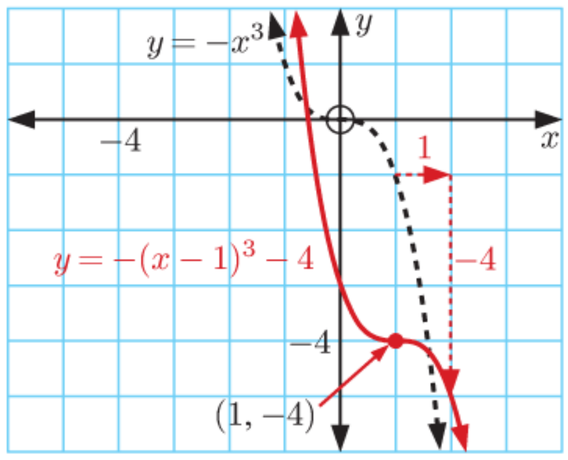
9 **a** $2n$ **b** 10 **c** 9 **d** 9

EXERCISE 5J

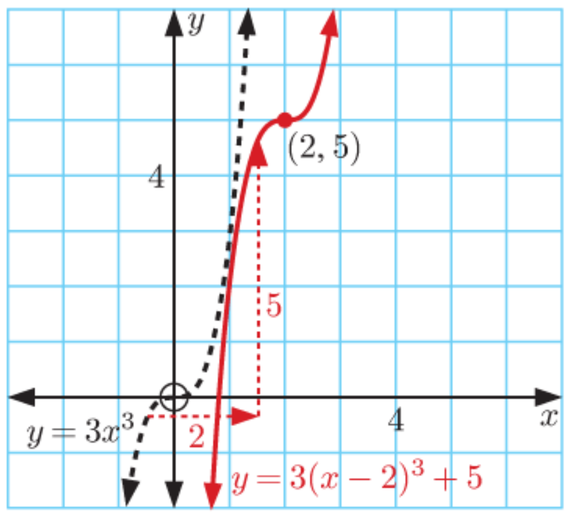
1 **a**



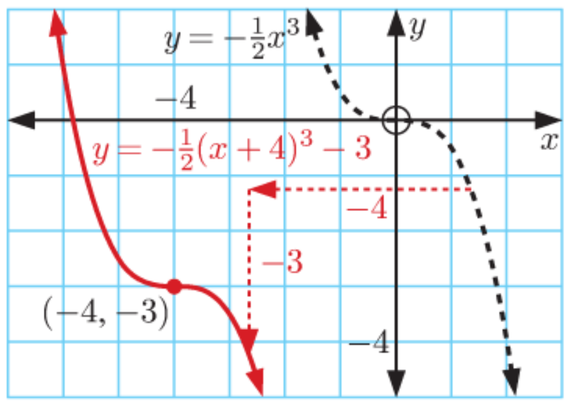
b



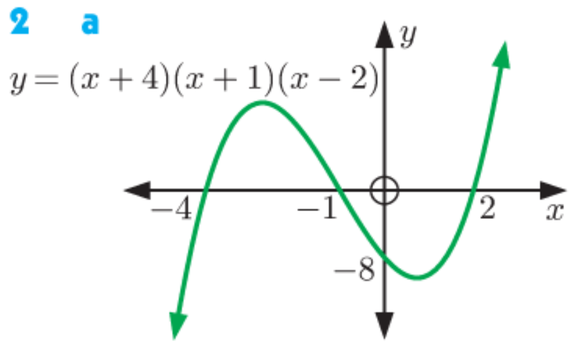
c



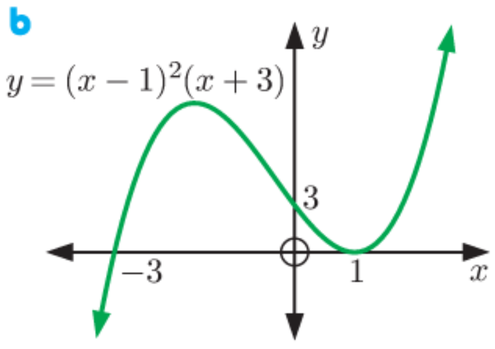
d



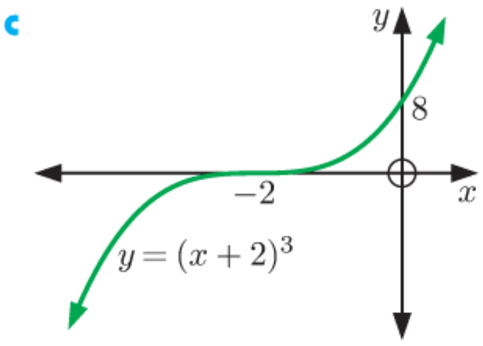
2 **a**



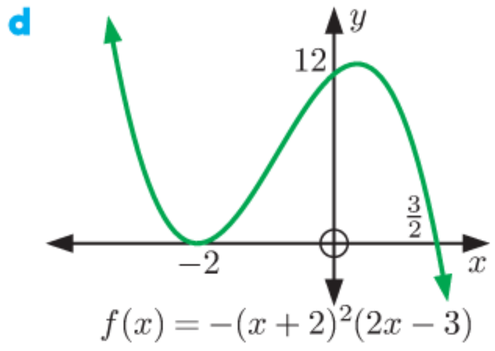
b



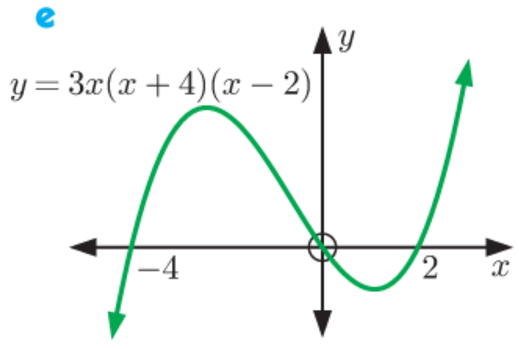
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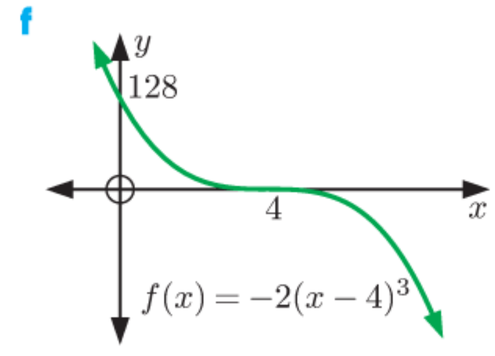
d



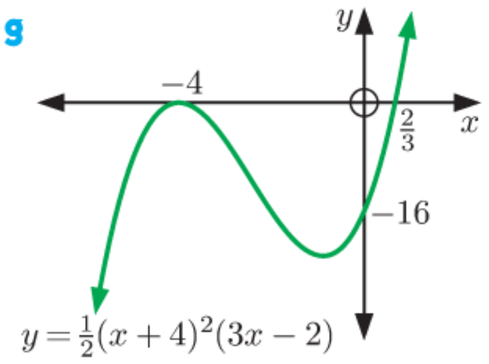
e



f

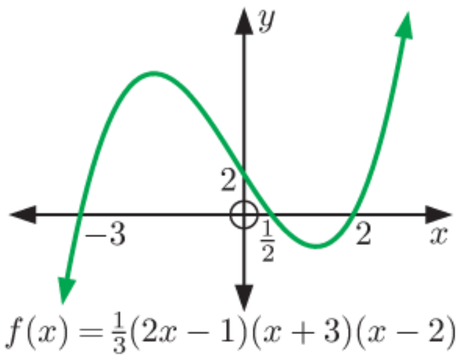


g



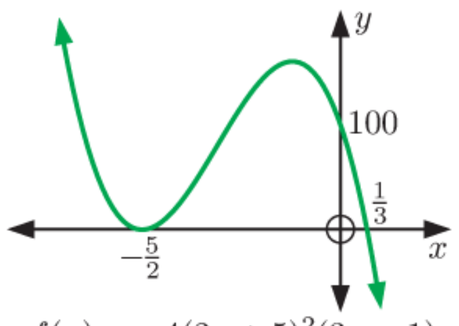
$y = \frac{1}{2}(x+4)^2(3x-2)$

h



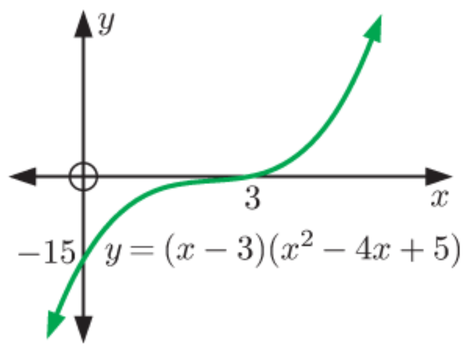
$f(x) = \frac{1}{3}(2x-1)(x+3)(x-2)$

i



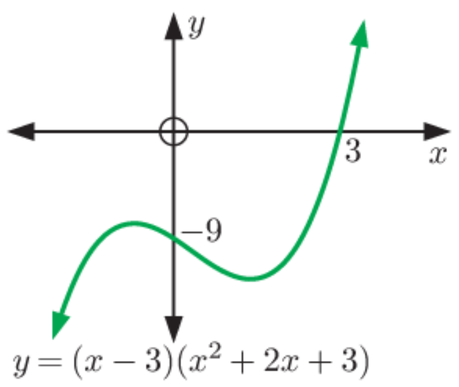
$f(x) = -4(2x+5)^2(3x-1)$

j



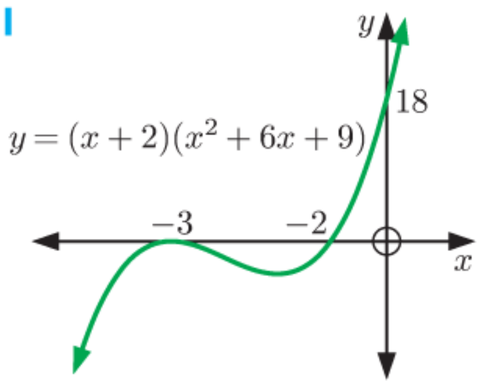
$y = (x-3)(x^2-4x+5)$

k



$y = (x-3)(x^2+2x+3)$

l



$y = (x+2)(x^2+6x+9)$

3 **a** $y = 2(x+1)(x-2)(x-3)$

b $y = (x+1)(x-3)^2$

c $y = 2(x+4)(x+2)(x-1)$

d $y = -2(x+3)(2x-1)(2x+1)$

e $y = \frac{1}{4}(x+4)^2(x-3)$

f $y = \frac{1}{10}(x+5)(x+2)(x-5)$

g $y = -\frac{1}{8}(x-2)^3$ **h** $y = \frac{1}{4}(x+4)(x-3)^2$

i $y = -2(x+3)(x+2)(2x+1)$

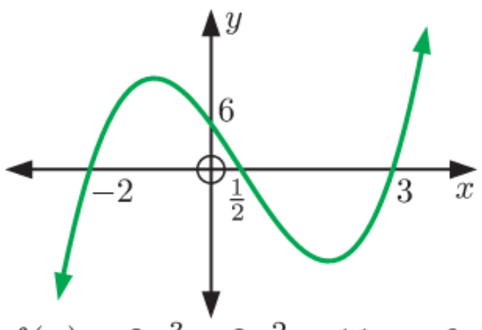
4 **a** $f(-2) = 0$

b $(x+2)(2x-1)(x-3)$

d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

e as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to -\infty$

c



$f(x) = 2x^3 - 3x^2 - 11x + 6$

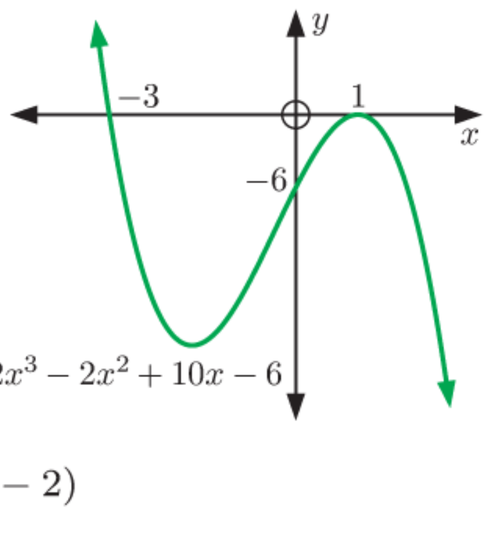
5 **a** $y = (x-3)(x-1)(x+2)$ **b** $y = 3x(x+2)(2x-1)$

c $y = (x-1)^2(x+2)$ **d** $y = -\frac{1}{3}(3x+2)^2(x-4)$

6 **a** $-2(x+3)(x-1)^2$

c as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to \infty$

b



$y = -2x^3 - 2x^2 + 10x - 6$

7 **a** $P(x) = 5(2x-1)(x+3)(x-2)$

b $P(x) = -2(x+2)^2(x-1)$

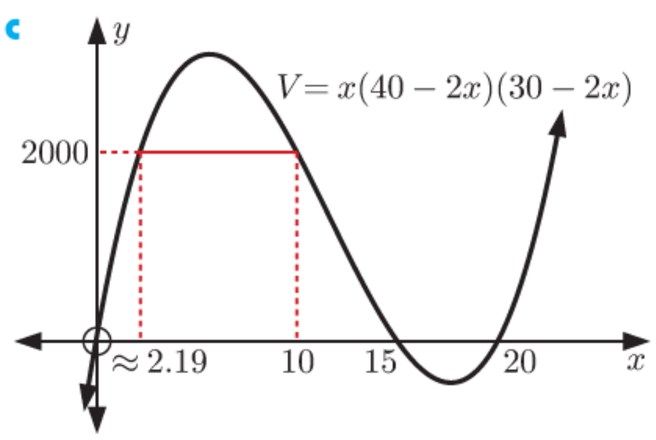
c $P(x) = (x-2)(2x^2-3x+2)$

8 **a** $a = 700$, the time at which the barrier has returned to its original position.

b $k = \frac{17}{7\,200\,000}$, $f(t) = \frac{17}{7\,200\,000}\,t(t-700)^2$

9 **b** $0 < x < 15$

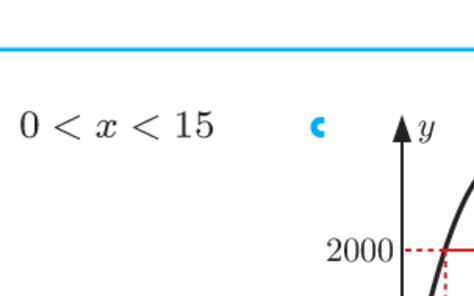
c



d 10 cm × 10 cm or ≈ 2.19 cm × 2.19 cm

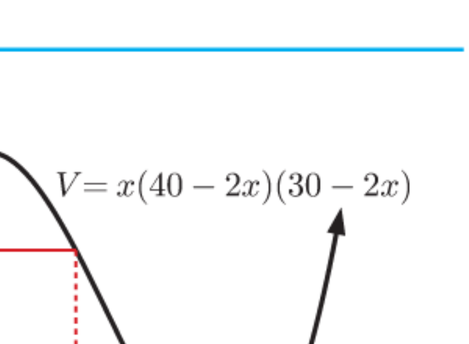
EXERCISE 5K

1 **a**

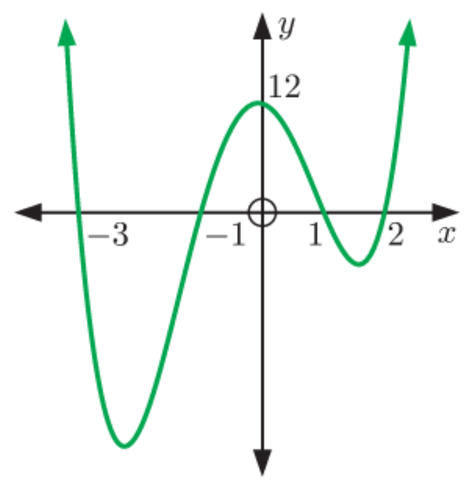


$y = x^2(x-1)(2x+3)$

b

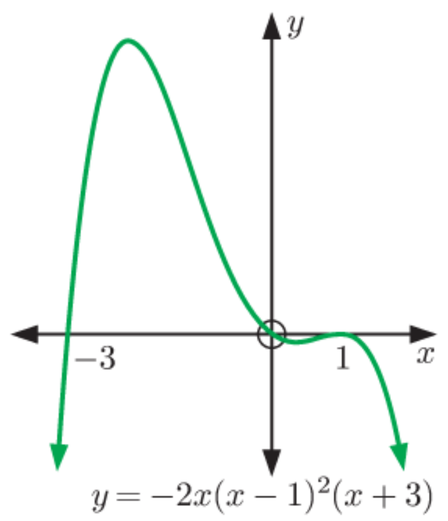


c



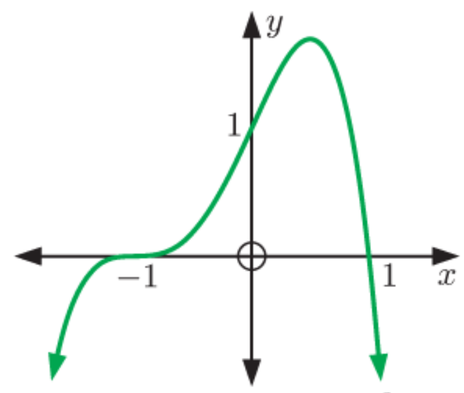
$f(x) = 2(x+3)(x+1)(x-1)(x-2)$

d



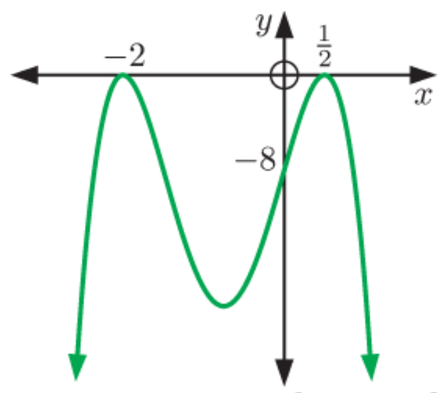
$y = -2x(x-1)^2(x+3)$

e



$f(x) = -(x-1)(x+1)^3$

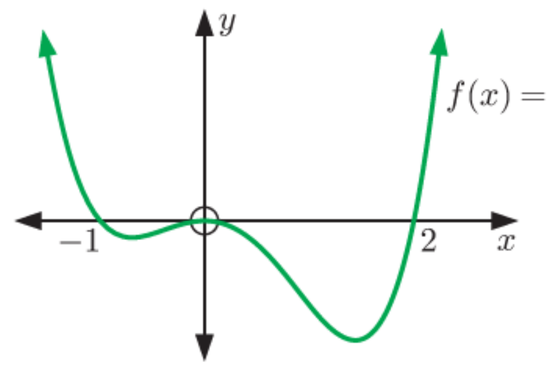
f



$f(x) = -2(x+2)^2(2x-1)^2$

2 **a** $x^2(x-2)(x+1)$

b



3 **a** $y = 2(x+1)^2(x-1)^2$

b $y = (x+3)(x+1)^2(3x-2)$

c $y = -2(x+2)(x+1)(x-2)^2$

d $y = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$

e $y = \frac{1}{4}(x+1)(x-4)^3$ **f** $y = x^2(x+2)(x-3)$

4 **a** $y = (x+4)(2x-1)(x-2)^2$

b $y = \frac{1}{4}(3x-2)^2(x+3)^2$

c $y = 2(2x+1)(2x-1)(x+2)(x-2)$

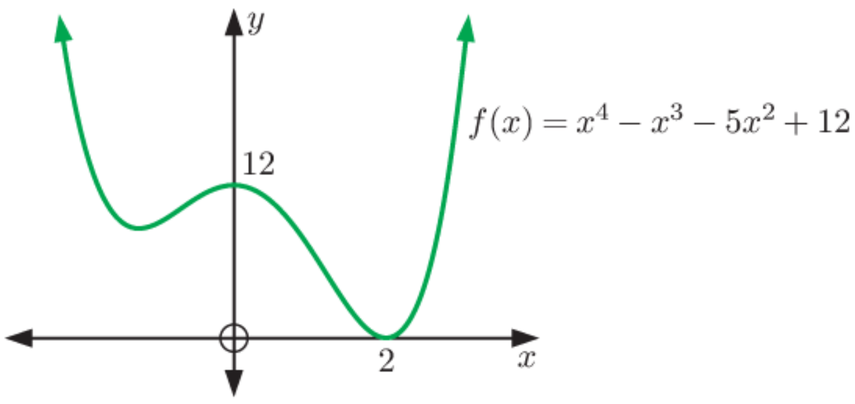
d $y = (x-1)^2(\frac{8}{3}x^2 + \frac{8}{3}x - 1)$

5 **a** $y = \frac{1}{2}x^4 + \frac{7}{2}x^3 + 6x^2 - 2x - 8$

b $y = -2x^4 - 4x^3 + 8x^2 + 4x - 6$

6 **a** $f(x) = (x-2)^2(x^2+3x+3)$ where x^2+3x+3 has $\Delta < 0$, so x^2+3x+3 has no real zeros.

b



7 **a** ≈ 877 joules s^{-1} **b** ≈ 900 joules s^{-1}

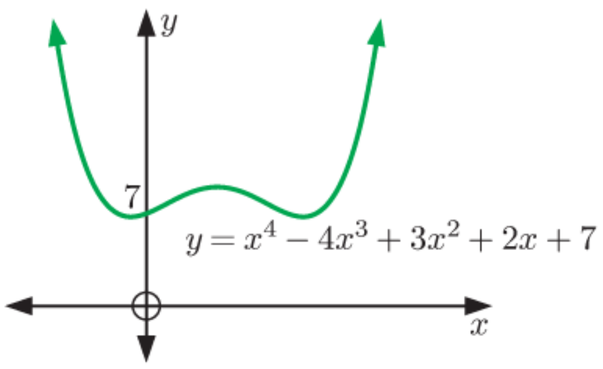
8 **a** $(x^2+x+1)(x^2-5x+7)$

b x^2+x+1 has $a > 0$ and $\Delta < 0$

x^2-5x+7 has $a > 0$ and $\Delta < 0$

$\therefore$ the graph lies entirely above the x-axis.

c



9 **a** $a = 120,\ b = 200$ **b** $k = 1\,000\,000$ **c** 6.75 m

EXERCISE 5L

1 **a** $-1,\ 2 \pm \sqrt{3}$ **b** $1,\ 1 \pm i$ **c** $\frac{7}{2},\ -1 \pm 2i$

d $\frac{1}{2},\ \pm i\sqrt{10}$ **e** $\pm\frac{1}{2},\ 3,\ -2$ **f** $2,\ 1 \pm 3i$

2 **a** $x = -2,\ \pm i\sqrt{3}$ **b** $x = -2,\ -\frac{1}{2},\ 1$

c $x = 2$ (treble root) **d** $x = -2,\ \frac{3}{2},\ 3$

e $x = -3,\ 2,\ 1 \pm \sqrt{2}$ **f** $x = -\frac{1}{2},\ 3,\ 2 \pm i$

3 **a** $x = 2,\ \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$ **b** $x = -3,\ \pm 2i$

c $x = \frac{1}{2},\ 2 \pm \sqrt{3}$ **d** $x = 2,\ 1 \pm 2i$

e $x = -\frac{1}{2},\ 1,\ \frac{3}{2}$ **f** $x = -2,\ \frac{2}{3},\ \pm i\sqrt{3}$

g $x = -1,\ \frac{1}{2},\ 1 \pm i\sqrt{3}$ **h** $x = -\frac{5}{2},\ \pm 2i$

4 **a** $x \approx -3.27,\ -0.860,\ 2.13$ **b** $x \approx -2.52,\ -1.18,\ 2.70$

5 ≈ 1.11 m or ≈ 9.94 m

EXERCISE 5M

1 **a** $(x-2)(x+3)(x+2)$ **b** $-3 \leqslant x \leqslant -2$ or $x \geqslant 2$

2 **a** $3\left(-\frac{2}{3}\right)^3 - 10\left(-\frac{2}{3}\right)^2 - 17\left(-\frac{2}{3}\right) - 6 = 0$ ✓

b $2 \pm \sqrt{7}$ **c** $x < -\frac{2}{3}$ or $2 - \sqrt{7} < x < 2 + \sqrt{7}$

3 **a** $x < -4$ or $1 < x < 2$

b $-3 \leqslant x \leqslant 1 - \sqrt{6}$ or $x \geqslant 1 + \sqrt{6}$

c $x \leqslant 2 - \sqrt{5}$ or $\frac{1}{2} \leqslant x \leqslant 2 + \sqrt{5}$

d $-1 - \sqrt{11} < x < -\frac{4}{3}$ or $x > -1 + \sqrt{11}$

e $x \leqslant -\sqrt{7}$ or $-\frac{1}{4} \leqslant x \leqslant \sqrt{7}$

f $\frac{-1-\sqrt{17}}{2} < x < \frac{6}{5}$ or $x > \frac{-1+\sqrt{17}}{2}$

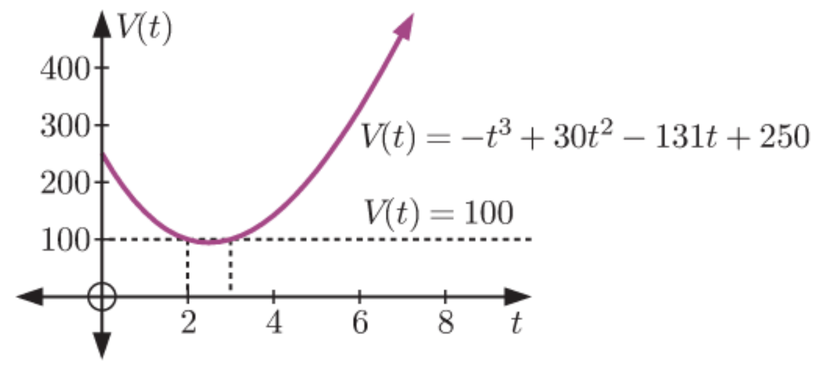
4 **a** $-2.09 < x < 0.572$ or $x > 2.51$

b $x < -1.66$ or $0.327 < x < 1.84$

c $-2.55 < x < 0.290$ or $x \geqslant 2.26$ **d** $x \leqslant \frac{3}{2}$

e $-1.20 < x < -0.125$ or $x > 1.33$ **f** $x < -0.924$

5 March



REVIEW SET 5A

1 **a** $2z^3 - z^2 + 2z + 6$ **b** $-2z^4 + 7z^3 - 7z^2 + 12z$

c $-2z^2 - 5z - 19 + \dfrac{57}{3-z}$

2 **a** $12x^4 - 9x^3 + 8x^2 - 26x + 15$

b $4x^4 - 4x^3 + 13x^2 - 6x + 9$

3 **a** $x^2 - 2x + 4 - \dfrac{8}{x+2}$ **b** $2x^2 + 2x - 1 + \dfrac{3}{x-1}$

c $x - 5 + \dfrac{19x+30}{(x+2)(x+3)}$

4 **a** yes **b** no **5** $(z^2 - 2z + 2)(z^2 + 2z + 2)$

6 $P(z) = a(z^4 - 6z^3 + 14z^2 - 10z - 7),\ a \neq 0$

7 $a = 7,\ b = 0$ *or* $a = 4,\ b = \pm\sqrt{3}$

8 **a** $a = -7,\ b = 12$ **b** $a = 3,\ b = 1$

9 $(2x-3)(x+5)(x-2)$ **10** $a = 7,\ b = -20$

11 "When a polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained, then $R = P(k)$."

12 7 **13** 267 214 **14** $a = -14,\ b = -24$

15 $k = 3,\ b = 27,\ x = 3, -3$; $k = -1,\ b = -5,\ x = -1, 5$

16 $k = -4$, zeros are $3 \pm 2i,\ -1 \pm \sqrt{2}$ **17** $\pm 2i,\ -1 \pm i$

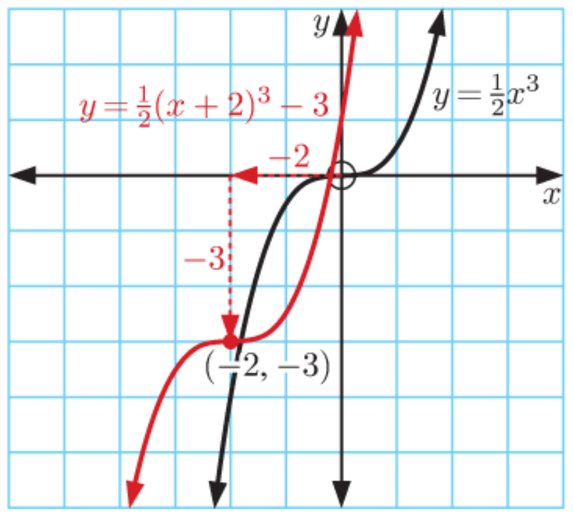
18 **a** sum $= \frac{4}{3}$, product $= \frac{8}{3}$ **b** sum $= 0$, product $= -5$

19 **a** $m = 1,\ n = \pm 2$

20 **Another hint:** Show that $(\alpha\beta)^3 + (\alpha\beta)^2 - 1 = 0$

21 $a = 0,\ b = 0$ or $a = 3,\ b = 18$

22

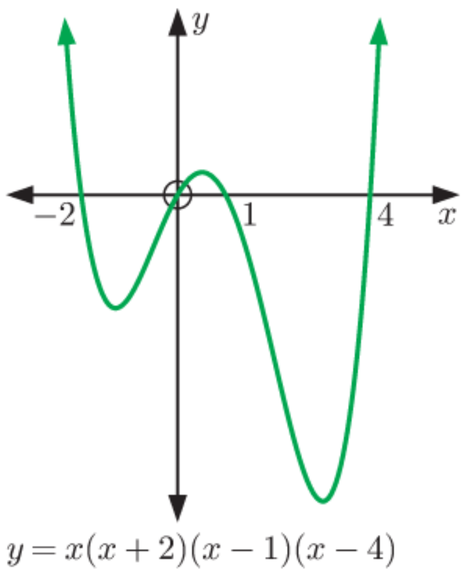


We translate $y = \frac{1}{2}x^3$ 2 units left and 3 units down.

23 **a** $y = (x+2)(x-2)(x-3)$

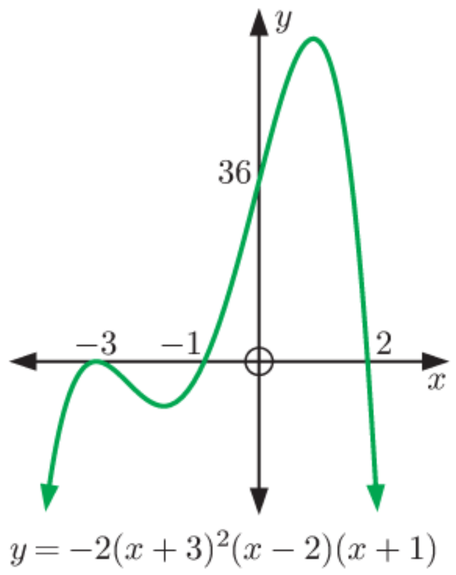
b $y = -2(x+1)^2(x-4)$

24 **a**



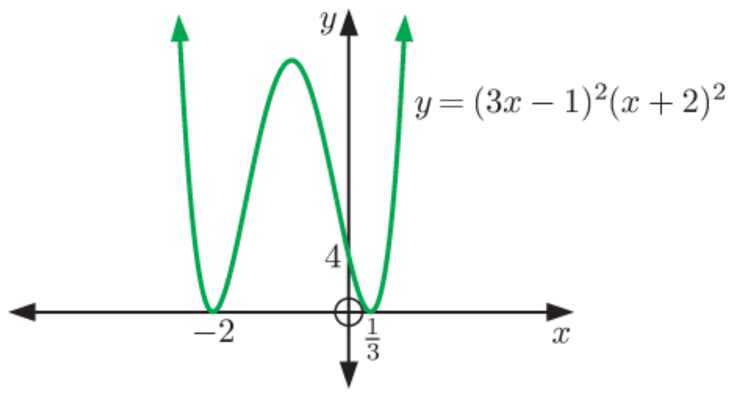
$y = x(x+2)(x-1)(x-4)$

b

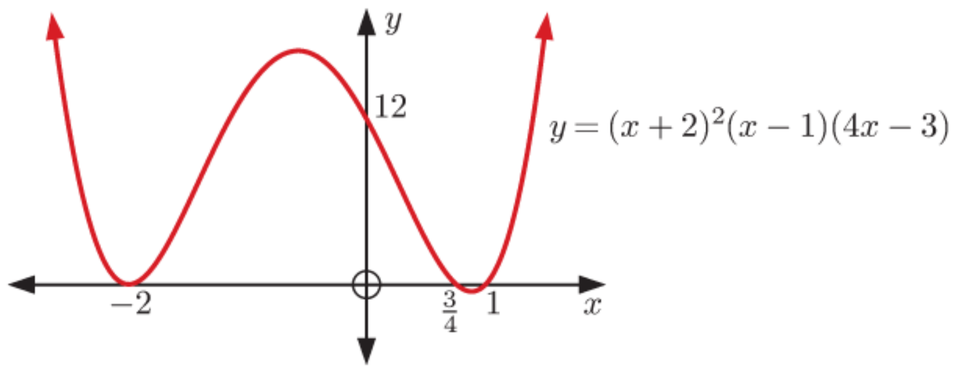


$y = -2(x+3)^2(x-2)(x+1)$

c



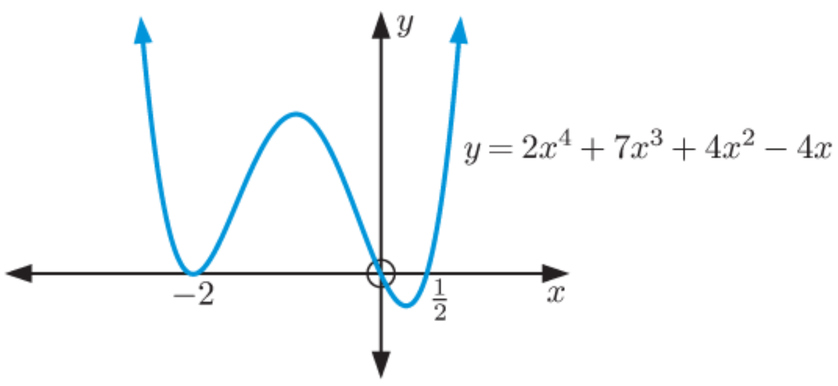
25 $P(x) = (x+2)^2(x-1)(4x-3)$



26 $P(x) = 2(x+4)(x-1)^2(x-3)$

27 **a** $P(x) = x(x+2)^2(2x-1)$

b



c $x \leqslant 0$ and $x \geqslant \frac{1}{2}$

28 **a** $x = -3,\ -\frac{1}{2},\ \frac{2}{3}$ **b** $x = -\frac{1}{3},\ 2,\ -1 \pm 3i$

29 **a** $x < -3$ or $1 - \sqrt{7} < x < 1 + \sqrt{7}$

b $-\sqrt{5} \leqslant x \leqslant -\frac{4}{5}$ or $x \geqslant \sqrt{5}$

REVIEW SET 5B

1 **a** $f(x) - g(x) = x^4 - 3x^3 - 3x^2 + 2x - 8$

b **i** 7 **ii** −2 **iii** −7

2 **a** $3x^5 + 13x^4 - 9x^3 - x^2 + 4x - 2$

b $x^6 - 4x^5 + 4x^4 + 2x^3 - 4x^2 + 1$

3 **a** $(3x-2)$ and $(x-3)$, zeros are $\frac{2}{3}$ and 3

b $(x-1)$, $(2x+1)$, and $(x^2 - 2x + 7)$,
zeros are $1,\ -\frac{1}{2},\ 1 \pm i\sqrt{5}$

5 **a** $P(x) = a(x+4)(x-1)(x-6),\ a \neq 0$

b $P(x) = a(3x-1)(x^2 - 6x + 4),\ a \neq 0$

c $P(x) = a(2x-1)(x^2+2),\ a \neq 0$

d $P(x) = a(x^2 - 2x + 2)(x^2 + 6x + 10),\ a \neq 0$

6 $a = 1,\ b = -1,\ c = 3$

7 **a** $x^2 + 3x - 2 + \dfrac{5}{2x+3}$ **b** $x^3 + x^2 + 2x + 2 + \dfrac{4}{x-1}$

c $x^2 - x + 4 + \dfrac{8 - 6x}{(x-1)(x+2)}$

8 $(3x-2)(2x-1)(x+5)$

9 $x = -\frac{1}{2}$ (repeated), and $x = 3$

10 Other factor is $(x^2 - 3x + 3)$, $\Delta < 0$ $\therefore$ no real roots

11 **a** $k > 0$ **b** $k \leqslant 0$ **12** 2 **13** $m = 7$

14 quotient is $x^2 + 3x - 9$, remainder is $5x + 17$,
$a = 4,\ b = -18$

15 **a** **Another hint:** $x^2 + 1 = (x+i)(x-i)$ is a factor of $x^{10} + 1$ if both $(x+i)$ and $(x-i)$ are factors.

b $1 - x$

16 $(2z+1)(z+i\sqrt{5})(z-i\sqrt{5})$ **17** $a = -8,\ b = 17$

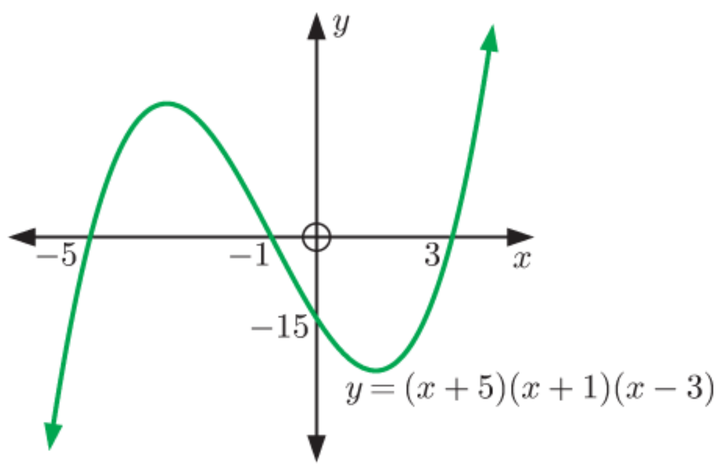
18 $a = -21$, other zeros are $5 + i,\ \frac{1}{2}$

19 **a** sum $= -\frac{3}{2}$, product $= -3$ **b** sum $= 0$, product $= \frac{3}{2}$

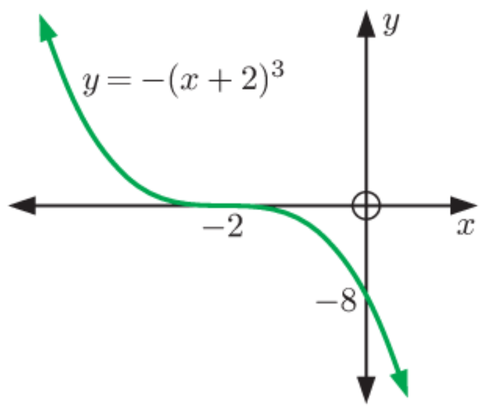
20 **b** If $m = \sqrt{3}$, coefficient is $-4 - 2\sqrt{3}$
If $m = -\sqrt{3}$, coefficient is $-4 + 2\sqrt{3}$

22 $P(x) = 2x^3 - 6x - 4$

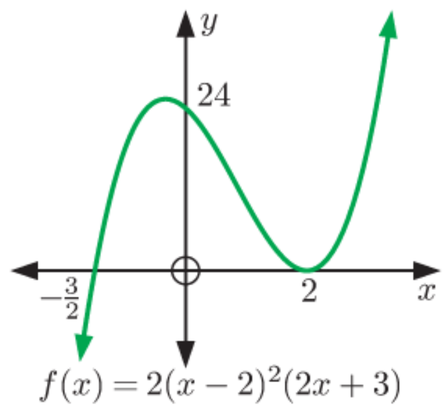
23 **a**



b



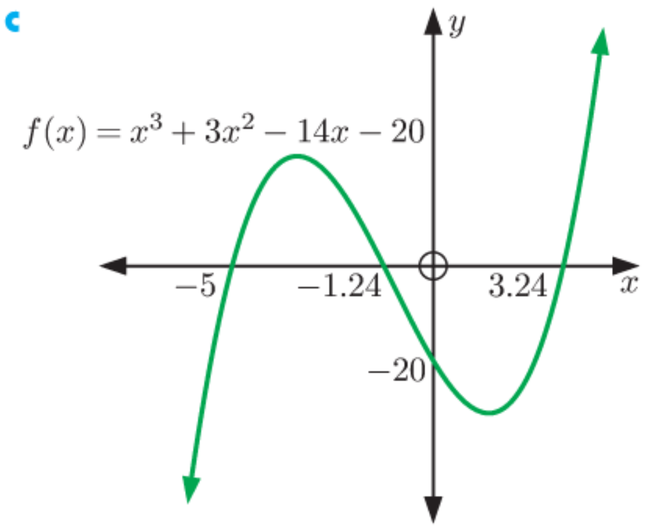
c



24 **a** $f(x) = (x+5)(x^2 - 2x - 4)$
$f(-5) = 0$ $\therefore$ -5 is an x-intercept.

b $1 - \sqrt{5} \approx -1.24,\quad 1 + \sqrt{5} \approx 3.24$

c



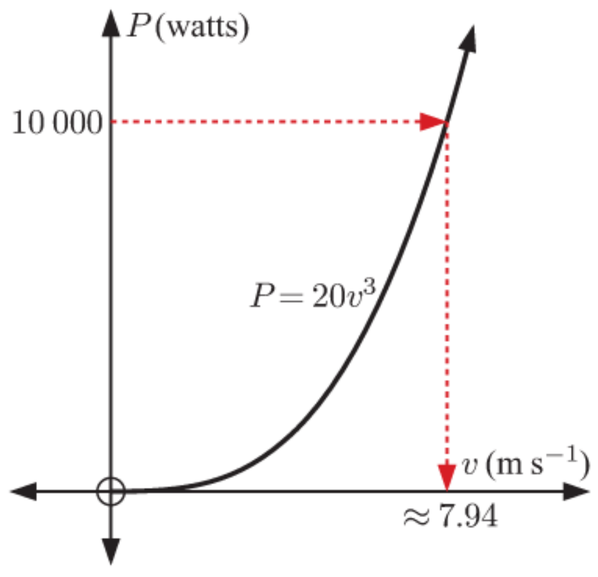
d as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -\infty$

25 **a** $0 \leqslant v < 100$

b 4320 watts

d ≈ 7.94 m s^{-1}

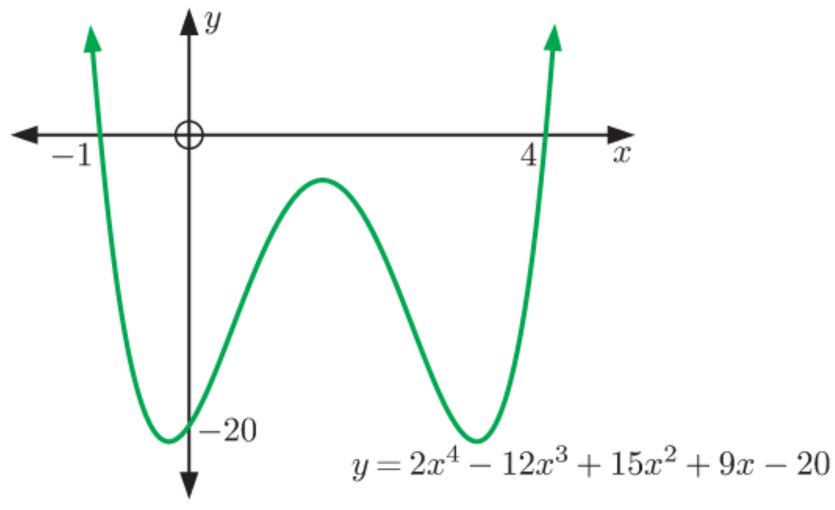
c



26 **a** $y = \frac{1}{2}(x-3)^3(x+2)$ **b** $y = -3(x+1)^2(x-2)^2$

27 $(x+1)(x-4)(2x^2-6x+5)$

$2x^2-6x+5$ has $\Delta < 0$, so $2x^2-6x+5$ has no real zeros.



28 **a** $-\frac{1}{2}$ and $\pm i\sqrt{5}$ **b** -7, -1, and 2

29 **a** $x \leqslant -1.10$ or $0.854 \leqslant x \leqslant 4.25$ **b** $x > 2.33$

EXERCISE 6A

4 **a** odd **b** neither **c** even **d** odd **e** even **f** neither

5 $(-1, 3)$, $(5, -2)$ **6** $(-4, -6)$, $(1, -2)$ **7** $a = -\frac{3}{2}$

8 $b = -1$ **9** **b** $b = 0$, $d = 0$ **c** $b = 0$, $d = 0$

10 **a** Even, the graph of the function is symmetric about the y-axis.

b Odd, the graph of the function has rotational symmetry about the origin.

c Odd, the graph has rotational symmetry about the origin.

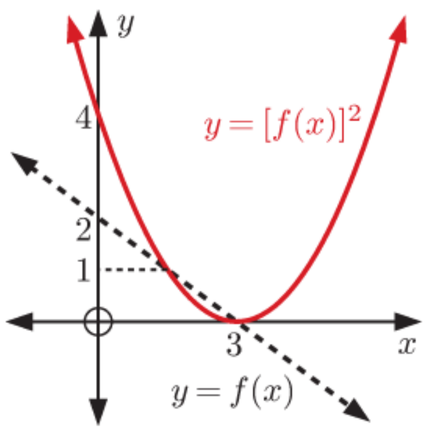
11 **a** $k = n\pi$, $n \in \mathbb{Z}$ **b** $k = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

c $k \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$

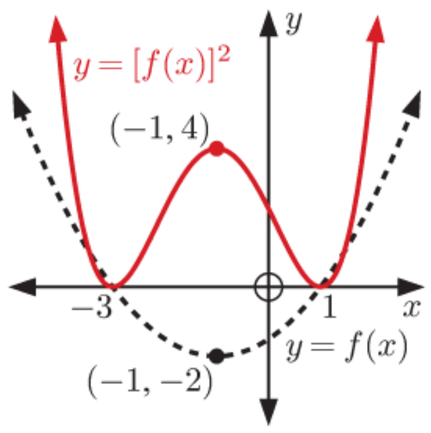
13 even **14** odd

EXERCISE 6B

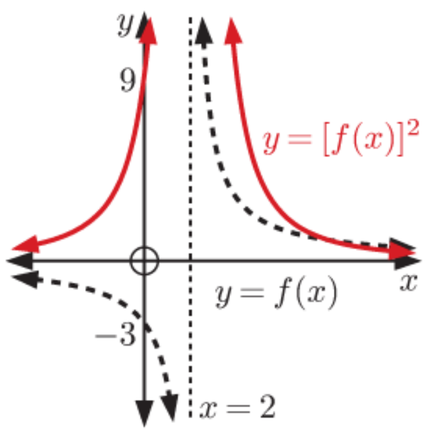
1 **a**



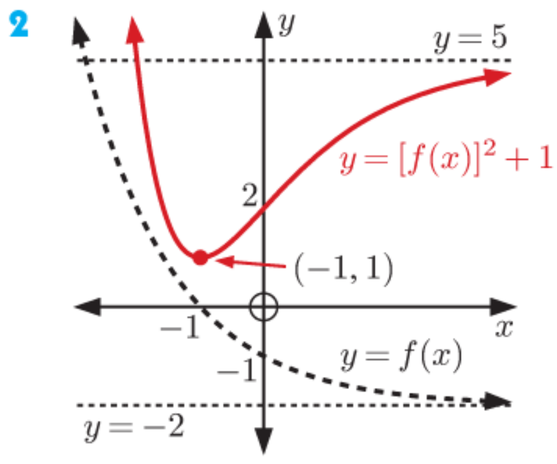
b



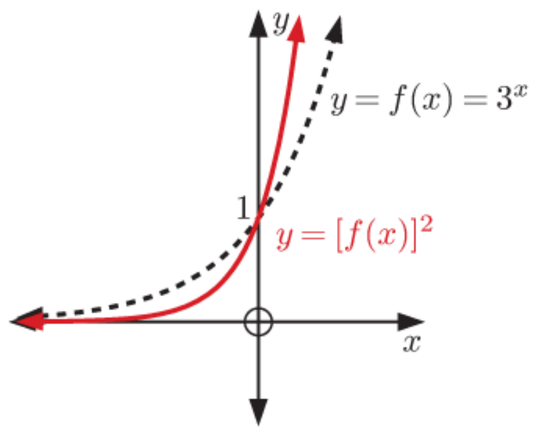
c



2

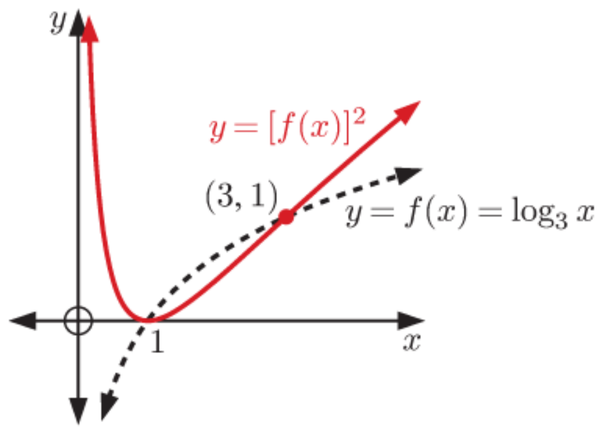


3 **a**, **b**



c A horizontal stretch with scale factor $\frac{1}{2}$.

4 **a**, **b**



c $(1, 0)$ and $(3, 1)$

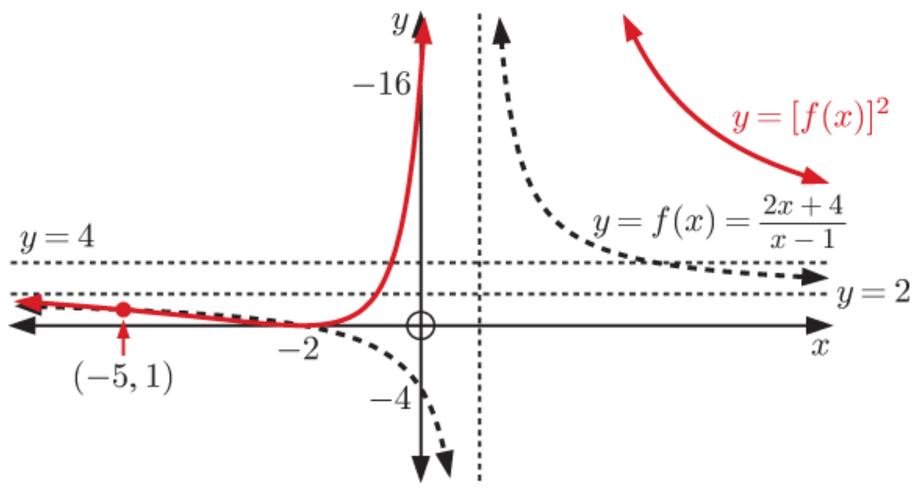
6 Domain is $\{x \mid 0 \leqslant x \leqslant 5\}$, Range is $\{y \mid 0 \leqslant y \leqslant 16\}$

7 **a** x-int. -2, y-int. -4, VA $x = 1$, HA $y = 2$

b x-int. -2, y-int. 16, VA $x = 1$, HA $y = 4$

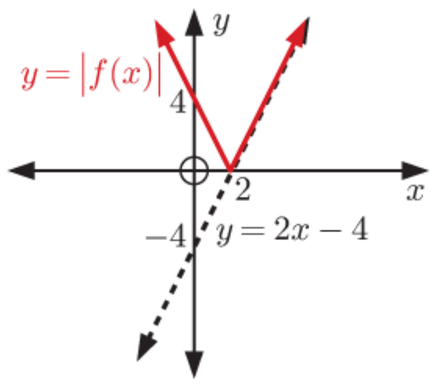
c $(-5, 1)$ and $(-2, 0)$

d

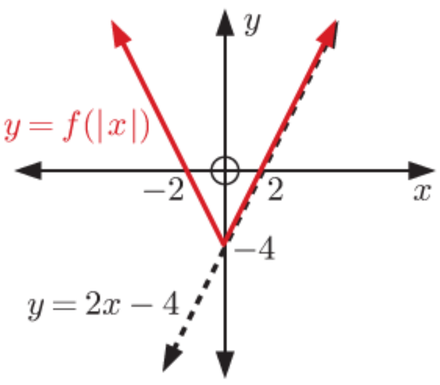


EXERCISE 6C.1

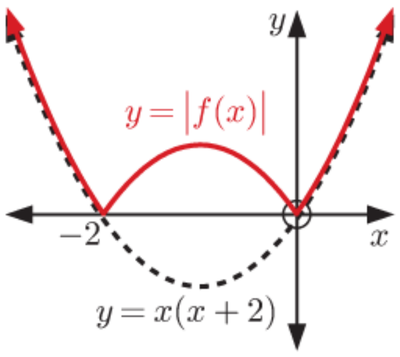
1 **a**



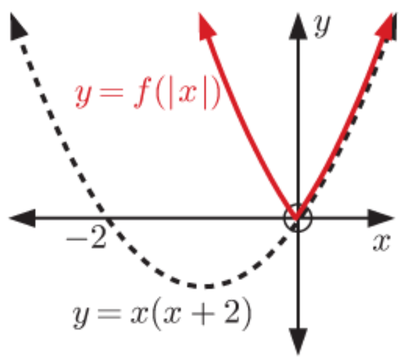
b



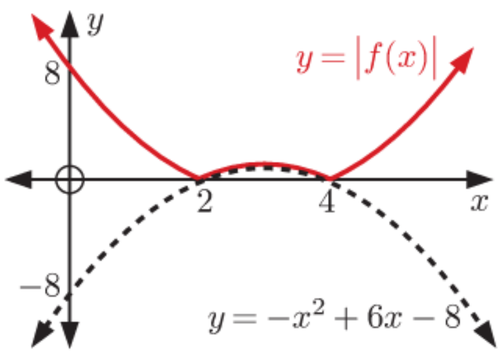
2 **a**



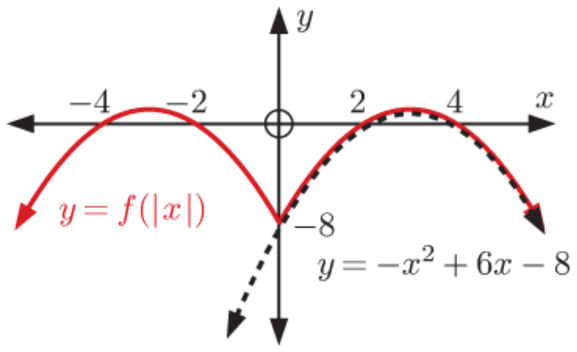
b



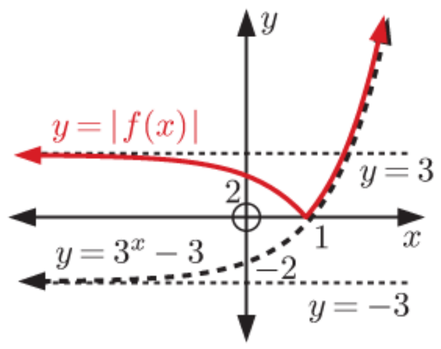
3 **a**



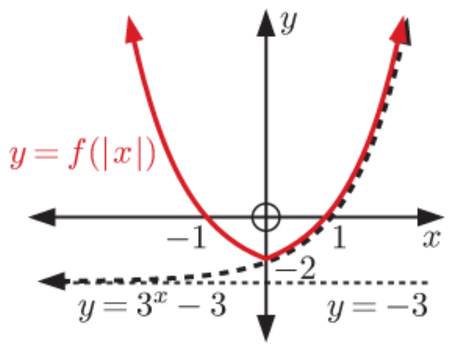
b



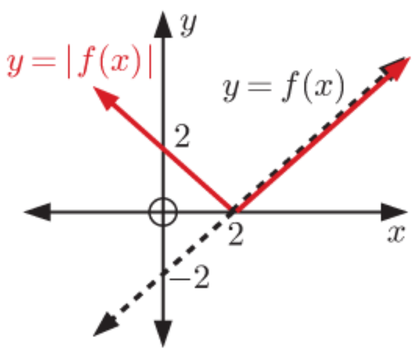
4 **a**



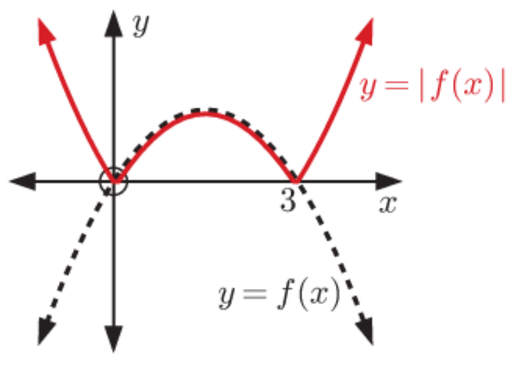
b



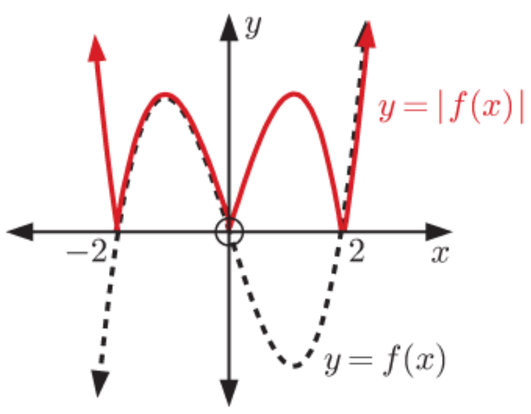
5 **a** **i**



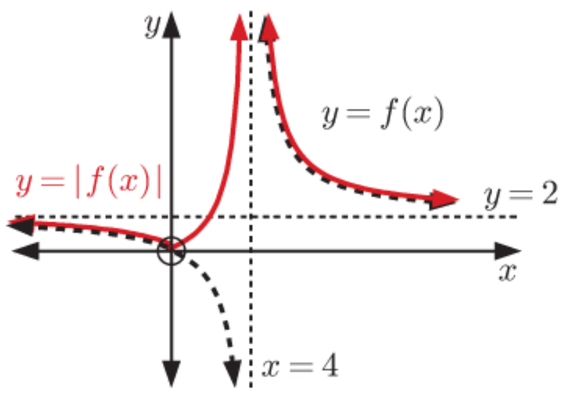
ii



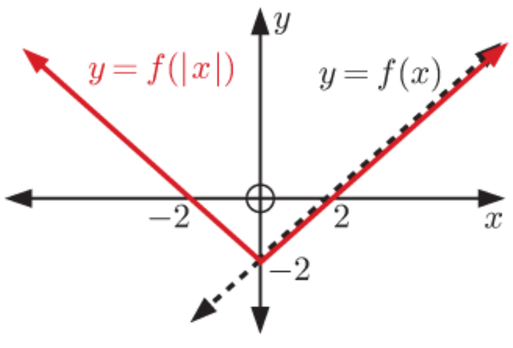
iii



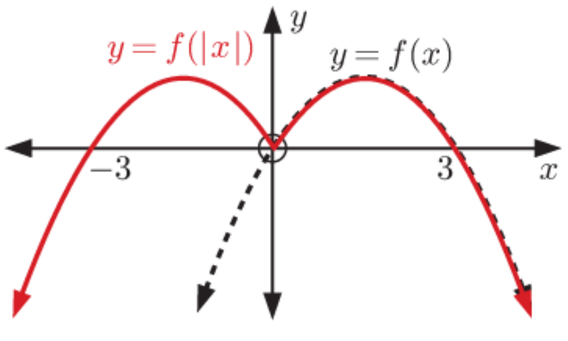
iv

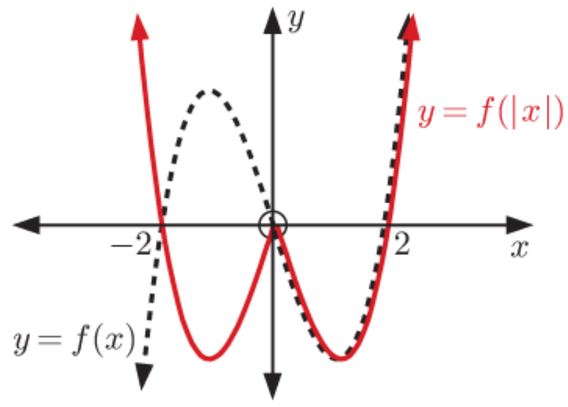


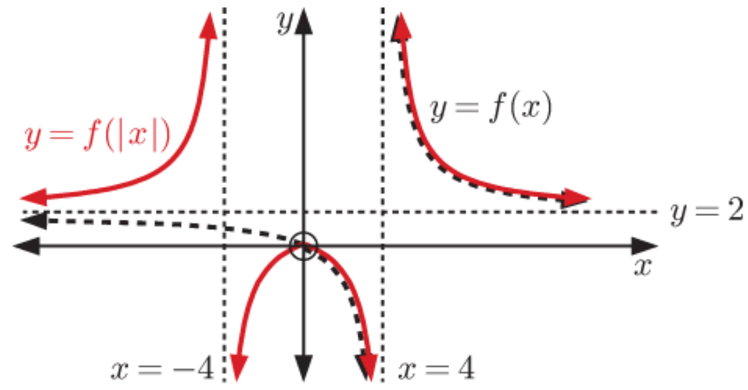
b **i**

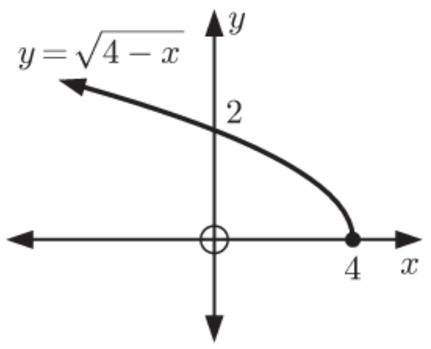


ii

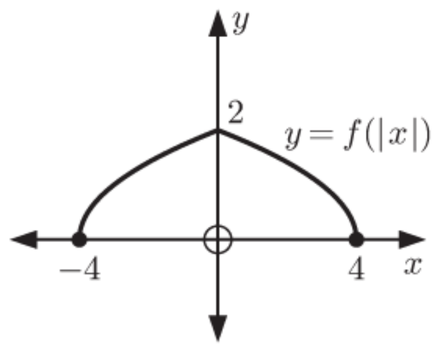


iii 

iv 

7 **a** 

Domain is $\{x \mid x \leqslant 4\}$,
Range is $\{y \mid y \geqslant 0\}$

b 

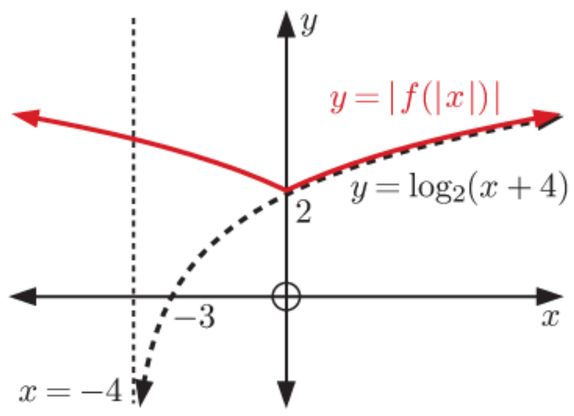
Domain is
$\{x \mid -4 \leqslant x \leqslant 4\}$,
Range is $\{y \mid 0 \leqslant y \leqslant 2\}$

8 **a** **i** $\{x \mid -6 \leqslant x \leqslant 6\}$ **ii** $\{y \mid 0 \leqslant y \leqslant 7\}$

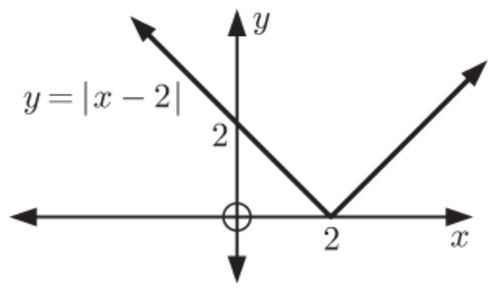
b No, we do not know the behaviour of $f(x)$ on $-2 \leqslant x < 0$, which is discarded when we find $f(|x|)$. This may or may not affect the range of $y = f(|x|)$.

9 **a** x-intercepts -3 and 4, y-intercept 2

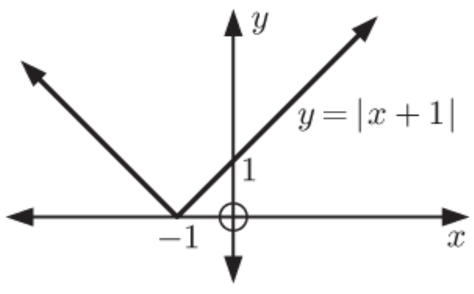
b x-intercepts 4 and -4, y-intercept -2

10 

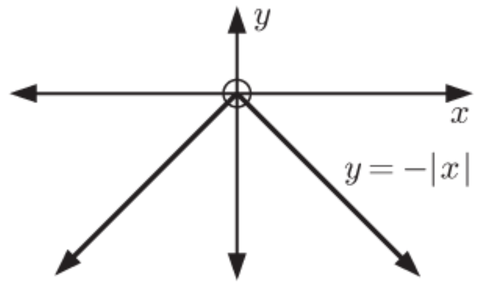
12 **a** $y = \begin{cases} x - 2, & x \geqslant 2 \\ 2 - x, & x < 2 \end{cases}$



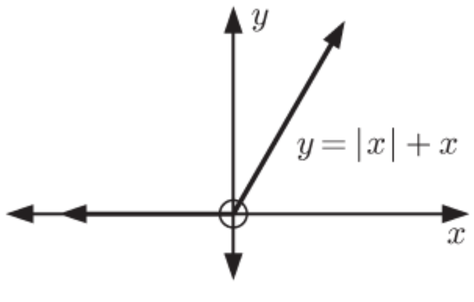
b $y = \begin{cases} x + 1, & x \geqslant -1 \\ -x - 1, & x < -1 \end{cases}$



c $y = \begin{cases} -x, & x \geqslant 0 \\ x, & x < 0 \end{cases}$

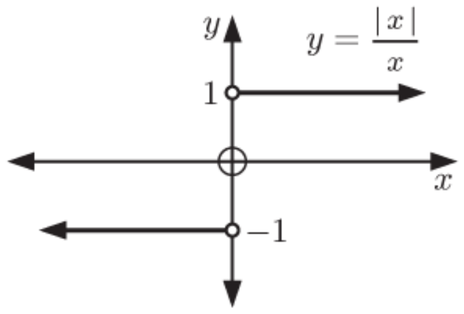


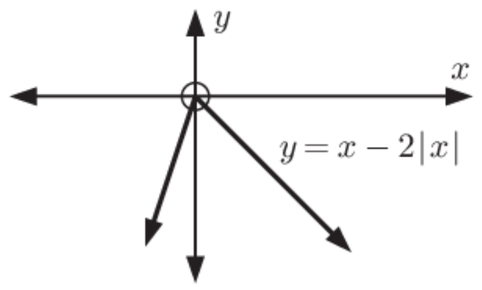
d $y = \begin{cases} 2x, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$



e $y = \begin{cases} 1, & x > 0 \\ \text{undefined}, & x = 0 \\ -1, & x < 0 \end{cases}$

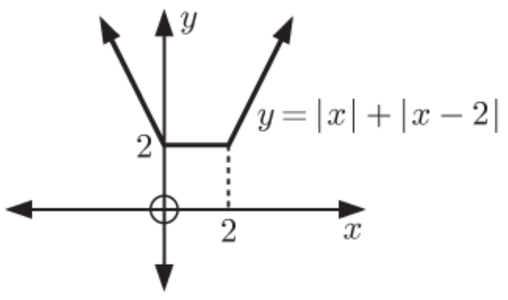
f $y = \begin{cases} -x, & x \geqslant 0 \\ 3x, & x < 0 \end{cases}$

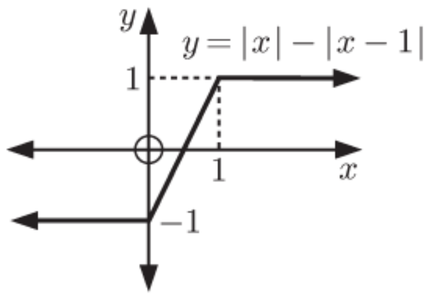




g $y = \begin{cases} 2x - 2, & x \geqslant 2 \\ 2, & 0 \leqslant x < 2 \\ 2 - 2x, & x < 0 \end{cases}$

h $y = \begin{cases} 1, & x \geqslant 1 \\ 2x - 1, & 0 \leqslant x < 1 \\ -1, & x < 0 \end{cases}$





EXERCISE 6C.2

1 **a** $x = \pm 3$ **b** no solutions **c** $x = 0$
d $x = 4$ or -2 **e** $x = -1$ or 7 **f** no solutions
g $x = 1$ or $\frac{1}{3}$ **h** $x = 0$ or 3 **i** $x = -2$ or $\frac{14}{5}$

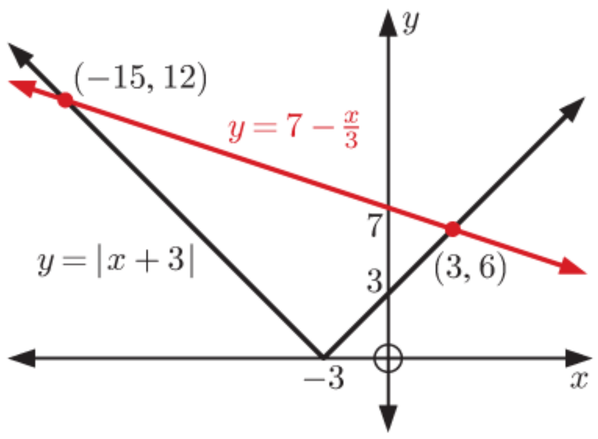
2 **a** $x = \frac{3}{2}$ or $\frac{3}{4}$ **b** $x = -2$ or $-\frac{4}{7}$ **c** $x = -1$ or 7

3 **a** In the case $\dfrac{3x+1}{x-1} = 3$ we get $3x + 1 = 3x - 3$ which has no solutions.

b $x = \frac{1}{3}$

5 **a** $x = \frac{3}{2}$ or $-\frac{1}{4}$ **b** $x = -\frac{4}{3}$ or -6 **c** $x = \frac{1}{2}$
d $x = \frac{5}{2}$ **e** $x = \frac{1}{2}$ or $-\frac{1}{2}$ **f** $x = \frac{2}{5}$ or -6

6 **a**



b $x = -15$ or 3

7 **a** $x = 1$ **b** $x = -\frac{4}{5}$ or 4
c $x \approx 0.714$ or $x = 5$ **d** $x \approx 2.69$
e $x \approx 1.28$ or 2.43 **f** $x \approx -1.91$, 0.304, or 2.09

EXERCISE 6C.3

1 **a** $-3 \leqslant x \leqslant 5$ **b** $x < -9$ or $x > 5$
c $1 < x < 2$ **d** $x \leqslant -\frac{7}{3}$ or $x \geqslant -1$
e $-\frac{3}{2} < x < \frac{5}{2}$ **f** $x < -\frac{9}{10}$ or $x > -\frac{7}{10}$

2 **a** $x < -2$ **b** $x < -1$ or $x > 1$
c all $x \in \mathbb{R}$ **d** $x \leqslant -\frac{7}{4}$ or $x \geqslant -\frac{1}{2}$
e $x < -10$ or $x > \frac{6}{5}$ **f** $\frac{7}{5} \leqslant x \leqslant 17$

3 **a** $\frac{3}{2} \leqslant x \leqslant 3$, $x \neq 2$ **b** $x \geqslant -\frac{1}{4}$, $x \neq 1$
c $x < -10$ or $x > 2$

4 **a** $1 < x < 3$ **b** $x \geqslant 3$ **c** $x < -2.79$ or $x > 2.30$

d $-\frac{5}{2} \leqslant x \leqslant \frac{7}{2}$ **e** $-2.24 \leqslant x \leqslant 2.11$
f $-2.80 \leqslant x < -1$ or $2 < x \leqslant 2.92$
g $-1 \leqslant x < -0.189$ or $0.254 < x < 0.937$
h $-3.87 \leqslant x < -2.97$ or $-1.72 < x < 2.19$ or $2.59 < x < 3.5$

EXERCISE 6D.1

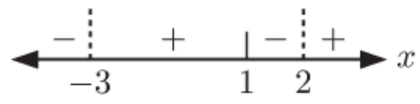
1 **a** vertical asymptotes $x = 2$ and $x = 6$, horizontal asymptote $y = 0$

b x-intercept $\frac{5}{4}$, y-intercept $-\frac{5}{12}$

2 **a** **i** vertical asymptotes $x = -3$ and $x = 2$, horizontal asymptote $y = 0$

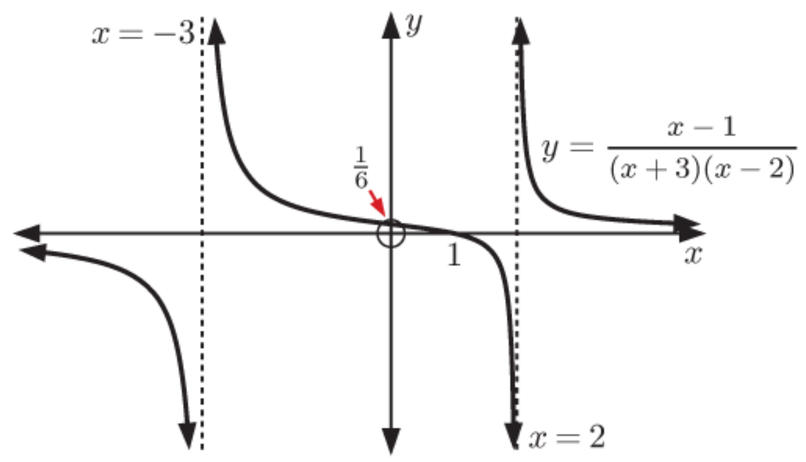
ii x-intercept 1, y-intercept $\frac{1}{6}$

iii



iv As $x \to -3^-$, $y \to -\infty$
As $x \to -3^+$, $y \to \infty$
As $x \to 2^-$, $y \to -\infty$
As $x \to 2^+$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

v



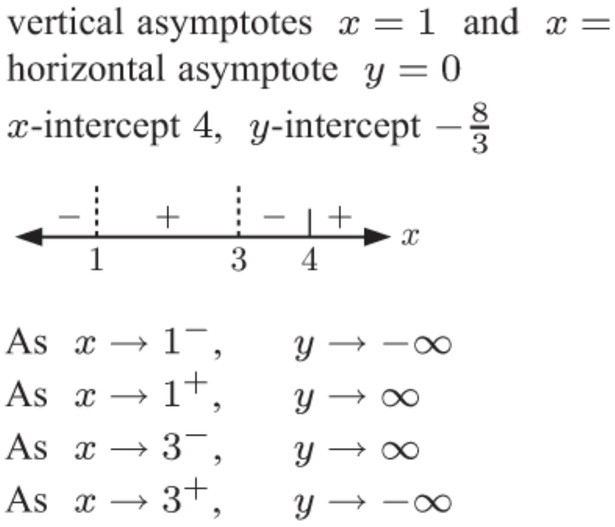
b **i** vertical asymptotes $x = 1$ and $x = 3$, horizontal asymptote $y = 0$

ii x-intercept 4, y-intercept $-\frac{8}{3}$

iii

iv As $x \to 1^-$, $y \to -\infty$
As $x \to 1^+$, $y \to \infty$
As $x \to 3^-$, $y \to \infty$
As $x \to 3^+$, $y \to -\infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

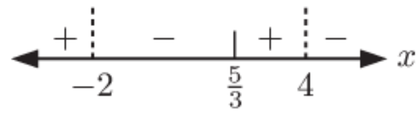
v



c **i** vertical asymptotes $x = -2$ and $x = 4$, horizontal asymptote $y = 0$

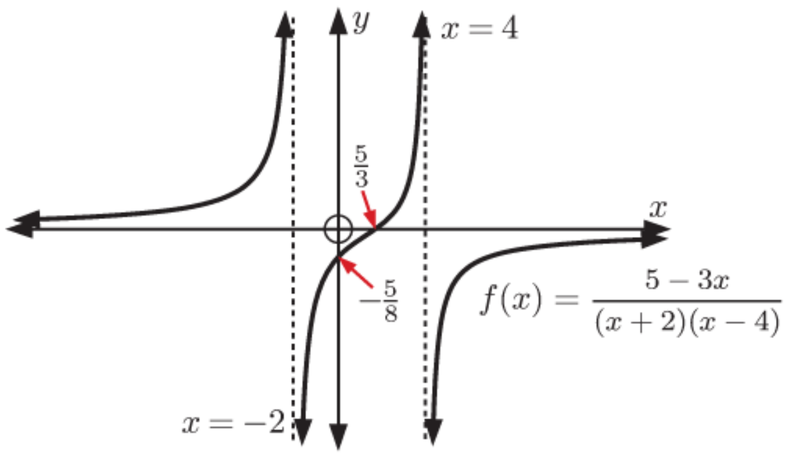
ii x-intercept $\frac{5}{3}$, y-intercept $-\frac{5}{8}$

iii



iv As $x \to -2^-$, $f(x) \to \infty$
As $x \to -2^+$, $f(x) \to -\infty$
As $x \to 4^-$, $f(x) \to \infty$
As $x \to 4^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^-$

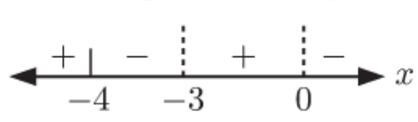
v



d **i** vertical asymptotes $x = -3$ and $x = 0$, horizontal asymptote $y = 0$

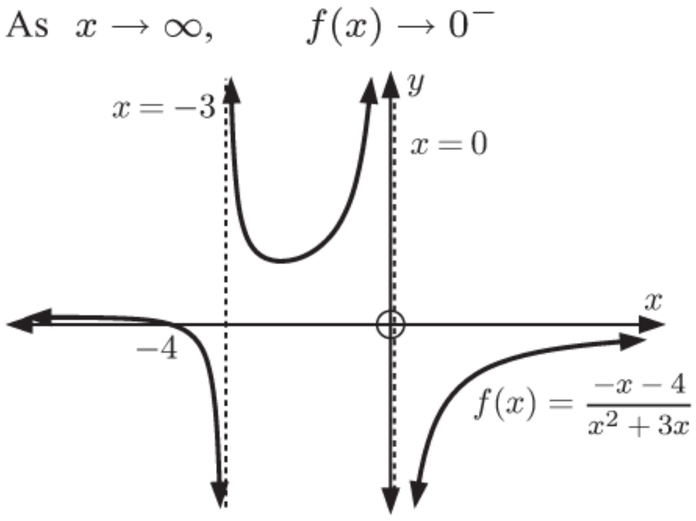
ii x-intercept -4, no y-intercept

iii



iv As $x \to -3^-$, $f(x) \to -\infty$
As $x \to -3^+$, $f(x) \to \infty$
As $x \to 0^-$, $f(x) \to \infty$
As $x \to 0^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^-$

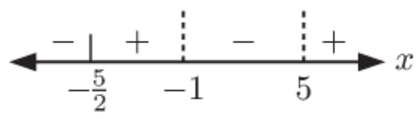
v



e **i** vertical asymptotes $x = -1$ and $x = 5$, horizontal asymptote $y = 0$

ii x-intercept $-\frac{5}{2}$, y-intercept -1

iii



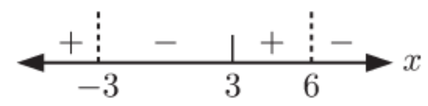
iv As $x \to -1^-$, $y \to \infty$
As $x \to -1^+$, $y \to -\infty$
As $x \to 5^-$, $y \to -\infty$
As $x \to 5^+$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

v

f **i** vertical asymptotes $x = -3$ and $x = 6$, horizontal asymptote $y = 0$

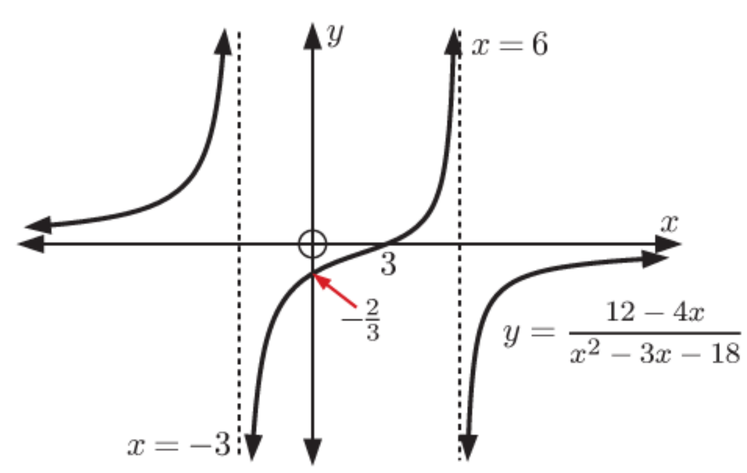
ii x-intercept 3, y-intercept $-\frac{2}{3}$

iii



iv As $x \to -3^-$, $y \to \infty$
As $x \to -3^+$, $y \to -\infty$
As $x \to 6^-$, $y \to \infty$
As $x \to 6^+$, $y \to -\infty$
As $x \to -\infty$, $y \to 0^+$
As $x \to \infty$, $y \to 0^-$

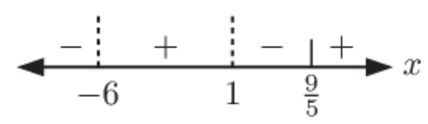
v



g **i** vertical asymptotes $x = -6$ and $x = 1$, horizontal asymptote $y = 0$

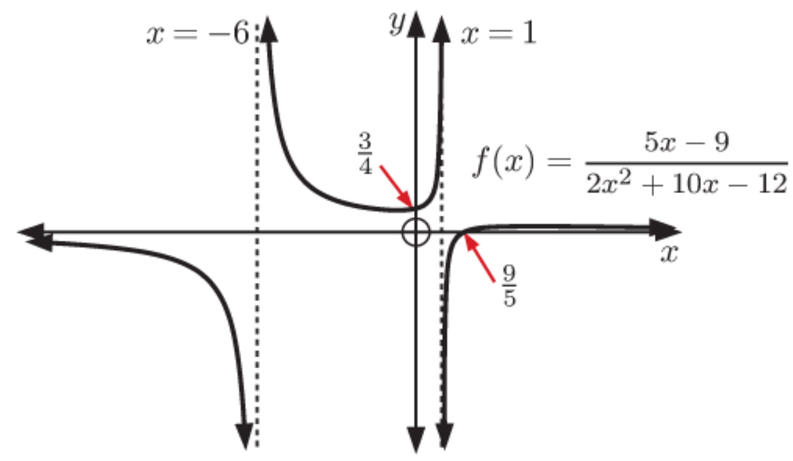
ii x-intercept $\frac{9}{5}$, y-intercept $\frac{3}{4}$

iii



iv As $x \to -6^-$, $f(x) \to -\infty$
As $x \to -6^+$, $f(x) \to \infty$
As $x \to 1^-$, $f(x) \to \infty$
As $x \to 1^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

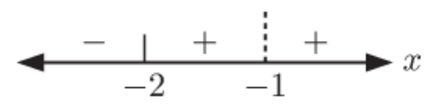
v



h **i** vertical asymptote $x = -1$, horizontal asymptote $y = 0$

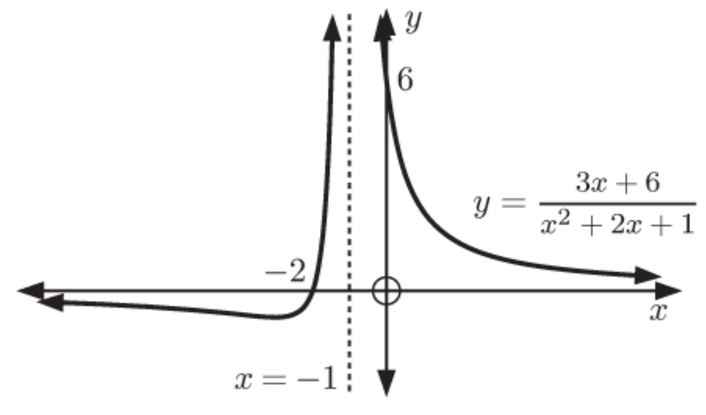
ii x-intercept -2, y-intercept 6

iii



iv As $x \to -1^-$, $y \to \infty$
As $x \to -1^+$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

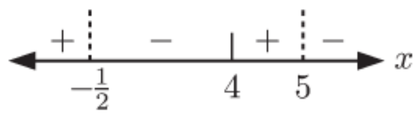
v



i **i** vertical asymptotes $x = -\frac{1}{2}$ and $x = 5$, horizontal asymptote $y = 0$

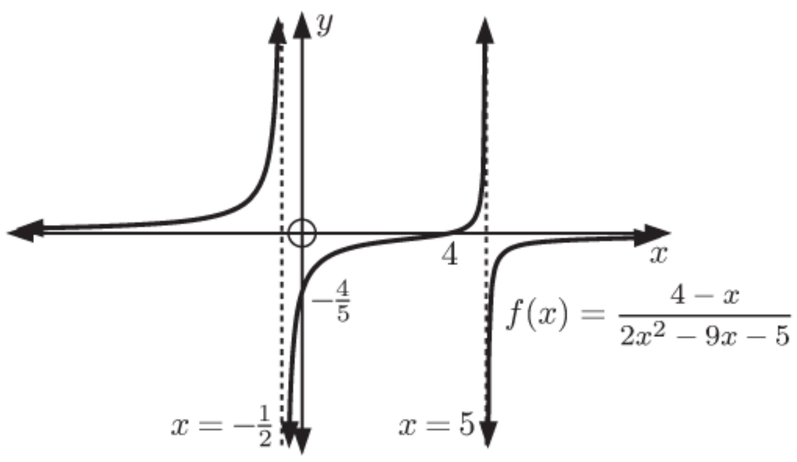
ii x-intercept 4, y-intercept $-\frac{4}{5}$

iii



iv As $x \to -\frac{1}{2}^-$, $f(x) \to \infty$
As $x \to -\frac{1}{2}^+$, $f(x) \to -\infty$
As $x \to 5^-$, $f(x) \to \infty$
As $x \to 5^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^-$

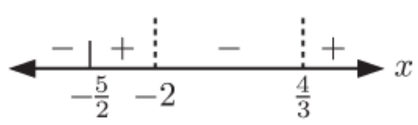
v



j **i** vertical asymptotes $x = -2$ and $x = \frac{4}{3}$, horizontal asymptote $y = 0$

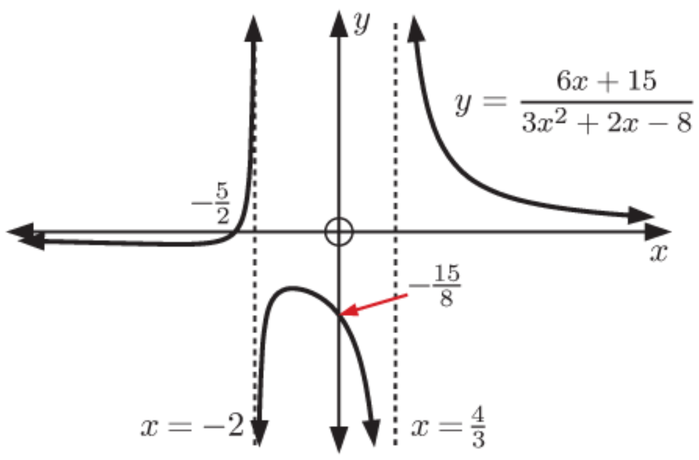
ii x-intercept $-\frac{5}{2}$, y-intercept $-\frac{15}{8}$

iii



iv As $x \to -2^-$, $y \to \infty$
As $x \to -2^+$, $y \to -\infty$
As $x \to \frac{4}{3}^-$, $y \to -\infty$
As $x \to \frac{4}{3}^+$, $y \to \infty$
As $x \to -\infty$, $y \to 0^-$
As $x \to \infty$, $y \to 0^+$

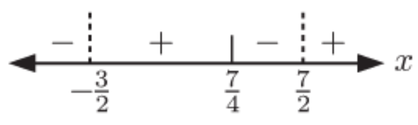
v



k **i** vertical asymptotes $x = -\frac{3}{2}$ and $x = \frac{7}{2}$, horizontal asymptote $y = 0$

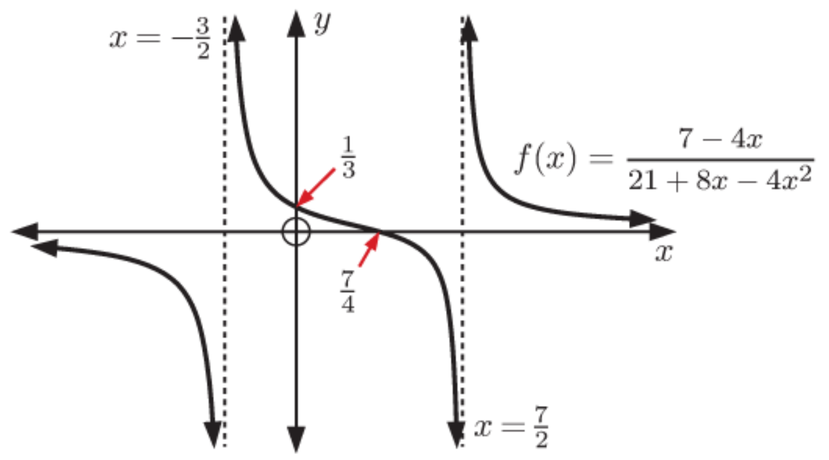
ii x-intercept $\frac{7}{4}$, y-intercept $\frac{1}{3}$

iii



iv As $x \to -\frac{3}{2}^-$, $f(x) \to -\infty$
As $x \to -\frac{3}{2}^+$, $f(x) \to \infty$
As $x \to \frac{7}{2}^-$, $f(x) \to -\infty$
As $x \to \frac{7}{2}^+$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

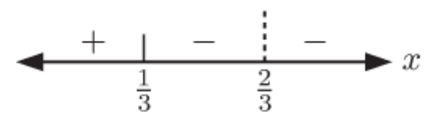
v



l **i** vertical asymptote $x = \frac{2}{3}$, horizontal asymptote $y = 0$

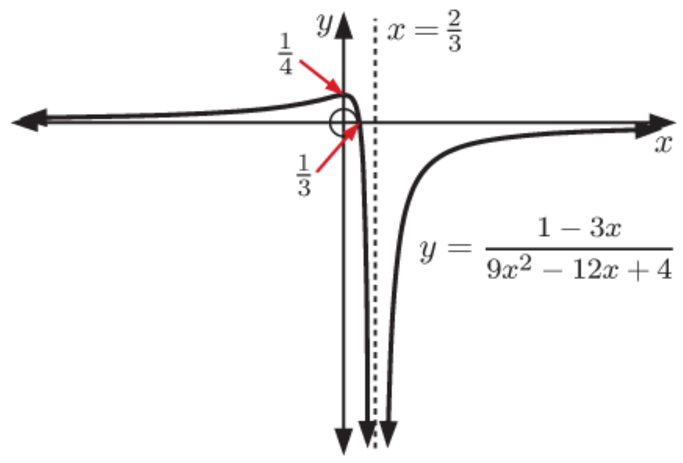
ii x-intercept $\frac{1}{3}$, y-intercept $\frac{1}{4}$

iii



iv As $x \to \frac{2}{3}^-$, $y \to -\infty$
As $x \to \frac{2}{3}^+$, $y \to -\infty$
As $x \to -\infty$, $y \to 0^+$
As $x \to \infty$, $y \to 0^-$

v



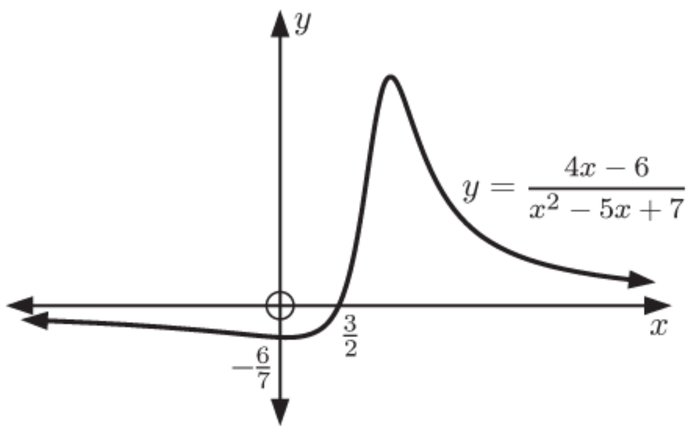
3 **a** $y = 0$

b $x^2 - 5x + 7 > 0$ for all x

$\therefore \quad y = \dfrac{4x-6}{x^2-5x+7}$ is defined for all x and has no vertical asymptotes.

c x-intercept $\frac{3}{2}$, y-intercept $-\frac{6}{7}$

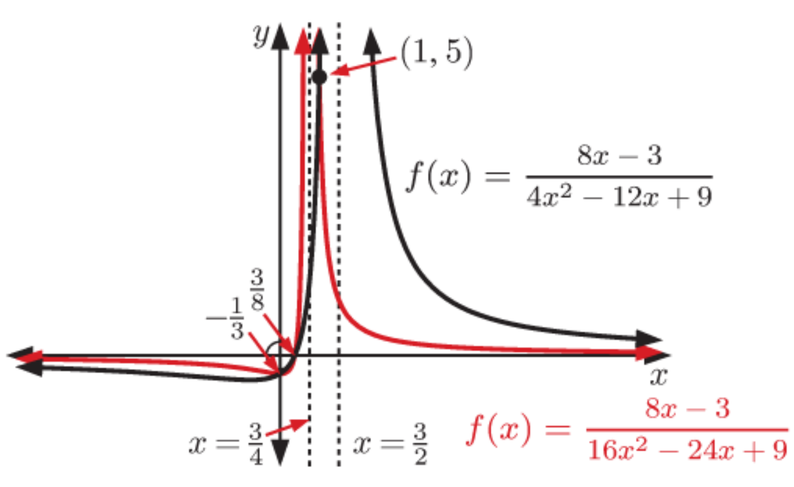
d



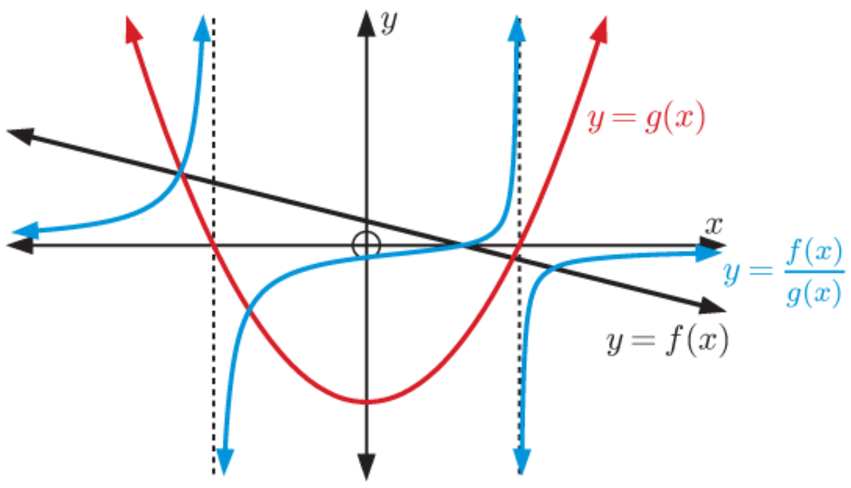
4 **a** $f(x) = \dfrac{8x-3}{4x^2-12x+9}$, vertical asymptote $x = \frac{3}{2}$

or $f(x) = \dfrac{8x-3}{16x^2-24x+9}$, vertical asymptote $x = \frac{3}{4}$

b



5



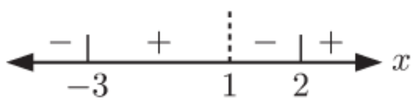
EXERCISE 6D.2

1 **a** $x = 1$ **b** x-intercepts -1 and $\frac{2}{3}$, y-intercept 2

c $y = 3x + 4 + \dfrac{2}{x-1}$, oblique asymptote $y = 3x + 4$

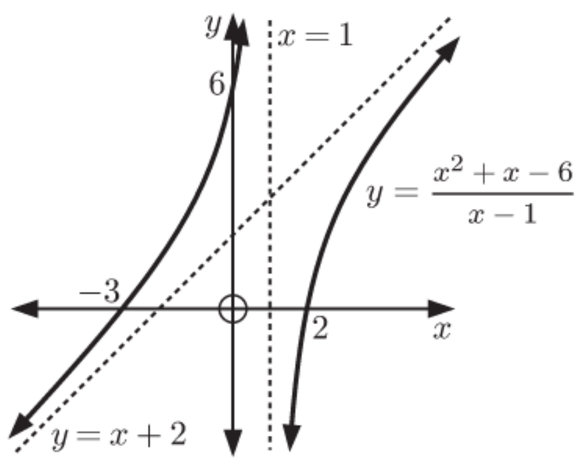
2 **a** **i** $x = 1$ **ii** x-intercepts -3 and 2, y-intercept 6

iii $y = x + 2$ **iv**



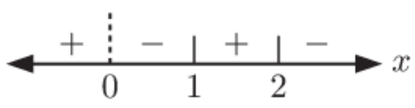
v As $x \to 1^-$, $y \to \infty$
As $x \to 1^+$, $y \to -\infty$
As $x \to -\infty$, $y \to x + 2^+$
As $x \to \infty$, $y \to x + 2^-$

vi



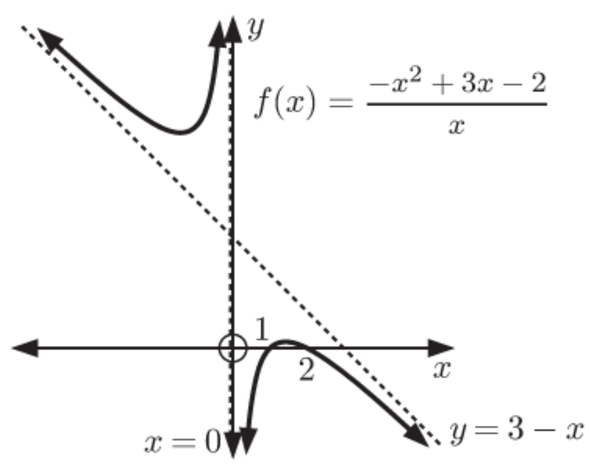
b **i** $x = 0$ **ii** x-intercepts 1 and 2, no y-intercept

iii $y = 3 - x$ **iv**



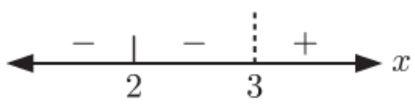
v As $x \to 0^-$, $f(x) \to \infty$
As $x \to 0^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 3 - x^+$
As $x \to \infty$, $f(x) \to 3 - x^-$

vi



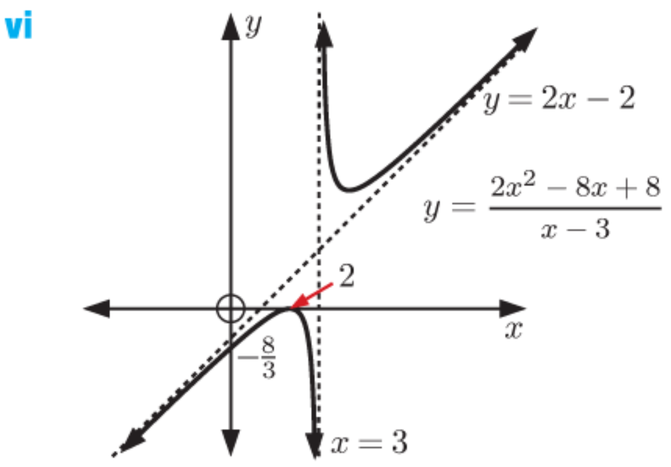
c **i** $x = 3$ **ii** x-intercept 2, y-intercept $-\frac{8}{3}$

iii $y = 2x - 2$ **iv**



v As $x \to 3^-$, $y \to -\infty$
As $x \to 3^+$, $y \to \infty$
As $x \to -\infty$, $y \to 2x - 2^-$
As $x \to \infty$, $y \to 2x - 2^+$

vi

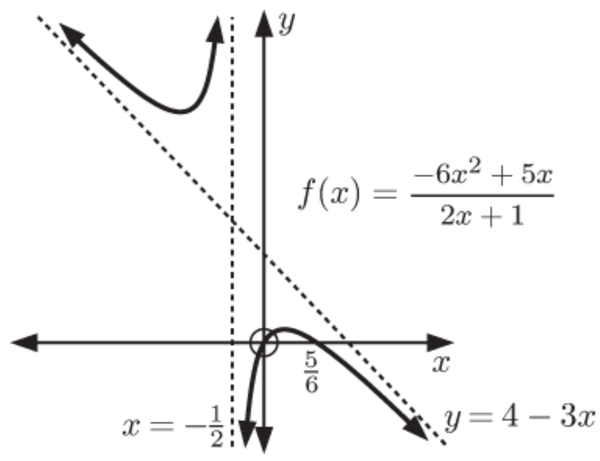


d **i** $x = -\frac{1}{2}$ **ii** x-intercepts 0 and $\frac{5}{6}$, y-intercept 0

iii $y = 4 - 3x$ **iv**

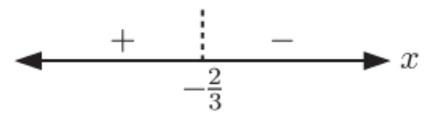
v As $x \to -\frac{1}{2}^-$, $f(x) \to \infty$
As $x \to -\frac{1}{2}^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 4 - 3x^+$
As $x \to \infty$, $f(x) \to 4 - 3x^-$

vi



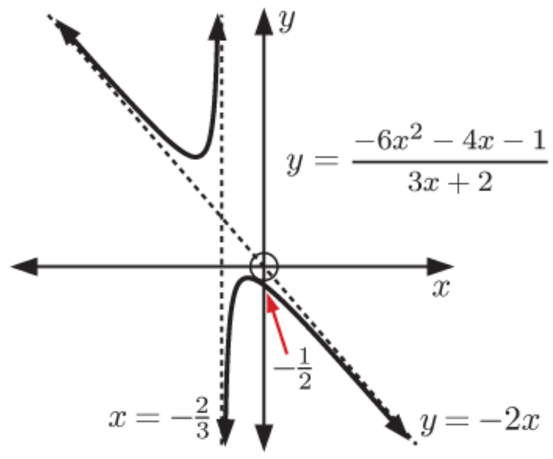
e **i** $x = -\frac{2}{3}$ **ii** no x-intercepts, y-intercept $-\frac{1}{2}$

iii $y = -2x$ **iv**



v As $x \to -\frac{2}{3}^-$, $y \to \infty$
As $x \to -\frac{2}{3}^+$, $y \to -\infty$
As $x \to -\infty$, $y \to -2x^+$
As $x \to \infty$, $y \to -2x^-$

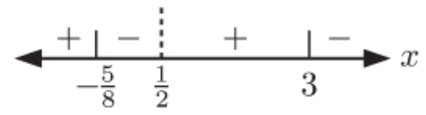
vi



f **i** $x = \frac{1}{2}$

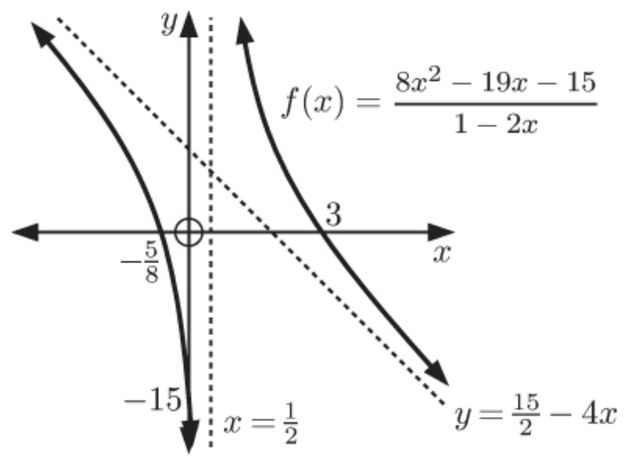
ii x-intercepts $-\frac{5}{8}$ and 3, y-intercept -15

iii $y = \frac{15}{2} - 4x$ **iv**



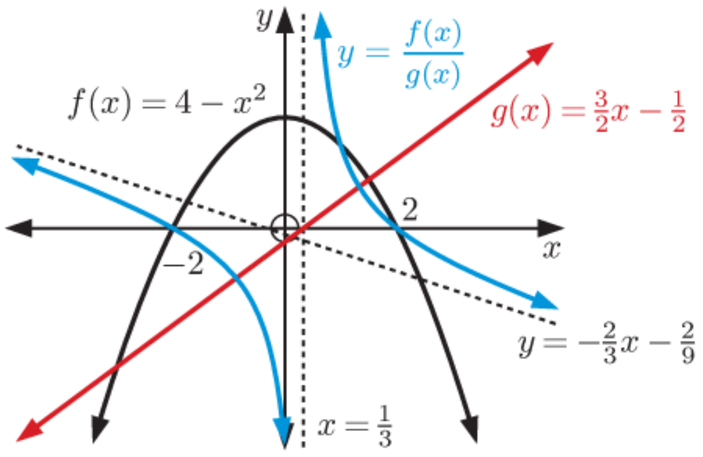
v As $x \to \frac{1}{2}^-$, $f(x) \to -\infty$
As $x \to \frac{1}{2}^+$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to \frac{15}{2} - 4x^-$
As $x \to \infty$, $f(x) \to \frac{15}{2} - 4x^+$

vi



3 **a** **i** $x = \frac{1}{3}$ **ii** $y = -\frac{2}{3}x - \frac{2}{9}$

b



4 $y = \dfrac{a}{d}x + \dfrac{b}{d} - \dfrac{ae}{d^2}$

EXERCISE 6E

2 **a** $\dfrac{1}{x+1} - \dfrac{1}{x+2}$ **b** $\dfrac{-2}{x-2} + \dfrac{3}{x-3}$ **c** $\dfrac{1}{x+2} + \dfrac{2}{x-4}$

d $\dfrac{1}{x+2} + \dfrac{2}{x-2}$ **e** $\dfrac{3}{4(x+1)} + \dfrac{5}{4(x-3)}$

f $\dfrac{-1}{x-2} - \dfrac{4}{x+6}$ **g** $\dfrac{1}{2x+3} + \dfrac{3}{x+1}$

h $\dfrac{-26}{7(3x+1)} + \dfrac{11}{7(x-2)}$ **i** $\dfrac{5}{4x-1} - \dfrac{2}{2x+5}$

4 $\dfrac{2}{x+2} - \dfrac{2}{x+3} - \dfrac{5}{(x+3)^2}$

5 **b** $\dfrac{-3}{x+3} + \dfrac{3}{x-1} + \dfrac{7}{(x-1)^2}$

6 **b** $\dfrac{4}{2x-1} + \dfrac{3x-2}{x^2+x+4}$

REVIEW SET 6A

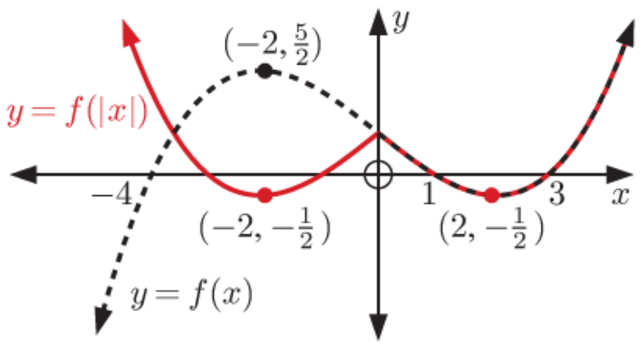
1 **a** odd **b** neither **c** even

2 **a** $k = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$ **b** $k = n\pi,\ n \in \mathbb{Z}$

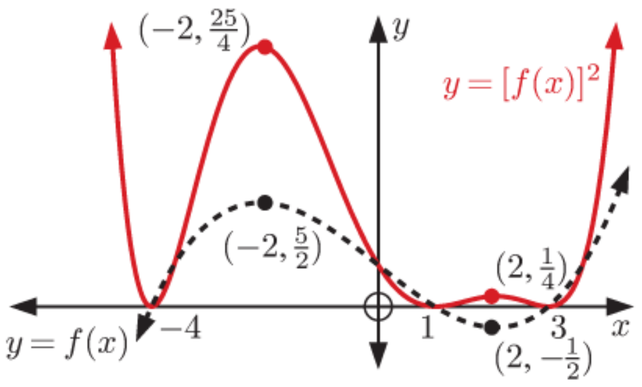
c $k \neq \dfrac{n\pi}{2},\ n \in \mathbb{Z}$

3 odd **4** $b = 3$

5 **a**



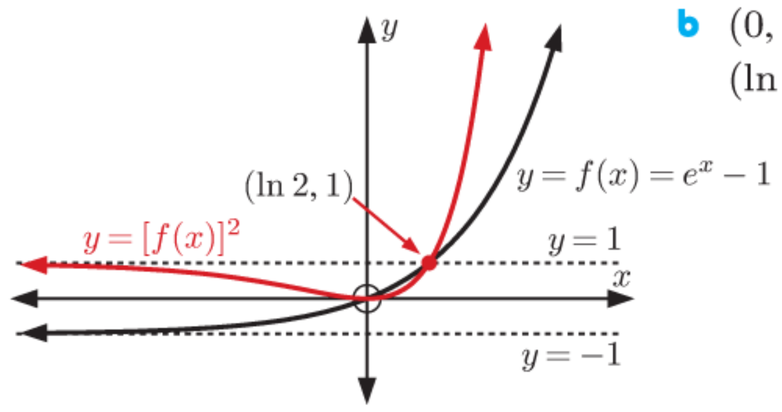
b



6 **a** $\{x \mid -5 < x < 5\}$ **b** $\{y \mid 0 \leqslant y \leqslant 5\}$

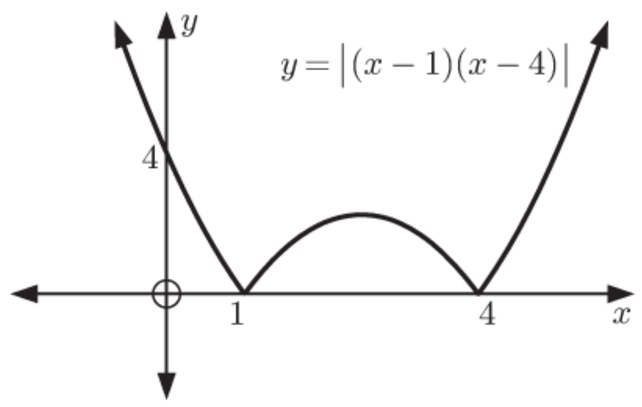
7 **a** odd **b** even **c** neither **d** even **e** even **f** odd

8 **a**

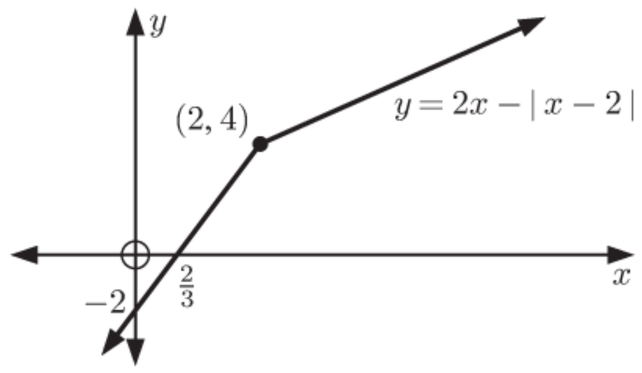


b (0, 0) and (ln 2, 1)

9 **a**



b



10 **a** $x = 1$ or 7 **b** $x = -6$ or $\frac{1}{2}$

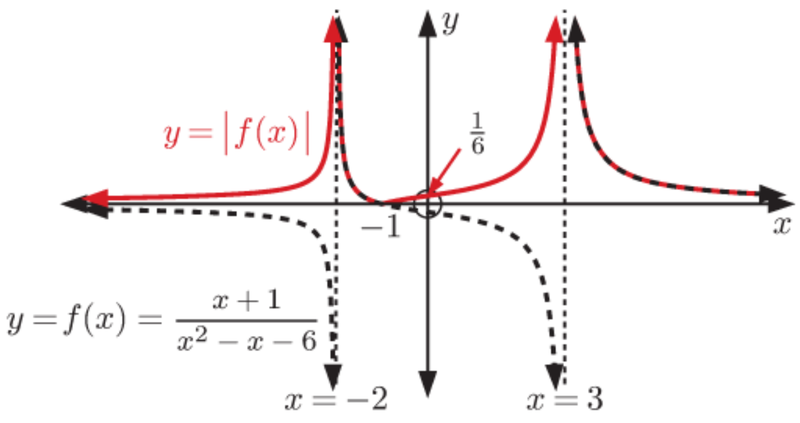
11 **a** $-9 \leqslant x \leqslant 3$ **b** $x < 1$ or $x > 2$ **c** $-\frac{7}{2} < x < \frac{3}{4}$

12 **a** $x < 1$ or $x > 9$ **b** $-7.14 \leqslant x \leqslant 1.91$ or $x \geqslant 3.53$

13 **a** vertical asymptotes $x = -2$ and $x = 3$, horizontal asymptote $y = 0$

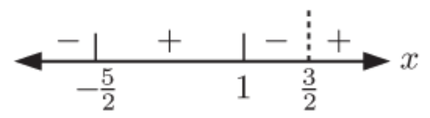
b x-intercept -1, y-intercept $-\frac{1}{6}$

c



14 **a** $x = \frac{3}{2}$ **b** x-intercepts $-\frac{5}{2}$ and 1, y-intercept $\frac{5}{3}$

c $y = x + 3$ **d**



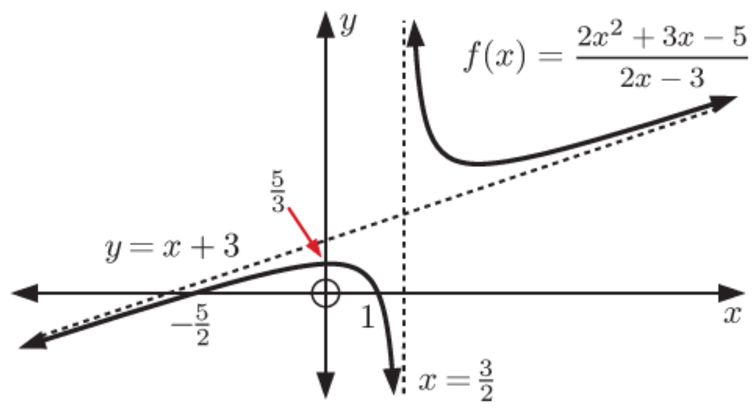
e As $x \to \frac{3}{2}^-$, $f(x) \to -\infty$

As $x \to \frac{3}{2}^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to x + 3^-$

As $x \to \infty$, $f(x) \to x + 3^+$

f



15 **a** $\dfrac{2}{3(x+5)} + \dfrac{1}{3(x-1)}$ **b** $\dfrac{-2}{2x+1} + \dfrac{2}{x+3}$

16 $\dfrac{-2}{x+4} + \dfrac{2}{x-1} + \dfrac{3}{(x-1)^2}$

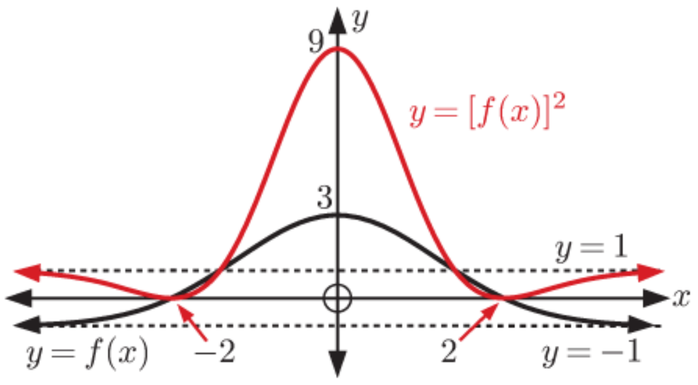
REVIEW SET 6B

1 $(2, 6)$ and $(-1, -4)$

2 **b** Let $f(x) = x + 3$, $g(x) = x - 3$, neither of which are odd nor even. $f(x) \times g(x) = x^2 - 9$ which is even.

3 $a = \frac{\pi}{4} + \frac{n\pi}{2}$, $n \in \mathbb{Z}$, $b = 0$

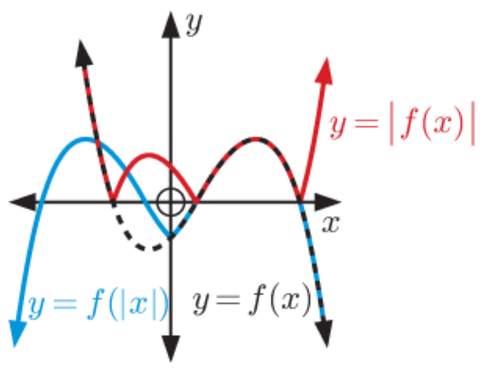
4



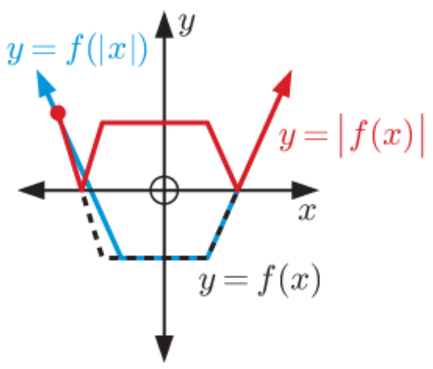
5 $|-x| = |x|$, $\therefore \quad f(|-x|) = f(|x|)$
So, $f(|x|)$ is even.

6 **a** 3 turning points, 2 x-intercepts
b cannot say about the x-intercepts or turning points
c 3, 4, or 5 turning points, 2 x-intercepts

7 **a**



b



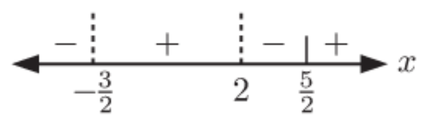
8 **a** $x = -\frac{4}{5}$ or $\frac{8}{5}$ **b** $x = 3$ or -1

9 **a** $x < -\frac{5}{3}$ or $x > \frac{1}{3}$ **b** $-\frac{1}{6} \leqslant x \leqslant \frac{9}{2}$
c $x \leqslant -8$ or $x \geqslant -\frac{8}{3}$

10 **a** $x \approx 0.315$, 1.78, or 11.2 **b** $x \geqslant \frac{1}{2}$

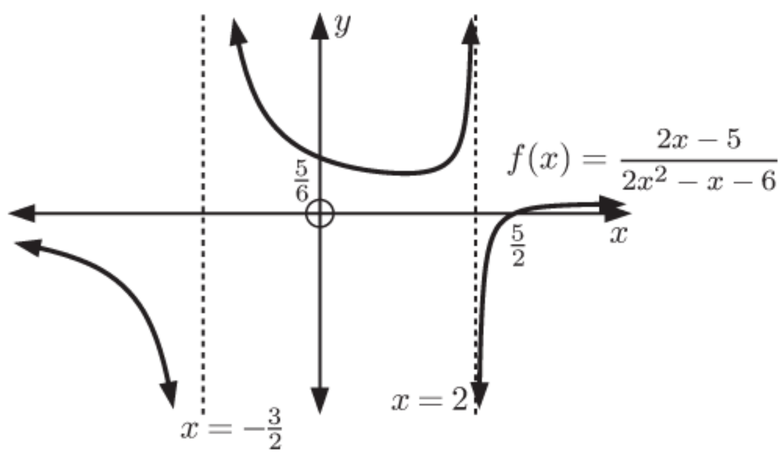
11 $a = 6$, $b = -12$, $d = -5$, $e = -12$

12 **a** vertical asymptotes $x = -\frac{3}{2}$ and $x = 2$,
horizontal asymptote $y = 0$
b x-intercept $\frac{5}{2}$, y-intercept $\frac{5}{6}$
c

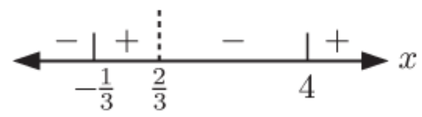


d As $x \to -\frac{3}{2}^-$, $f(x) \to -\infty$
As $x \to -\frac{3}{2}^+$, $f(x) \to \infty$
As $x \to 2^-$, $f(x) \to \infty$
As $x \to 2^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

e

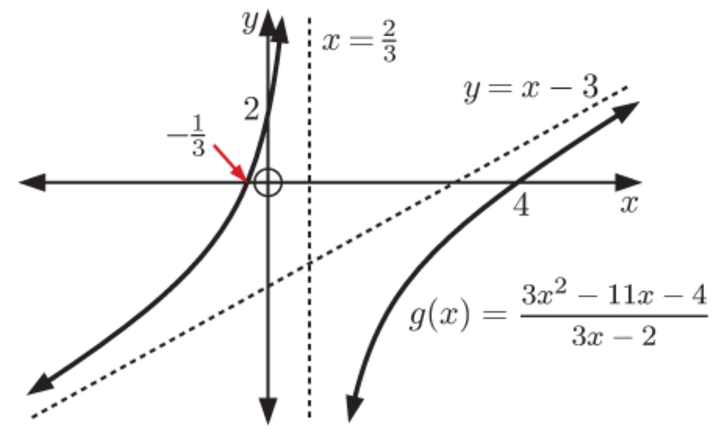


13 **a** $x = \frac{2}{3}$ **b** x-intercepts $-\frac{1}{3}$ and 4, y-intercept 2
c $y = x - 3$ **d**



e As $x \to \frac{2}{3}^-$, $g(x) \to \infty$
As $x \to \frac{2}{3}^+$, $g(x) \to -\infty$
As $x \to -\infty$, $g(x) \to x - 3^+$
As $x \to \infty$, $g(x) \to x - 3^-$

f



14 **a** $\dfrac{-3}{x-3} + \dfrac{1}{x-1}$ **b** $\dfrac{-10}{x+3} + \dfrac{11}{x+4}$

15 **b** $\dfrac{1}{2x-3} + \dfrac{4}{x+2} - \dfrac{5}{(x+2)^2}$

16 **a** **i** even **ii** odd
b $\dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2} = f(x)$

EXERCISE 7A

1 **a** 12 routes **b** 36 routes

2 30 meal combinations **3** $26^5 = 11\,881\,376$ codes

4 $26^3 \times 10^4 = 175\,760\,000$ number plates

EXERCISE 7B

1 **a** 5 **b** 7 **c** 11 **d** 14

2 **a** 3 **b** 9 **3** 9 routes

EXERCISE 7C

1 **a** 2 **b** 6 **c** 24 **d** 120 **e** 720 **f** 3 628 800

2 **a** $4!$ **b** $7!$ **c** $\dfrac{6!}{4!}$ **d** $\dfrac{8!}{5!}$ **e** $\dfrac{10!}{6!}$
f $\dfrac{15!}{11!}$ **g** $\dfrac{9!}{3!6!}$ **h** $\dfrac{13!}{4!9!}$ **i** $\dfrac{15!}{5!10!}$

3 **a** 7 **b** 56 **c** 132 **d** 120 **e** 45 **f** 4950

4 **a** n, $n \geqslant 1$ **b** $(n+2)(n+1)$, $n \geqslant 0$
c $(n+1)n$, $n \geqslant 1$

5 **a** $4! \times 6$ **b** $10! \times 10$ **c** $5! \times 43$ **d** $10! \times 131$
e $7! \times 81$ **f** $6! \times 62$ **g** $11! \times 10$ **h** $8! \times 32$

6 **a** $11!$ **b** $9!$ **c** $8!$ **d** 9
e 34 **f** $n+1$ **g** $(n-1)!$ **h** $(n+1)!$

EXERCISE 7D.1

1 **a** W, X, Y, Z
b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY
c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZW, YZX, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX

2 **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

2 at a time: 20 3 at a time: 60

3 $5! = 120$ ways

4 **a** $\frac{6!}{3!} = 120$ **b** $\frac{7!}{2!} = 2520$ **c** $\frac{26!}{11!} \approx 1.01 \times 10^{19}$

5 72 ways **6** 1680 ways **7** 336 ways

8 **a** 64 numbers **b** 24 numbers

9 **a** 12 **b** 24 **c** 36 **10** **a** 120 **b** 6 **c** 6

11 **a** 360 **b** 60 **c** 12

12 **a** 294 **b** 108 **c** 42 **d** 150

13 **a** 24 **b** 6 **c** 18 **d** 12 **e** 6

14 **a** 48 **b** 24 **c** 15

15 **a** 48 **b** 72 **16** **a** 3 628 800 **b** 241 920

17 **a** 360 **b** 336 **c** 288

18 **a** 15 120 **b** 720

19 **a** 3 628 800 **b** **i** 151 200 **ii** 3600 **iii** 33 600

EXERCISE 7D.2

1 AAAB, AABA, ABAA, BAAA

$\frac{4!}{3!1!} = 4$ distinguishable permutations ✓

2 AABB, ABAB, ABBA, BAAB, BABA, BBAA

$\frac{4!}{2!2!} = 6$ distinguishable permutations ✓

3 **a** 60 **b** 30 **c** 30 **4** 90

5 **a** 10 **b** 2 **6** **a** 4 **b** 12 **c** 8

EXERCISE 7E

1 **a** 3 **b** 15 **c** 5 **d** 21

2 **a** PQ, PR, PS, PT, QR, QS, QT, RS, RT, ST **b** $\binom{5}{2} = \frac{5!}{2!3!} = 10$ ✓

3 **a** ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF

b $\binom{6}{4} = 15$ ✓

4 **a** **i** 4 **ii** 7 **iii** 10

b $\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n(n-1)!}{(n-1)!} = n$

5 **a** **i** 1 **ii** 1 **iii** 1 **iv** 1

b **i** $\binom{n}{0} = \frac{n!}{0!n!} = 1$ **ii** $\binom{n}{n} = \frac{n!}{n!0!} = 1$

6 **a** **i** $\binom{7}{2} = 21$, $\binom{7}{5} = 21$

ii $\binom{10}{3} = 120$, $\binom{10}{7} = 120$

b $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

7 $k = 3$ or 6

8 **a** $\frac{1}{2}n^2 - \frac{1}{2}n$, $n \in \mathbb{Z}^+$, $n \geqslant 2$

b $\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{11}{24}n^2 - \frac{1}{4}n$, $n \in \mathbb{Z}^+$, $n \geqslant 2$

9 6435 different teams

10 **a** 80 730 different committees

b 14 950 different committees

11 10 different selections

12 **a** $\binom{12}{5} = 792$ teams

b **i** $\binom{2}{2}\binom{10}{3} = 120$ **ii** $\binom{2}{1}\binom{10}{4} = 420$

13 $\binom{3}{3}\binom{1}{0}\binom{11}{6} = 462$

14 **a** 364 **b** **i** 20 **ii** 120 **iii** 168 **iv** 56

15 **a** 1820 **b** 756 **c** 1470

16 **a** The answers for **1** are given an order. The answers for **2** are ticks without specifying order.

b question **1** **c** **i** 210 ways **ii** 35 ways

d 7350 ways

17 **a** $\binom{6}{2}\binom{3}{1}\binom{7}{2} = 945$ **b** $\binom{6}{2}\binom{10}{3} = 1800$

c $\binom{16}{5} - \binom{9}{0}\binom{7}{5} = 4347$

18 $\binom{20}{2} - 20 = 170$ **19** 11 200 ways

20 **a** 46 200 ways **b** 700 ways

21 **a** **i** $\binom{12}{2} = 66$ **ii** $\binom{11}{1} = 11$

b **i** $\binom{12}{3} = 220$ **ii** $\binom{11}{2} = 55$

22 $\binom{9}{4} = 126$

23 **a** Selecting the different committees of 4 from 5 males and 6 females in all possible ways.

b .... $= \binom{m+n}{r}$

24 **a** $\frac{\binom{12}{6}}{2} = 462$ **b** $\frac{\binom{12}{4}\binom{8}{4}\binom{4}{4}}{3!} = 5775$

25 $\binom{10}{2}\binom{7}{2} = 945$

26 $\binom{10}{2}\binom{9}{2} + \binom{9}{2}\binom{8}{2} + \binom{10}{2}\binom{8}{2} + \binom{10}{2}\binom{9}{1}\binom{8}{1}$
$+ \binom{10}{1}\binom{9}{2}\binom{8}{1} + \binom{10}{1}\binom{9}{1}\binom{8}{2} = 12\,528$

27 **a** $\binom{52}{13} = 635\,013\,559\,600$ **b** $\binom{48}{9} = 1\,677\,106\,640$

c $\frac{\binom{48}{9}}{\binom{52}{13}} = \frac{11}{4165} \approx 0.002\,64$

REVIEW SET 7A

1 48 **2** **a** 8! **b** $\frac{10!}{7!}$

3 **a** $n(n-1)$, $n \geqslant 2$ **b** $n+2$

4 PQ, PR, PS, QP, QR, QS, RP, RQ, RS, SP, SQ, SR

5 **a** 840 numbers **b** 360 numbers

6 **a** JKL, JKM, JKN, JKO, JLM, JLN, JLO, JMN, JMO, JNO, KLM, KLN, KLO, KMN, KMO, KNO, LMN, LMO, LNO, MNO

b $\binom{6}{3} = \frac{6!}{3!3!} = 20$ ✓

7 **a** 120 **b** **i** 6 **ii** 2 **iii** 48

8 **a** $\binom{6}{3} = 20$ **b** 12 **9** **a** 210 **b** **i** 30 **ii** 30

10 **a** $\binom{12}{4} = 495$

b $\binom{5}{2}\binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{5}{1}\binom{4}{1} + \binom{4}{2}\binom{5}{1}\binom{3}{1} = 270$

c $\binom{9}{2} = 36$

11 **a** 45 **b** 37 128 **c** 3 628 800

12 **a** $\binom{52}{8} = 752\,538\,150$ **b** $\left[\binom{13}{2}\right]^4 = 37\,015\,056$

c ≈ 0.0492

REVIEW SET 7B

1 6 paths **2** **a** 72 **b** 56

3 **a** 7! **b** $1 + (n+1)n = n^2 + n + 1$

4 **a** 6 **b** 6 **c** 36 **5** 336 ways

6 **a** $\binom{8+7+6}{5} = \binom{21}{5} = 20\,349$ committees

b **i** $\binom{8}{5} = 56$ committees **ii** 14 140 committees

7 **a** $\binom{9}{5} = \dfrac{9!}{5!4!} = 126$

$\binom{9}{2} \times \binom{7}{3} = 36 \times 35 = 1260 \neq 126$

b Miles did not take into account the number of ways that the two selected groups of people could be ordered in the final group of 5, which is $\binom{5}{2} = \binom{5}{3} = 10$. He would need to divide the RHS by 10 to get the correct result.

8 4320

9 **a** 43 758 teams **b** 11 550 teams **c** 41 283 teams

10 **a** 1680 **b** **i** 180 **ii** 300

11 **a** $\binom{16}{8} = 12\,870$ **b** 2 627 625

12 $\binom{6}{2}\binom{7}{1}\binom{8}{1} + \binom{7}{2}\binom{6}{1}\binom{8}{1} + \binom{8}{2}\binom{6}{1}\binom{7}{1} = 3024$

EXERCISE 8A

1 **a** $p^3 + 3p^2q + 3pq^2 + q^3$ **b** $x^3 + 3x^2 + 3x + 1$

c $x^3 - 9x^2 + 27x - 27$ **d** $8 + 12x + 6x^2 + x^3$

e $64 - 48x + 12x^2 - x^3$ **f** $27x^3 - 27x^2 + 9x - 1$

g $8x^3 + 60x^2 + 150x + 125$ **h** $x^6 - 3x^4 + 3x^2 - 1$

i $8a^3 - 12a^2b + 6ab^2 - b^3$ **j** $x\sqrt{x} - 3x + 3\sqrt{x} - 1$

k $27x^3 - 9x^2 + x - \frac{1}{27}$ **l** $8x^3 + 12x + \dfrac{6}{x} + \dfrac{1}{x^3}$

2 **a** $1 + 4x + 6x^2 + 4x^3 + x^4$

b $x^4 - 8x^3 + 24x^2 - 32x + 16$

c $81 - 108x + 54x^2 - 12x^3 + x^4$

d $1 + 8x + 24x^2 + 32x^3 + 16x^4$

e $16x^4 - 96x^3 + 216x^2 - 216x + 81$

f $16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4$

g $x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$

h $16x^4 - 32x^2 + 24 - \dfrac{8}{x^2} + \dfrac{1}{x^4}$

i $x^4 + 4x^3\sqrt{x} + 6x^3 + 4x^2\sqrt{x} + x^2$

3 **a** **i** $a^3 - 3a^2b + 3ab^2 - b^3$

ii $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

b The terms are the same, except for their signs. The signs in the expansions of $(a+b)^3$ and $(a+b)^4$ are all positive, whereas the signs in the expansions of $(a-b)^3$ and $(a-b)^4$ start with a positive and then alternate $(a > 0,\ b > 0)$.

4 **a** 1 5 10 10 5 1

b $(a+b)^5 = a^5 + 5a^2b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

c **i** $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

ii $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$

iii $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

iv $x^5 - 5x^3 + 10x - \dfrac{10}{x} + \dfrac{5}{x^3} - \dfrac{1}{x^5}$

5 **a** 1 6 15 20 15 6 1

b $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

c **i** $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $x^6 + 6x^4 + 15x^2 + 20 + \dfrac{15}{x^2} + \dfrac{6}{x^4} + \dfrac{1}{x^6}$

6 **a** $7 + 5\sqrt{2}$ **b** $161 + 72\sqrt{5}$ **c** $232 - 164\sqrt{2}$

7 **a** $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b 65.944 160 601 201

8 $2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$

9 **a** 270 **b** 4320

EXERCISE 8B

1 **a** $1^{11} + \binom{11}{1}(2x)^1 + \binom{11}{2}(2x)^2 + \ldots + \binom{11}{10}(2x)^{10} + (2x)^{11}$

b $(3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right)^1 + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + \ldots$

$\ldots + \binom{15}{14}(3x)^1\left(\frac{2}{x}\right)^{14} + \left(\frac{2}{x}\right)^{15}$

c $(2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right)^1 + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + \ldots$

$\ldots + \binom{20}{19}(2x)^1\left(-\frac{3}{x}\right)^{19} + \left(-\frac{3}{x}\right)^{20}$

2 **a** $T_6 = \binom{15}{5}(2x)^{10}5^5$ **b** $T_4 = \binom{9}{3}(x^2)^6y^3$

c $T_4 = \binom{10}{3}x^7(y^2)^3$ **d** $T_{10} = \binom{17}{9}x^8\left(-\frac{2}{x}\right)^9$

e $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\frac{1}{x}\right)^8$ **f** $T_7 = \binom{16}{6}\left(\sqrt{x}\right)^{10}1^6$

4 **a** $T_{r+1} = \binom{8}{r}x^{8-r}2^r$ **b** 448

5 **a** $T_{r+1} = \binom{7}{r}x^{7-r}b^r$ **b** $b = -2$

6 **a** $T_{r+1} = \binom{12}{r}\left(\frac{1}{2}x\right)^{12-r}\left(\frac{1}{3}\right)^r$

b **i** $\frac{55}{39\,366}$ **ii** $\frac{11}{972}$ **iii** $\frac{55}{3456}$

7 **a** $\binom{15}{5}2^5$ **b** $\binom{9}{3}(-3)^3$ **c** $\binom{15}{10}(\frac{1}{3})^5(-2)^{10}$

8 **a** $\binom{10}{5}3^52^5$ **b** $\binom{6}{3}2^3(-3)^3$ **c** $\binom{6}{3}2^3(-3)^3$

d $\binom{12}{4}2^8(-1)^4$

9 $T_3 = \binom{6}{2}(-2)^2x^8y^8$ **10** $n = 9,\ T_4 = 84x^3$

11 **a** $a = \pm\frac{1}{2}$ **b** $a = \pm 2$ **c** $a = \frac{1}{\sqrt{2}}$ **d** $a = 2$

12 **a** $\binom{6}{3}(-3)^3 + 4\binom{6}{2}(-3)^2 = 0$

b $2\binom{9}{3}3^6x^6 - \binom{9}{4}3^5x^6 = 91\,854x^6$ **c** $4\binom{4}{2} = 24$

d $3(-2)^1 - 4\binom{3}{2}(-2)^2 + 4(-2)^3 = -86$

e $3\binom{6}{1}(-2) + \binom{6}{2}(-2)^2 = 24$ **f** $3\binom{4}{2}(-2)^2 = 72$

13 $n = 6$ and $k = -2$

14 **a**

```
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
```

b **i** 2 **ii** 4 **iii** 8 **iv** 16 **v** 32

c The sum of the numbers in row n of Pascal's triangle is 2^n.

e **i** **Hint:** Let $x = 1$, in the expansion of $(1+x)^n$.

ii **Hint:** Let $x = -1$, in the expansion of $(1+x)^n$.

iii **Hint:** Let $x = 1$, in the expansion of $(1+x)^{2n+1}$.

f $\displaystyle\sum_{r=0}^{n} 2^r\binom{n}{r} = 3^n$

15 **a** $(3+x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + \ldots + \binom{n}{n-1}3^1x^{n-1} + x^n$

b 4^n

16 $1 + 10x + 35x^2 + 40x^3 - 30x^4$

17 When we expand $(a+b)(a+b)^n$ and collect like terms, we can use Pascal's rule $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ to simplify the coefficients.

18 $(-1)^{100} = 1$

19 **Hint:** Write out $(1+x)^n(1+x)^n$ and $(1+x)^{2n}$, then equate coefficients of x^n.

EXERCISE 8C

1 **a** $\frac{1}{2}$ **b** $\frac{3}{8}$ **c** 3 **d** 1 **e** -1 **f** $\frac{5}{81}$

2 **a** $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots.$ **b** $-1 < x < 1$

c ≈ 0.909 (using technology, $\frac{1}{1.1} = 0.\overline{90}$)

3 **a** $\dfrac{1}{\sqrt{4-x}} = \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \frac{5}{2048}x^3 + \ldots.$

b $-4 < x < 4$

c $\approx 0.527\,031$ (using technology, $\frac{1}{\sqrt{3.6}} \approx 0.527\,046$)

4 **a** $\sqrt[3]{1+3x} = 1 + x - x^2 + \frac{5}{3}x^3 - \frac{10}{3}x^4 + \ldots.$

b $-\frac{1}{3} < x < \frac{1}{3}$

c **i** $\approx 1.091\,667$ **ii** $\approx 1.091\,333$

d $\sqrt[3]{1.3} \approx 1.091\,393$

5 **a** $\dfrac{1}{(1-2x)^2} = 1 + 4x + 12x^2 + 32x^3 + \ldots.$

b The expansion is valid for $-\frac{1}{2} < x < \frac{1}{2}$.

c $\frac{1}{(0.96)^2} \approx 1.085\,056$

6 **a** $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots.$

b The expansion is valid for $-1 < x < 1$.

c The series only converges provided $-1 < x < 1$.

d $\approx 6.082\,763$

7 **a** $(1+x)^{\frac{1}{5}} = 1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 + \ldots.$

b $\approx 1.001\,992$

c The series only converges provided $-1 < x < 1$, but to estimate $\sqrt[5]{2}$ we would need to substitute $x = 1$.

8 **a** $\dfrac{1}{(1+2x)^2} = 1 - 4x + 12x^2 - 32x^3 + \ldots.$ **b** 75

9 **a** $\dfrac{3}{x+1} + \dfrac{-2}{x-2}$

b $\dfrac{x-8}{x^2-x-2} = 4 - \frac{5}{2}x + \frac{13}{4}x^2 - \frac{23}{8}x^3 + \ldots.$

c $-1 < x < 1$ **d** $\approx 3.882\,766$

10 **a** **i** $(1+3x)^{\frac{1}{4}} = 1 + \frac{3}{4}x - \frac{27}{32}x^2 + \frac{189}{128}x^3 + \ldots.$
provided $-\frac{1}{3} < x < \frac{1}{3}$

ii $(2-x)^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{2}}\left(1 + \frac{1}{8}x + \frac{5}{128}x^2 + \frac{15}{1024}x^3 + \ldots.\right)$
provided $-2 < x < 2$

b $\sqrt[4]{\dfrac{1+3x}{2-x}} = \frac{1}{\sqrt[4]{2}}\left(1 + \frac{7}{8}x - \frac{91}{128}x^2 + \frac{1449}{1024}x^3 + \ldots.\right)$

11 **a** $\dfrac{1}{\sqrt[4]{1+x}} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \ldots.$

b $-1 < x < 1$

c $\dfrac{1}{\sqrt[4]{1+3x^2}} = 1 - \frac{3}{4}x^2 + \frac{45}{32}x^4 - \frac{405}{128}x^6 + \ldots.$

REVIEW SET 8A

1 **a** $x^3 + 9x^2 + 27x + 27$

b $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

2 **a** $T_5 = \binom{9}{4}(2x)^5 3^4$ **b** $T_4 = \binom{9}{3} y^6(-3z)^3$

c $T_8 = \binom{12}{7}(3x)^5\left(-\frac{1}{x}\right)^7$ **d** $T_5 = \binom{10}{4}(2x)^6\left(-\frac{1}{\sqrt{x}}\right)^4$

3 **a** $170 + 78\sqrt{3}$ **b** $x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3$

4 $64.964\,808$

5 **a** $(a+b)^6$
$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

b **i** $x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

ii $64 + \dfrac{192}{x} + \dfrac{240}{x^2} + \dfrac{160}{x^3} + \dfrac{60}{x^4} + \dfrac{12}{x^5} + \dfrac{1}{x^6}$

6 **a** $\binom{12}{6}2^6(-3)^6$ **b** $8\binom{6}{2} - 6\binom{6}{1} = 84$

7 $1 + 24\binom{5}{1}(-3) + 16\binom{5}{2}(-3)^2 = 1081$ **8** $c = 3$

9 **a** $2^n + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \binom{n}{3}2^{n-3}x^3 + \ldots.$
$\ldots. + \binom{n}{n-1}2^1x^{n-1} + x^n$

b 3^n **Hint:** Let $x = 1$ in **a**.

10 $a = \pm 4$

11 **a** $1 + x + x^2 + x^3 + \ldots.$ **b** $-1 < x < 1$

c $\approx 1.052\,625$ (using technology, $\frac{1}{0.95} \approx 1.052\,632$)

12 **a** $\dfrac{1}{\sqrt{x+3}} = \dfrac{1}{\sqrt{3}} - \dfrac{1}{6\sqrt{3}}x + \dfrac{1}{24\sqrt{3}}x^2 - \dfrac{5}{432\sqrt{3}}x^3 + \ldots.$

b $-3 < x < 3$

c **i** $\approx 0.427\,667$ (using technology, $\frac{1}{\sqrt{5}} \approx 0.447\,214$)

ii $\approx 0.559\,014$ (using technology, $\frac{1}{\sqrt{3.2}} \approx 0.559\,017$)

Our estimates are more accurate for x closer to 0.

13 **a** $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots.$

b The expansion is valid for $-1 < x < 1$.

c $\approx 3.872\,986$

14 **a** $\dfrac{1}{(1-3x)^2} = 1 + 6x + 27x^2 + 108x^3 + \ldots.$ **b** 91

REVIEW SET 8B

1 **a** $x^3 - 6x^2y + 12xy^2 - 8y^3$

b $81x^4 + 216x^3 + 216x^2 + 96x + 16$

2 $\binom{6}{3}2^3 5^3 = 20\,000$

3 **a** $\binom{6}{4}2^2(-1)^4$ **b** $\binom{9}{3}(-6)^3$ **c** 0

4 **a** $792 - 560\sqrt{2}$ **b** $8x^4 + 36x^3 + 42x^2 + 19x + 3$

5 **a** $(-4)^{10}$ **b** $3\binom{6}{4}5^2 2^4 - 5\binom{6}{5}2^5$

6 **a** $\binom{9}{2}3^7 = 78\,732$ **b** $\binom{9}{6}3^3 = 2268$

7 $8\binom{4}{3}(-2)^3 + \binom{4}{1}(-2) = -264$

8 $k = -\frac{1}{4},\ n = 16$ **9** $k = 180$ **10** $q = \pm\sqrt{\frac{3}{35}}$

11 **a** $a = \pm\frac{1}{4}$ **b** $a = \pm 2\sqrt{2}$

12 **a** $(1+2x)^{-\frac{3}{2}} = 1 - 3x + \frac{15}{2}x^2 - \frac{35}{2}x^3 + \ldots.$

b $-\frac{1}{2} < x < \frac{1}{2}$ **c** $\approx 0.866\,563$

13 $\dfrac{1+x^2}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{11}{8}x^2 - \frac{13}{16}x^3 + \ldots.$ provided $-1 < x < 1$

14 **a** $\left(1+\frac{x}{2}\right)^{\frac{1}{3}} = 1+\frac{1}{6}x-\frac{1}{36}x^2+\frac{5}{648}x^3+\ldots.$
provided $-2<x<2$

b $\left(1-\frac{x}{3}\right)^{-\frac{1}{3}} = 1+\frac{1}{9}x+\frac{2}{81}x^2+\frac{14}{2187}x^3+\ldots.$
provided $-3<x<3$

c $\left(\frac{6+3x}{6-2x}\right)^{\frac{1}{3}} = 1+\frac{5}{18}x+\frac{5}{324}x^2+\frac{265}{17\,496}x^3+\ldots.$
provided $-2<x<2$

EXERCISE 9A

1 **a** The cat is not black. **b** x is not prime.
c The tree is not deciduous.

2 **a** False, x may be -3. **b** True, $3^2=9$.
c False, $x^2=9$ does not imply that $x=3$.

3 **a** True, as the square root of any positive number is real.
b False: for example, $\sqrt{0}=0\in\mathbb{R}$ but 0 is not positive.
c False, $\sqrt{x}\in\mathbb{R} \not\Rightarrow x$ is positive.

4 **a** "If Socrates is an animal, then Socrates is a cat." **b** false

5 **a** equivalent **b** not equivalent

6 We need to turn cards D and 3. We do not need to turn cards K and 7.

EXERCISE 9B

1 **b** If $x^2-x-6=0$ then $x=-2$ *or* 3.

2 **Hint:** Use Pythagoras' theorem.

4 **Hint:** Let the middle number be x.

6 **Hint:** $(a-b)^2\geqslant 0$ for all $a, b\in\mathbb{R}$.

7 **Hint:** $\sin 2\theta = 2\sin\theta\cos\theta$

8 **Hint:** The 3-digit number "abc" has value $100a+10b+c$.

9 **a** $4x^2=3x \not\Rightarrow 4x=3$
b $(x+3)(2-x)=4 \not\Rightarrow x+3=4 \vee 2-x=4$

EXERCISE 9C

1 **a** **Hint:** $(a+b)^2-(a-b)^2$
$=[a+b+(a-b)][a+b-(a-b)]$

2 $a=14$

3 **Hint:** $(x-y)^5+(x-y)^3=(x-y)^3[(x-y)^2+1]$

4 **a** n^4+4 **b** $n=\pm 1$

5 **b** **i** 121 and 81 **ii** 676 and 576

6 **a** $(a-b)(a+b)=b(a-b) \not\Rightarrow a+b=b$

b $\frac{4x-40}{6-x}=\frac{4x-40}{13-x} \not\Rightarrow 6-x=13-x$

7 **a** $6x-12=3(x-2) \not\Rightarrow 6x-12+3(x-2)=0$,
$6x-12+3(x-2)=0 \not\Rightarrow 12x-24=0$
b $x(x-6)=3(-3) \not\Rightarrow x=3 \vee x-6=-3$

8 **b** No, $\frac{1}{n+1}+\frac{1}{n^2+n}$ is undefined for $n=0,-1$ while $\frac{1}{n}$ is only undefined for $n=0$.

EXERCISE 9D

1 **Hint:** Let $x=0.\overline{9}$, $\therefore\ 10x=9.\overline{9}=9+x$ and so on.

3 **Hint:** $u_1=4-\sqrt{2}$, $r=\frac{1}{3-\sqrt{2}}$

6 **Hint:** Let the 2 odd integers be $2a+1$ and $2b+1$, $a, b\in\mathbb{Z}$.

7 **Hint:** Let $p=2a+1$, $q=2b+1$, $a, b\in\mathbb{Z}$.

EXERCISE 9E

3 **Hint:** $n^3+2n=(n^3-n)+3n$, then use the result of **2**.

5 **Hint:** Let $n=7k$, $7k+1$, ..., $7k+6$, and show that n^2 never leaves remainder 3 when divided by 7.

7 Box **C**

8 **Hint:** Let the distance between consecutive points be 1 unit. For a given selection of 5 points, consider the distance around the circle to the next chosen point.

EXERCISE 9F

1 **a** Let $a=1$, $b=1$, $2^2\neq 1+1$
b Let $p=7$, $2p+1=15$ which is not prime.
c Let $k=31$, $6k-1=185=5\times 37$
$6k+1=187=11\times 17$

2 **a** Let $p_1=2$, $p_2=7$, then $p=2\times 7+1=15$, which is not prime.
b If $n=6$, then
$p=2\times 3\times 5\times 7\times 11\times 13+1=30\,031=59\times 509$

3 For $n=6$ there are 31 regions. $31\neq 2^5$

4 333 333 331 is not prime.

5 **a**

n	2^n-1
1	1
2	3
3	7
4	15
5	31

n	2^n-1
6	63
7	127
8	255
9	511
10	1023

$n=2, 3, 5, 7$

b $n=11$, $2^{11}-1=2047=23\times 89$

EXERCISE 9G

1 **a** If you do not get wet then it is not raining.
b If it is not Thursday then penguins cannot fly.
c If you are not aged 18 or over then you do not drink alcohol.
d If k is prime then k is not divisible by 5.
e If a and b are not even then $a^2+b^2\neq c^2$.

3 **Hint:** The contrapositive is "if a and b are rational then the product ab is rational".

4 **Hint:** The contrapositive is "if n is a perfect square then $n\equiv 0$ or $1 \bmod 4$".

EXERCISE 9H

1 **a** **Hint:** Suppose $\log_2 5=\frac{p}{q}$, where $p, q\in\mathbb{Z}$, $q\neq 0$.

b **Hint:** Suppose $\log_3 5=\frac{p}{q}$, where $p, q\in\mathbb{Z}$, $q\neq 0$.

c **Hint:** If p^2 is a multiple of 3, then p must be a multiple of 3.

d **Hint:** If p^3 is a multiple of 2, then p must be a multiple of 2.

2 **Hint:** If p^2 is a multiple of 4, then p is not necessarily a multiple of 4.

4 **Hint:** Use **Exercise 9G** question **4**.

5 No, we could find another solution by reversing the 2s and 3s. So, the solution is not unique, and the sudoku would not be well-posed.

	3	2						
	2	3						

6 No (**Hint:** Consider the first player employing the same strategy S.)

7 **Hint:** Suppose there is a largest prime p, then consider $N = 4(3 \times 7 \times 11 \times .... \times p) - 1$.

REVIEW SET 9A

3 **a** The boy does not have blue eyes.

b x is not larger than 4.

4 **a** True, by definition.

b False, the period of f may be $\frac{p}{2}$ or $\frac{p}{3}$, and so on.

c False, $f(x+p) = f(x)$ for all $x \not\Rightarrow f$ is periodic with period p.

5 **Hint:** $(3a-b)^2 \geqslant 0$ for all $a, b \in \mathbb{R}$.

6 **b** **ii** The LHS is undefined for $\theta = \dfrac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$ while the RHS is defined for all $\theta \in \mathbb{R}$.

8 **a** **i** 9, composite **ii** 35, composite

iii 91, composite **iv** 189, composite

c For all $k \in \mathbb{Z}^+$, $k^3 + (k+1)^3$ will always have factors $(2k+1)$ and (k^2+k+1) and is hence composite.

9 **Hint:** $n^3 - n = n(n+1)(n-1)$

10 **b** **Hint:** Expand $(10m+k)^5$.

11 If $p = 5$, $5! + 1 = 121$ which is not prime.

12 **Hint:** The contrapositive is "if either a or b are divisible by n then ab is divisible by n".

15 **b** **Hint:** $4n^2 - m^2 = (2n+m)(2n-m)$

REVIEW SET 9B

2 **Hint:** $(\sqrt{a} - \sqrt{b})^2 \geqslant 0$

3 **a** true **b** If $\sin x$ is positive then x is acute. **c** false

4 **a** not equivalent **b** equivalent

5 **Hint:** $k^3 + k^2 - k - 1 = (k^2-1)(k+1)$

6 **Hint:** Divide the triangle into 3 smaller triangles, then find the sum of their areas.

7 **Hint:** "ab" has value $10a + b$, "ba" has value $10b + a$.

8 $\left(3 - \frac{5}{2}\right)^2 = \left(2 - \frac{5}{2}\right)^2 \not\Rightarrow 3 - \frac{5}{2} = 2 - \frac{5}{2}$

10 10 and 15 both divide 30, but 150 does not divide 30.

11 **Hint:** Consider the cases $a < b$ and $a \geqslant b$ and the value of $|a-b|$ in each case.

12 **Hint:** The contrapositive is "if n is odd then n^3 is odd".

14 **c** **Hint:** Use your answer to part **b**.

15 **a** definitely a THOG **b** definitely not a THOG

c definitely not a THOG **d** definitely a THOG

EXERCISE 10A

1 **a** The nth term of the sequence 3, 7, 11, 15, 19, ... is $4n-1$ for $n \in \mathbb{Z}^+$.

b The nth term of the sequence 25, 22, 19, 16, ... is $28 - 3n$ for $n \in \mathbb{Z}^+$.

c The nth term of the sequence 8, 12, 18, 27, ... is $8 \times \left(\frac{3}{2}\right)^{n-1}$ for $n \in \mathbb{Z}^+$.

2 **a** $3^n > 1 + 2n$ for all $n \in \mathbb{Z}^+$, $n \geqslant 2$.

b $11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$.

c $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

d $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

3 **a** $2 + 4 + 6 + 8 + 10 + ... + 2n = n(n+1)$,

$\sum_{i=1}^{n} 2^i = n(n+1)$ for all $n \in \mathbb{Z}^+$.

b $1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + ... + n \times n! = (n+1)! - 1$,

$\sum_{i=1}^{n} i \times i! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$.

c $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$,

$\sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all $n \in \mathbb{Z}^+$.

d $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + ...$ to n terms $= \frac{n}{6n+4}$,

$\sum_{i=1}^{n} \frac{1}{(3i-1)(3i+2)} = \frac{n}{6n+4}$ for all $n \in \mathbb{Z}^+$.

e $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) .... \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$,

$\prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{Z}^+$.

f $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$,

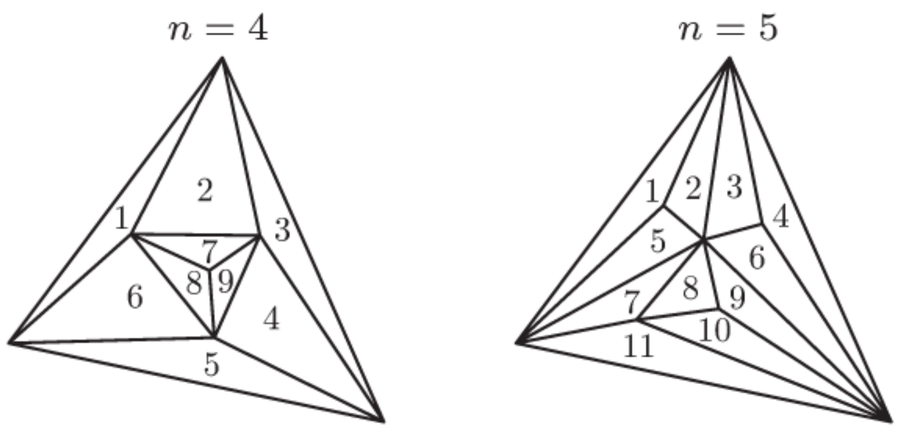
$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ for all $n \in \mathbb{Z}^+$.

g $\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right) .... \left(1 - \frac{2}{n+2}\right)$

$= \frac{2}{(n+1)(n+2)}$,

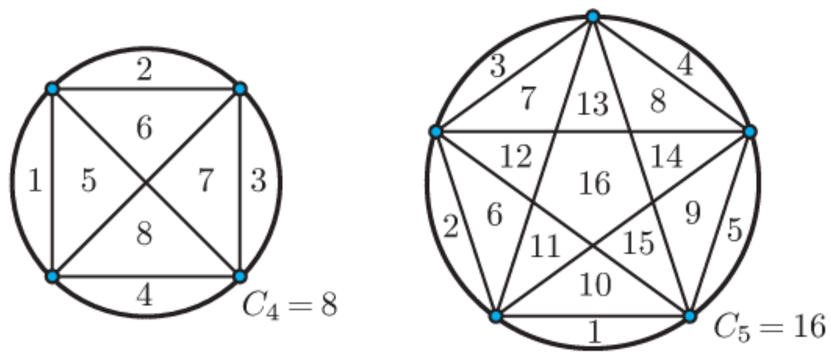
$\prod_{i=1}^{n} \left(1 - \frac{2}{i+2}\right) = \frac{2}{(n+1)(n+2)}$ for all $n \in \mathbb{Z}^+$.

4 **a**



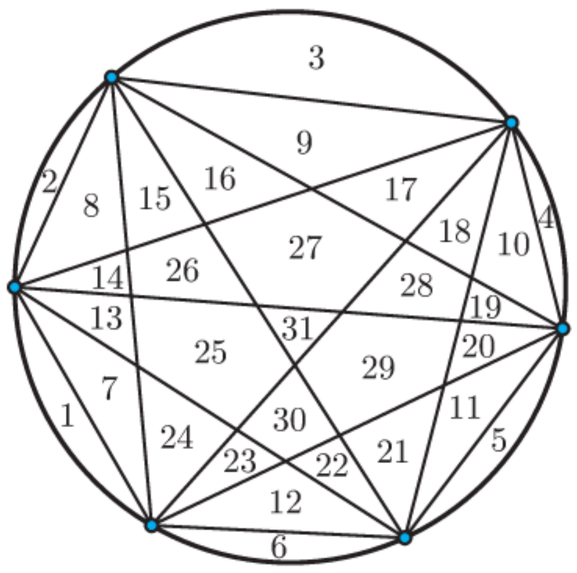
b *Proposition*: The maximum number of triangles for n points within the original triangle is given by $T_n = 2n + 1$, $n \in \mathbb{Z}^+$.

5 **a**



b *Conjecture*: The number of regions for n points placed around a circle is given by $C_n = 2^{n-1}$, $n \in \mathbb{Z}^+$.

c $n = 6$



No. By the conjecture we expect $2^5 = 32$ regions, but there are only 31.

EXERCISE 10B

11 **b** $\frac{11}{210}$ **12** **b** $\frac{7}{360}$ **13** **c** $\frac{3\,628\,799}{3\,628\,800}$

15 **a** $a = 3,\ b = 4$ **c** $\frac{11}{2}$

18 **a** *Conjecture*: $u_n = n^2$ for all $n \in \mathbb{Z}^+$ **c** $u_{20} = 400$

19 **a** *Conjecture*: $u_n = \dfrac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

c $u_{50} = \frac{50}{101}$

20 **a** $A_1 = 2,\ B_1 = 1;\quad A_2 = 7,\ B_2 = 4;$
$A_3 = 26,\ B_3 = 15;\quad A_4 = 97,\ B_4 = 56$

c For $n = 1, 2, 3, 4,\ (A_n)^2 - 3(B_n)^2 = 1$
Conjecture: $(A_n)^2 - 3(B_n)^2 = 1$ for all $n \in \mathbb{Z}^+$

22 **a** $\dfrac{1 - \sin^n x}{1 - \sin x},\quad \sin x \neq 1$

b $\dfrac{1}{1 - \sin x}$ if $-1 < \sin x < 1$,
not convergent for $\sin x = \pm 1$

c $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

23 **b** **i** $\sin 8x$ **ii** $\dfrac{\sin 20x}{2\sin x}$ **iii** $\dfrac{\sin 2nx}{2\sin x}$

24 **b** 0

29 **b** **i** $\dfrac{\sin(2^4 x)}{2^4}$ **ii** $\dfrac{\sin(2^6 x)}{2^6}$

c $\sin x \cos x \cos 2x \ldots . \cos 2^n x = \dfrac{\sin(2^{n+1} x)}{2^{n+1}}$

31 **f** **Hint:** Show that P_1 and P_2 are true. Then consider the cases where k is odd or even to prove that if P_{k-1} and P_k are true, P_{k+1} is true.

REVIEW SET 10A

7 **a** $u_2 = 9,\ u_3 = 27,\ u_4 = 81$

Conjecture: $u_n = 3^n,\ n \in \mathbb{Z},\ n \geqslant 0$

EXERCISE 11A

1 **a** It contains a product of two variables, xy.

b It contains the squared term $-2x_3^2$.

c It contains the square root term $-\sqrt{y}$.

2 **a** $x = \dfrac{t+3}{8},\ y = t,$ where $t \in \mathbb{R}$

b $x_1 = 2s - t + 10,\ x_2 = s,\ x_3 = t,$ where $s, t \in \mathbb{R}$

c $x = -2 - s + 2t,\ y = s,\ z = t,$ where $s, t \in \mathbb{R}$

3 **a** inconsistent; $x + y$ cannot be equal to both 2 *and* 3 simultaneously.

b consistent; $x = \frac{1}{2},\ y = 0$ is a solution.

c consistent; $x = 1,\ y = 1,\ z = 3$ is a solution.

4 **a** $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & -1 \\ -1 & 1 & 2 \end{array}\right)$, overspecified

b $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & 1 & 2 & -4 \end{array}\right)$, underspecified

c $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 8 \\ 2 & 1 & -3 & 0 \end{array}\right)$, neither underspecified nor overspecified

5 **a** $\begin{cases} 2x - y + 3z = 7 \\ x + 3y - z = 5 \end{cases}$ **b** $\begin{cases} 3x + y - 2z = -3 \\ -y + 2z = 8 \\ 2x - z = 2 \end{cases}$

c $\begin{cases} 4x - y + 2z = -3 \\ -x + 3y + 3z = 2 \\ 2x + y - 2z = 5 \end{cases}$

EXERCISE 11B

1 **a** $\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 2 & 1 & -2 \end{array}\right)$

b **i** $\sim \left(\begin{array}{cc|c} 2 & -6 & 6 \\ 2 & 1 & -2 \end{array}\right)$ **ii** $\sim \left(\begin{array}{cc|c} 0 & -7 & 8 \\ 2 & 1 & -2 \end{array}\right)$

iii $\sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 7 & 0 & -3 \end{array}\right)$

2 **a** $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 3 & a \end{array}\right)$ **b** $\sim \left(\begin{array}{cc|c} 3 & 3 & 12 \\ 3 & 3 & a \end{array}\right)$

c $a = 12$

3 **a** $\left(\begin{array}{cc|c} 2 & -1 & 5 \\ -1 & a & 3 \end{array}\right)$ **b** $\sim \left(\begin{array}{cc|c} 2 & -1 & 5 \\ 2 & -2a & -6 \end{array}\right)$

c $a = \frac{1}{2}$

4 **a** $\left(\begin{array}{ccc|c} 1 & 1 & 1 & p \\ 1 & 0 & 2 & q \\ 2 & 1 & 3 & r \end{array}\right)$ **b** $\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & p \\ 2 & 1 & 3 & p+q \\ 2 & 1 & 3 & r \end{array}\right)$

c $p + q = r$

EXERCISE 11C

1 **a** $\left(\begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 1 & -3 \end{array}\right)$ **b** $\sim \left(\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 7 & -7 \end{array}\right)$

c $x = -1,\ y = -1$

2 **a** The lines are coincident, the system has infinitely many solutions.

b The lines are neither coincident nor parallel, the system has a unique solution.

c The lines are parallel, the system has no solutions.

d If $a = 4,$ the lines are coincident, the system has infinitely many solutions.
If $a \neq 4,\ a \in \mathbb{R},$ the lines are parallel, the system has no solutions.

3 **a** $x = 1,\ y = 3$ **b** $x = 4,\ y = -3$

c $x = -1,\ y = -2$ **d** $x = 2,\ y = -3$

4 **a** The equations represent coincident lines, which meet at infinitely many points.
$\therefore$ the system has infinitely many solutions.

b $\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 6 & 8 \end{array}\right)$
$\sim \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right)$ $R_2 - 2R_1 \to R_2$

The second equation is a multiple of the first, so we obtain a row of zeros when we use row operations. The second equation adds nothing to aid the solution of the system.

c $x = 4 - 3t,\ y = t,$ where $t \in \mathbb{R}$

d $x = s,\ y = \dfrac{4-s}{3},$ where $s \in \mathbb{R}$

e **Hint:** Let $\left(s', \dfrac{4-s'}{3}\right)$ be a point in the second solution set.

5 **a**
$$\left(\begin{array}{cc|c} 1 & -5 & 8 \\ 2 & -10 & a \end{array}\right)$$
$$\sim \left(\begin{array}{cc|c} 1 & -5 & 8 \\ 0 & 0 & a-16 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

b When $a \neq 16$, the second row gives $0x + 0y \neq 0$ which is not possible.
$\therefore$ there are no solutions.

c $x = 5t + 8,\ y = t,$ where $t \in \mathbb{R}$

6 • If $a \neq 6$, the unique solution is $x = \dfrac{4a-3b}{a-6},\ y = \dfrac{b-8}{a-6}$.

• If $a = 6,\ b = 8,$ the lines are coincident and infinitely many solutions exist.
The solution set is $x = 4 - 3t,\ y = t,\ t \in \mathbb{R}$.

• If $a = 6,\ b \neq 8,$ the lines are parallel and the system has no solutions.

7 **a** $c = 2$ **b** $a + b = -1,\ 2a + b = 2$

c $a = 3,\ b = -4,\ y = 3x^2 - 4x + 2$

8 **a**
$$\left(\begin{array}{cc|c} a & b & m \\ c & d & n \end{array}\right)$$
$$\sim \left(\begin{array}{cc|c} a & b & m \\ 0 & bc-ad & cm-an \end{array}\right) \quad cR_1 - aR_2 \to R_2$$

b For a unique solution, the left hand side of the second row must not all be zero.
$\therefore\ ad \neq bc$

c For infinitely many solutions, the second row must be a row of zeros.
$\therefore\ bc - ad = 0$ and $cm - an = 0$

EXERCISE 11D

1 **a** $x = 4,\ y = -2,\ z = 1$ **b** $x = 4,\ y = -3,\ z = 2$

c no solutions

d $x = 1 - \frac{11}{3}t,\ y = \frac{2}{3}t,\ z = t,$ where $t \in \mathbb{R}$

e no solutions **f** $x = 4,\ y = -2,\ z = 1$

2 **a** $y = x^2 + 3x - 6$ **b** $y = -2x^2 + 4x + 9$

c $y = \frac{1}{2}x^2 + 3x - 4$

3 We find that $a = 0,\ b = -2,$ and $c = 5$. So the three points lie on the line $y = -2x + 5,$ but not on a quadratic.

4 **a** $P(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1,\ n \geqslant 0$ **b** 79 pieces

5 **a**
$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 0 & k-8 \end{array}\right)$$
b $k \neq 8$

c The system has infinitely many solutions if the last row is all zeros. This occurs when $k = 8$.
The solutions have the form $x = 1 - \frac{9}{5}t,\ y = \frac{2}{5}t + 1,$ $z = t,$ where $t \in \mathbb{R}$.

d In row echelon form, row 3 reads $0x + 0y + 0z = k - 8$.
From **b** and **c** the system has no solutions if $k \neq 8$ or infinitely many solutions if $k = 8$.
$\therefore$ the system never has a unique solution.

6 **a**
$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & 0 & k-13 & -k+13 \end{array}\right)$$

b The system has infinitely many solutions if the last row is all zeros. This occurs when $k = 13$.
The solutions have the form $x = 1 - \frac{4}{3}t,\ y = \frac{5}{3}t + 2,$ $z = t,$ where $t \in \mathbb{R}$.

c $x = \frac{7}{3},\ y = \frac{1}{3},\ z = -1$ for all $k \neq 13,\ k \in \mathbb{R}$

7 **a**
$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right)$$

b The system has infinitely many solutions if the last row is all zeros. This occurs when $a = -1$.
The solutions have the form $x = \dfrac{19-6t}{7},\ y = \dfrac{-5t-11}{7},$ $z = t,$ where $t \in \mathbb{R}$.

c $x = \frac{1}{7}a + 2,\ y = \frac{2}{7}a - 2,\ z = 1,\ a \neq -1,\ a \in \mathbb{R}$

8 **a**
$$\sim \left(\begin{array}{ccc|c} 1 & 4 & m & -m \\ 0 & -4m & 1-m^2-m & m^2+m+1 \\ 0 & 0 & -(m-3)(m-1) & (m-3)(m+1) \end{array}\right)$$

b If $m = 1,$ the last row gives $0x + 0y + 0z = -4$ which is not possible.

c $x = \dfrac{4+2t}{3},\quad y = \dfrac{-13-11t}{12},\quad z = t,\quad t \in \mathbb{R}$

d $x = 0,\quad y = \dfrac{m}{2(m-1)},\quad z = \dfrac{m+1}{1-m}$

REVIEW SET 11A

1 **a** $x = \dfrac{-2-t}{5},\ y = t,$ where $t \in \mathbb{R}$

b $x = 1 + 4s - 3t,\ y = s,\ z = t,$ where $s, t \in \mathbb{R}$

2 **a**
$$\left(\begin{array}{cc|c} 3 & -2 & 4 \\ 1 & 5 & -8 \end{array}\right),$$
neither underspecified nor overspecified

b
$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & -1 \\ 3 & -2 & 0 & 6 \end{array}\right),$$
underspecified

c
$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 5 & -1 & 2 \\ -3 & 1 & -7 \end{array}\right),$$
overspecified

3 **a**
$$\left(\begin{array}{cc|c} 4 & -6 & -1 \\ a & 2 & 3 \end{array}\right)$$
b
$$\sim \left(\begin{array}{cc|c} 4 & -6 & -1 \\ -3a & -6 & -3 \end{array}\right)$$

c $a = -\frac{4}{3}$

4 **a** $x = 7,\ y = -2$ **b** $x = -1,\ y = 4$

5 **a** $k \neq -15,\ k \in \mathbb{R}$
The unique solution is $x = \frac{3}{2},\ y = 0$.

b If $k = -15,$ there are infinitely many solutions of the form
$x = \dfrac{3+5t}{2},\ y = t,$ where $t \in \mathbb{R}$.

6 **a** $x = 1 + 2t,\ y = t,\ z = 0,\ t \in \mathbb{R}$

b $x = 4,\ y = -2,\ z = 1$

c $x = t,\ y = \frac{1}{3} - 2t,\ z = t,$ where $t \in \mathbb{R}$

7 **a**
$$\begin{cases} -3a + b + 13d = 39 \\ b - 2d = 0 \\ 3a + 3b - d = 1 \end{cases}$$
b $y = \dfrac{-3x+4}{x+2}$

8 **a**
$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & 7 & -5 & 19 \\ 0 & 0 & k-5 & 28 \end{array}\right)$$

b When $k = 5,$ the system is inconsistent and therefore has no solutions.

c **i** $z = \dfrac{28}{k-5}$, $y = \dfrac{19}{7} + \dfrac{20}{k-5}$, $x = \dfrac{8}{7} + \dfrac{4}{k-5}$,
$k \in \mathbb{R},\ k \neq 5$

ii $x = -\frac{6}{7},\ y = -\frac{51}{7},\ z = -14$

REVIEW SET 11B

1 **a** consistent; $x = \frac{2}{3},\ y = 0$ is a solution

b inconsistent; the lines represented by each equation are parallel.

2 **a** $\begin{cases} 5x - y + 4z = 3 \\ x \quad\quad - 2z = -1 \end{cases}$ **b** $\begin{cases} 2x \quad\quad - 3z = -2 \\ -x + 6y \quad\quad = 3 \\ 3x + 3y - 4z = -7 \end{cases}$

3 **a** $\left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 5 & 1 & -1 & b \\ 2 & 4 & -10 & c \end{array}\right)$

b $\sim \left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 4 & -10 & b-3a \\ 2 & 4 & -10 & c \end{array}\right)$ **c** $b - 3a = c$

4 **a** $a = 2;\quad x = -\left(\dfrac{t+1}{4}\right),\ y = t,$ where $t \in \mathbb{R}$

b When $a \neq 2$, the system is inconsistent and therefore has no solutions.

5 • If $a = -8,\ b = 20$, there are infinitely many solutions of the form $x = 5 + 2t,\ y = t,$ where $t \in \mathbb{R}$.
• If $a = -8,\ b \neq -20$, there are no solutions.
• If $a \neq -8$, the system has the unique solution
$x = 5 + 2\left(\dfrac{b-20}{a+8}\right),\ y = \dfrac{b-20}{a+8}$.

6 **a** $x = 4,\ y = -3,\ z = 2$

b no solution, system is inconsistent

7 **a** $\begin{cases} -2a + 4b + c = -20 \\ a + 3b + c = -10 \end{cases}$
$\therefore\ a = 2 - t,\ b = -4 - 3t,\ c = 10t,$ where $t \in \mathbb{R}$.

b There are three unknowns and only two equations. The system is underspecified.

c **i** $q = 2$; a circle passing through the points $(-2,\ 4)$, $(1,\ 3)$, and $(4,\ 2)$ cannot be constructed because the points are collinear.

ii $x^2 + y^2 - 2x - 16y + 40 = 0$

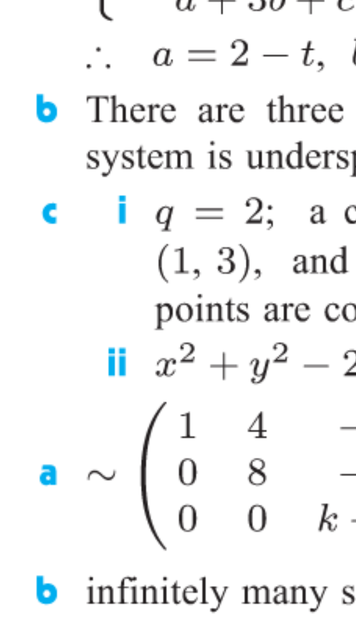
8 **a** $\sim \left(\begin{array}{ccc|c} 1 & 4 & -1 & k \\ 0 & 8 & -1 & 3+2k \\ 0 & 0 & k-3 & 3-k \end{array}\right)$

b infinitely many solutions when $k = 3$:
$x = \dfrac{t-3}{2},\ y = \dfrac{t+9}{8},\ z = t,$ where $t \in \mathbb{R}$

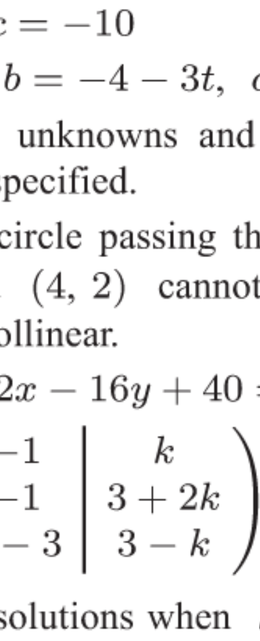
c $x = -2,\ y = \dfrac{k+1}{4},\ z = -1$

EXERCISE 12A.1

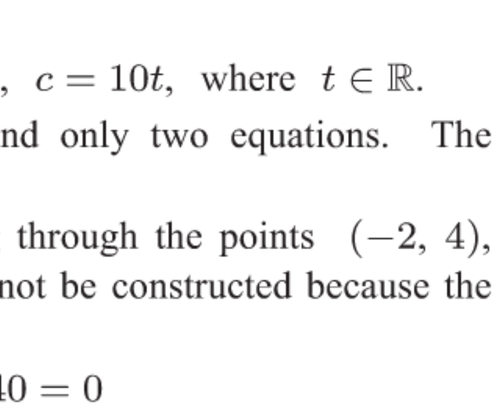
1 **a**



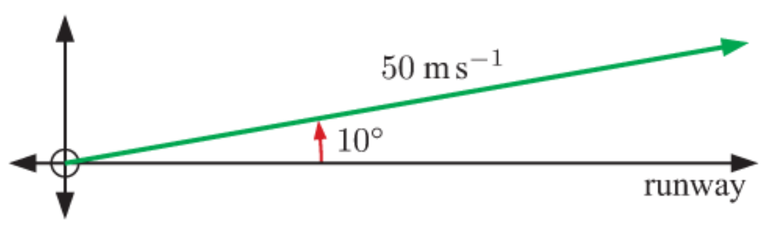
b



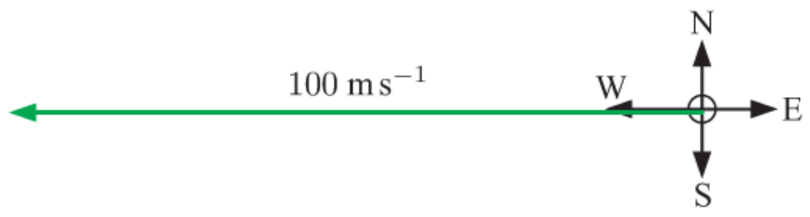
c



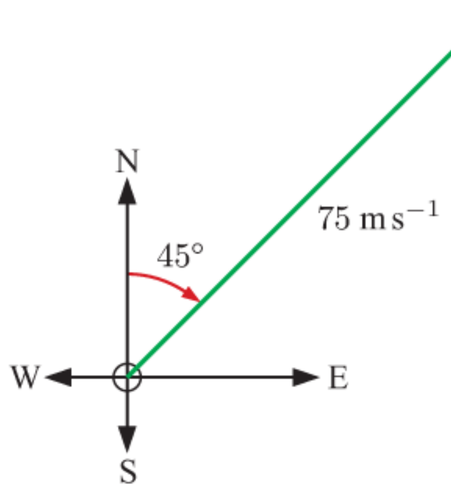
d



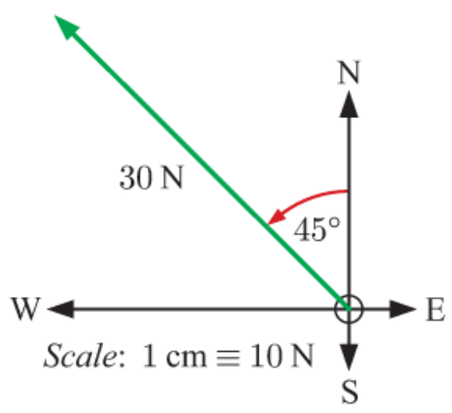
2 **a**



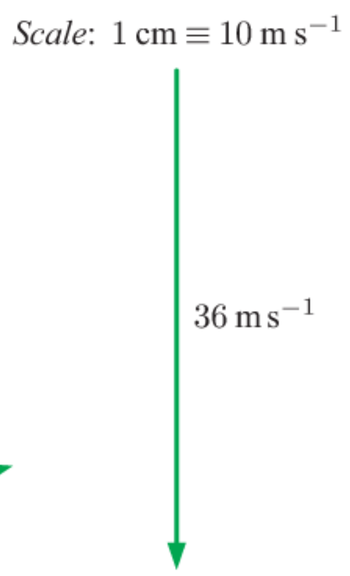
b



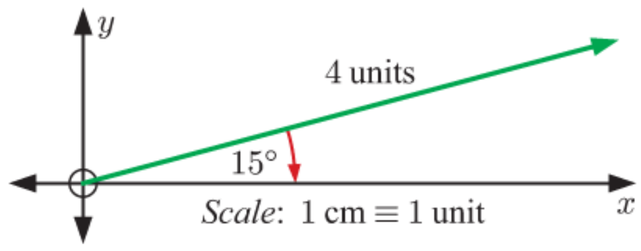
3 **a**



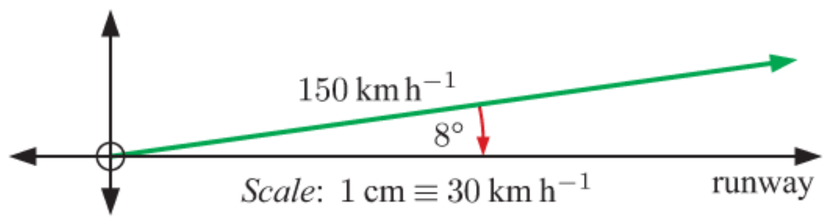
b



c



d



EXERCISE 12A.2

1 **a** **p**, **q**, **s**, **t** **b** **p**, **q**, **r**, **t** **c** **p** and **r**, **q** and **t**
d **q**, **t** **e** **p** and **q**, **p** and **t**

2 **a** true **b** true **c** false **d** false **e** true **f** false

3 **a** **i** $\overrightarrow{BC}$ **ii** $\overrightarrow{ED}$

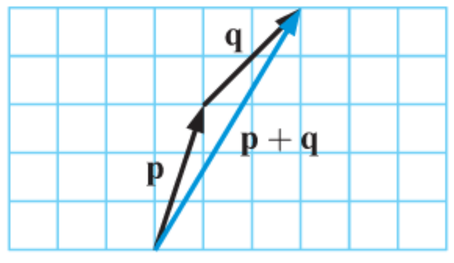
b **i** $\overrightarrow{FE}$, $\overrightarrow{BC}$

ii $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$, $\overrightarrow{AF}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{DC}$

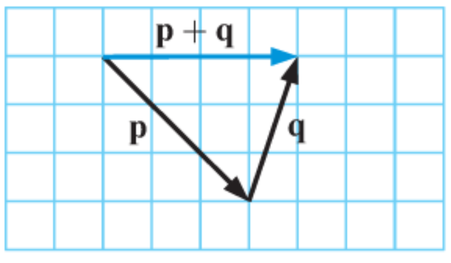
c $\overrightarrow{FC}$ (or $\overrightarrow{CF}$)

EXERCISE 12B.1

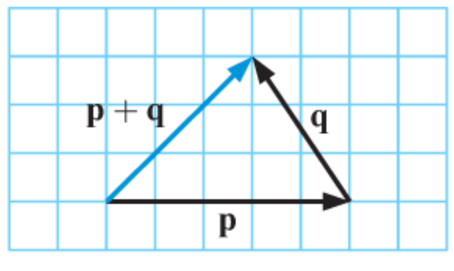
1 **a**



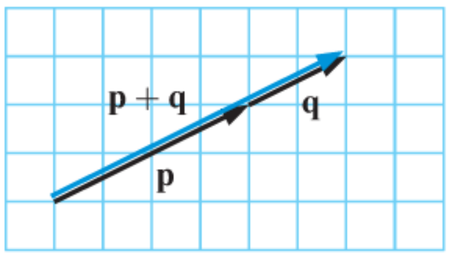
b



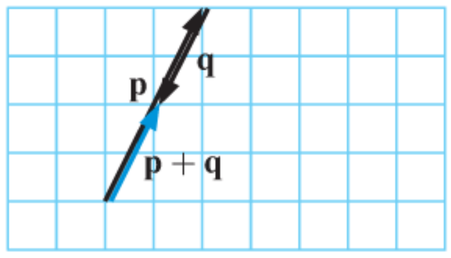
c



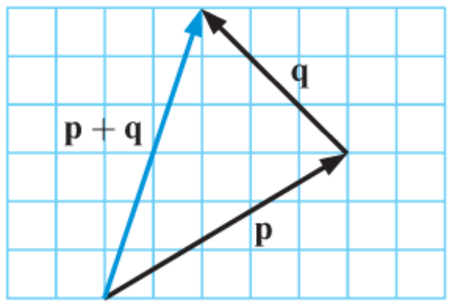
d



e

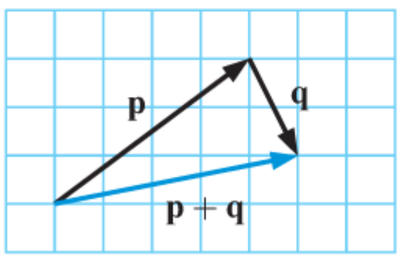


f

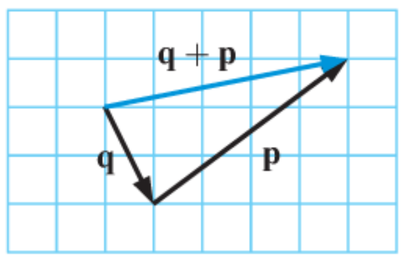


2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{BD}$ **c** 0 **d** $\overrightarrow{AD}$ **e** $\overrightarrow{AD}$ **f** 0

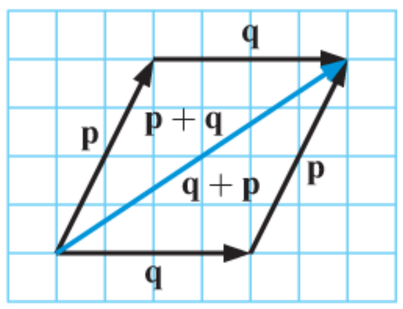
3 **a** **i**



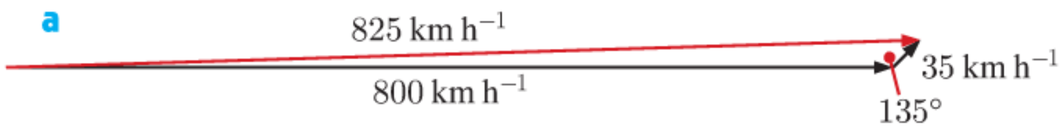
ii



b yes



5 **a**



Scale: $1\text{ cm} \equiv 125\text{ km h}^{-1}$

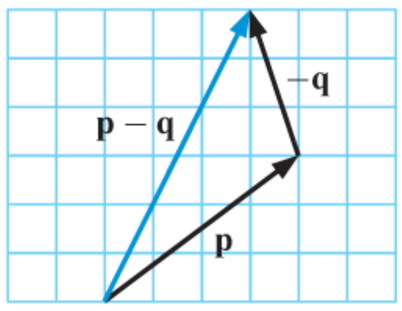
b We use vector addition.

c $\approx 825\text{ km h}^{-1}$, 88° east of north

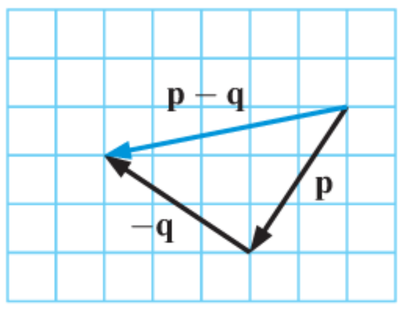
6 **a, b** ≈ 70.0 N, in the direction $\approx 357°$

EXERCISE 12B.2

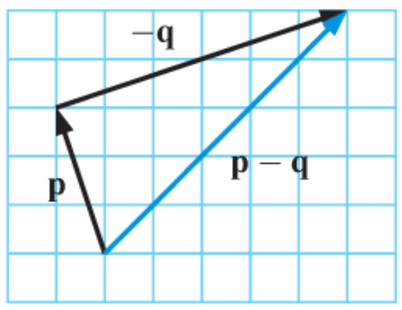
1 **a**



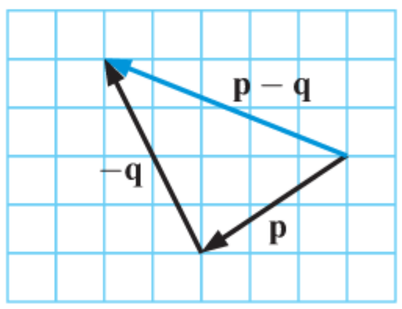
b



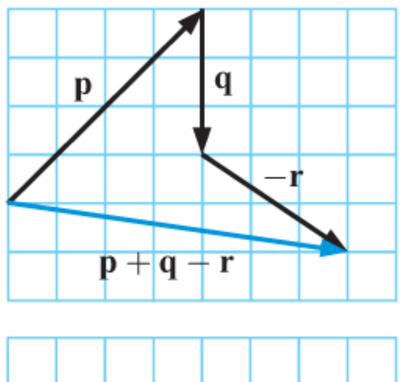
c



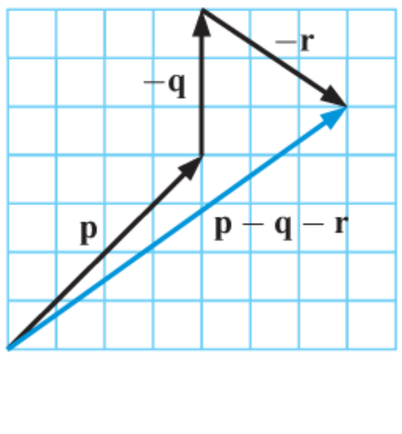
d



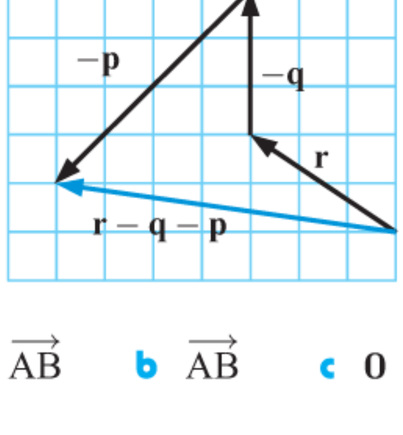
2 **a**



b



c



3 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{AB}$ **c** 0 **d** $\overrightarrow{AD}$ **e** 0 **f** $\overrightarrow{AD}$

EXERCISE 12B.3

1 **Note:** Other answers are possible.

a $\mathbf{t} = \mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} = -\mathbf{s} - \mathbf{t}$

c $\mathbf{r} = -\mathbf{p} - \mathbf{q} - \mathbf{s}$ **d** $\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$

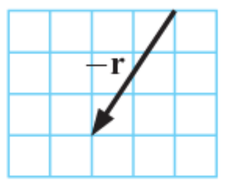
e $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$ **f** $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$

2 **a** **i** $\mathbf{r} + \mathbf{s}$ **ii** $-\mathbf{t} - \mathbf{s}$ **iii** $\mathbf{r} + \mathbf{s} + \mathbf{t}$

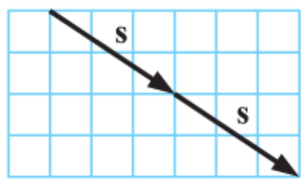
b **i** $\mathbf{p} + \mathbf{q}$ **ii** $\mathbf{q} + \mathbf{r}$ **iii** $\mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 12B.4

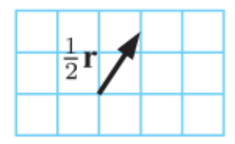
1 **a**



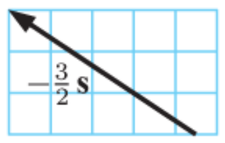
b



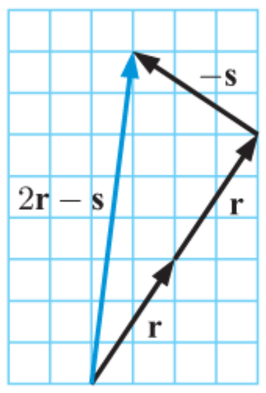
c



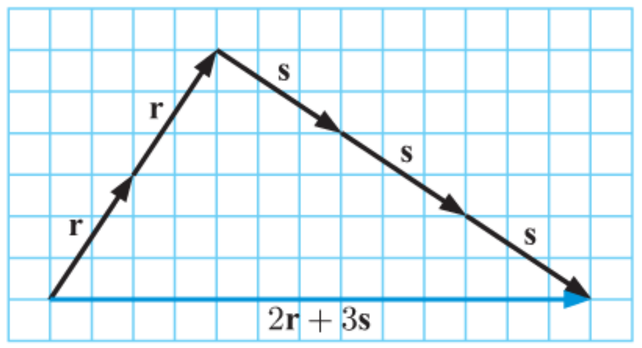
d



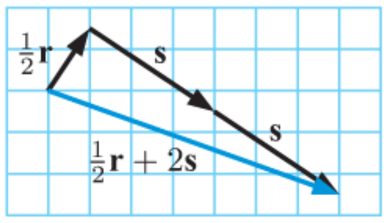
e



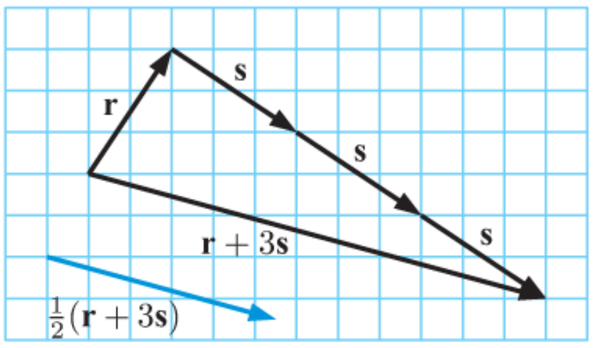
f



g



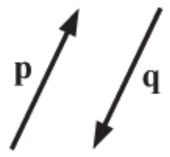
h



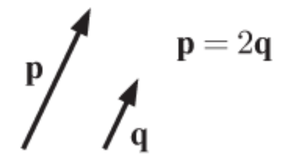
2 **a**



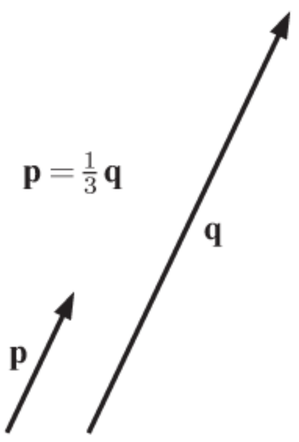
b



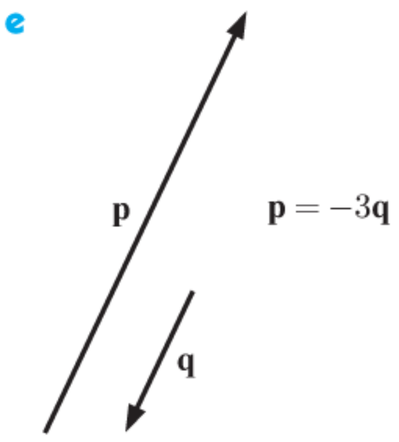
c



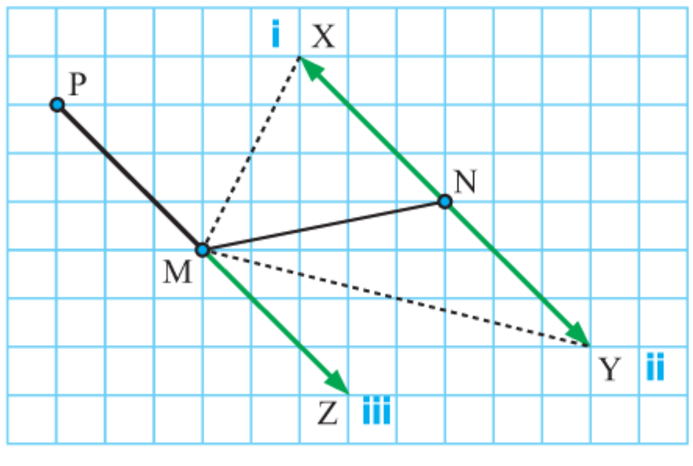
d



e



3 **a**



b a parallelogram

4 **a** $-\mathbf{p}$ **b** $\mathbf{p}+\mathbf{q}$ **c** $\frac{1}{2}(\mathbf{p}+\mathbf{q})$ **d** $\frac{1}{2}(\mathbf{q}-\mathbf{p})$

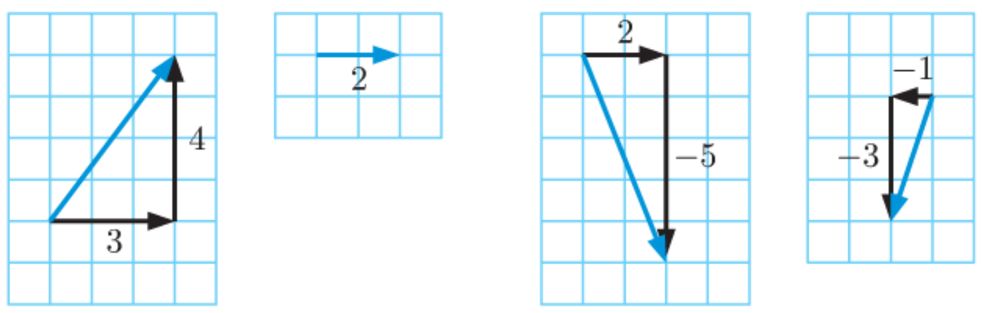
5 **a** $\mathbf{b}$ **b** $2\mathbf{b}$ **c** $\mathbf{b}-\mathbf{a}$ **d** $\mathbf{b}-\mathbf{a}$

6 **a** $\overrightarrow{BD}=\frac{1}{2}\mathbf{a}$ **b** $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ **c** $\overrightarrow{BA}=-\mathbf{b}+\mathbf{a}$
d $\overrightarrow{OD}=\mathbf{b}+\frac{1}{2}\mathbf{a}$ **e** $\overrightarrow{AD}=\mathbf{b}-\frac{1}{2}\mathbf{a}$ **f** $\overrightarrow{DA}=\frac{1}{2}\mathbf{a}-\mathbf{b}$

EXERCISE 12C

1 **a** $\begin{pmatrix}7\\3\end{pmatrix}$, $7\mathbf{i}+3\mathbf{j}$ **b** $\begin{pmatrix}-6\\0\end{pmatrix}$, $-6\mathbf{i}$
c $\begin{pmatrix}2\\-5\end{pmatrix}$, $2\mathbf{i}-5\mathbf{j}$ **d** $\begin{pmatrix}0\\6\end{pmatrix}$, $6\mathbf{j}$
e $\begin{pmatrix}-6\\3\end{pmatrix}$, $-6\mathbf{i}+3\mathbf{j}$ **f** $\begin{pmatrix}-5\\-5\end{pmatrix}$, $-5\mathbf{i}-5\mathbf{j}$

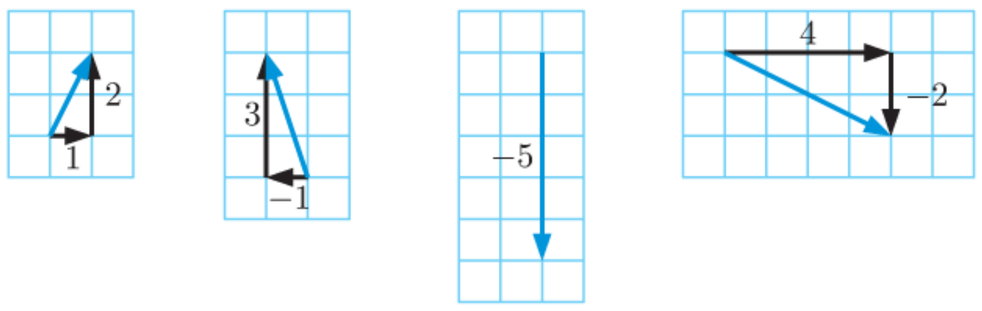
2 **a** $3\mathbf{i}+4\mathbf{j}$ **b** $2\mathbf{i}$ **c** $2\mathbf{i}-5\mathbf{j}$ **d** $-\mathbf{i}-3\mathbf{j}$



3 **a** **i** $\begin{pmatrix}-4\\-1\end{pmatrix}$, $-4\mathbf{i}-\mathbf{j}$ **ii** $\begin{pmatrix}-1\\-5\end{pmatrix}$, $-\mathbf{i}-5\mathbf{j}$
iii $\begin{pmatrix}2\\0\end{pmatrix}$, $2\mathbf{i}$ **iv** $\begin{pmatrix}3\\-4\end{pmatrix}$, $3\mathbf{i}-4\mathbf{j}$
v $\begin{pmatrix}-3\\4\end{pmatrix}$, $-3\mathbf{i}+4\mathbf{j}$ **vi** $\begin{pmatrix}3\\5\end{pmatrix}$, $3\mathbf{i}+5\mathbf{j}$
vii $\begin{pmatrix}5\\-3\end{pmatrix}$, $5\mathbf{i}-3\mathbf{j}$ **viii** $\begin{pmatrix}2\\1\end{pmatrix}$, $2\mathbf{i}+\mathbf{j}$
ix $\begin{pmatrix}-4\\-1\end{pmatrix}$, $-4\mathbf{i}-\mathbf{j}$

b $\overrightarrow{BA}=\overrightarrow{ED}$, $\overrightarrow{AB}=\overrightarrow{DE}$, $\overrightarrow{AD}=\overrightarrow{BE}$, $\overrightarrow{DA}=\overrightarrow{EB}$

4 **a** $\begin{pmatrix}1\\2\end{pmatrix}$ **b** $\begin{pmatrix}-1\\3\end{pmatrix}$ **c** $\begin{pmatrix}0\\-5\end{pmatrix}$ **d** $\begin{pmatrix}4\\-2\end{pmatrix}$



5 $\begin{pmatrix}0\\0\end{pmatrix}$

6 **a** $a=3,\ b=5$ **b** $a=4,\ b=6$ **c** $x=4,\ y=1$

EXERCISE 12D

1 **a** 5 units **b** 5 units **c** 2 units
d $\sqrt{8}$ units **e** 3 units

2 **a** $\sqrt{2}$ units **b** 13 units **c** $\sqrt{17}$ units
d 3 units **e** $|k|$ units

3 **a** unit vector **b** unit vector **c** not a unit vector
d unit vector **e** not a unit vector

4 **a** $k=\pm1$ **b** $k=\pm1$ **c** $k=0$
d $k=\pm\frac{1}{\sqrt{2}}$ **e** $k=\pm\frac{\sqrt{3}}{2}$

5 $p=\pm3$

EXERCISE 12E

1 **a** $\begin{pmatrix}-2\\6\end{pmatrix}$ **b** $\begin{pmatrix}-2\\6\end{pmatrix}$ **c** $\begin{pmatrix}-1\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-1\\-1\end{pmatrix}$
e $\begin{pmatrix}-5\\-3\end{pmatrix}$ **f** $\begin{pmatrix}-5\\-3\end{pmatrix}$ **g** $\begin{pmatrix}-6\\4\end{pmatrix}$ **h** $\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\begin{pmatrix}-3\\7\end{pmatrix}$ **b** $\begin{pmatrix}-4\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-8\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-6\\9\end{pmatrix}$
e $\begin{pmatrix}0\\-5\end{pmatrix}$ **f** $\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\mathbf{a}+\mathbf{0}=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}a_1+0\\a_2+0\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}=\mathbf{a}$

b $\mathbf{a}-\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}a_1-a_1\\a_2-a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

4 **a** $\begin{pmatrix}-3\\-15\end{pmatrix}$ **b** $\begin{pmatrix}-1\\2\end{pmatrix}$ **c** $\begin{pmatrix}0\\14\end{pmatrix}$ **d** $\begin{pmatrix}5\\-3\end{pmatrix}$
e $\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$ **f** $\begin{pmatrix}-7\\7\end{pmatrix}$ **g** $\begin{pmatrix}5\\11\end{pmatrix}$ **h** $\begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

5 **a** **i** $\begin{pmatrix}8\\-1\end{pmatrix}$ **ii** $\begin{pmatrix}8\\-1\end{pmatrix}$ **iii** $\begin{pmatrix}8\\-1\end{pmatrix}$

b The vector expressions are equal, as each consists of the sum of 2 **p**s and 3 **q**s. Each expression is equal to $2\mathbf{p}+3\mathbf{q}$.

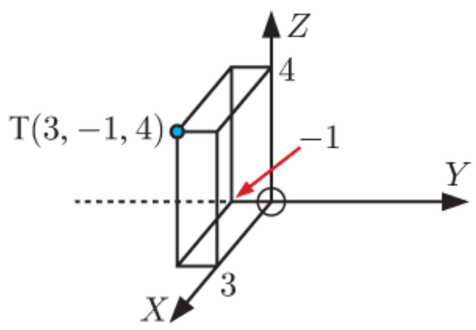
6 **a** $k=6$ **b** $k=-\frac{7}{5}$

7 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units
d $\sqrt{10}$ units **e** $\sqrt{29}$ units **f** $2\sqrt{13}$ units
g 11 units **h** $\sqrt{29}$ units

8 **a** **i** $\sqrt{10}$ units **ii** $2\sqrt{10}$ units **iii** $2\sqrt{10}$ units
iv $3\sqrt{10}$ units **v** $3\sqrt{10}$ units

EXERCISE 12F

1 **a**



b $\overrightarrow{OT}=\begin{pmatrix}3\\-1\\4\end{pmatrix}$

c $OT=\sqrt{26}$ units

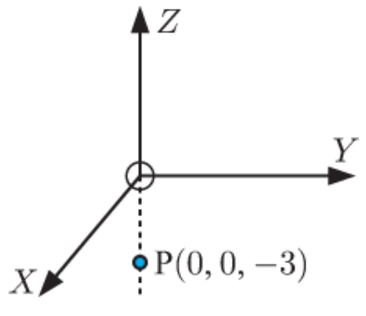
2 **a** $\overrightarrow{OP}=\begin{pmatrix}2\\0\\4\end{pmatrix}=2\mathbf{i}+4\mathbf{k}$

b $\overrightarrow{OP}=\begin{pmatrix}-3\\1\\2\end{pmatrix}=-3\mathbf{i}+\mathbf{j}+2\mathbf{k}$

c $\overrightarrow{OP}=\begin{pmatrix}0\\-2\\0\end{pmatrix}=-2\mathbf{j}$

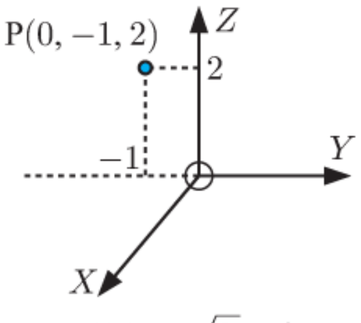
3 **a** 3 units **b** 5 units **c** $\sqrt{21}$ units **d** 3 units

4 **a**



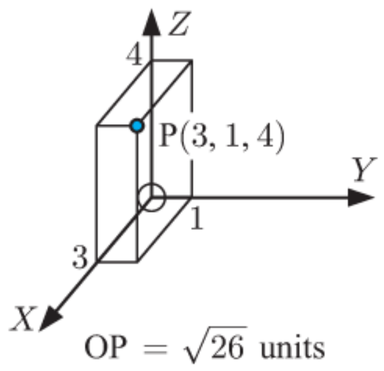
OP = 3 units

b



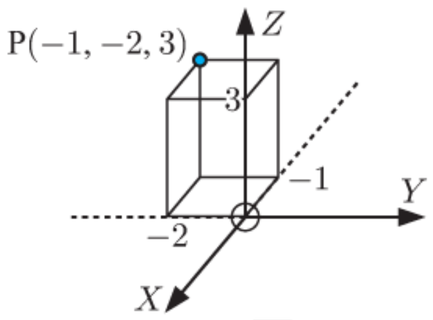
OP = $\sqrt{5}$ units

c



OP $= \sqrt{26}$ units

d



OP $= \sqrt{14}$ units

5 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $|\mathbf{a}| = \sqrt{21}$ units

b $\mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$, $|\mathbf{b}| = 5$ units

c $\mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $|\mathbf{c}| = \sqrt{13}$ units

6 **a** $a = 5,\ b = 6,\ c = -6$ **b** $a = 4,\ b = 2,\ c = 1$

c $a = 2,\ b = \frac{16}{5},\ c = \frac{26}{5}$

7 **a**, **c**, and **d** are unit vectors.

8 **a** $k = \pm\frac{\sqrt{11}}{4}$ **b** $k = \pm\frac{2}{3}$ **c** $k = \pm\frac{1}{\sqrt{3}}$

d $k = \pm\sqrt{14}$

9 **a** $m = \pm 3$ **b** $m = \pm\sqrt{11}$

10 $5 \leqslant \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| \leqslant \sqrt{34}$

EXERCISE 12G

1 **a** $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$

d $\begin{pmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{pmatrix}$ **e** $\begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ **f** $\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

2 **a** $\sqrt{10}$ **b** $\sqrt{6}$ **c** $2\sqrt{10}$ **d** $2\sqrt{10}$

e $-3\sqrt{6}$ **f** $3\sqrt{6}$ **g** $3\sqrt{2}$ **h** $\sqrt{14}$

3 **a** $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -8 \\ 8 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ -10 \\ 3 \end{pmatrix}$ **d** $\begin{pmatrix} -14 \\ 24 \\ 9 \end{pmatrix}$

e $\sqrt{38}$ **f** $\sqrt{3}$ **g** $\begin{pmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{pmatrix}$ **h** $\begin{pmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}$

4 **a** $a = \frac{1}{3},\ b = 2,\ c = 1$ **b** $a = 1,\ b = -1,\ c = 2$

c $a = 4,\ b = -1,\ c = -19$

d $a = 0,\ b = -1,\ c = -1$ *or* $a = -\frac{1}{2},\ b = 0,\ c = -1$

EXERCISE 12H

1 **a** $\mathbf{x} = \frac{1}{2}\mathbf{q}$ **b** $\mathbf{x} = 2\mathbf{n}$ **c** $\mathbf{x} = -\frac{1}{3}\mathbf{p}$

d $\mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$ **e** $\mathbf{x} = \frac{1}{5}(4\mathbf{s} - \mathbf{t})$ **f** $\mathbf{x} = 12\mathbf{m} - 3\mathbf{n}$

2 **a** $\mathbf{x} = \begin{pmatrix} 1 \\ -\frac{5}{2} \\ 2 \end{pmatrix}$ **b** $\mathbf{x} = \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix}$ **c** $\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ -1 \end{pmatrix}$

3 **a** $\mathbf{x} = \begin{pmatrix} 4 \\ -6 \\ -5 \end{pmatrix}$ **b** $\mathbf{x} = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{pmatrix}$ **c** $\mathbf{x} = \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{pmatrix}$

EXERCISE 12I

1 **a** $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

2 **a** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ **c** $\sqrt{17}$ units

3 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$ **c** $\overrightarrow{BC} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

4 **a** $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

5 **a** M(1, 4) **b** $\overrightarrow{CA} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $\overrightarrow{CM} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

6 **a** $\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ **d** $\begin{pmatrix} -5 \\ -4 \\ -2 \end{pmatrix}$

7 $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$, AB $= \sqrt{29}$ units

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{26}$ units, $|\overrightarrow{BA}| = \sqrt{26}$ units

9 **a** $\overrightarrow{NM} = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$ **b** $\overrightarrow{MN} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

c MN $= \sqrt{42}$ units

10 **a** $\overrightarrow{AB} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ **b** $\sqrt{50}$ units

11 **a** $\sqrt{13}$ units **b** $\sqrt{14}$ units **c** 3 units

12 $\overrightarrow{AC} = -\mathbf{i} - 2\mathbf{k}$

13 **a** $\begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

14 **a** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$, $|\overrightarrow{AB}| = \sqrt{16 + (k-3)^2}$ units

b $k = 0$ or 6

15 **a** A$(0, y, 0)$ **b** $(0, 2, 0)$ and $(0, -4, 0)$

16 **a** B(4, 2) **b** C(2, 2) **17** **a** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **b** Q(3, 3)

18 **a** $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ **c** D$(-1, -2)$

20 C$(5, 1, -8)$, D$(8, -1, -13)$, E$(11, -3, -18)$

21 B$(0, 3, 5)$, radius $= \sqrt{3}$ units

22 **a** $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$, AB $= \sqrt{17}$ units

b $\overrightarrow{AC} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$, AC $= \sqrt{30}$ units

c $\overrightarrow{CB} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$, $CB = \sqrt{35}$ units

d ABC is scalene, and not right angled.

24 **a** right angled **b** straight line (not a triangle)

25 **a** $|\overrightarrow{AB}| = \sqrt{158}$ units, $|\overrightarrow{BC}| = \sqrt{129}$ units, $|\overrightarrow{AC}| = \sqrt{29}$ units, and $29 + 129 = 158$

b area ≈ 30.6 units^2

26 **a** $r = 2,\ s = 4,\ t = -7$ **b** $r = -4,\ s = 0,\ t = 3$

27 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$

b ABCD is a parallelogram.

28 **a** S$(-2,\ 8,\ -3)$ **b** midpoints are at $(-\frac{1}{2},\ 3,\ 1)$

29 **a** parallelogram **b** not a parallelogram

30 **a** R$(3,\ 1,\ 6)$ **b** X$(2,\ -1,\ 0)$

EXERCISE 12J

1 **a** parallel **b** not parallel **c** parallel
d parallel **e** parallel **f** not parallel

2 **a** $a = 3$ **b** $a = -6$

3 **a** $r = 3,\ s = -9$ **b** $r = -6,\ s = -4$

4 **a** $\overrightarrow{AB} \parallel \overrightarrow{CD}$, $|\overrightarrow{AB}| = 3|\overrightarrow{CD}|$

b $\overrightarrow{RS} \parallel \overrightarrow{KL}$, $|\overrightarrow{RS}| = \frac{1}{2}|\overrightarrow{KL}|$, $\overrightarrow{RS}$ and $\overrightarrow{KL}$ are in opposite directions.

c $\overrightarrow{AB} \parallel \overrightarrow{BC}$, $|\overrightarrow{AB}| = 2|\overrightarrow{BC}|$, A, B, and C are collinear.

5 $\overrightarrow{QS} = 2\overrightarrow{PR}$, PR : QS $= 1 : 2$

6 **a** $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

7 **a** $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ **b** $\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$ **c** $-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

8 **a** $\frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{k})$ **b** $\frac{1}{\sqrt{6}}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

c $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

9 **a** $\begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$ **c** $\sqrt{2}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

d $\frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

10 **a** $\pm\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ **b** $\pm\frac{2}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

11 **a** $\overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$ **b** $\overrightarrow{OB} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$

c B$(3 + 2\sqrt{2},\ 2 - 2\sqrt{2})$

12 **a** X$(4,\ -1)$ **b** X$(-\frac{13}{5},\ 3,\ \frac{14}{5})$

c X$\left(2 + \frac{4}{\sqrt{6}},\ -1 - \frac{4}{\sqrt{6}},\ 4 + \frac{8}{\sqrt{6}}\right)$

14 **a** $\overrightarrow{AB} = -\frac{5}{3}\overrightarrow{BC}$ **b** $\overrightarrow{PQ} = -\frac{3}{5}\overrightarrow{QR}$ **c** $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{BC}$

d $\overrightarrow{PQ} = -\frac{1}{2}\overrightarrow{QR}$

15 **a** $a = 7,\ b = -1$ **b** $a = -\frac{7}{2},\ b = -\frac{21}{2}$

16 **a** $\begin{pmatrix} x - X \\ y - Y \\ z - Z \end{pmatrix}$ **b** $(x - X)^2 + (y - Y)^2 + (z - Z)^2 = r^2$

EXERCISE 12K

1 **a** 11 **b** 6 **c** 2 **d** -9

2 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

3 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

5 **a** 1 **b** 1 **c** 3 **d** 0 **e** 0 **f** 5

6 **a** **i** 23 **ii** 12 **iii** -7 **iv** -2

8 **Hint:** $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b})$

9 $\mathbf{a} \bullet \mathbf{b}$ is a scalar and so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is the scalar product of a scalar and a vector, which is meaningless.

10 16

11 **Hint:** Let $\mathbf{a} = k\mathbf{b}$ where k is a scalar, so $|\mathbf{a}| = |k||\mathbf{b}|$.

EXERCISE 12L

1 **a** 11 **b** $\approx 49.8°$

2 **a** $\approx 71.6°$ **b** $\approx 177°$ **c** $\approx 18.4°$ **d** $45°$

4 **Hint:** Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} = 0$.

5 **a** $\approx 69.0°$ **b** $\approx 94.1°$ **c** $\approx 114°$ **d** $\approx 72.5°$

6 **a** $t = 6$ **b** $t = -8$ **c** $t = 0$ or 2

d $t = -\frac{3}{2}$ **e** $t = -\frac{5}{6}$ **f** $t = 0$ or -3

7 $r = -2,\ s = 5,\ t = -4$

8 **Note:** The negative of these vectors are also valid answers.

a $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

9 $\begin{pmatrix} -\frac{15}{\sqrt{13}} \\ \frac{10}{\sqrt{13}} \end{pmatrix}$ and $\begin{pmatrix} \frac{15}{\sqrt{13}} \\ -\frac{10}{\sqrt{13}} \end{pmatrix}$

10 ≈ 3.089 **11** **a** 5 **b** -9

14 **a** $\mathbf{a} \bullet \mathbf{b} \neq 0$ **b** $|\mathbf{a}| = 12$ **15** **a** $\mathbf{c} = \mathbf{d}$ **b** $\mathbf{c} = -\mathbf{d}$

16 $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

17 **Hint:** Choose a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a and b are integers.

Solve for c such that $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$.

18 $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$, $\therefore$ $B\widehat{A}C$ is a right angle

19 **b** $|\overrightarrow{AB}| = \sqrt{14}$ units, $|\overrightarrow{BC}| = \sqrt{14}$ units, ABCD is a rhombus

c 0; the diagonals of a rhombus are perpendicular.

20 $A\widehat{B}C \approx 62.5°$, the exterior angle $\approx 117.5°$

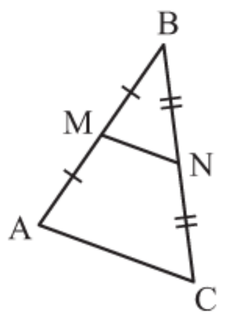
21 **a** $\approx 54.7°$ **b** $60°$ **c** $\approx 35.3°$

22 **a** $\approx 30.3°$ **b** $\approx 54.2°$

23 **a** M$(\frac{3}{2},\ \frac{5}{2},\ \frac{3}{2})$ **b** $\approx 51.5°$

EXERCISE 12M

3 **Hint:**



M and N are the midpoints of [AB] and [BC] respectively.

Let $\overrightarrow{AM} = \overrightarrow{MB} = \mathbf{a}$ and $\overrightarrow{BN} = \overrightarrow{NC} = \mathbf{b}$.

Find $\overrightarrow{MN}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

4 **a** **i** $\overrightarrow{PC} = \mathbf{r}$, $\overrightarrow{DP} = \mathbf{s}$ **ii** $\overrightarrow{AB} = \mathbf{r} + \mathbf{s}$, $\overrightarrow{DC} = \mathbf{r} + \mathbf{s}$

b $\overrightarrow{AB} \parallel \overrightarrow{DC}$, $|\overrightarrow{AB}| = |\overrightarrow{DC}|$

c If the diagonals of a quadrilateral bisect each other then the quadrilateral is a parallelogram.

6 **a** $P(\cos\theta, \sin\theta)$

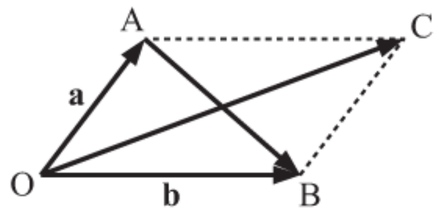
b $\overrightarrow{BP} = \begin{pmatrix} \cos\theta + 1 \\ \sin\theta \end{pmatrix}$, $\overrightarrow{AP} = \begin{pmatrix} \cos\theta - 1 \\ \sin\theta \end{pmatrix}$

c $\overrightarrow{AP} \bullet \overrightarrow{BP} = 0$

d The angle in a semi-circle is a right angle.

7 **a** **Hint:** Square both sides.

b Consider the parallelogram. Find $\overrightarrow{AB}$ and $\overrightarrow{OC}$, and so on.



9 **Hint:** Let parallelogram OABC have sides $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Find $\overrightarrow{AC}$ and $\overrightarrow{OB}$.

10 **a** $\overrightarrow{AB} = -\mathbf{r} + \mathbf{s}$, $\overrightarrow{OQ} = \frac{3}{4}\mathbf{r} + \frac{3}{4}\mathbf{s}$

13 **a** $\overrightarrow{OX} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\overrightarrow{BZ} = \frac{1}{2}\mathbf{a} - \mathbf{b}$ **b** $\overrightarrow{OP} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

c $\overrightarrow{YP} = \frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{b}$, $\overrightarrow{PA} = \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b}$

EXERCISE 12N.1

1 **a** $\begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix}$

d $-\mathbf{i} - \mathbf{j} - \mathbf{k}$ **e** $\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ **f** $\begin{pmatrix} -8 \\ -5 \\ 1 \end{pmatrix}$

2 **a** $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix}$ **b** $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$

c $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

3 **a** $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

$\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all vectors $\mathbf{a}$.

b $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

c **i** $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

ii $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

5 **a** $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ **b** 17

6 **a** $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ **b** $5\mathbf{j}$ **c** $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ **d** $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

8 **a** $\mathbf{a} \times \mathbf{b}$ **b** $\mathbf{0}$ **c** $2(\mathbf{b} \times \mathbf{a})$ **d** 0

10 **a** $k\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$

b $k\begin{pmatrix} 6 \\ 22 \\ -15 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$

c $(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})n$, $n \in \mathbb{R}$, $n \neq 0$

d $(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})n$, $n \in \mathbb{R}$, $n \neq 0$

11 $k\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$, $\frac{\sqrt{10}}{6}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$ and $-\frac{\sqrt{10}}{6}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$

12 **a** $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

EXERCISE 12N.2

1 **a** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

2 **a** $\mathbf{a} \bullet \mathbf{b} = -1$, $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$ **b** $\cos\theta = -\frac{1}{\sqrt{28}}$

c $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$ **d** $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$

4 **a** $\mathbf{p} \bullet \mathbf{q} = 0 \quad \therefore \quad \mathbf{p} \perp \mathbf{q}$ **b** $\begin{pmatrix} -4 \\ -17 \\ -5 \end{pmatrix}$

c $|\mathbf{p} \times \mathbf{q}| = \sqrt{330}$, $|\mathbf{p}| = \sqrt{30}$, $|\mathbf{q}| = \sqrt{11}$

$\therefore \quad |\mathbf{p}||\mathbf{q}| = \sqrt{30 \times 11} = \sqrt{330} = |\mathbf{p} \times \mathbf{q}|$

5 **a** $\overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$ **b** $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

EXERCISE 12N.3

1 **a** $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$

c $\frac{\sqrt{83}}{2}$ units^2

2 **a** $\frac{\sqrt{101}}{2}$ units^2 **b** $\frac{\sqrt{133}}{2}$ units^2 **c** $\frac{\sqrt{69}}{2}$ units^2

3 **a** $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units^2

b **i** $\overrightarrow{BO} = -\mathbf{b}$, $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

ii $\frac{1}{2}|-\mathbf{b} \times (\mathbf{a} - \mathbf{b})|$ units^2

4 $x = 2 \pm 2\sqrt{33}$

5 **a** $\overrightarrow{CB} = \overrightarrow{OA}$ and $\overrightarrow{AB} = \overrightarrow{OC}$

$\therefore$ opposite pairs of sides are parallel and equal in length.

b $\sqrt{222}$ units^2

6 $8\sqrt{2}$ units^2 **7** **a** D$(-4, 1, 3)$ **b** $\sqrt{307}$ units^2

8 **a** PQ = QR = RS = SP

All sides are equal in length. $\therefore$ PQRS is a rhombus.

b $\sqrt{1197}$ units^2

9 **a** **i** $|\mathbf{b} \times \mathbf{c}|$ units^2 **ii** $|\mathbf{a}| \sin\theta$ units

b **Hint:** Let ϕ be the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

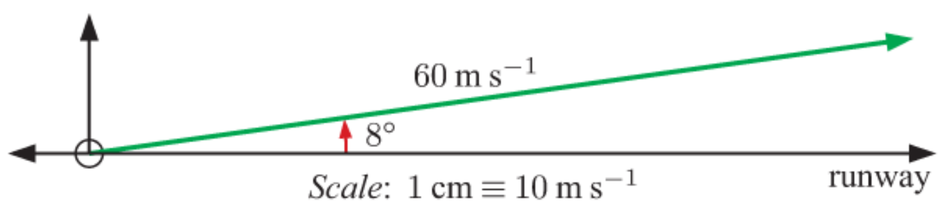
c 9 units^3

10 **a** **Hint:** Volume $= \frac{1}{3}$ area of base $\times$ perpendicular height

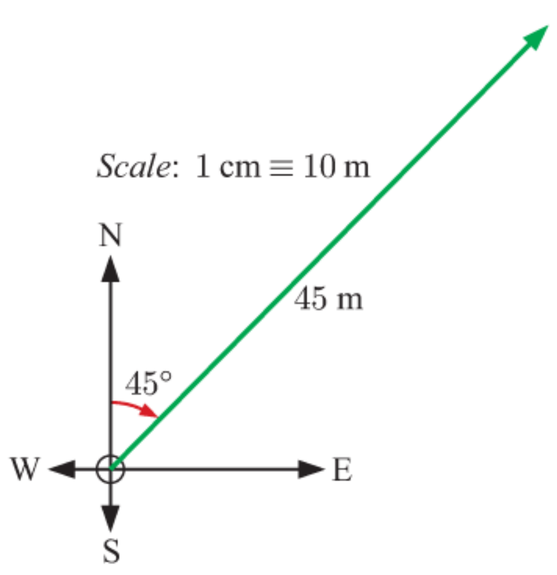
b $3\frac{1}{2}$ units^3

REVIEW SET 12A

1 **a**



b



2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AD}$

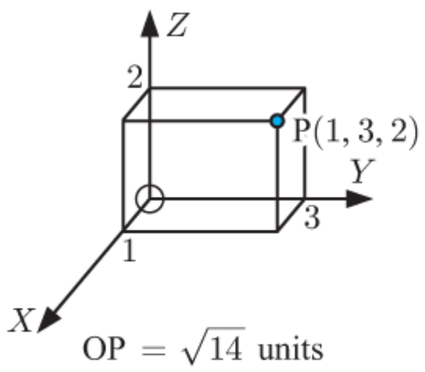
3 **Note:** Other answers are possible.

a $\mathbf{q} = \mathbf{p} + \mathbf{r}$ **b** $\mathbf{l} = \mathbf{k} - \mathbf{j} + \mathbf{n} - \mathbf{m}$

4 **a** $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $4\mathbf{i} + 3\mathbf{j}$ **b** $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $-\mathbf{i} - 5\mathbf{j}$

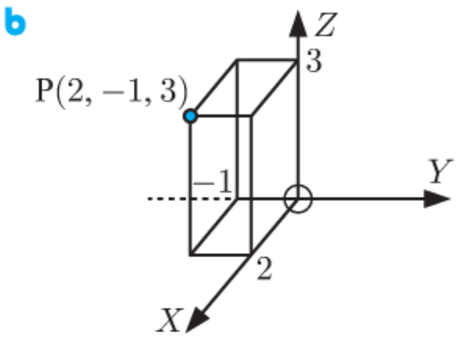
c $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $-4\mathbf{i} + 2\mathbf{j}$

5 **a**



OP $= \sqrt{14}$ units

b



OP $= \sqrt{14}$ units

6 $\begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$ **7** **a** $\mathbf{p} + \mathbf{q}$ **b** $\frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

8 **a** $\begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}$ **b** $\sqrt{14}$ **c** $\begin{pmatrix} 24 \\ -5 \\ -8 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$

e $\begin{pmatrix} 3\sqrt{53} \\ -\sqrt{53} \\ 2\sqrt{53} \end{pmatrix}$ **f** $\sqrt{65}$

9 $a = 1,\ b = 1,\ c = -5$ **10** **a** 5 units **b** $2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$

11 $m = 5,\ n = -\frac{1}{2}$ **13** **a** **i** $\mathbf{p} + \mathbf{q}$ **ii** $\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

14 **a** **i** $\mathbf{q} + \mathbf{r}$ **ii** $\mathbf{r} + \mathbf{q}$

b [DB] and [AC] are parallel and equal in length.

15 $-\frac{5}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ **16** **a** -13 **b** -36

17 **a** $\mathbf{a} \bullet \mathbf{b} = -4$ **b** $\mathbf{b} \bullet \mathbf{c} = 10$ **c** $\mathbf{a} \bullet \mathbf{c} = -10$

18 **a** 8 **b** $\approx 62.2°$ **19** $\approx 64.0°$ **20** $t = -4$

21 $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$ **22** $\approx 40.7°$

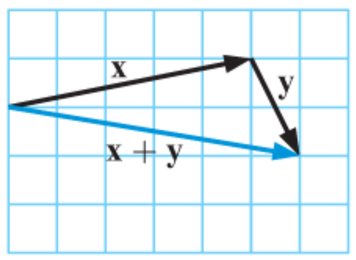
23 **a** $\begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix}$ **b** 0 **c** $\begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$ **24** $\pm\frac{3}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$

26 $\frac{\sqrt{66}}{2}$ units^2

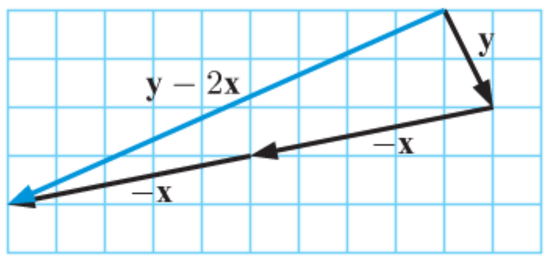
27 $S = \frac{1}{2}[|\mathbf{a} \times \mathbf{b}| + |\mathbf{a} \times \mathbf{c}| + |\mathbf{b} \times \mathbf{c}| + |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|]$

REVIEW SET 12B

1 **a**



b



2 **a** $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$

3 **a** $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $5\mathbf{i} + 2\mathbf{j}$ **b** $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $-3\mathbf{i} + 3\mathbf{j}$

4 **a** $\mathbf{r} + \mathbf{q}$ **b** $-\mathbf{p} + \mathbf{r} + \mathbf{q}$ **c** $\mathbf{r} + \frac{1}{2}\mathbf{q}$ **d** $-\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$

5 **a** $\begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$ **c** $\sqrt{74}$ **6** $\begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$

7 **a** $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$ **b** $\sqrt{162}$ units **c** $\sqrt{61}$ units

8 $k = \pm\frac{1}{2}$ **9** $a = -2,\ b = 0$

10 $\pm\frac{4}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **11** **a** $\overrightarrow{CD} = (k-1)\mathbf{b} - \mathbf{a}$

12 **b** $\overrightarrow{AD} = k\mathbf{q}$ **d** It is parallel to the other two sides.

13 C(2, 2, −7), D(5, 3, −11), E(8, 4, −15)

14 X(6, 5, −13) **15** $\mathbf{v} \bullet \mathbf{w} = \pm 6$

17 $\overrightarrow{OT} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

18 **a** $\overrightarrow{AC} = -\mathbf{p} + \mathbf{r}$, $\overrightarrow{BC} = -\mathbf{q} + \mathbf{r}$

19 **a** $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$ **b** $\approx 41.8°$ **20** $\approx 61.6°$

21 $t = 2 \pm \sqrt{2}$ **22** $\approx 80.3°$ **23** $\approx 16.1°$

24 $k\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix},\ k \neq 0$ **25** $\mathbf{u} \bullet \mathbf{v} = \pm\sqrt{199}$

27 **a** $\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$ **b** $C\left(-\frac{24}{5\sqrt{2}}, -\frac{6}{\sqrt{2}}, -\frac{18}{5\sqrt{2}}\right)$

c $\frac{5}{\sqrt{2}}$ units^2 **d** $\frac{10}{\sqrt{2}}$ units^3

EXERCISE 13A

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = 1 - \lambda,\ y = 4 + 2\lambda,\ \lambda \in \mathbb{R}$ **iii** $2x + y = 6$

b **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 5 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = 5 - 2\lambda,\ y = 2 + 5\lambda,\ \lambda \in \mathbb{R}$ **iii** $5x + 2y = 29$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 7 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = -6 + 3\lambda,\ y = 7\lambda,\ \lambda \in \mathbb{R}$ **iii** $7x - 3y = -42$

d **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = \lambda,\ y = 2 + 3\lambda,\ \lambda \in \mathbb{R}$ **iii** $y = 2 + 3x$

e **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = 3 + 4\lambda,\ y = 2\lambda,\ \lambda \in \mathbb{R}$ **iii** $x - 2y = 3$

f **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -11 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = -2 + 6\lambda,\ y = 5 - 11\lambda,\ \lambda \in \mathbb{R}$

iii $11x + 6y = 8$

2 **a** $x = 4 - \lambda,\ y = -3 + 2\lambda,\ \lambda \in \mathbb{R}$

b Points are: (4, −3), (3, −1), (2, 1), (5, −5), (7, −9)

3 **a** (0, 8)

b It is parallel to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and in the opposite direction.

c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \end{pmatrix},\ \mu \in \mathbb{R}$

4 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = 1 + 2\lambda,\ y = 3 + \lambda,\ z = -7 + 3\lambda,\ \lambda \in \mathbb{R}$

iii $\dfrac{x-1}{2} = y-3 = \dfrac{z+7}{3}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

ii $x = \lambda, \ y = 1+\lambda, \ z = 2-2\lambda, \ \lambda \in \mathbb{R}$

iii $x = y-1 = \dfrac{-z+2}{2}$

c **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$

ii $x = -2+\lambda, \ y = 2, \ z = 1, \ \lambda \in \mathbb{R}$

iii $y = 2, \ z = 1$

d **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

ii $x = 2\lambda, \ y = 2-\lambda, \ z = -1+3\lambda, \ \lambda \in \mathbb{R}$

iii $\dfrac{x}{2} = -y+2 = \dfrac{z+1}{3}$

e **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

ii $x = 3, \ y = 2, \ z = -1+\lambda, \ \lambda \in \mathbb{R}$

iii $x = 3, \ y = 2$

5 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \ \lambda \in \mathbb{R}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$

d $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R}$

6 **a** $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

7 **a** $(0, 4, 1)$ **b** $(4, 0, 9)$ **c** $(-\frac{1}{2}, \frac{9}{2}, 0)$

8 **a** (x_0, y_0, z_0) **b** $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

c $\dfrac{x-x_0}{l} = \dfrac{y-y_0}{m} = \dfrac{z-z_0}{n}, \quad l, m, n \neq 0$

9 $(0, 7, 3)$ and $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$

10 **a** $\sqrt{3\lambda^2+2}$ units **b** $\lambda = 0$

11 **a** L_2: $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \ s \in \mathbb{R}$

b L_3: $\dfrac{x}{-4} = \dfrac{y}{-16} = \dfrac{z+2}{3}$

EXERCISE 13B

1 $\approx 75.7°$ **3** $\approx 75.5°$

4 **a** $\approx 28.6°$ **b** **i** $a = -\frac{48}{7}$ **ii** $\approx 82.0°$

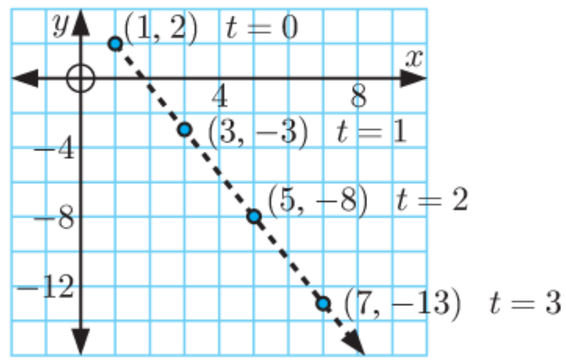
5 $\approx 76.7°$ **6** $\approx 49.9°$

7 **a** $\approx 78.7°$ **b** $\approx 63.4°$ **c** $\approx 63.4°$ **d** $\approx 71.6°$

EXERCISE 13C

1 **a** $(1, 2)$

b



c $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

d $\sqrt{29}$ cm s^{-1}

e $-\frac{8}{\sqrt{29}} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

2 **a** **i** $(-4, 3, 0)$ **ii** $\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$ **iii** ≈ 14.3 m s^{-1}

b **i** $(3, 0, 4)$ **ii** $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ **iii** 3 m s^{-1}

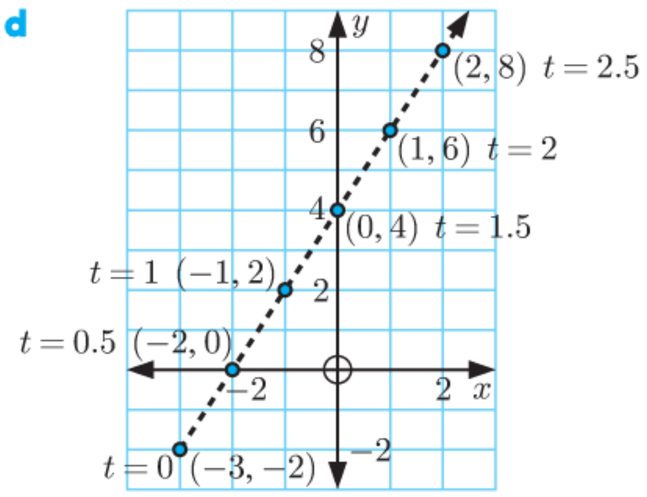
3 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \quad t \geqslant 0$ **b** $(8, -4.5)$

c 45 minutes

4 **a** $\begin{pmatrix} -3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad t \geqslant 0$ **b** $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

c **i** $t = 1.5$ s **ii** $t = 0.5$ s

d



5 **a** $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 7.2 \end{pmatrix}$ **c** $\begin{pmatrix} \frac{30\sqrt{85}}{17} \\ \frac{35\sqrt{85}}{17} \end{pmatrix}$

d $\begin{pmatrix} 2\sqrt{66} \\ \frac{7\sqrt{66}}{2} \\ \frac{\sqrt{66}}{2} \end{pmatrix}$ **e** $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$

6 **a** A is at $(4, 5)$, B is at $(1, -8)$

b Yacht A: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; Yacht B: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c Yacht A: $\sqrt{5}$ km h^{-1}; Yacht B: $\sqrt{5}$ km h^{-1}

d $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

7 **a** $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\therefore \ x_1(t) = -5+3t, \quad y_1(t) = 4-t$

b $\sqrt{10}$ km min^{-1}

c a minutes later, $(t-a)$ minutes have elapsed.

$\therefore \ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t-a) \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

$\therefore \ x_2(t) = 15-4(t-a), \quad y_2(t) = 7-3(t-a)$

d $\approx 55.3°$

e Torpedo is fired at 1:35:28 pm and the explosion occurs at 1:37:42 pm.

8 **a** $\begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}$ **b** (100, −87, 45) **c** 810 m **d** $\approx 19.5°$

9 **a** $\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$ **b** $\approx 19.2 \text{ km h}^{-1}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}, \ t \in \mathbb{R}$

d 1 hour **e** $\approx 8.98°$

EXERCISE 13D

1 **a** $\frac{3\sqrt{5}}{5}$ units **b** $\frac{\sqrt{2}}{2}$ units **c** $\frac{9\sqrt{2}}{2}$ units **d** 0 units

2 **a** $6\mathbf{i} - 6\mathbf{j}$ **b** $\begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix}$ **c** when $t = \frac{3}{4}$ hours

d $t = 0.84$ and position is (0.96, 0.72)

3 **a** $\begin{pmatrix} -120 \\ -40 \end{pmatrix}$ **b** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \end{pmatrix} + t \begin{pmatrix} -120 \\ -40 \end{pmatrix}$

c (80, 60)

d The aircraft is exactly 100 km from O at 1:00 pm.

e at 1:45 pm and $d_{\min} = 10\sqrt{10}$ km **f** 2:30 pm

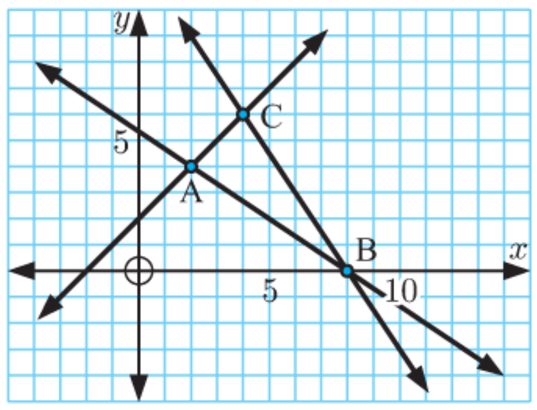
4 **a** (2, −1, 4) **b** $\sqrt{27}$ units

5 **a** $(2, \frac{1}{2}, \frac{5}{2})$ **b** $\sqrt{\frac{3}{2}}$ units

6 **a** $\begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix}$ **b** **i** after 40 seconds **ii** $\sqrt{18}$ m

EXERCISE 13E

1 **a**

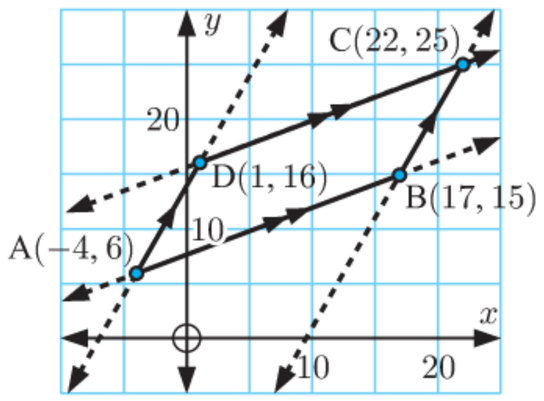


b A(2, 4), B(8, 0), C(4, 6)

c BC = BA $= \sqrt{52}$ units

$\therefore$ $\triangle$ABC is isosceles.

2 **a**



b A(−4, 6), B(17, 15), C(22, 25), D(1, 16)

3 **a** A(2, 3), B(8, 6), C(5, 0)

b AB = BC $= \sqrt{45}$ units, AC $= \sqrt{18}$ units

4 **a** P(10, 4), Q(3, −1), R(20, −10)

b $\overrightarrow{PQ} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} 10 \\ -14 \end{pmatrix}$, $\overrightarrow{PQ} \bullet \overrightarrow{PR} = 0$

c $\widehat{QPR} = 90°$ **d** 74 units^2

5 **a** A(2, 5), B(18, 9), C(14, 25), D(−2, 21)

b **i** $\overrightarrow{AC} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$ and $\overrightarrow{DB} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$

ii $|\overrightarrow{AC}| = \sqrt{544}$ units, $|\overrightarrow{DB}| = \sqrt{544}$ units

iii 0

iv midpoint of [AC]: (8, 15), midpoint of [DB]: (8, 15)

c Diagonals are perpendicular and equal in length, and as their midpoints are both (8, 15), ABCD is a square.

EXERCISE 13F.1

1 **a** intersecting **b** parallel **c** intersecting **d** coincident **e** intersecting **f** parallel

2 $\sim \left(\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{15}{4} \\ 0 & 1 & -1 \end{array}\right)$ The lines are intersecting.

3 **a** The second equation is a multiple of the first. The lines are coincident.

b It gives no more information than the first. Gives the same solutions for x and y.

c **i** $x = t, \ y = \dfrac{3-t}{2}, \ t \in \mathbb{R}$

ii $y = s, \ x = 3 - 2s, \ s \in \mathbb{R}$

4 **b** The system is inconsistent and so has no solutions.

$\therefore$ the lines are parallel.

5 **b** The lines are coincident. Infinitely many solutions exist of the form $x = t, \ y = \dfrac{5-2t}{3}, \ t \in \mathbb{R}$.

6 **b** If $k \neq 4$, the system is inconsistent and so has no solutions. The lines are parallel.

If $k = 4$, the system has infinitely many solutions of the form $x = t, \ y = 3t - 2, \ t \in \mathbb{R}$. The lines are coincident.

7 **a** $\left(\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 0 & k-16 \end{array}\right)$

b $k = 16$, solutions are $x = \dfrac{t+8}{3}, \ y = t, \ t \in \mathbb{R}$

c $k \neq 16$, the lines are parallel.

8 **a** $\left(\begin{array}{cc|c} 1 & 2 & \frac{1}{4} \\ 0 & 2a+8 & -21 \end{array}\right)$ **b** $a \neq -4$

c When $a = -4$, the last row is $0 \ 0 \mid -21$.

$\therefore$ no solutions exist, the lines are parallel.

9 **a** A unique solution exists for $m \neq 2$ or -2.

If $m \neq \pm 2$, $x = \dfrac{6}{m+2}$, $y = \dfrac{6}{m+2}$

The lines intersect at a point.

b If $m = 2$, there are infinitely many solutions of the form $x = t, \ y = 3 - t, \ t \in \mathbb{R}$. The lines are coincident.

If $m = -2$, there are no solutions. The lines are parallel.

EXERCISE 13F.2

1 **a** Intersect at (1, 2, 3), angle $\approx 10.9°$.

b Skew, angle $\approx 62.7°$. **c** Parallel, $\therefore$ angle $= 0°$.

d Skew, angle $\approx 11.4°$.

e Intersect at (−4, 7, −7), angle $\approx 40.2°$.

f Parallel, $\therefore$ angle $= 0°$. **g** Coincident, $\therefore$ angle $= 0°$.

2 Line 1 and line 2 are parallel.

Line 1 and line 3 are skew with angle $\approx 48.2°$.

Line 2 and line 3 are skew with angle $\approx 48.2°$.

EXERCISE 13F.3

1 **a** $\frac{2}{\sqrt{26}}$ units **b** $2\sqrt{2}$ units

2 **a** 0 units **b** $\frac{111}{\sqrt{1498}} \approx 2.87$ units

c $\frac{\sqrt{2769}}{13} \approx 4.05$ units **d** $\frac{3\sqrt{2}}{4}$ units **e** 0 units

f $\frac{\sqrt{6180}}{10} \approx 7.86$ units **g** 0 units

EXERCISE 13G

1 **a** $2x - y + 3z = 8$ **b** $3x + 4y + z = 19$

2 **a** $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

3 **a** $y = 0$ **b** $z = 4$

4 **Note:** Many vector forms exist.

a **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \ s, t \in \mathbb{R}$

ii $-2x + 6y + z = 18$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \ s, t \in \mathbb{R}$

ii $-5x + 3y + 12z = 12$

c **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}, \ s, t \in \mathbb{R}$

ii $-y + z = 3$

5 $3x + 7y + z = 3$

6 $x + 3y + z = 10$ (Other answers are possible.)

7 **a** $y - 3z = -7$ **b** $x - z = -2$ **c** $3x - y = 1$

8 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

9 yes, the X-axis at $(2, 0, 0)$ **10** $y - 2z = 8$

11 **a** $(-1, 3, 1)$ **b** $6x - 8y - 5z = -35$

12 **a** $k = -\frac{3}{2}$ **b** B is $(3, 6, -\frac{11}{2})$ or $(-1, -2, \frac{5}{2})$

13 $x = 2 - t, \ y = -1 + 3t, \ z = 3 - 3t, \ t \in \mathbb{R}$; $(-1, 8, -6)$

14 **a** $x = 1 + t, \ y = -2 + 2t, \ z = 4 - 5t, \ t \in \mathbb{R}$

b **i** $(0, -4, 9)$ **ii** $(1, -2, 4)$ **iii** $(-5, -14, 34)$

15 **a** $x = 1 + 2t, \ y = 5 - 5t, \ z = 3 + t, \ t \in \mathbb{R}$

b $(3, 0, 4)$

16 **a** $(-1, -1, 4)$; 3 units **b** $(0, 1, -3)$; $2\sqrt{11}$ units

c $(-\frac{1}{7}, -\frac{26}{7}, -\frac{17}{7})$; $2\sqrt{\frac{3}{7}}$ units

17 $(1, -3, 0)$ **18** **a** $x + 5y + 3z = -10$ **b** $(3, -2, -1)$

19 **a** $\mathrm{N}(\frac{17}{5}, \frac{6}{5}, 1)$, $d = \frac{2}{\sqrt{5}}$ units

b $\mathrm{N}(\frac{1}{6}, \frac{5}{6}, -\frac{1}{3})$, $d = \frac{5}{\sqrt{6}}$ units

21 **d** **i** $\frac{10}{\sqrt{6}}$ units **ii** $2\sqrt{3}$ units

22 **a** $\frac{19}{2\sqrt{6}}$ units **b** $\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$ units **23** $\frac{26}{\sqrt{138}}$ units

24 $2x - y + 2z = -1$ and $2x - y + 2z = 11$

EXERCISE 13H

1 **a** $\approx 13.1°$ **b** $0°$ (the line and plane are parallel)

c $\approx 11.3°$ **d** $\approx 30.7°$

2 **b** 1 unit **3** **a** $\approx 36.0°$ **b** $\approx 11.3°$ **c** $\approx 51.7°$

4 **a** $\approx 23.6°$ **b** C$(0, -8, 0)$

c **i** $3x - 10y - 5z = -15$ **ii** $\approx 30.1°$

5 **a** $\approx 83.7°$ **b** $\approx 84.8°$ **c** $\approx 86.2°$ **d** $\approx 73.2°$

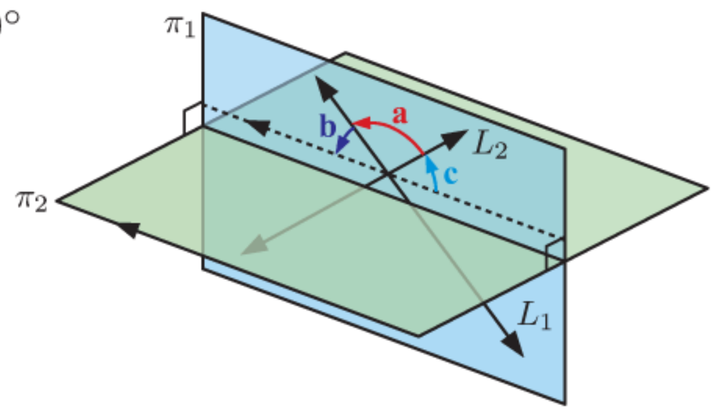
e $\approx 62.3°$

6 $k = 4$

7 **a** $k = -2, \ (2, 3, 7)$ **b** $8x + 11y - 2z = 35$

d **i** $\approx 54.0°$ **ii** $\approx 72.6°$ **iii** $\approx 53.4°$

e $180°$



8 **a** $\approx 55.5°$ **b** $k = 5$ **c** B$(6, 0, 17)$

d **ii** l is parallel to both P_1 and P_2, so l must be perpendicular to the normals of each plane.

$\therefore$ l has direction $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$.

e $32x - 65y + 41z = 385$ **f** **i** $\approx 36°$ **ii** $\approx 19.5°$

g sum $\approx 55.5°$, this is equal to the angle between P_1 and P_2.

EXERCISE 13I

1 **a** 3 interpretations: the planes are intersecting, parallel, or coincident.

b **i** $a_1 = ka_2, \ b_1 = kb_2, \ c_1 = kc_2, \ d_1 \neq kd_2$ for some k

ii $a_1 = ka_2, \ b_1 = kb_2, \ c_1 = kc_2, \ d_1 = kd_2$ for some k

c **i** Planes meet in a line

$x = -2 + 3t, \ y = t, \ z = 5, \ t \in \mathbb{R}$

ii Planes meet in a line

$x = \frac{8 - 2t}{3}, \ y = \frac{t - 1}{3}, \ z = t, \ t \in \mathbb{R}$

iii Planes are coincident

$\therefore \ x = 6 - 2s + 3t, \ y = s, \ z = t, \ s, t \in \mathbb{R}$

2 **a** If $k = -2$, planes are coincident with infinitely many solutions.

If $k \neq -2$, planes meet in a line with infinitely many solutions.

b If $k = 16$, planes are coincident with infinitely many solutions.

If $k \neq 16$, planes are parallel with no solutions.

4 **a** $x = 1, \ y = -2, \ z = 4$; meet at the point $(1, -2, 4)$

b Meet in a line $x = \frac{9 - t}{3}, \ y = \frac{6 + 5t}{3}, \ z = t, \ t \in \mathbb{R}$

c Meet in a line $x = 3t - 3, \ y = t, \ z = 5t - 11, \ t \in \mathbb{R}$

d No solutions as 2 planes are parallel and intersected by the 3rd plane.

e Two planes are coincident and the other cuts obliquely at the line $x = \frac{5}{2} + \frac{1}{2}t, \ y = -\frac{3}{2} + \frac{3}{2}t, \ z = t, \ t \in \mathbb{R}$

f $x = 3, \ y = -2, \ z = 0$; meet at the point $(3, -2, 0)$

5 **a** No solution, the line of intersection of any two planes is parallel to the third plane.

b $x = \frac{2 - 2t}{3}, \ y = \frac{8t - 11}{3}, \ z = t, \ t \in \mathbb{R}$

6 If $k = 5$, there are infinitely many solutions. The planes meet in a line $x = -10t, \ y = -1 - 7t, \ z = t, \ t \in \mathbb{R}$.

If $k \neq 5$, there are no solutions. The line of intersection of any two planes is parallel to the third.

7 If $a = 2$, there are no solutions. The line of intersection of any two planes is parallel to the third.

If $a \neq 2$, there is a unique solution $x = \dfrac{a-14}{4(a-2)}$, $y = \dfrac{18-3a}{4(a-2)}$, $z = \dfrac{3}{a-2}$. The planes meet at a point.

8 **a** $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 9-k & 7 \end{array}\right)$

b If $k \neq 2$, then R_2 and R_3 of the reduced matrix in **a** are inconsistent with one another.

Two of the planes are parallel if $k = 9$ or $k = -6$. For the remaining values of k for which the system is inconsistent, the line of intersection of any two planes is parallel to the third plane.

c $k = 2$; the planes intersect in a line.

9 $\left(\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -3 & -1-2m & 5 \\ 0 & 0 & (m+1)(m+5) & -7(m+1) \end{array}\right)$

a If $m = -5$, no solution. The line of intersection of any two planes is parallel to the third.

b If $m = -1$, infinitely many solutions. Two planes are coincident and the third meets in a line.

c **i** If $m \neq -5$ or -1, there is a unique solution. The planes meet at a point.

10 **a** $\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 2 & a & 6-a \\ 0 & 0 & a(a-4) & (a+8)(a-4) \end{array}\right)$

b $a = 0$; the line of intersection of any two planes is parallel to the third plane.

c $a = 4$; $x = \dfrac{5-5t}{2}$, $y = 1-2t$, $z = t$, $t \in \mathbb{R}$

The planes intersect in a line.

d $a \neq 0$ or 4; $x = -\dfrac{a}{2} - \dfrac{12}{a}$, $y = -1-a$, $z = \dfrac{a+8}{a}$

The planes meet at a point.

When $a = 2$, the solution is $x = -7$, $y = -3$, $z = 5$.

The planes meet at $(-7, -3, 5)$.

11 **a** $\left(\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right)$

b $x = \dfrac{19-6t}{7}$, $y = \dfrac{-5t-11}{7}$, $z = t$, $t \in \mathbb{R}$

The planes meet in a line.

c $x = \frac{1}{7}a + 2$, $y = \frac{2}{7}a - 2$, $z = 1$

The planes meet at a point.

12 They meet at the point $\left(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29}\right)$.

REVIEW SET 13A

1 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $t \in \mathbb{R}$

b $x = -6 + 4t$, $y = 3 - 3t$, $t \in \mathbb{R}$ **c** $3x + 4y = -6$

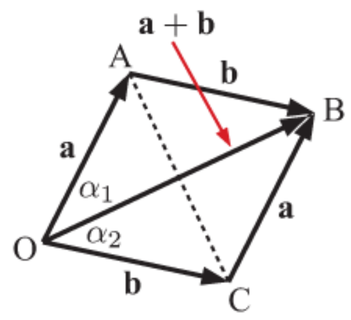
2 $m = 10$

3 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

b $C\left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$, or

$C\left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right)$

4 **a** OABC is a rhombus. So, its diagonals bisect its angles.



b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

c $(7, 3\frac{3}{4}, -3\frac{1}{4})$

5 **a** A$(5, 2)$, B$(6, 5)$, C$(8, 3)$

b $|\overrightarrow{AB}| = \sqrt{10}$ units, $|\overrightarrow{BC}| = \sqrt{8}$ units, $|\overrightarrow{AC}| = \sqrt{10}$ units

c isosceles

6 **a** **i** $-6\mathbf{i} + 10\mathbf{j}$ **ii** $-5\mathbf{i} - 15\mathbf{j}$

iii $(-6-5t)\mathbf{i} + (10-15t)\mathbf{j}$, $t \geqslant 0$

b $t = 0.48$ hours

c Shortest distance ≈ 8.85 km, so it will miss the reef.

7 **a** $x_1(t) = 2 + t$, $y_1(t) = 4 - 3t$, $t \geqslant 0$

b $x_2(t) = 13 - t$, $y_2(t) = (3-2a) + at$, $t \geqslant 2$

c interception occurred at 2:22:30 pm

d bearing $\approx 192.7°$, ≈ 4.54 km min^{-1} **e** $\approx 31.2°$

8 $(\frac{8}{3}, \frac{7}{3}, \frac{4}{3})$

9 **a** $5x + y + 4z = 3$ **b** $x = 5t$, $y = t$, $z = 4t$, $t \in \mathbb{R}$

c $(\frac{5}{14}, \frac{1}{14}, \frac{2}{7})$

10 **b** $\approx 28.6°$ **c** 14 units

11 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$, $\lambda \in \mathbb{R}$

b $-4x + 5z = 24$ **c** $(-5, 2, 9)$ and $(11, 2, -11)$

12 $(6, -1, -10)$

13 **a** They do not meet, the line is parallel to the plane.

b $\frac{16}{\sqrt{14}}$ units

14 **a** 3 units **b** $(1, 2, 4)$ **c** $\sqrt{116}$ units

15 **a** $\mathbf{n} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ **b** D$(-1, -1, 2)$ **c** $\sqrt{140}$ units^2

d $(\frac{1}{6}, \frac{5}{6}, \frac{2}{3})$

16 **a** A$(2, -1, 0)$ **c** $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$, $u \in \mathbb{R}$

d $3x - y + 2z = 7$ **e** $\sqrt{14}$ units^2

17 **a** $14x + 29y - 4z = 32$ **b** $\approx 55.9°$ **c** $r = \frac{2 \pm \sqrt{10}}{2}$

18 **a** $\approx 15.8°$ **b** $\approx 65.9°$

19 **a** $a = 2$, $b = 1$, $c = 6$ **b** $(3, -1, 1)$

c $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$

e $44x + 45y + 13z = 100$ **f** $\approx 29.2°$

20 **a** $5\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ **b** $m = 1$ **c** $x - y - z = 0$

d $t = 2$ **e** $\approx 22.0°$

21 If $k = -2$, the planes meet in the line
$x = \frac{4}{3},\ y = -\frac{11}{3} + t,\ z = t,\ t \in \mathbb{R}$.
If $k \neq -2$ the planes meet at the point $(\frac{4}{3}, -\frac{14}{3}, -1)$.

22 **b** $k = -1$

c $t(p+10) = q+2$ has infinitely many solutions for t when $p+10 = 0$ and $q+2 = 0,\ \therefore\ p = -10,\ q = -2$

REVIEW SET 13B

1 **a** $(5, 2)$ **b** $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 10 \end{pmatrix},\ s \in \mathbb{R}$

2 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = 2 + 4\lambda,\ y = -3 + 2\lambda,\ z = 1 - \lambda,\ \lambda \in \mathbb{R}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix},\ \lambda \in \mathbb{R}$

ii $x = -1 + 6\lambda,\ y = 6 - 8\lambda,\ z = 3 - 3\lambda,\ \lambda \in \mathbb{R}$

3 $\approx 8.13°$

4 **a** (KL) is parallel to (MN) as $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ is parallel to $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$

b (KL) is perpendicular to (NK) as $\begin{pmatrix} 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \end{pmatrix} = 0$

and (NK) is perpendicular to (MN) as $\begin{pmatrix} 4 \\ 10 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 2 \end{pmatrix} = 0$

c K(7, 17), L(22, 11), M(33, −5), N(3, 7) **d** 261 units^2

5 $\approx 30.5°$

6 **a** $|\overrightarrow{AB}| = 3\sqrt{3}$ units

b A lies on the line $\mathbf{r}$ where $\lambda = -3$ and B lies on $\mathbf{r}$ where $\lambda = 0$ $\therefore$ the line between A and B is the same as line $\mathbf{r}$, so it can be described by $\mathbf{r}$.

c $\approx 70.5°$

7 $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$

8 **a** $(-4, 3, 1)$ **b** $(28, 27, -7)$ **c** $\begin{pmatrix} 8 \\ 6 \\ -2 \end{pmatrix}$

d $\approx 10.2\ \text{m s}^{-1}$

9 **a** $(-10, 5, 12)$ **b** $\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$ **c** $\approx 15.2\ \text{m s}^{-1}$

d 30 seconds **e** $(170, 425, 0)$ **f** $(50, 145, 8)$

10 **a** A(3, −4) and B(4, 3)

b for A $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, for B $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ **c** $\approx 82.9°$

d at $t = 1.5$ hours

11 **a** Intersecting at $(4, 3, 1)$, angle $\approx 44.5°$.

b Skew, angle $\approx 71.2°$.

c Intersecting at $(-1, 2, 3)$, angle $\approx 27.0°$.

12 **a** $4x + 2y + z = 3$ **b** $\approx 64.1°$

13 N(−2, 7, −4) **14** $\frac{17}{3}$ units

15 **a** $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$ **b** $(-1, 3, 1)$ **c** $6x - 8y - 5z = -35$

16 $\frac{31}{\sqrt{110}}$ units

17 **a** $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $|\overrightarrow{PQ}| = \sqrt{26}$ units, $\overrightarrow{QR} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

b $x = 2 + \lambda,\ y = 4\lambda,\ z = 1 - 3\lambda,\ \lambda \in \mathbb{R}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}$

(Other answers are possible.)

18 **b** $(-3, -2, 5)$

c **i** $\approx 45.5°$ $(\approx 45.47°)$ **ii** $\approx 45.5°$ $(\approx 45.51°)$

19 **b** $2\sqrt{2}$ units

c $(16, -14, 14)$ on line 1 and $(16, -12, 16)$ on line 2

20 $\approx 7.82°$

21 **b** **i** $p \neq 0,\ p \neq -2$, 3 planes meet at a point

ii $p = 0$ or -2, $q \neq 0$, 3 planes have no common point of intersection, planes 2 and 3 are parallel

iii $p = 0$ or -2, $q = 0$, planes 2 and 3 are coincident

c $p = -2,\ q = 0$: $x = 2t - 2,\ y = t,\ z = 1,\ t \in \mathbb{R}$

$p = 0,\ q = 0$: $x = -2t,\ y = \dfrac{t-1}{2},\ z = t,\ t \in \mathbb{R}$

22 **e** The three planes have no common point of intersection. The line of intersection of any two planes is parallel to the third plane.

23 **a** $\left(\begin{array}{ccc|c} 1 & -3 & 2 & -5 \\ 0 & 10 & -4-k & 25 \\ 0 & 0 & k+4 & -5 \end{array}\right)$

b No solutions if $k = -4$. Two planes are parallel and intersected by the third plane.

c **i** Unique solution when $k \neq -4$.

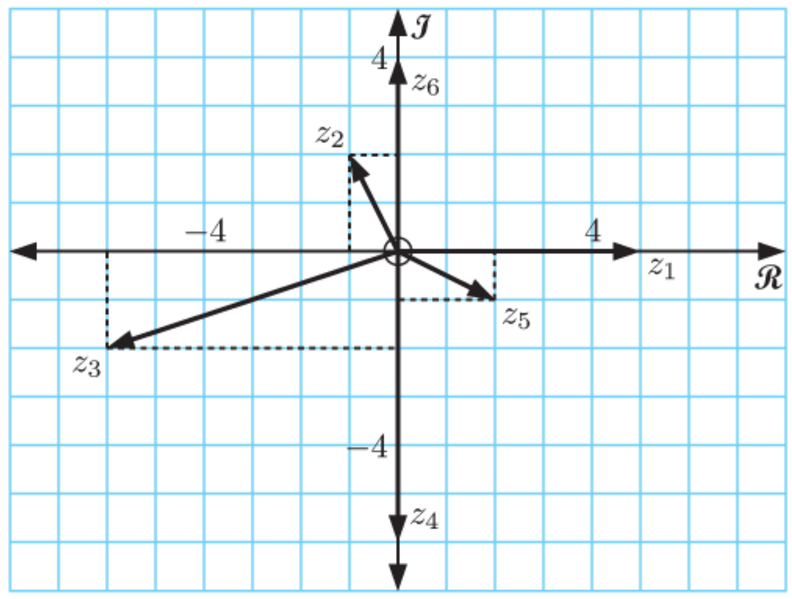
ii $x = 1 + \dfrac{10}{k+4},\ y = 2,\ z = \dfrac{-5}{k+4}$

The planes meet at a point.

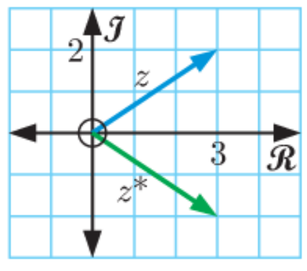
iii $x = 3,\ y = 2,\ z = -1$

EXERCISE 14A

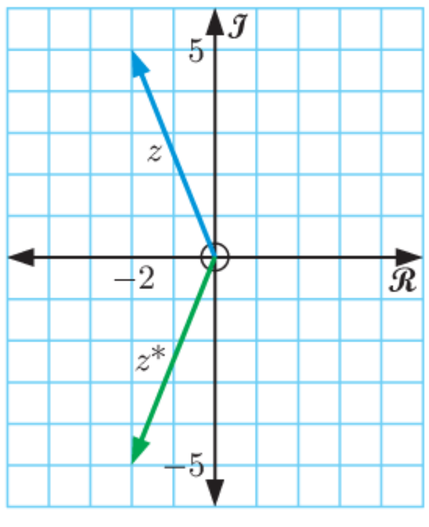
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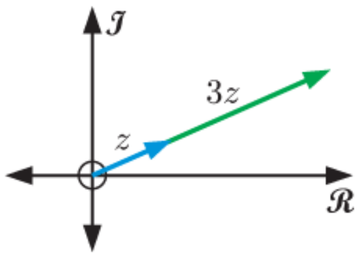
2 **a**



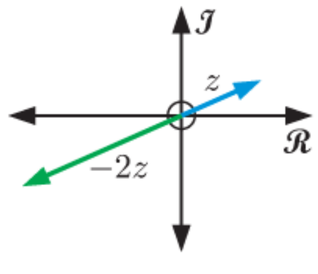
b



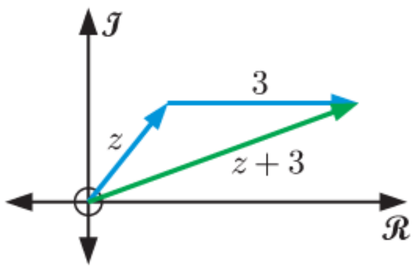
3 **a**



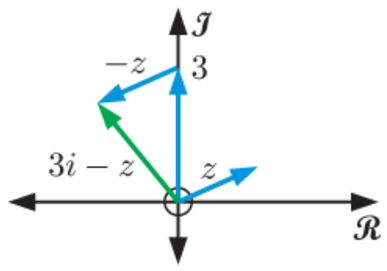
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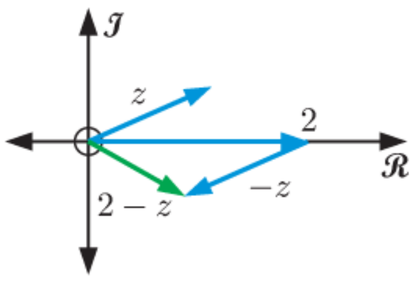
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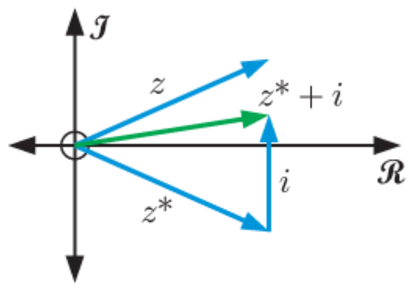
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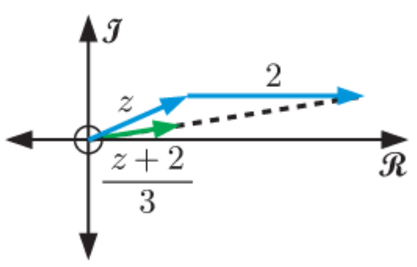
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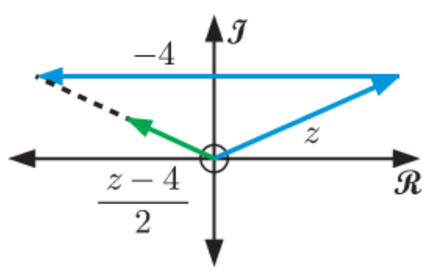
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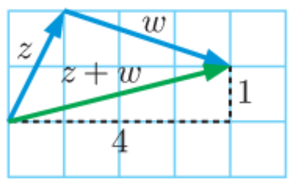
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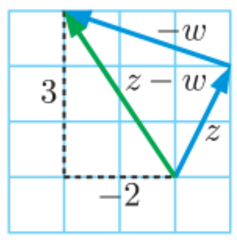
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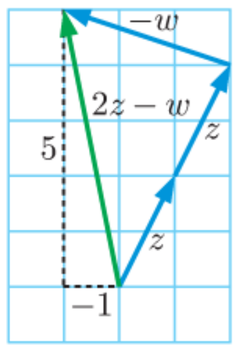
4 **a** $4+i$



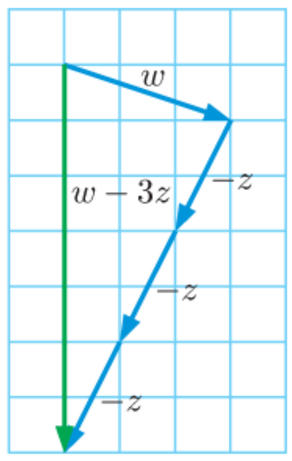
b $-2+3i$



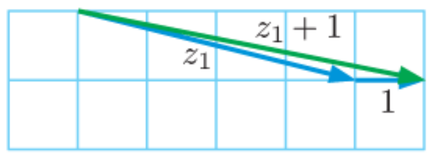
c $-1+5i$



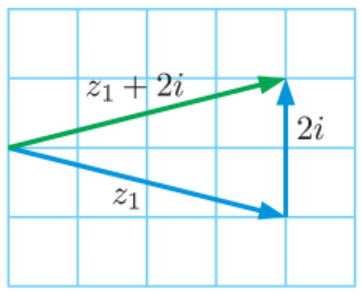
d $-7i$



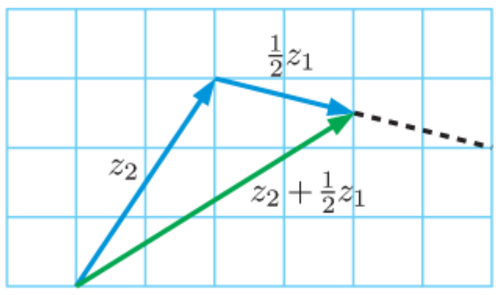
5 **a** $5-i$



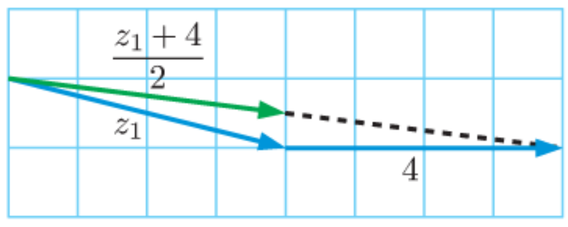
b $4+i$



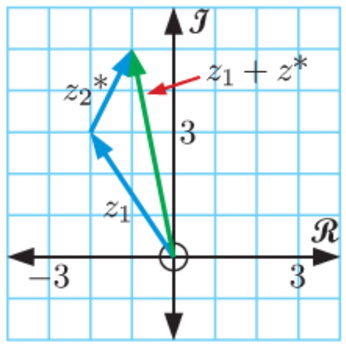
c $4+\frac{5}{2}i$



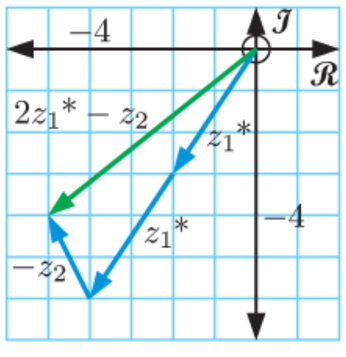
d $4-\frac{1}{2}i$



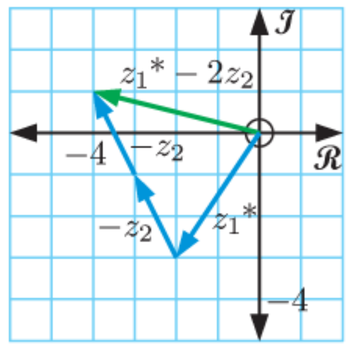
6 **a** $-1+5i$



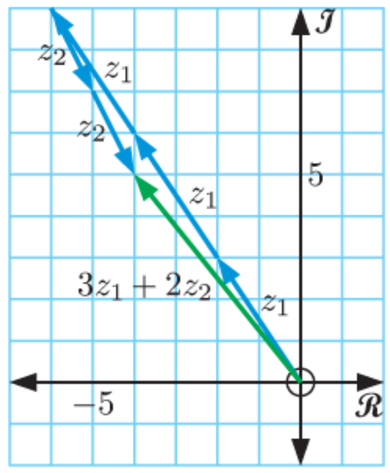
b $-5-4i$



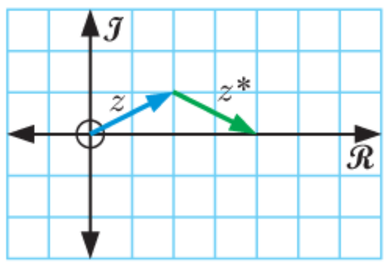
c $-4+i$



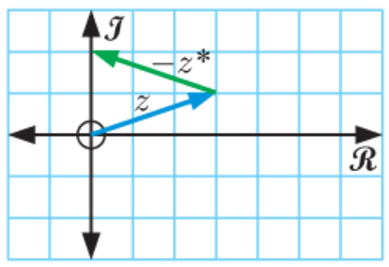
d $-4+5i$



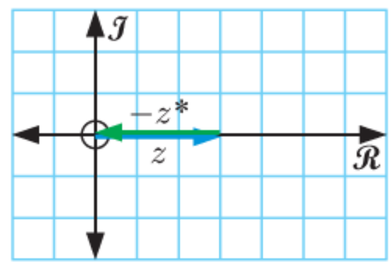
7



8

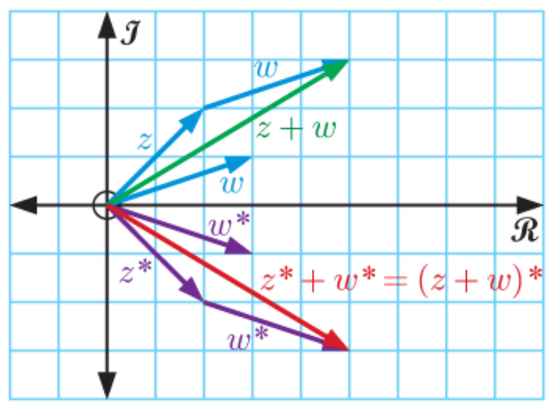


$z-z^*$ is purely imaginary when $\mathcal{Im}(z) \neq 0$

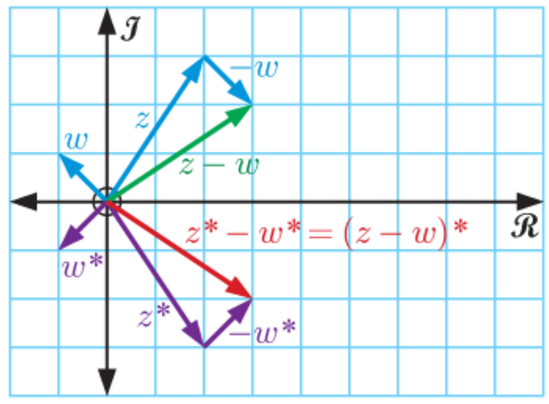


$z-z^*=0$

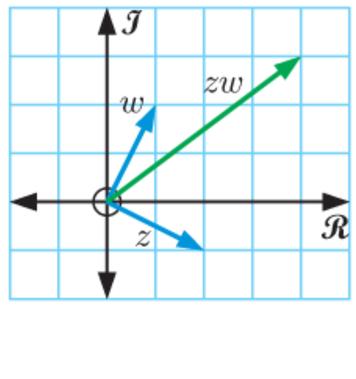
9 **a**



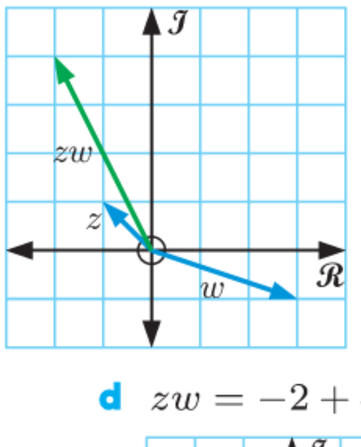
b



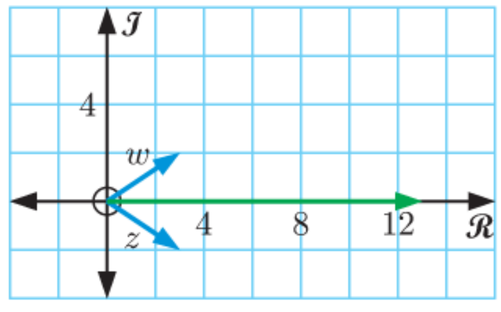
10 **a** $zw=4+3i$



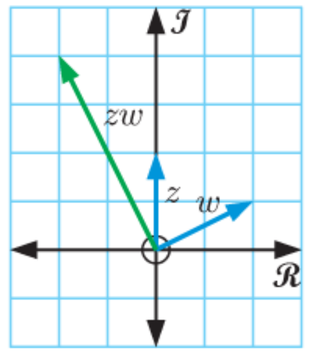
b $zw=-2+4i$



c $zw=13$



d $zw=-2+4i$



EXERCISE 14B

1 **a** 5 **b** 13 **c** $2\sqrt{17}$ **d** 3 **e** 4

2 **a** $|z| = \sqrt{5}$, $\arg z \approx 0.464$
b $|z| = \sqrt{5}$, $\arg z \approx -0.464$
c $|z| = \sqrt{5}$, $\arg z \approx 2.68$
d $|z| = \sqrt{5}$, $\arg z \approx -2.68$

3 **a** $|w| = \sqrt{13}$, $\arg w \approx 0.588$
b $|w| = \sqrt{13}$, $\arg w \approx 2.55$
c $|w| = \sqrt{13}$, $\arg w \approx -0.983$
d $|w| = \sqrt{13}$, $\arg w \approx 2.16$

4 $a = -\sqrt{3}$

5 **a** $\sqrt{5}$ **b** $\sqrt{5}$ **c** 5 **d** 5 **e** $5\sqrt{2}$ **f** $5\sqrt{2}$
g $\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$ **i** 5 **j** 5 **k** $5\sqrt{5}$ **l** $5\sqrt{5}$

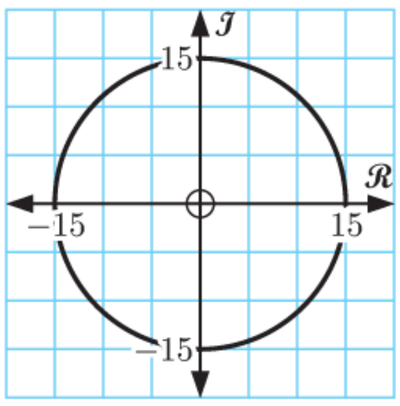
7 **a** 1 **b** $|r|$ **9** **c** $\left(\frac{5}{4}\right)^{30} \approx 808$

10 **a** 6 **b** 4 **c** $2\sqrt{5}$ **d** 2 **e** $\frac{1}{2}$ **f** $\frac{4}{3}$

11 **a** $w = \left(\dfrac{a^2+b^2-1}{a^2+(b-1)^2}\right) + \left(\dfrac{2a}{a^2+(b-1)^2}\right)i$ **b** 0

12 $z = 2 + 6i$ **13** **a** 8 **b** 6 **c** 10 **d** 10

14 **a** 15 **b**



16 **a** $4 + 5i$ **b** $|z+w| = \sqrt{41}$, $\arg(z+w) \approx 0.896$
c $\theta \approx 0.353$

17 **a** $y = x$ **b** $y = \frac{1}{\sqrt{3}}x + 1,\ x > 0$ **c** $7x^2 + 16y^2 = 112$

EXERCISE 14C

1 **a** **i** $4\sqrt{2}$ units **ii** $(1, 4)$
b **i** $5\sqrt{5}$ units **ii** $(-\frac{3}{2}, 2)$

2 **a** $\sqrt{32 - 32\cos\frac{\pi}{5}}$
b perimeter $= \sqrt{32 - 32\cos\frac{\pi}{5}} + 8$ units
area $= 8\sin\frac{\pi}{5}$ units^2

3 **a** reflection in the $\mathcal{R}$-axis **b** rotation of π about O
c reflection in the $\mathcal{I}$-axis **d** clockwise rotation of $\frac{\pi}{2}$ about O

EXERCISE 14D.1

1 **a** $4\operatorname{cis}0$ **b** $4\operatorname{cis}\frac{\pi}{2}$ **c** $6\operatorname{cis}\pi$ **d** $3\operatorname{cis}\left(-\frac{\pi}{2}\right)$
e $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **f** $2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **g** $2\operatorname{cis}\frac{5\pi}{6}$ **h** $4\operatorname{cis}\frac{\pi}{6}$

2 0, no angle exists with the positive real axis.

3 $k\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ if $k > 0$, $-k\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ if $k < 0$,
not possible if $k = 0$

4 **a** $2i$ **b** $4\sqrt{2} + 4\sqrt{2}i$ **c** $2\sqrt{3} + 2i$
d $1 - i$ **e** $-\frac{\sqrt{3}}{2} + \frac{3}{2}i$ **f** -5

5 **a** $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ **b** 1 **c** 1

6 **a** $\approx -1.41 + 1.01i$ **b** $\approx 1.27 - 3.06i$
c $\approx -2.55 - 1.25i$

7 **a** $\approx 5\operatorname{cis}(-0.927)$ **b** $\approx 13\operatorname{cis}(-1.97)$
c $\approx 17.7\operatorname{cis}(2.29)$

8 **a** $2\operatorname{cis}\frac{\pi}{4}$ **b** $\approx \sqrt{19}\operatorname{cis}(-2.50)$

EXERCISE 14D.2

1 **a** $\operatorname{cis}3\theta$ **b** $\operatorname{cis}2\theta$ **c** $\operatorname{cis}3\theta$ **d** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$
e $\sqrt{2} + i\sqrt{2}$ **f** 8 **g** $-2i$ **h** -4 **i** $4i$

2 **a** $\operatorname{cis}\pi = -1$ **b** $\operatorname{cis}\pi = -1$ **c** $\operatorname{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

3 **a** $|z| = 2$, $\arg z = \theta$ **b** $2\operatorname{cis}(-\theta)$
c $2\operatorname{cis}(\theta + \pi)$ **d** $2\operatorname{cis}(\pi - \theta)$

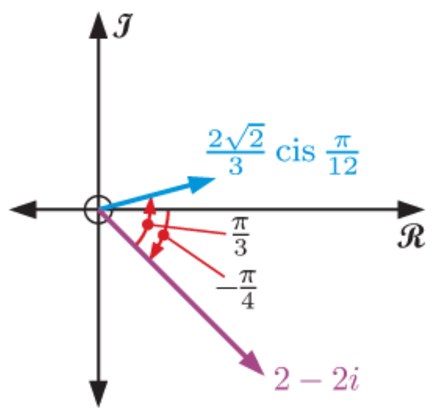
4 **a** $\operatorname{cis}\left(-\frac{\pi}{2}\right)$ **b** $r\operatorname{cis}\left(\theta - \frac{\pi}{2}\right)$
c A clockwise rotation through $\frac{\pi}{2}$ about the origin.

5 **a** $\operatorname{cis}\pi$ **b** $-z = r\operatorname{cis}(\theta + \pi)$
c An anticlockwise rotation through π about the origin.

6 z^* is the reflection of z in the $\mathcal{R}$-axis.
$\therefore\ z^* = r\operatorname{cis}(-\theta)$

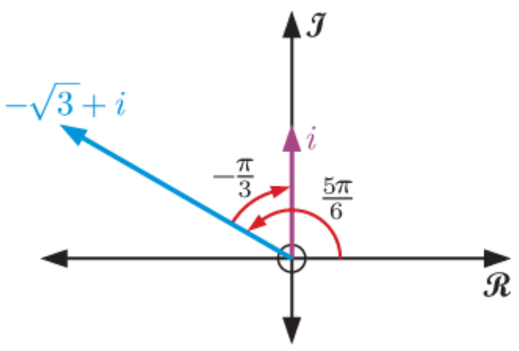
EXERCISE 14D.3

1 **a** $z = 2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **c**
b $\frac{2\sqrt{2}}{3}\operatorname{cis}\frac{\pi}{12}$



d z has been dilated with scale factor $\frac{1}{3}$, then rotated anticlockwise through $\frac{\pi}{3}$ about the origin.

2 **a** $z = 2\operatorname{cis}\frac{5\pi}{6}$ **c**
b $\operatorname{cis}\frac{\pi}{2} = i$



d z has been dilated with scale factor $\frac{1}{2}$, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

3 **a** $z = \sqrt{2}\operatorname{cis}\frac{3\pi}{4}$, $w = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ **b** $2\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$
c z has been dilated with scale factor 2, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

4 **a** $\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$
b $\cos\frac{11\pi}{12} = \frac{-\sqrt{2}-\sqrt{6}}{4}$, $\sin\frac{11\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

7 **a** $|-z| = 3$, $\arg(-z) = \theta - \pi$
b $|z^*| = 3$, $\arg(z^*) = -\theta$
c $|iz| = 3$, $\arg(iz) = \theta + \frac{\pi}{2}$
d $|(1+i)z| = 3\sqrt{2}$, $\arg[(1+i)z] = \theta + \frac{\pi}{4}$
e $\left|\frac{z}{i}\right| = 3$, $\arg\left(\frac{z}{i}\right) = \theta - \frac{\pi}{2}$
f $\left|\frac{z}{1-i}\right| = \frac{3}{\sqrt{2}}$, $\arg\left(\frac{z}{1-i}\right) = \theta + \frac{\pi}{4}$

8 **a** $|z-1| = 2\sin\frac{\phi}{2}$, $\arg(z-1) = \frac{\phi}{2} + \frac{\pi}{2}$
b $z - 1 = \left(2\sin\frac{\phi}{2}\right)\operatorname{cis}\left(\frac{\phi}{2} + \frac{\pi}{2}\right)$,
$(z-1)^* = \left(2\sin\frac{\phi}{2}\right)\operatorname{cis}\left(-\frac{\phi}{2} - \frac{\pi}{2}\right)$

9 **a** $z_2 - z_1 = \overrightarrow{AB}, \quad z_3 - z_2 = \overrightarrow{BC}$

b $\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = 1, \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \frac{2\pi}{3}$ **c** 1

10 **a** $a(x^2 + 2x + 4) = 0, \ a \neq 0$

b $a(x^2 - 2x + 2) = 0, \ a \neq 0$

EXERCISE 14E

1 **a** -1 **b** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **c** $-i$

2 **a** $\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}, \quad \sqrt{2}e^{i\frac{3\pi}{4}}$ **b** $\frac{3\sqrt{3}}{2} - \frac{3}{2}i, \quad 3e^{-i\frac{\pi}{6}}$

c $-1 + \sqrt{3}i, \quad 2\,\text{cis}\,\frac{2\pi}{3}$

4 **a** $\frac{\theta}{2}$ **b** $4\theta + \frac{\pi}{2}$ **c** $\frac{2}{5}\theta - \frac{\pi}{2}$ **d** $\frac{\pi}{2} - \frac{1}{3}\theta$

5 **a** $\left|\sqrt{w}\right| = \sqrt{3}, \quad \arg(\sqrt{w}) = \frac{\phi}{2}$

b $\left|\frac{z}{w^2}\right| = \frac{2}{9}, \quad \arg\left(\frac{z}{w^2}\right) = \theta - 2\phi$

c $\left|w^3 \times \sqrt{z}\right| = 27\sqrt{2}, \quad \arg(w^3 \times \sqrt{z}) = 3\phi + \frac{1}{2}\theta$

6 **a** $\approx 0.540 + 0.841i$ **b** $\approx 0.455 + 0.891i$ **c** $e^{-\frac{\pi}{2}}$

d -1

EXERCISE 14F

2 **a** $32\,\text{cis}\,2\pi = 32$ **b** $\text{cis}\,3\pi = -1$

c $64\,\text{cis}\,\frac{3\pi}{2} = -64i$ **d** $\sqrt{5}\,\text{cis}\,\frac{\pi}{14} \approx 2.18 + 0.498i$

e $2\,\text{cis}\,\frac{\pi}{6} = \sqrt{3} + i$ **f** $32\,\text{cis}\,\frac{\pi}{3} = 16 + 16\sqrt{3}i$

3 **a** $128 - 128i$ **b** $1024 + 1024\sqrt{3}i$

c $\frac{1}{524\,288}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$ **d** $\frac{1}{64}(1 - i)$

e $\sqrt{2}\cos\left(-\frac{\pi}{12}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{12}\right)$ **f** $\frac{1}{64}(-\sqrt{3} - i)$

4 **a** $|z|^{\frac{1}{2}}\,\text{cis}\,\frac{\theta}{2}$ **b** $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$ **c** true **5** $\text{cis}\,3\theta$

6 **a** $z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}, \quad z^n = 2^{\frac{n}{2}}\,\text{cis}\,\frac{n\pi}{4}$

b **i** $n = 4k, \ k \in \mathbb{Z}$ **ii** $n = 2 + 4k, \ k \in \mathbb{Z}$

7 **a** $\left|z^3\right| = 8, \quad \arg(z^3) = 3\theta$

b $\left|iz^2\right| = 4, \quad \arg(iz^2) = 2\theta + \frac{\pi}{2}$

c $\left|\frac{1}{z}\right| = \frac{1}{2}, \quad \arg\left(\frac{1}{z}\right) = -\theta$

d $\left|-\frac{i}{z^2}\right| = \frac{1}{4}, \quad \arg\left(-\frac{i}{z^2}\right) = -2\theta - \frac{\pi}{2}$

9 **a** $\overrightarrow{AB} \equiv z_2 - z_1, \quad \overrightarrow{BC} \equiv z_3 - z_2$

Hint: Notice that $\overrightarrow{BC}$ is a 90° rotation of $\overrightarrow{BA}$ about B.

b $\overrightarrow{OD} \equiv z_1 + z_3 - z_2$

10 **b** $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

c **i** $x = \frac{1}{\sqrt{2}}, \ \cos\frac{5\pi}{12}, \ \cos\frac{11\pi}{12}$

ii $x = \tan\frac{\pi}{9}, \ \tan\frac{4\pi}{9}, \ \tan\frac{7\pi}{9}$

11 **a** $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

b $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

12 **a** **iii** $\left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

b **Hint:** When $n = 1$, $2i\sin\theta = z - \frac{1}{z}$. Now cube both sides.

13 **a** $r^6\,\text{cis}\,6\theta$

c **i** $k = 6, \quad \left|z \times z^2 \times z^3 \times \ldots. \times z^6\right| = 2^{21}$

ii **Hint:** $k(k+1)$ is even for all $k \in \mathbb{Z}$.

15 $\dfrac{\cos n\theta - \cos\theta - \cos[(n+1)\theta] + 1}{2 - 2\cos\theta}$

16 **b** $2^n\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$

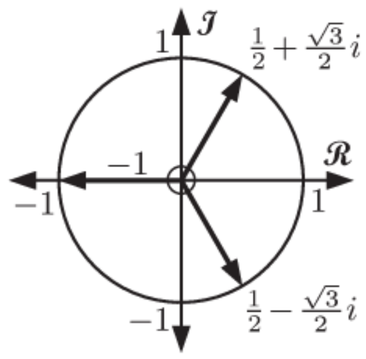
EXERCISE 14G.1

1 $1, \ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

2 **a** $z = \sqrt{3} - i, \ 2i, \ -\sqrt{3} - i$

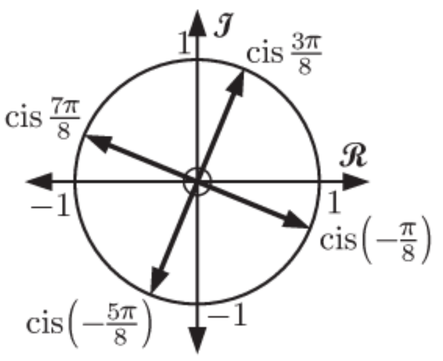
b $z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i, \ 3i, \ -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

3 $-1, \ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

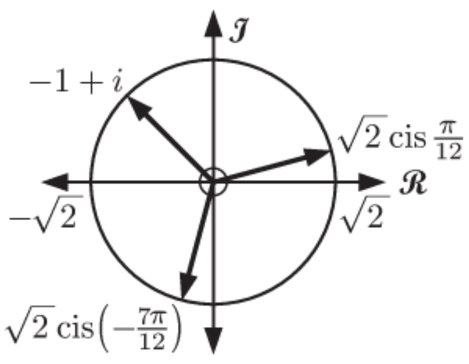


4 **a** $z = \pm 2, \ \pm 2i$ **b** $z = \sqrt{2} \pm i\sqrt{2}, \ -\sqrt{2} \pm i\sqrt{2}$

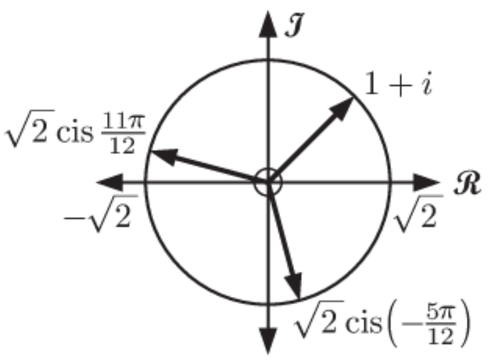
5 $\text{cis}\,\frac{3\pi}{8}, \ \text{cis}\,\frac{7\pi}{8}, \ \text{cis}\left(-\frac{\pi}{8}\right), \ \text{cis}\left(-\frac{5\pi}{8}\right)$



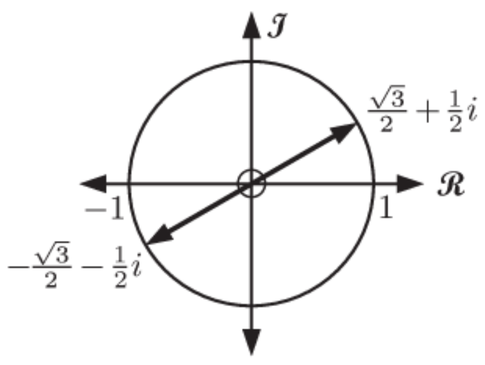
6 **a** $z = \sqrt{2}\,\text{cis}\,\frac{\pi}{12}, \ \sqrt{2}\,\text{cis}\,\frac{3\pi}{4} = -1 + i, \ \sqrt{2}\,\text{cis}\left(-\frac{7\pi}{12}\right)$



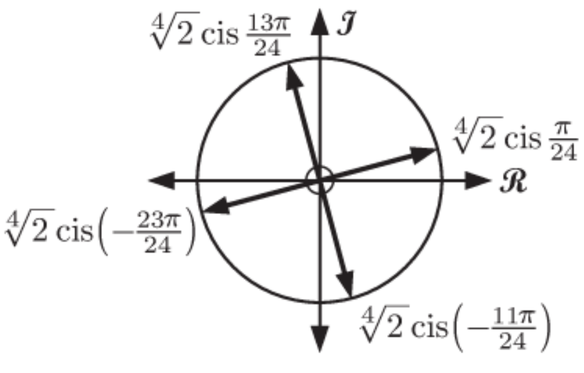
b $z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4} = 1 + i, \ \sqrt{2}\,\text{cis}\,\frac{11\pi}{12}, \ \sqrt{2}\,\text{cis}\left(-\frac{5\pi}{12}\right)$



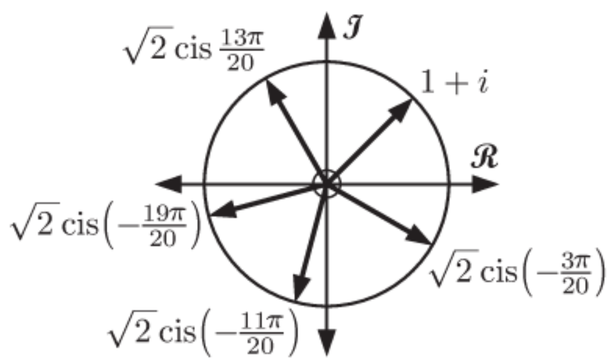
c $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$



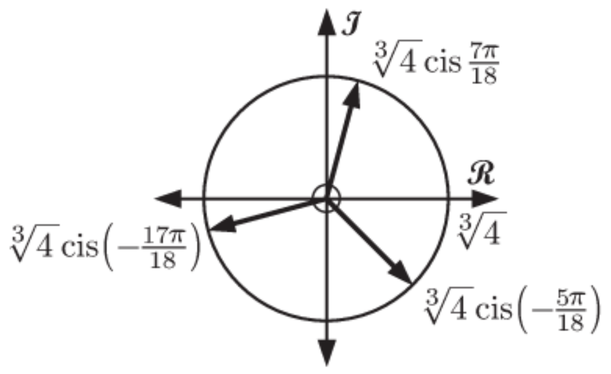
d $z = \sqrt[4]{2}\,\text{cis}\,\frac{\pi}{24}, \ \sqrt[4]{2}\,\text{cis}\,\frac{13\pi}{24}, \ \sqrt[4]{2}\,\text{cis}\left(-\frac{11\pi}{24}\right), \ \sqrt[4]{2}\,\text{cis}\left(-\frac{23\pi}{24}\right)$



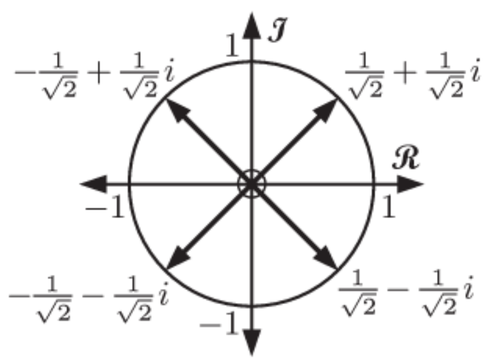
e $z = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{20}\right)$, $\sqrt{2}\,\text{cis}\,\frac{\pi}{4} = 1 + i$, $\sqrt{2}\,\text{cis}\,\frac{13\pi}{20}$, $\sqrt{2}\,\text{cis}\left(-\frac{11\pi}{20}\right)$, $\sqrt{2}\,\text{cis}\left(-\frac{19\pi}{20}\right)$



f $z = \sqrt[3]{4}\,\text{cis}\left(-\frac{5\pi}{18}\right)$, $\sqrt[3]{4}\,\text{cis}\,\frac{7\pi}{18}$, $\sqrt[3]{4}\,\text{cis}\left(-\frac{17\pi}{18}\right)$



7 **a** $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, or $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



b $z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$

8 **a** $|z| = 1$ and $\arg z = -\frac{2\pi}{3}$ **b** $z^3 = \text{cis}\,(-2\pi) = 1$

c Simplifies to $2(z + z^*) - 5$ where $z + z^*$ is always real.

9 **a** $16\,\text{cis}\left(-\frac{\pi}{2}\right)$ **b** **i** $2\,\text{cis}\,\frac{7\pi}{8}$ **ii** $2\cos\frac{7\pi}{8} + [2\sin\frac{7\pi}{8}]i$

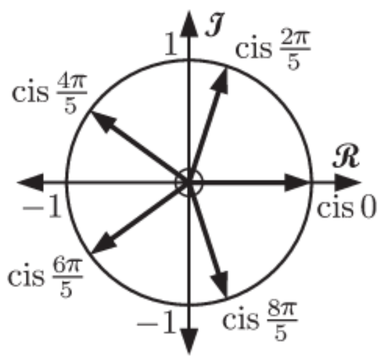
EXERCISE 14G.2

1 **a** **i** $z = w^n - 3$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

ii $z = 2w^n + 1$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

iii $z = \dfrac{1 - w^n}{2}$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

3 **a** $z = \text{cis}\,0 = 1$, $\text{cis}\,\frac{2\pi}{5}$, $\text{cis}\,\frac{4\pi}{5}$, $\text{cis}\,\frac{6\pi}{5}$, $\text{cis}\,\frac{8\pi}{5}$

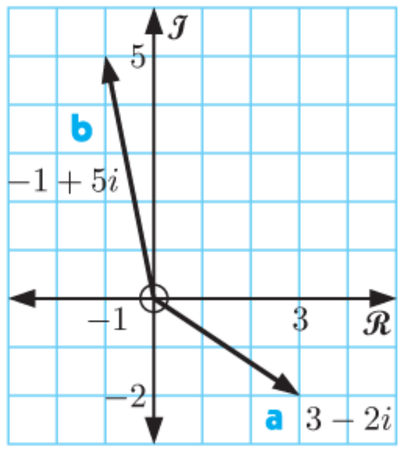


c $1 - w^5$

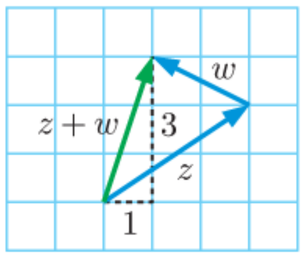
4 **b** **ii** **Hint:** The LHS is a geometric series.

REVIEW SET 14A

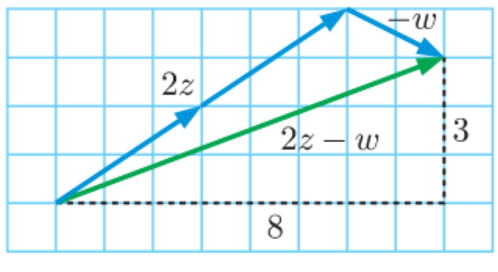
1



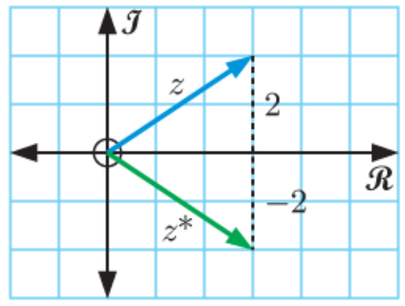
2 **a** $1 + 3i$



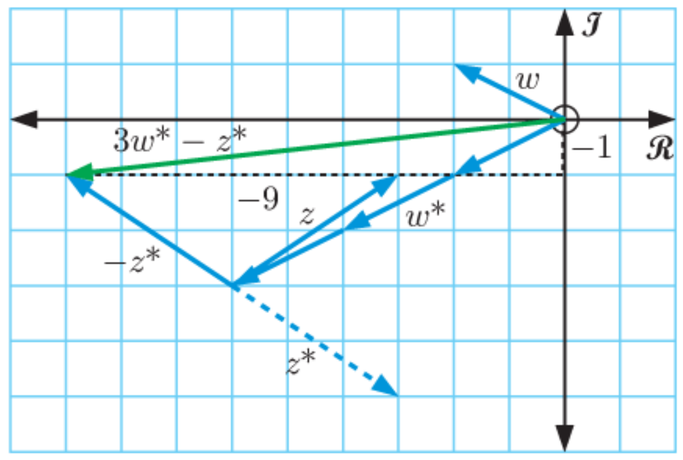
b $8 + 3i$



c $3 - 2i$



d $-9 - i$



3 **a** $|z| = \sqrt{29}$, $\arg z \approx 0.381$

b $|z| = \sqrt{29}$, $\arg z \approx 2.76$

c $|z| = \sqrt{29}$, $\arg z \approx -0.381$

d $|z| = \sqrt{29}$, $\arg z \approx 1.95$

4 **a** 25 **b** 20 **c** $5\sqrt{10}$ **d** $\frac{1}{5}$ **e** $\frac{3}{25}$ **f** 1

5 **a** A reflection in the $\mathcal{R}$-axis, followed by a stretch with scale factor 2.

b A rotation of π about O, followed by a stretch with scale factor $\frac{1}{2}$.

c An anticlockwise rotation of $\frac{\pi}{2}$ about O, followed by a reflection in the $\mathcal{R}$-axis.

7 Real part is $16\sqrt{3}$. Imaginary part is 16.

9 $2\sqrt{3} - 2i$, $-2\sqrt{3} - 2i$, $4i$

10 **a** $\left|(2z)^{-1}\right| = \frac{1}{2}$, $\arg\left[(2z)^{-1}\right] = -\theta$

b $|1 - z| = 2\sin\frac{\theta}{2}$, $\arg(1 - z) = \frac{\theta}{2} - \frac{\pi}{2}$

11 $-1 + i\sqrt{3} = 2\,\text{cis}\,\frac{2\pi}{3}$, $m = \dfrac{3k}{2}$, $k \in \mathbb{Z}$ **12** $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

13 **a** $5 + 2i$ **b** $2\sqrt{2}$ **c** 17^5 **d** ≈ -2.03

14 **a** $x = 0$, $y > 1$ **b** $3x^2 + 3y^2 - 20x + 12 = 0$

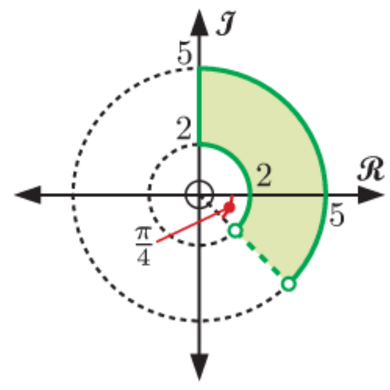
15 $4\,\text{cis}\left(-\frac{\pi}{3}\right)$, $n = 3(k + \frac{1}{2})$, $k \in \mathbb{Z}$ **16** $\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$, -3

17 **a** $\left|z^3\right| = 64$, $\arg\left(z^3\right) = 3\theta$

b $\left|\frac{1}{z}\right| = \frac{1}{4}$, $\arg\left(\frac{1}{z}\right) = -\theta$

c $|iz^*| = 4$, $\arg(iz^*) = \frac{\pi}{2} - \theta$

18



20 $1 + z = 2\cos\frac{\alpha}{2}\,\text{cis}\,\frac{\alpha}{2}$, $|1 + z| = 2\cos\frac{\alpha}{2}$, $\arg(1 + z) = \frac{\alpha}{2}$

21 **b** $\left|\dfrac{z_2 - z_1}{z_3 - z_2}\right| = 1$, $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \frac{\pi}{3}$

22 $1, w, w^2, w^3, w^4$ where $w = \text{cis}\,\frac{2\pi}{5}$

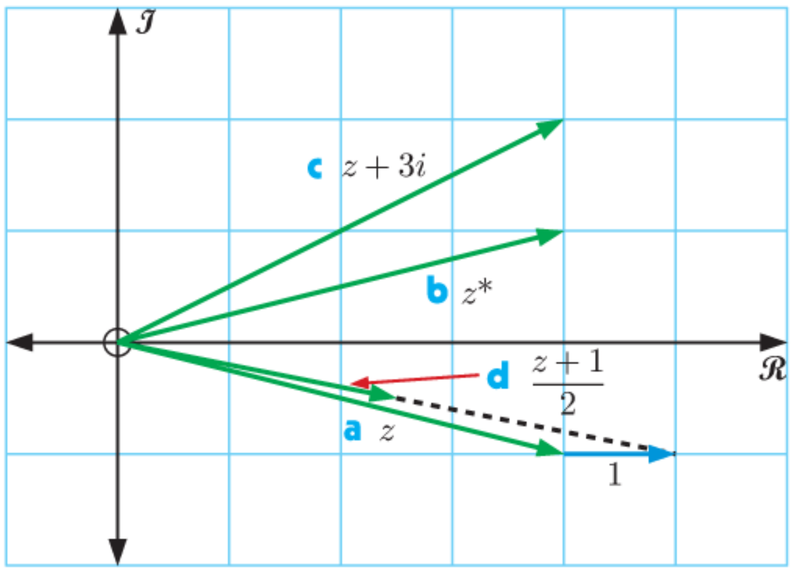
a $z = \frac{3}{2}$, $w + \frac{1}{2}$, $w^2 + \frac{1}{2}$, $w^3 + \frac{1}{2}$, $w^4 + \frac{1}{2}$, $w = \text{cis}\,\frac{2\pi}{5}$

b $z = 0$, $w - 1$, $w^2 - 1$, $w^3 - 1$, $w^4 - 1$, $w = \text{cis}\,\frac{2\pi}{5}$

c $z = \dfrac{w + 1}{w - 1}$, $\dfrac{w^2 + 1}{w^2 - 1}$, $\dfrac{w^3 + 1}{w^3 - 1}$, $\dfrac{w^4 + 1}{w^4 - 1}$, $w = \text{cis}\,\frac{2\pi}{5}$

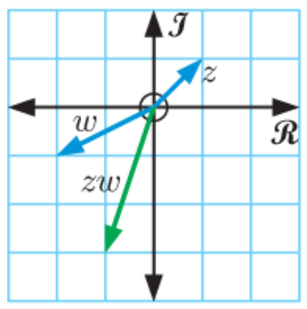
REVIEW SET 14B

1



2 **a** 12 **b** 8 **c** $\frac{\sqrt{2}}{4}$ **d** $4\sqrt{10}$

3 $zw = -1 - 3i$



4 **a** $\sqrt{10}$ **b** $\sqrt{10}$ **c** 10 **d** 10 **e** 10
f 1 **g** 10 **h** $10\sqrt{10}$

5 $x^2 + y^2 + 6x = 0$

7 **a** $5\,\text{cis}\left(-\frac{\pi}{2}\right)$ **b** $4\,\text{cis}\left(-\frac{\pi}{3}\right)$ **c** $-k\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$, $k < 0$

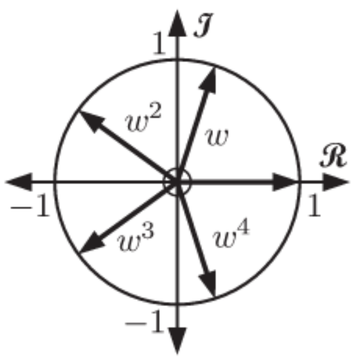
8 $b = \frac{1}{\sqrt{3}}$

9 **b** $(1-i)z = 4\,\text{cis}\left(\alpha - \frac{\pi}{4}\right)$, $\arg[(1-i)z] = \alpha - \frac{\pi}{4}$

10 **a** $\left|\frac{z_1^2}{z_2^2}\right| = 1$, $\arg\left(\frac{z_1^2}{z_2^2}\right) = \pi$

11 $\mathcal{Re}\left(\left(\frac{z}{w}\right)^4\right) = \frac{a}{2b}$, $\mathcal{Im}\left(\left(\frac{z}{w}\right)^4\right) = -\frac{a\sqrt{3}}{2b}$

12 **a** 1, w, w^2, w^3, and w^4 where $w = \text{cis}\,\frac{2\pi}{5}$



c 31

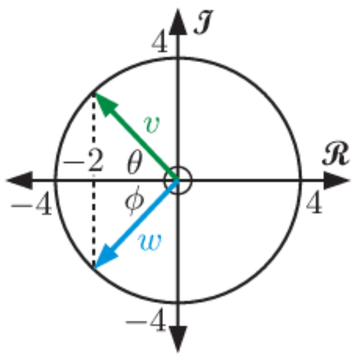
13 **a** $(\text{cis}\,\theta)^3$ **b** $(\text{cis}\,\theta)^{-2}$ **c** $(\text{cis}\,\theta)^{-1}$

14 $z = 2^{0.3}\,\text{cis}\,\frac{\pi}{20}$, $2^{0.3}\,\text{cis}\,\frac{9\pi}{20}$, $2^{0.3}\,\text{cis}\,\frac{17\pi}{20}$,
$2^{0.3}\,\text{cis}\left(-\frac{7\pi}{20}\right)$, $2^{0.3}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

18 **a** $\frac{5\pi}{12}$ **b** $-\frac{11\pi}{12}$

20 **a** $a(z^2 - 2\cos\frac{2\pi}{5}z + 1) = 0$, $a \neq 0$
b $a(z^2 + z - 1) = 0$, $a \neq 0$

22 **b** **i**



ii $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$
$\therefore \arg v = \frac{2\pi}{3}$

iii $\arg w = -\frac{2\pi}{3}$

iv $\arg\left(\frac{v^m w}{i}\right) = \frac{\pi(4m-7)}{6}$

v $m = \frac{7}{4}$

EXERCISE 15A

1 **a** 7 **b** 7 **c** 11 **d** 16 **e** 0 **f** 5

2 **a** 5 **b** 7 **c** c **3** **a** -2 **b** 7 **c** -1

4 As $x \neq 0$, we can cancel the xs, to give

$$\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1.$$

5 **a**

x	$\frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.999 99	3.999 99

x	$\frac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.000 01	4.000 01

$x^2 - 4$

b $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

6 **a** -3 **b** 5 **c** -1 **d** 6 **e** -4 **f** -8
g 1 **h** 2 **i** 5

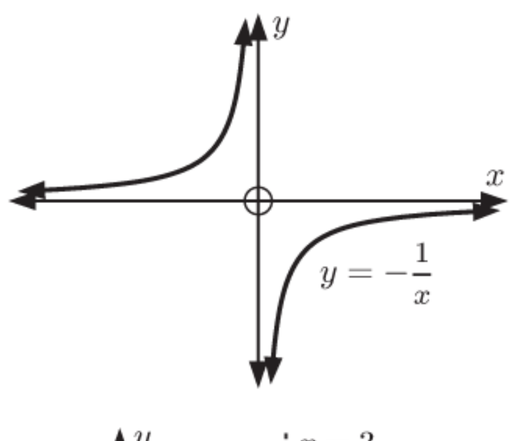
7 $\lim_{x\to a} f(x) = \lim_{x\to a} l$

$\therefore \lim_{x\to a} f(x) - \lim_{x\to a} l = 0$

$\therefore \lim_{x\to a} (f(x) - l) = 0$

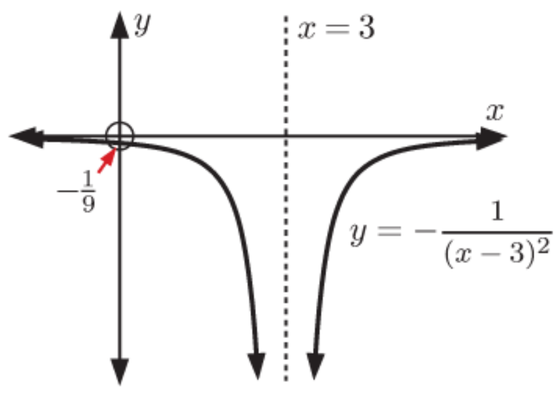
EXERCISE 15B

1 **a**



b **i** -1
ii 1
iii does not exist

2 **a**

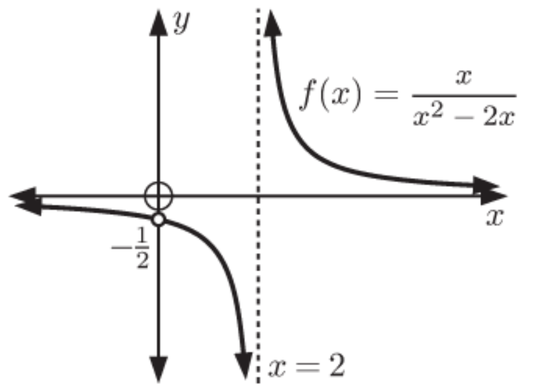


b **i** $-\frac{1}{9}$
ii does not exist
iii $-\frac{1}{4}$

3 **a** -1 **b** 0

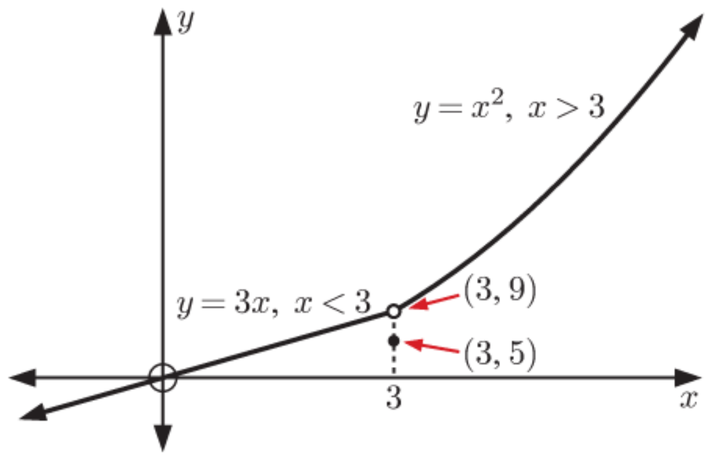
4 **a** $x = 0$ or 2

b



c **i** does not exist
ii -1
iii -2

5 **a**



b 9

c yes, $\lim_{x\to 3} f(x) = 9$ since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 9$

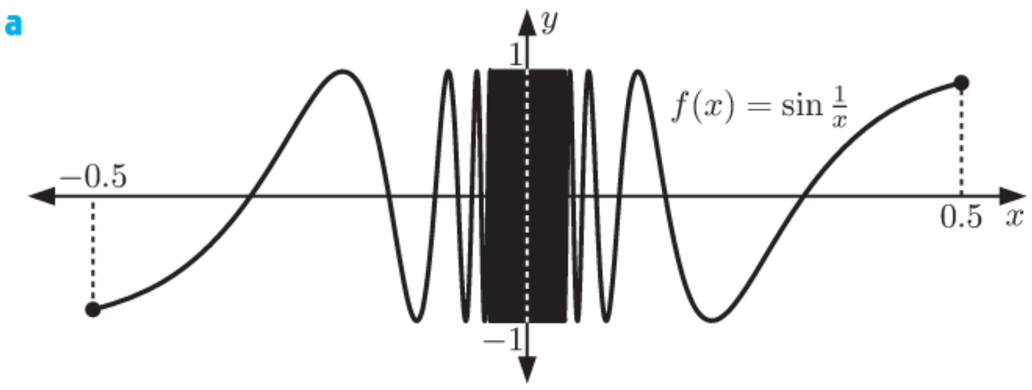
6 No, since $\lim_{x\to 0^-} \sqrt{x}$ does not exist.

7 **a** -1 **b** 1 **c** does not exist

8 **a** 0 **b** 1 **c** does not exist

9 **a** 0 **b** 0 **c** does not exist **d** does not exist

10 **a**



b $f(x)$ does not approach any value as $x \to 0$ from above or below. It oscillates over values between -1 and 1.

c $\lim_{x\to 0} f(x)$ does not exist since neither $\lim_{x\to 0^-} f(x)$ nor $\lim_{x\to 0^+} f(x)$ exist.

11 **a** **i** $\frac{1}{2}$ **ii** 0 **iii** 0

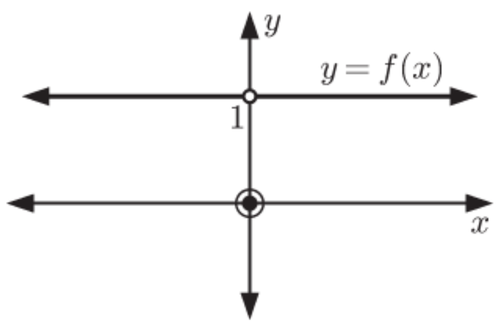
b **i**, **ii** The limits do not exist. As $x \to \frac{1}{2}$ and $x \to \sqrt{2}$ from either side, the function oscillates between x and 0, never approaching any value.

iii $\lim_{x\to 0} f(x) = 0$. As $x \to 0$ from either side, the function oscillates between x and 0, but eventually approaches 0. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$.

12 **a** 0

b $u_1 = \dfrac{x^2}{1+x^2}$, $r = \dfrac{1}{1+x^2}$, $f(x) = 1$ for all $x \neq 0$

c 1 **d**



EXERCISE 15C

1 0

2 **a** 3 **b** $-\frac{2}{3}$ **c** -1 **d** 1 **e** $\frac{1}{4}$ **f** 0
g 0 **h** does not exist **i** does not exist

3 **a** **i** as $x \to 0^-$, $f(x) \to -\infty$
as $x \to 0^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0^-$
as $x \to \infty$, $f(x) \to 0^+$
vertical asymptote $x = 0$, horizontal asymptote $y = 0$

ii $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to \infty} f(x) = 0$

b **i** as $x \to -3^-$, $f(x) \to \infty$
as $x \to -3^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to 3^+$
as $x \to \infty$, $f(x) \to 3^-$
vertical asymptote $x = -3$,
horizontal asymptote $y = 3$

ii $\lim_{x\to -\infty} f(x) = 3$, $\lim_{x\to \infty} f(x) = 3$

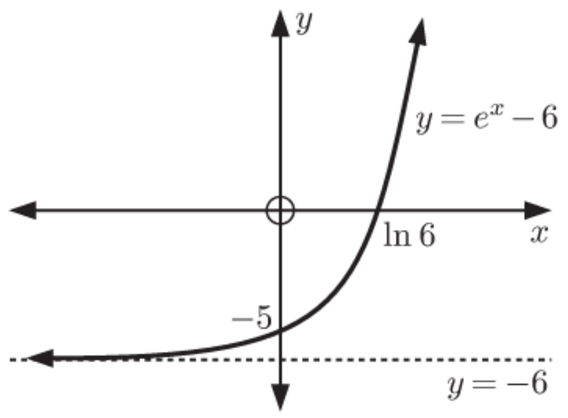
c **i** as $x \to -\frac{2}{3}^-$, $f(x) \to -\infty$
as $x \to -\frac{2}{3}^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -\frac{2}{3}^-$
as $x \to \infty$, $f(x) \to -\frac{2}{3}^+$
vertical asymptote $x = -\frac{2}{3}$,
horizontal asymptote $y = -\frac{2}{3}$

ii $\lim_{x\to -\infty} f(x) = -\frac{2}{3}$, $\lim_{x\to \infty} f(x) = -\frac{2}{3}$

d **i** as $x \to 1^-$, $f(x) \to \infty$,
as $x \to 1^+$, $f(x) \to -\infty$,
as $x \to -\infty$, $f(x) \to -1^+$
as $x \to \infty$, $f(x) \to -1^-$
vertical asymptote $x = 1$,
horizontal asymptote $y = -1$

ii $\lim_{x\to -\infty} f(x) = -1$, $\lim_{x\to \infty} f(x) = -1$

4 **a**



b **i** $\lim_{x\to -\infty} (e^x - 6) = -6$

ii $\lim_{x\to \infty} (e^x - 6)$ does not exist
$y = -6$ is a horizontal asymptote of $y = e^x - 6$.

5 **a**

x	xe^{-x}
10	$\approx 0.000\,454$
50	$\approx 9.64 \times 10^{-21}$
100	$\approx 3.72 \times 10^{-42}$
200	$\approx 2.77 \times 10^{-85}$

b $\lim_{x\to \infty} xe^{-x} = 0$

c yes

6 **a** $\lim_{x\to -\infty} (2e^{-x} - 3)$ does not exist
$\lim_{x\to \infty} (2e^{-x} - 3) = -3$

b $\lim_{x\to -\infty} (e^{-x} + 5e^{\frac{1}{x}})$ does not exist
$\lim_{x\to \infty} (e^{-x} + 5e^{\frac{1}{x}}) = 5$

7 $\frac{1}{2}$

EXERCISE 15D

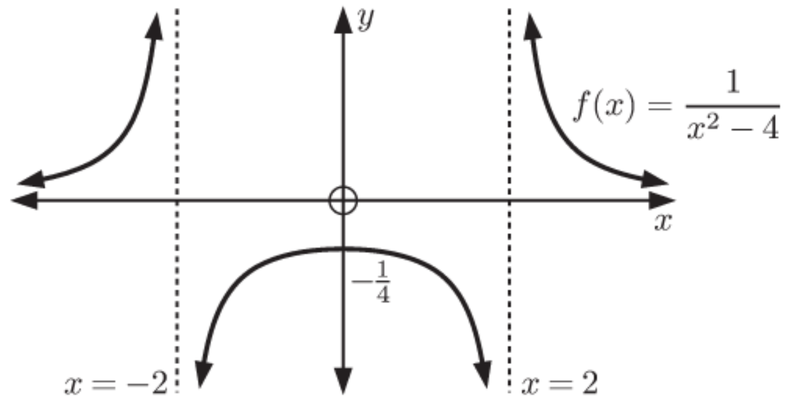
1 **a** 2 **b** 1 **c** 1 **d** 0 **e** 4
f $\frac{1}{2}$ **g** 3π **h** $\frac{7}{4}$ **i** $\frac{1}{2}$

2 **a** does not exist **b** 0 **c** $\frac{1}{2}$

EXERCISE 15E

1 **a** $x = 6$ **b** $x = 5$ **c** $x \neq 5$ or 6

2 **a**



b $f(x)$ is continuous for all $x \in \mathbb{R},\ x \neq \pm 2$

3 **a** $a > 0$ **b** $x \geqslant 0$

4 **a** f has an essential discontinuity at $x = -2$.
f is continuous for all $x \in \mathbb{R},\ x \neq -2$.

b f has a removable discontinuity at $x = -1$.
f is continuous for all $x \in \mathbb{R},\ x \neq -1$.
We could define the function $g(x) = x$.

c f has an essential discontinuity at $x = -2$ and a removable discontinuity at $x = 4$.
f is continuous for all $x \in \mathbb{R},\ x \neq -2$ or 4.
We could define the function $g(x) = \dfrac{1}{x+2}$.

d f has an essential discontinuity at $x = 0$.
f is continuous for all $x \in \mathbb{R},\ x \neq 0$.

e f has an essential discontinuity at $x = 3$.
f is continuous for all $x \in \mathbb{R},\ x \neq 3$.

f f has removable discontinuities at $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.
f is continuous for all $x \in \mathbb{R},\ x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.
We could define the function $g(x) = \sin x$.

5 **a** $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)} \quad \{x \neq 1\}$
$= 3$
$\therefore$ f is continuous on $\mathbb{R}$ if $k = 3$.

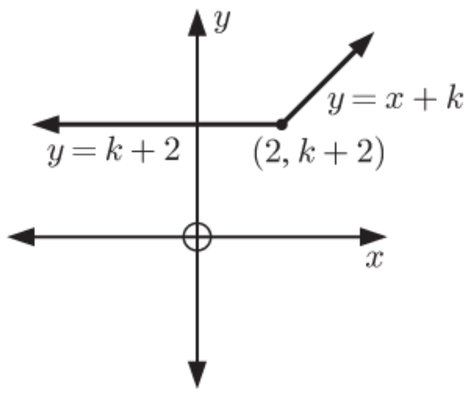
b $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
$\therefore$ f is continuous on $\mathbb{R}$ if $k = 1$.

c $f(2) = 2^2 = 4$
$\therefore$ f is continuous on $\mathbb{R}$ if $k = 3$.

d $f(x) = \dfrac{1}{x}$ has an essential discontinuity at $x = 0$.
$\therefore$ there is no $k \in \mathbb{R}$ such that f is continuous on $\mathbb{R}$.

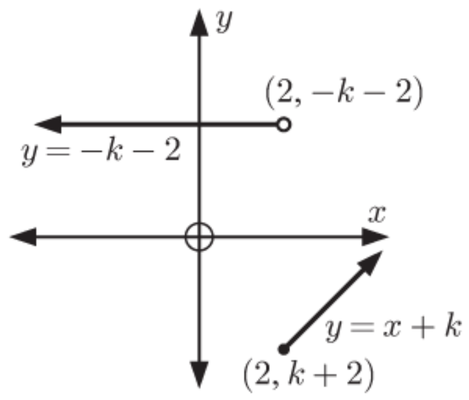
e $y = kx$ always passes through $(0, 0)$.
$\therefore$ f is continuous on $\mathbb{R}$ for all $k \in \mathbb{R}$.

f If $k + 2 \geqslant 0$:
$k \geqslant -2$,
and $f(x)$ has graph:



$\therefore$ $f(x)$ is continuous for all $x \in \mathbb{R}$

If $k + 2 < 0$:
$k < -2$,
and $f(x)$ has graph:



$\therefore$ $f(x)$ is discontinuous at $x = 2$

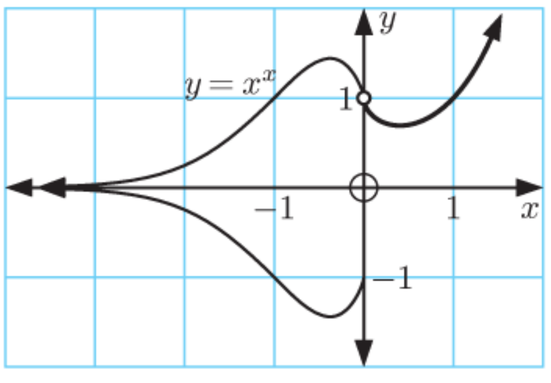
So, if $k \geqslant -2,\ k \in \mathbb{R},\ f$ is continuous on $\mathbb{R}$.
If $k < -2,\ k \in \mathbb{R},\ f$ is continuous for all $x \in \mathbb{R},\ x \neq 2$.

6 **a, b** **Hint:** Show that while $\lim\limits_{n \to \infty} x_n = a$,

$$\lim_{n \to \infty} g(x_n) \neq g(a).$$

c The Dirichlet function is continuous nowhere.

7 **a**

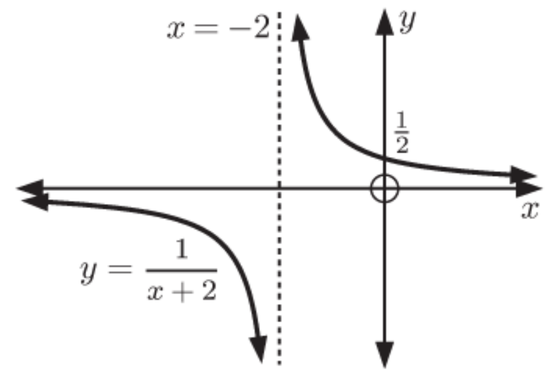


b continuous for $x > 0$, discontinuous for $x \leqslant 0$

REVIEW SET 15A

1 **a** -1 **b** -1 **c** 8

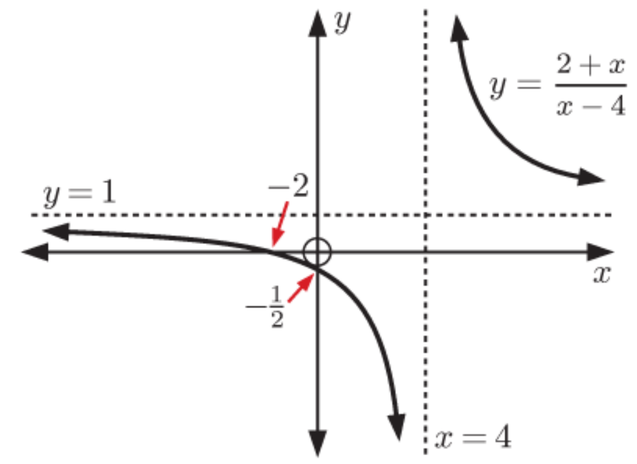
2 **a**



b **i** $\frac{1}{2}$ **ii** $\frac{1}{4}$ **iii** does not exist

3 **a** 4 **b** $\frac{2}{3}$ **c** π

4 **a**

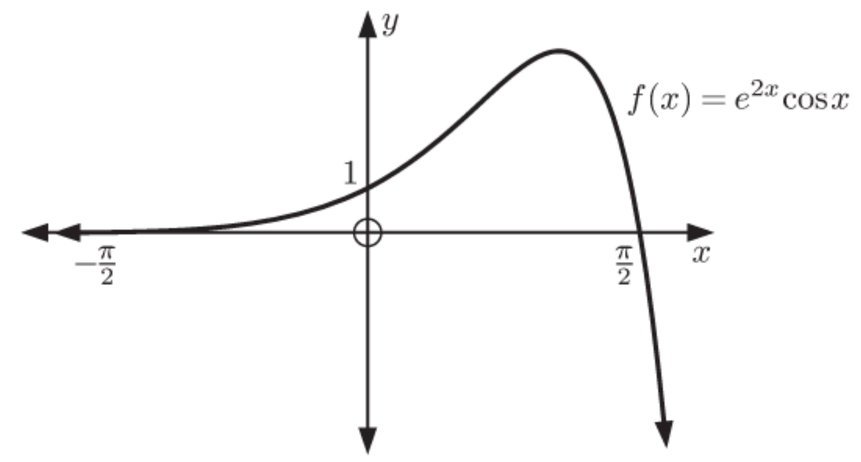


b as $x \to 4^-,\ y \to -\infty$, as $x \to -\infty,\ y \to 1^-$,
as $x \to 4^+,\ y \to \infty$, as $x \to \infty,\ y \to 1^+$
vertical asymptote $x = 4$, horizontal asymptote $y = 1$

c $\lim\limits_{x \to -\infty} \dfrac{2+x}{x-4} = 1$, $\lim\limits_{x \to \infty} \dfrac{2+x}{x-4} = 1$

5 **Hint:** First show that $\sin \frac{\theta}{2} = \dfrac{b}{2r}$.

6 **a**



b **i** 0 **ii** does not exist

7 **a** f has an essential discontinuity at $x = 0$.
f is continuous for all $x \in \mathbb{R},\ x \neq 0$.

b g has an essential discontinuity at $x = \frac{3}{4}$ and a removable discontinuity at $x = \frac{1}{2}$.
g is continuous for all $x \in \mathbb{R},\ x \neq \frac{1}{2}$ or $\frac{3}{4}$.

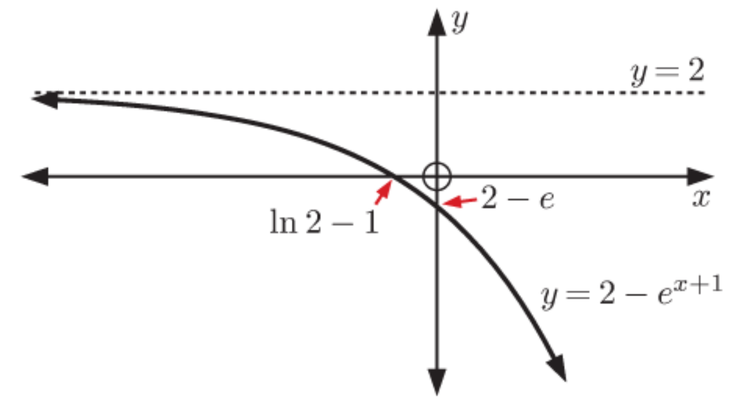
We could define the function $h(x) = \dfrac{1}{4x-3}$.

REVIEW SET 15B

1 **a** -3 **b** 3 **c** -1

2 **a** $x = 4, -3$ **b** **i** $-\frac{1}{7}$ **ii** does not exist

3 **a**



b $\lim_{x \to -\infty} (2 - e^{x+1}) = 2$,

$\lim_{x \to \infty} (2 - e^{x+1})$ does not exist

c $y = 2$

4 **a** Area of sector $= \frac{1}{2}r^2\theta$, area of triangle $= \frac{1}{2}r^2 \sin\theta$

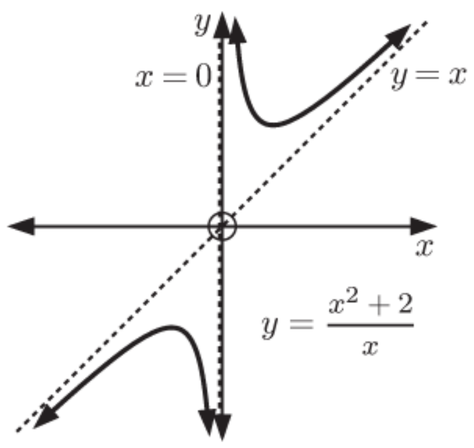
$\therefore$ shaded area $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

5 $k = -1 \pm \sqrt{2}$

6 **a** vertical asymptote $x = 0$, oblique asymptote $y = x$

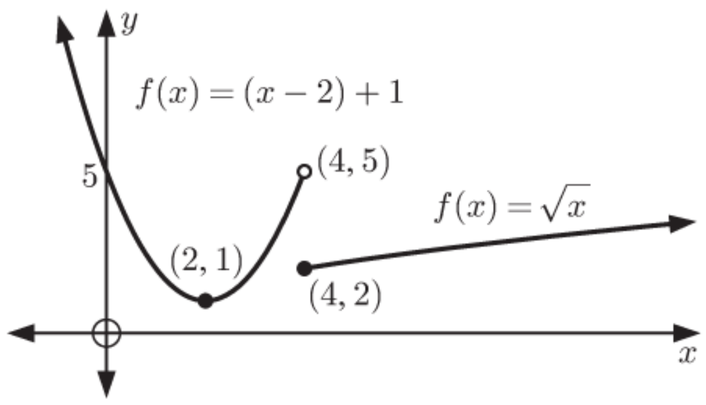
b



c **i** does not exist **ii** 0

d As $|x| \to \infty$, $\dfrac{x^2+2}{x} \to x$. This is shown by the oblique asymptote $y = x$.

7 **a**



b $\lim_{x \to 4^-} f(x) = 5$, $\lim_{x \to 4^+} f(x) = 2$

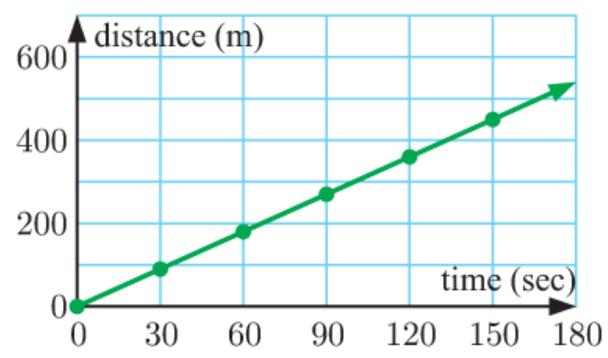
c essential discontinuity, $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$

8 **a** $x = n\pi$, $n \in \mathbb{Z}$ **b** $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

EXERCISE 16A.1

1 **a** Yes. The distance increases by the same amount each time interval.

b



c 3 m per s

2 **a** 3 **b** -2 **c** $-\frac{1}{4}$ **3** $\frac{5}{2}$

EXERCISE 16A.2

1 **a** No. The graph is not a straight line.

b **i** 60 km per hour **ii** 100 km per hour

2 **a** 100 m per hour **b** 100 m per hour (downwards)

3 **a** $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** $-\frac{5}{4}$ **d** -2

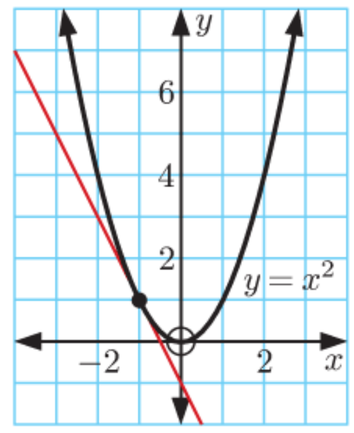
4 **a** **i** 3 **ii** 2.5 **iii** 2.1 **iv** 2.01 **v** 2.001

b The average rate of change approaches 2.

EXERCISE 16B

1 **a** 0.5 m s^{-1} **b** 2 m s^{-1} **2** **a** 1 **b** 4

3 **a**, **b**



c -2

EXERCISE 16C

1 **a** $(3+h)^2$ **b** $\dfrac{(3+h)^2 - 9}{(3+h) - 3} = 6 + h$ for $h \neq 0$

c **i** 7 **ii** 6.5 **iii** 6.1 **iv** 6.01 **d** 6

2 **a** **i** 2 **ii** 8

b

x-coordinate	*Gradient of tangent to* $f(x) = x^2$
1	2
2	4
3	6
4	8

c $2a$

3 **a** 5 **b** 3 **c** -1 **d** 4

EXERCISE 16D

1 **a** $f(0) = 4$ **b** $f'(0) = -1$ **2** $f'(2) = 1$

3 **a** positive **b** negative **c** negative **d** positive

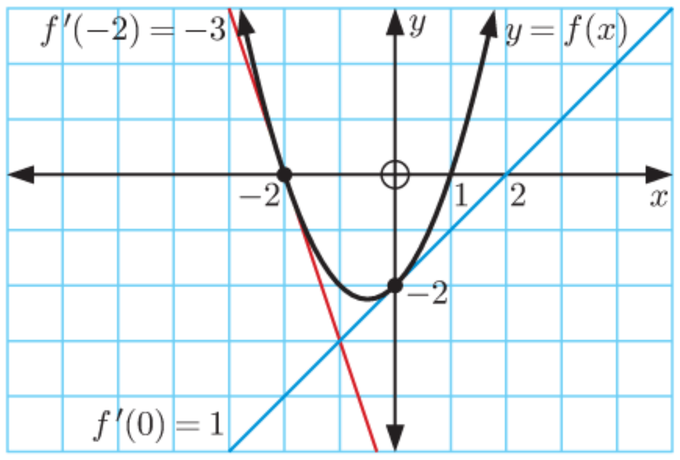
4 **a** **i** $f'(-2) = -3$

At $x = -2$, the derivative function is -3, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3.

ii $f'(0) = 1$

At $x = 0$, the derivative function is 1, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1.

b



EXERCISE 16E

1 **a** $f'(x) = 1$ **b** $f'(x) = 0$ **c** $f'(x) = 3x^2$

2 **a** $f'(x) = 2$ **b** $f'(x) = 2x - 3$ **c** $f'(x) = -2x + 5$

3 **a** $\dfrac{dy}{dx} = -1$ **b** $\dfrac{dy}{dx} = 4x + 1$ **c** $\dfrac{dy}{dx} = 3x^2 - 4x$

4 **a** 12 **b** 108

5 **a** $f'(x) = -\dfrac{1}{x^2}$

b $f'(-1) = -1$; the tangent at the point where $x = -1$ has gradient -1.

$f'(3) = -\frac{1}{9}$; the tangent at the point where $x = 3$ has gradient $-\frac{1}{9}$.

6 **a** **i** 3 **ii** -1 **b** $f'(x) = -2x + 3$

c $f'(0) = 3$, $f'(2) = -1$

7 **a** $\dfrac{dy}{dx} = 3x^2 - 3$ **b** $(-1, 2)$ and $(1, -2)$

8 **a** $f'(x) = -\frac{1}{2}x + 1$

b $f'(-2) = 2, \quad f'(3) = -\frac{1}{2}$

Gradients of tangents are 2 and $-\frac{1}{2}$, and $2 \times -\frac{1}{2} = -1$

$\therefore$ tangents are perpendicular.

9 **a** $f'(x) = m$, a constant function

b $f'(x) = 2ax + b$, a linear function

c $f'(x) = 3ax^2 + 2bx + c$, a quadratic function

10 $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, $y = \sqrt{x}$ is differentiable for all $x > 0$.

11 **a**

$f(x)$	x^4	x^3	x^2	x^1	$x^{\frac{1}{2}}$	x^0	x^{-1}
$f'(x)$	$4x^3$	$3x^2$	$2x$	1	$\frac{1}{2}x^{-\frac{1}{2}}$	0	$-x^{-2}$

b If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

12 $f'(x) = -\sin x$ **13** **c** $f'(x) = -\sin x$

EXERCISE 16F

1 $f(x)$ is not continuous at $x = 0$.

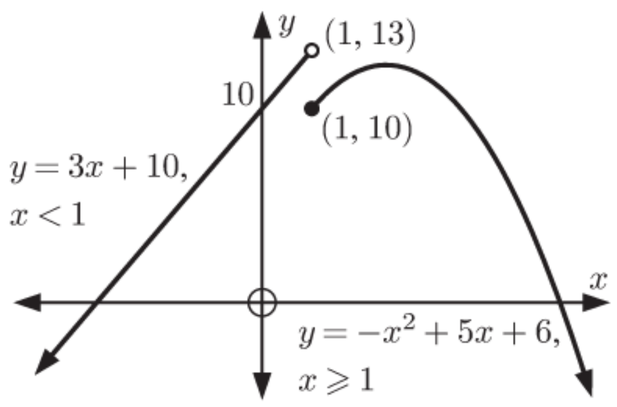
$\therefore$ $f(x)$ is not differentiable at $x = 0$.

2 $f(x)$ is not continuous at $x = 0$.

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

4 $f(x) = \begin{cases} -x^2 + 5x + 6, & x \geqslant 1 \\ 3x + 10, & x < 1 \end{cases}$

a



b **i** $f'_-(1) = 3$ **ii** $f'_+(1) = -2(1) + 5 = 3$

c No, although $f'_-(1) = f'_+(1)$, $f(x)$ is not continuous at $x = 1$.

5 $c = 2, \quad d = -1$ **6** $a = \frac{1}{4}, \quad b = -1$

REVIEW SET 16A

1 $\frac{4}{3}$ **2** $f'(3) = -\frac{1}{2}$

3 **b**

h	$\dfrac{f(3+h) - f(3)}{h}$
0.1	12.2
0.01	12.02
0.001	12.002
0.0001	12.0002

c $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h} = 12$

The gradient of the tangent to $y = 2x^2$ at $(3, 18)$ is 12.

4 **a** $f'(x) = 2x + 2$ **b** $\dfrac{dy}{dx} = -6x$

5 **a** $\dfrac{dy}{dx} = 4x$ **b** gradient $= 16$ **c** $x = -3$

6 **a** $f'(x) = 3x^2 - 6x$ **b** $f'(-1) = 9, \quad f'(3) = 9$

Gradients of tangents are both 9.

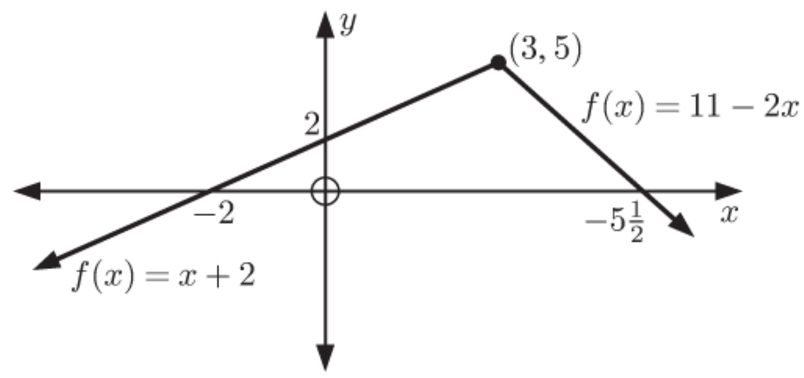
$\therefore$ the tangents are parallel.

7 **a** $\dfrac{dy}{dx} = 9 - 3x^2$ **b** $(-\sqrt{3}, -6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$

8 $f(x)$ is not continuous at $x = 2$.

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

9 **a**

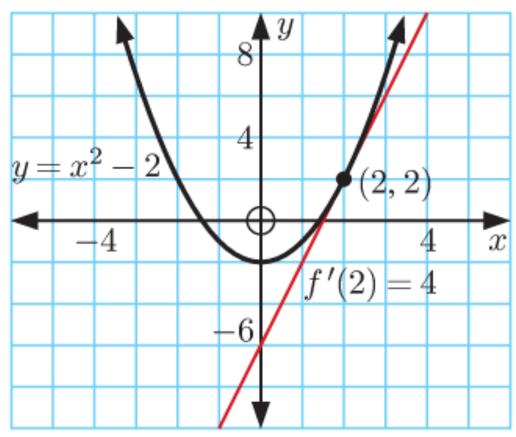


b no

REVIEW SET 16B

1 $\frac{1}{2}$ **2** **a** 2°C per hour **b** $-2\frac{1}{2}$°C per hour

3 **a, b**



c 4

4 **a** $f'(x) = 4x^3 - 2$

b $f'(-2) = -34$

The gradient of $f'(x)$ at the point where $x = -2$ is -34.

5 **a** $\dfrac{dy}{dx} = 2x + 5$ **b** $(-4, -6)$

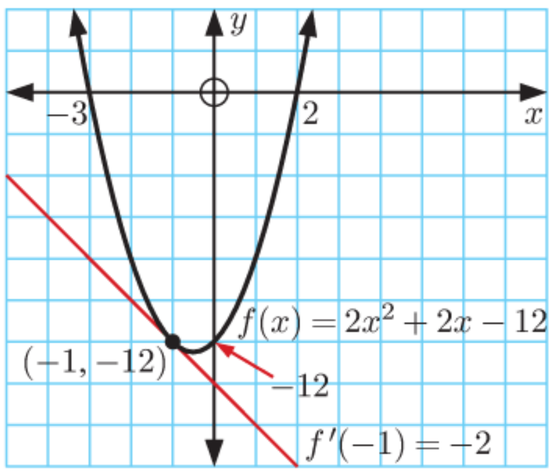
6 **a** **i** 447.2 m **ii** 432.8 m **b** $f'(t) = -9.6t$ m s^{-1}

c **i** 9.6 m s^{-1} **ii** 19.2 m s^{-1}

7 **a** $f'(x) = 4x + 2$

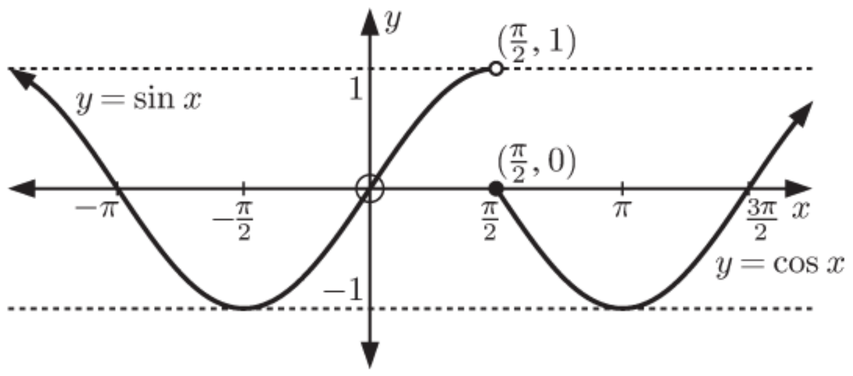
b $(-1, -12)$

c



8 **c** $f'(x) = \cos x$

9 **a**



b $f(x)$ is not continuous at $x = \frac{\pi}{2}$.

$\therefore$ $f(x)$ is not differentiable at $x = \frac{\pi}{2}$.

EXERCISE 17A

1 **a** $3x^2$ **b** $8x^7$ **c** $11x^{10}$ **d** 6 **e** $6x^2$

f $14x$ **g** $15x^4$ **h** $30x^5$ **i** 5 **j** $2x$

k $2x + 1$ **l** $2x + 3$ **m** $4x + 1$ **n** $6x - 7$ **o** $-4x$

p $2x^3 - 12x$ **q** $3x^2 - 8x + 6$ **r** $6x^2 + 1$

s $-1 - 12x^2$ **t** $\frac{3}{5}x^2 - 7x$

2 **a** $-\frac{2}{x^3}$ **b** $-\frac{5}{x^6}$ **c** $-\frac{3}{x^2}$ **d** $-\frac{12}{x^4}$

e $\frac{28}{x^5}$ **f** $2-\frac{6}{x^3}$ **g** $2x+\frac{6}{x^2}$ **h** $\frac{6}{x^4}$

i $-\frac{4}{x^3}-\frac{36}{x^5}$ **j** $3+\frac{1}{x^2}-\frac{4}{x^3}$ **k** $\frac{16}{x^3}-\frac{12}{x^4}$

l $-\frac{2}{5x^3}$ **m** $4+\frac{1}{4x^2}$ **n** $1+\frac{3}{x^2}$

o $2x-\frac{4}{x^2}$ **p** $-\frac{2}{x^2}+\frac{10}{x^3}$

3 **a** $\frac{1}{2\sqrt{x}}$ **b** $\frac{1}{3\sqrt[3]{x^2}}$ **c** $-\frac{1}{2x\sqrt{x}}$

d $3x^2-\frac{1}{4\sqrt{x}}$ **e** $-\frac{2}{x^3}+\frac{3}{\sqrt{x}}$ **f** $2-\frac{1}{2\sqrt{x}}$

g $\frac{3}{2}\sqrt{x}$ **h** $-\frac{3}{2x^2\sqrt{x}}$ **i** $4x+\frac{3}{2x\sqrt{x}}$

j $-\frac{1}{2x\sqrt{x}}+\frac{4}{x^2}$ **k** $\frac{1}{2\sqrt{x}}-\frac{5}{2x\sqrt{x}}$ **l** $-\frac{7}{2x\sqrt{x}}-\frac{3\sqrt{x}}{2}$

m $6x-\frac{3\sqrt{x}}{2}$ **n** $\frac{-10}{x^3\sqrt{x}}$ **o** $2+\frac{9}{2x^2\sqrt{x}}$

p $\frac{5\sqrt[3]{x^2}}{3}-\frac{2}{3\sqrt[3]{x}}-\frac{2}{3x\sqrt[3]{x}}$

4 **a** $\frac{dy}{dx}=2\pi x$ **b** $\frac{dy}{dx}=6x+\frac{16}{x^3}$

c $\frac{dy}{dx}=\frac{3}{\sqrt{x}}-\frac{5}{x^2}$ **d** $\frac{dy}{dx}=12\pi x^2$

e $\frac{dy}{dx}=7.5x^2-2.8x$ **f** $\frac{dy}{dx}=10$

g $\frac{dy}{dx}=2x-1$ **h** $\frac{dy}{dx}=12x-1$ **i** $\frac{dy}{dx}=2x-10$

j $\frac{dy}{dx}=8x-4$ **k** $\frac{dy}{dx}=6x^2-6x-5$

l $\frac{dy}{dx}=\frac{3\sqrt{x}}{2}-\frac{3}{\sqrt{x}}-\frac{9}{2x\sqrt{x}}$

5 **a** $f'(x)=-3+6x-3x^2$

b $f'(x)=81x^2-\frac{81\sqrt{x}}{2}+\frac{3}{2x^2\sqrt{x}}$

6 **a** $\frac{dy}{dt}=2t^3-\frac{1}{3}$ **b** $\frac{dy}{dt}=3t^{-\frac{3}{2}}=\frac{3}{t\sqrt{t}}$

c $\frac{dT}{dt}=\frac{1}{3}t^{-\frac{2}{3}}+4t^{-3}=\frac{1}{3\sqrt[3]{t^2}}+\frac{4}{t^3}$

d $\frac{dP}{du}=-5u^{-2}-15u^{\frac{1}{2}}=-\frac{5}{u^2}-15\sqrt{u}$

7 **a** 4 **b** 22 **c** $-\frac{16}{729}$ **d** -7 **e** $\frac{3}{2}$

f $\frac{13}{4}$ **g** $\frac{1}{8}$ **h** -11

8 **a** $b=3,\ c=-4$ **b** $b=2,\ c=-3$

9 $f'_-(2)=16\neq f'_+(2)=14$

$\therefore$ not differentiable at $x=2$.

10 $\frac{dy}{dx}=4+\frac{3}{x^2}$, $\frac{dy}{dx}$ is the gradient function of $y=4x-\frac{3}{x}$ from which the gradient of the tangent at any point can be found.

11 **a** $\{x \mid x>0\}$ **b** $f'(x)=\frac{1}{2\sqrt{x}}+\frac{2}{x\sqrt{x}}$

c $\{x \mid x>0\}$

d $f'(1)=2.5$ The gradient of the tangent to the curve $f(x)=\sqrt{x}-\frac{4}{\sqrt{x}}$ at $x=1$ is 2.5.

12 **a** $\frac{dS}{dt}=4t+4$ m s^{-1}, $\frac{dS}{dt}$ is the instantaneous rate of change in the car's position at the time t.

b When $t=3$, $\frac{dS}{dt}=16$ m s^{-1}. This is the instantaneous rate of change in position at the time $t=3$ seconds.

13 When $x=1000$, $\frac{dC}{dx}=7$. When 1000 toasters per week are being produced, the cost of production is increasing by £7 per toaster.

14 **a** The gradient of the tangent is $\frac{dy}{dx}\approx\frac{\delta y}{\delta x}$

$\therefore\ \delta y\approx\frac{dy}{dx}\times\delta x$

b If δx is too large, $\frac{\delta y}{\delta x}$ will not accurately estimate the gradient of the tangent.

c **i** ≈ 25.1 **ii** ≈ 65.92 **iii** ≈ 26.46 **iv** ≈ 14.4

EXERCISE 17B.1

1 **a** $g(f(x))=(2x+7)^2$ **b** $g(f(x))=2x^2+7$

c $g(f(x))=\sqrt{3-4x}$ **d** $g(f(x))=3-4\sqrt{x}$

e $g(f(x))=\frac{2}{x^2+3}$ **f** $g(f(x))=\frac{4}{x^2}+3$

2 **Note:** There may be other answers.

a $g(x)=x^3,\ f(x)=3x+10$

b $g(x)=x^5,\ f(x)=7-2x$

c $g(x)=\frac{1}{x},\ f(x)=2x+4$

d $g(x)=\sqrt{x},\ f(x)=x^2-3x$

e $g(x)=\frac{1}{x^4},\ f(x)=5x-1$

f $g(x)=\frac{10}{x^3},\ f(x)=3x-x^2$

EXERCISE 17B.2

1 **a** $u^{-2},\ u=2x-1$ **b** $u^{\frac{1}{2}},\ u=x^2-3x$

c $2u^{-\frac{1}{2}},\ u=2-x^2$ **d** $u^{\frac{1}{3}},\ u=x^3-x^2$

e $4u^{-3},\ u=3-x$ **f** $10u^{-1},\ u=x^2-3$

2 **a** $\frac{dy}{dx}=4(2x+3)=8x+12$ **b** $\frac{dy}{dx}=8x+12$

3 **a** $\frac{dy}{dx}=8(4x-5)$ **b** $\frac{dy}{dx}=2(5-2x)^{-2}$

c $\frac{dy}{dx}=\frac{1}{2}(3x-x^2)^{-\frac{1}{2}}(3-2x)$

d $\frac{dy}{dx}=-12(1-3x)^3$ **e** $\frac{dy}{dx}=-18(5-x)^2$

f $\frac{dy}{dx}=\frac{1}{3}(2x^3-x^2)^{-\frac{2}{3}}(6x^2-2x)$

g $\frac{dy}{dx}=-60(5x-4)^{-3}$

h $\frac{dy}{dx}=5(x^2-5x+8)^4(2x-5)$

i $\frac{dy}{dx}=6\left(x^2-\frac{2}{x}\right)^2\left(2x+\frac{2}{x^2}\right)$

4 **a** $-\frac{1}{\sqrt{3}}$ **b** -18 **c** -8 **d** -4 **e** $-\frac{3}{32}$ **f** 0

5 $a = 3,\ b = 1$ **6** $a = 2,\ b = 1$ **7** $a = 2,\ b = 3$

8 **a** $\dfrac{dy}{dx} = 3x^2$, $\dfrac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$ **Hint:** Substitute $y = x^3$

b $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = \dfrac{dy}{dy}$ $\{\text{chain rule}\}$ $= 1$

EXERCISE 17C

1 **a** $f'(x) = 2x - 1$ **b** $f'(x) = 4x + 2$

c $f'(x) = 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}$

d $f'(x) = 2x + 2$

e $f'(x) = (x^2-1)^{\frac{1}{2}} + x^2(x^2-1)^{-\frac{1}{2}}$

f $f'(x) = (x+1)^2 + 2x(x+1)$

2 **a** $\dfrac{dy}{dx} = 2x(2x-1) + 2x^2$

b $\dfrac{dy}{dx} = 4(2x+1)^3 + 24x(2x+1)^2$

c $\dfrac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2(3-x)^{-\frac{1}{2}}$

d $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$

e $\dfrac{dy}{dx} = 10x(3x^2-1)^2 + 60x^3(3x^2-1)$

f $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$

3 **a** -48 **b** $406\frac{1}{4}$ **c** $\frac{13}{3}$ **d** $\frac{11}{2}$

4 **b** $x = 3$ or $\frac{3}{5}$

c Domain of $\dfrac{dy}{dx}$ is $\{x \mid x > 0\}$. Domain of original function is $\{x \mid x \geqslant 0\}$. $\dfrac{dy}{dx}$ is undefined when $x = 0$.

5 $x = -1$ and $x = -\frac{5}{3}$ **6** $x = 1$ and $x = -\frac{1}{3}$

7 **a** $x = \frac{2}{3}$ **b** $x = 0$ **8** $a = -\frac{7}{4}$ and $a = -\frac{20}{9}$

EXERCISE 17D

1 **a** $\dfrac{dy}{dx} = \dfrac{7}{(2-x)^2}$ **b** $\dfrac{dy}{dx} = \dfrac{2x^2+2x}{(2x+1)^2}$

c $\dfrac{dy}{dx} = \dfrac{-x^2-3}{(x^2-3)^2}$ **d** $\dfrac{dy}{dx} = \dfrac{2x+1}{2\sqrt{x}(1-2x)^2}$

e $\dfrac{dy}{dx} = \dfrac{3x^2-6x+9}{(3x-x^2)^2}$ **f** $\dfrac{dy}{dx} = \dfrac{2-3x}{2(1-3x)^{\frac{3}{2}}}$

2 **a** $\dfrac{4}{(3-x)^2}$ **b** $\dfrac{-3x^2-3}{(x^2-1)^2}$ **c** $\dfrac{4x^3-3x^2}{(2x-1)^2}$

d $\dfrac{2x-20}{(x-5)^{\frac{3}{2}}}$ **e** $\dfrac{3x^2+3}{2\sqrt{x}(3-x^2)^2}$ **f** $\dfrac{-x^3-6x}{(x^2+3)^{\frac{3}{2}}}$

3 **a** 1 **b** 1 **c** $-\frac{7}{324}$ **d** $-\frac{28}{27}$ **4** $f'(x) = \dfrac{x-2}{2(x-1)^{\frac{3}{2}}}$

5 **a** $\dfrac{dy}{dx} = -\dfrac{1}{(x+1)^2}$

b At $x = -2$, $\dfrac{dy}{dx} = -1$. At $x = 0$, $\dfrac{dy}{dx} = -1$.

So, the gradient of each tangent is -1

$\therefore$ the tangents are parallel.

6 **b** **i** never $\{\dfrac{dy}{dx}$ is undefined at $x = -1\}$

ii $x \leqslant 0$ and $x = 1$

7 **b** **i** $x = -3$ and $x = 2$ **ii** $x = -\frac{1}{2}$

8 **b** **i** $x = -2 \pm \sqrt{11}$ **ii** $x = -2$

EXERCISE 17E.1

1 **a** $f'(x) = 4e^{4x}$ **b** $f'(x) = e^x$ **c** $f'(x) = -2e^{-2x}$

d $f'(x) = \frac{1}{2}e^{\frac{x}{2}}$ **e** $f'(x) = -e^{-\frac{x}{2}}$ **f** $f'(x) = 2e^{-x}$

g $f'(x) = 2e^{\frac{x}{2}} + 3e^{-x}$ **h** $f'(x) = \dfrac{e^x - e^{-x}}{2}$

i $f'(x) = -2xe^{-x^2}$ **j** $f'(x) = -\dfrac{e^{\frac{1}{x}}}{x^2}$

k $f'(x) = 20e^{2x}$ **l** $f'(x) = 40e^{-2x}$

m $f'(x) = 2e^{2x+1}$ **n** $f'(x) = \frac{1}{4}e^{\frac{x}{4}}$

o $f'(x) = -4xe^{1-2x^2}$ **p** $f'(x) = -0.02e^{-0.02x}$

2 **a** $e^x + xe^x$ **b** $3x^2e^{-x} - x^3e^{-x}$ **c** $\dfrac{xe^x - e^x}{x^2}$

d $\dfrac{1-x}{e^x}$ **e** $2xe^{3x} + 3x^2e^{3x}$ **f** $\dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $20e^{-0.5x} - 10xe^{-0.5x}$ **h** $\dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** $\dfrac{dy}{dx} = 4e^x(2+e^x)^3$ **b** $\dfrac{dy}{dx} = \dfrac{e^x}{2\sqrt{e^x-1}}$

c $\dfrac{dy}{dx} = \frac{3}{2}(e^x+e^{-x})^{\frac{1}{2}}(e^x-e^{-x})$

d $\dfrac{dy}{dx} = -e^{2x}(e^{2x}+2)^{-\frac{3}{2}}$

4 **a** 108 **b** -1 **c** $\frac{9}{\sqrt{19}}$ **d** $-\dfrac{4}{e^3}$

5 **a** $\{x \mid x \leqslant \ln 6\}$ **b** **i** P$(\ln 2, 2)$ **ii** $-\frac{1}{2}$

6 $k = -9$ **7** **a** $\dfrac{dy}{dx} = 2^x \ln 2$

8 P$(0, 0)$ or P$\left(2, \dfrac{4}{e^2}\right)$ **9** **c** $S(x)$ **d** $\dfrac{1}{[C(x)]^2}$

EXERCISE 17E.2

1 **a** $\dfrac{dy}{dx} = 2^x \ln 2$ **b** $\dfrac{dy}{dx} = 5^x \ln 5$

c $\dfrac{dy}{dx} = 3^x \ln 3 + 2 \times 7^x \ln 7 + e^x$

d $\dfrac{dy}{dx} = 2^x + x\,2^x \ln 2$ **e** $\dfrac{dy}{dx} = \dfrac{2^x \ln 2}{x} - \dfrac{2^x}{x^2}$

f $\dfrac{dy}{dx} = \dfrac{1 - x\ln 3}{3^x}$

2 **a** $\dfrac{dy}{dx} = -\dfrac{\ln 4}{4^x}$ **b** $\dfrac{dy}{dx} = 2^{x+3}\ln 2 + 2 \times 9^x \ln 3$

c $\dfrac{dy}{dx} = 5^{x^3+x} \ln 5 \times (3x^2+1)$ **d** $\dfrac{dy}{dx} = \dfrac{3x^2 - x^3 \ln 6}{6^x}$

e $\dfrac{dy}{dx} = \dfrac{3(1 - 5x\ln 2)}{2^{5x-1}}$

f $\dfrac{dy}{dx} = \dfrac{4e^{2x}\sqrt{x}(1 - x\ln 5) - 3^{\sqrt{x}}(\ln 3 - 4x\sqrt{x}\ln 5)}{2\sqrt{x}\,5^{x^2}}$

3 **a** $32 \ln 2$ **b** $-3 \ln 3$ **c** $30 \ln 2$ **d** $\frac{3}{25}(2 - 3\ln 5)$

4 P$(-1, \frac{3}{4})$

EXERCISE 17F.1

1 **a** $\frac{dy}{dx} = \frac{1}{x}$ **b** $\frac{dy}{dx} = \frac{2}{2x+1}$ **c** $\frac{dy}{dx} = \frac{1-2x}{x-x^2}$

d $\frac{dy}{dx} = -\frac{2}{x}$ **e** $\frac{dy}{dx} = 2x\ln x + x$ **f** $\frac{dy}{dx} = \frac{1-\ln x}{2x^2}$

g $\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$ **h** $\frac{dy}{dx} = \frac{2\ln x}{x}$

i $\frac{dy}{dx} = \frac{1}{2x\sqrt{\ln x}}$ **j** $\frac{dy}{dx} = \frac{e^{-x}}{x} - e^{-x}\ln x$

k $\frac{dy}{dx} = \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$ **l** $\frac{dy}{dx} = \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$

m $\frac{dy}{dx} = \frac{4}{1-x}$ **n** $\frac{dy}{dx} = \ln(x^2+1) + \frac{2x^2}{x^2+1}$

o $\frac{dy}{dx} = \frac{1-2\ln x}{x^3}$

2 **a** $f'(x) = \frac{1}{x}$ **b** $\ln(kx) = \ln k + \ln x$

So $f'(x) = \frac{1}{x}$ since $\ln k$ is constant

3 **a** $\frac{dy}{dx} = \ln 5$ **b** $\frac{dy}{dx} = \frac{3}{x}$ **c** $\frac{dy}{dx} = \frac{4x^3+1}{x^4+x}$

d $\frac{dy}{dx} = \frac{1}{x-2}$ **e** $\frac{dy}{dx} = \frac{6}{2x+1}[\ln(2x+1)]^2$

f $\frac{dy}{dx} = \frac{1-\ln(4x)}{x^2}$ **g** $\frac{dy}{dx} = -\frac{1}{x}$

h $\frac{dy}{dx} = \frac{1}{x\ln x}$ **i** $\frac{dy}{dx} = \frac{-1}{x(\ln x)^2}$

4 **a** $\frac{dy}{dx} = \frac{1}{2x-1}$ **b** $\frac{dy}{dx} = \frac{-2}{2x+3}$

c $\frac{dy}{dx} = 1 + \frac{1}{2x}$ **d** $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2(2-x)}$

e $\frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x-1}$ **f** $\frac{dy}{dx} = \frac{2}{x} + \frac{1}{3-x}$

5 **a** $f'(x) = \frac{9}{3x-4}$ **b** $f'(x) = \frac{1}{x} + \frac{2x}{x^2+1}$

c $f'(x) = \frac{2x+2}{x^2+2x} - \frac{1}{x-5}$

d $f'(x) = \frac{3}{x} - \frac{1}{x+4} - \frac{1}{x-1}$

6 2 **7** $a = 3,\ b = \frac{1}{e}$ **8** $a = 4,\ b = e^3$ **9** $(-3, \ln 6)$

10 $a = 1,\ b = 2$

EXERCISE 17F.2

1 **a** $f'(x) = \frac{1}{x\ln 2}$ **b** $f'(x) = \frac{1}{x\ln 10}$

c $f'(x) = \frac{1}{x\ln 3} - \frac{1}{2x\ln 5}$ **d** $f'(x) = \log_3 x + \frac{1}{\ln 3}$

e $f'(x) = \frac{\frac{x^2-3x}{x\ln 5} - (2x-3)\log_5 x}{(x^2-3x)^2}$

f $f'(x) = \frac{4(\log_2 x)^3}{x\ln 2}$

2 **a** $\frac{dy}{dx} = \frac{3}{(3x-1)\ln 2}$ **b** $\frac{dy}{dx} = \frac{8x+3}{(4x^2+3x)\ln 5}$

c $\frac{dy}{dx} = 3^x\left[\ln 3 \times \log_3(2x) + \frac{1}{x\ln 3}\right]$

d $\frac{dy}{dx} = -\frac{1}{2x\ln 5}$ **e** $\frac{dy}{dx} = \frac{1}{x\ln x \times \ln 2}$

f $\frac{dy}{dx} = \frac{\frac{1-x^2}{(4-x)\ln 10} - 2x\log(4-x)}{(x^2-1)^2}$

3 **a** $\frac{1}{2\ln 2}$ **b** $18 + \frac{9}{2\ln 3}$ **c** $\frac{1-2\ln 2}{4e^2\ln 2}$ **d** $\frac{21}{50}$

4 **a** $\frac{dy}{dx} = -\frac{2}{x\ln 3}$

b $\frac{dy}{dx} = \frac{1}{(x+5)\ln 2} + \frac{1}{(x-1)\ln 2}$

c $\frac{dy}{dx} = \frac{1}{x\ln 10} + \frac{1}{\ln 10}$

d $\frac{dy}{dx} = -\frac{2}{(2x-1)\ln 5} - \frac{1}{(x+2)\ln 5}$

e $\frac{dy}{dx} = \frac{2x-1}{(x^2-x)\ln 2} - \frac{1}{(x+3)\ln 2}$

f $\frac{dy}{dx} = \frac{3}{(4x-4\sqrt{x})\ln 2}$

5 $a = 1,\quad b = -\frac{5}{3}$

EXERCISE 17G.1

1 **a** $\frac{dy}{dx} = 2\cos 2x$ **b** $\frac{dy}{dx} = \cos x - \sin x$

c $\frac{dy}{dx} = -3\sin 3x - \cos x$ **d** $\frac{dy}{dx} = \cos(x+1)$

e $\frac{dy}{dx} = 2\sin(3-2x)$ **f** $\frac{dy}{dx} = 5\sec^2 5x$

g $\frac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2} + 3\sin x$ **h** $\frac{dy}{dx} = 4\cos x + 2\sin 2x$

i $\frac{dy}{dx} = -3\sin 6x - 20\cos 4x$

2 **a** $2x - \sin x$ **b** $\sec^2 x - 3\cos x$ **c** $e^x\cos x - e^x\sin x$

d $-e^{-x}\sin x + e^{-x}\cos x$ **e** $\frac{\cos x}{\sin x}$

f $2e^{2x}\tan x + e^{2x}\sec^2 x$ **g** $3\pi\sec^2 \pi x$ **h** $-\frac{1}{2}\sin\frac{x}{2}$

i $6\sec^2 2x$ **j** $\cos x - x\sin x$ **k** $\frac{x\cos x - \sin x}{x^2}$

l $\tan x + x\sec^2 x$ **m** $3^x(\ln 3\sin 2x + 2\cos 2x)$

n $-\frac{2\tan 2x}{\ln 10}$ **o** $\frac{2x\cos 2x - x\sin x - 3\cos x - 3\sin 2x}{x^4}$

3 **a** $2x\cos(x^2)$ **b** $-\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$ **c** $-\frac{\sin x}{2\sqrt{\cos x}}$

d $2\sin x\cos x$ **e** $-3\sin x\cos^2 x$

f $-\sin x\sin 2x + 2\cos x\cos 2x$ **g** $\sin x\sin(\cos x)$

h $-12\sin 4x\cos^2 4x$

4 **a** $-\operatorname{cosec} x\cot x$ **b** $-\operatorname{cosec} x\cot x$

5 **a** $\sec x\tan x$ **b** $-\operatorname{cosec}^2 x$ **6** **a** $-\frac{9}{8}$ **b** 0

7 **a** $f'(x) = 0$ **b** $f(x) = 2(\cos^2 x + \sin^2 x) + 1$
$= 2(1) + 1$
$= 3$, a constant

8 **a** tangent **B**

b $\frac{dy}{dx} = -\sin x + 4\cos 2x$

When $x = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{3}{2}$ When $x = \frac{7\pi}{6}$, $\frac{dy}{dx} = \frac{5}{2}$

which is $> \frac{3}{2}$ ✓

9 $a = \left(\frac{4}{\pi}\right)^{\pi},\quad n = \pi$

10 **a** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

However, $f'_-(0) = 5 \neq f'_+(0) = 1$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

b $f'(x) = \begin{cases} \cos x, & x > 0 \\ 2x + 5, & x < 0 \end{cases}$

11 **b** 1 **c** $\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}} = 1$

$\therefore \quad e^{i\theta} = \cos\theta + i\sin\theta$

EXERCISE 17G.2

2 **a** $-4\operatorname{cosec} 4x \cot 4x$ **b** $2\sec 2x \tan 2x$

c $-\frac{1}{3}\operatorname{cosec}^2\left(\frac{x}{3}\right)$ **d** $-8\operatorname{cosec}^2 2x \cot 2x$

e $-12\cot^2\left(\frac{x}{2}\right)\operatorname{cosec}^2\left(\frac{x}{2}\right)$ **f** $\frac{3}{4}\sqrt{\sec\frac{3x}{2}}\tan\frac{3x}{2}$

3 **a** $\dfrac{dy}{dx} = \sec x(1 + x\tan x)$

b $\dfrac{dy}{dx} = e^x(\cot x - \operatorname{cosec}^2 x)$ **c** $\dfrac{dy}{dx} = 8\sec 2x \tan 2x$

d $\dfrac{dy}{dx} = -e^{-x}\left(\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)\right)$

e $\dfrac{dy}{dx} = x\operatorname{cosec} x(2 - x\cot x)$

f $\dfrac{dy}{dx} = \sqrt{\operatorname{cosec} x}\left[1 - \frac{x}{2}\cot x\right]$ **g** $\dfrac{dy}{dx} = \tan x$

h $\dfrac{dy}{dx} = \operatorname{cosec}(x^2)\left[1 - 2x^2\cot(x^2)\right]$

i $\dfrac{dy}{dx} = \dfrac{-\sqrt{x}\operatorname{cosec}^2 x - \frac{1}{2}x^{-\frac{1}{2}}\cot x}{x} \equiv -\dfrac{\cos x \sin x + 2x}{2x\sqrt{x}\sin^2 x}$

EXERCISE 17H

4 **a** $\dfrac{2}{1 + 4x^2}$ **b** $\dfrac{-3}{\sqrt{1 - 9x^2}}$ **c** $\dfrac{1}{\sqrt{16 - x^2}}$

d $\dfrac{-1}{\sqrt{25 - x^2}}$ **e** $\dfrac{2x}{1 + x^4}$ **f** -1

5 **a** $\dfrac{dy}{dx} = \arcsin x + \dfrac{x}{\sqrt{1 - x^2}}$

b $\dfrac{dy}{dx} = e^x \arccos x - \dfrac{e^x}{\sqrt{1 - x^2}}$

c $\dfrac{dy}{dx} = -e^{-x}\arctan x + \dfrac{e^{-x}}{1 + x^2}$

d $\dfrac{dy}{dx} = 3x^2 \arcsin x + \dfrac{x^3}{\sqrt{1 - x^2}}$

e $\dfrac{dy}{dx} = 2x \arctan 3x + \dfrac{3(x^2 + 1)}{1 + 9x^2}$

f $\dfrac{dy}{dx} = 3^x\left(\ln 3 \arccos 2x - \dfrac{2}{\sqrt{1 - 4x^2}}\right)$

6 $\{x \mid -1 \leqslant x \leqslant 0 \text{ or } 1 \leqslant x \leqslant 2\}$, $f'(x) = \dfrac{2x - 1}{\sqrt{1 - (1 + x - x^2)^2}}$

7 **a** $\mathrm{P}(\sqrt{3}, \frac{\pi}{6})$ **b** $\frac{1}{4}$

8 $f'(x) = 0$, $\arcsin x + \arccos x = \frac{\pi}{2}$, a constant

(**Hint:** $\sin\theta = \cos(\frac{\pi}{2} - \theta)$)

9 **b** $\dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}$ **c** $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{a^2 - x^2}}$, $-a < x < a$

EXERCISE 17I

1 **a** $f''(x) = 6$ **b** $f''(x) = \dfrac{3}{2x^{\frac{5}{2}}}$

c $f''(x) = 12x - 6$ **d** $f''(x) = \dfrac{12 - 6x}{x^4}$

e $f''(x) = 24 - 48x$ **f** $f''(x) = \dfrac{20}{(2x - 1)^3}$

2 **a** $\dfrac{d^2y}{dx^2} = -6x$ **b** $\dfrac{d^2y}{dx^2} = 2 - \dfrac{30}{x^4}$ **c** $\dfrac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{5}{2}}$

d $\dfrac{d^2y}{dx^2} = \dfrac{8}{x^3}$ **e** $\dfrac{d^2y}{dx^2} = 6(x^2 - 3x)(5x^2 - 15x + 9)$

f $\dfrac{d^2y}{dx^2} = 2 + \dfrac{2}{(1 - x)^3}$ **g** $\dfrac{d^2y}{dx^2} = 9e^{3x}$

h $\dfrac{d^2y}{dx^2} = \dfrac{-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}}{x^3}$

i $\dfrac{d^2y}{dx^2} = -\dfrac{x^3 - 3x^2 - 6x - 6}{x^3e^x}$

3 **a** 9 **b** 10 **c** 12 **d** 6

5 **a** $x = 1$

b $x = 0, \pm\sqrt{6}$

6

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

8 **b** $f''(x) = 3\sin x \cos 2x + 6\cos x \sin 2x$

9 **a** $\dfrac{d^2y}{dx^2} = \dfrac{1}{x^2}$ **b** $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$ **c** $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^2}(1 - \ln x)$

10 **a** 0 **b** 3 **c** 0 **d** 6

11 **Hint:** Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute into the equation.

16 **a** **i** $f'(x) = -e^{-x}(x + 1)$ **ii** $f''(x) = e^{-x}(x)$

iii $f'''(x) = -e^{-x}(x - 1)$ **iv** $f^{(4)}(x) = e^{-x}(x - 2)$

b $f^{(n)}(x) = (-1)^n e^{-x}(x - n + 2)$

EXERCISE 17J

1 **a** $2\dfrac{dy}{dx}$ **b** $-3\dfrac{dy}{dx}$ **c** $6y^2\dfrac{dy}{dx}$ **d** $\dfrac{4}{y^2}\dfrac{dy}{dx}$

e $4y^3\dfrac{dy}{dx}$ **f** $\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}$ **g** $-\dfrac{2}{y^3}\dfrac{dy}{dx}$

h $y + x\dfrac{dy}{dx}$ **i** $2xy + x^2\dfrac{dy}{dx}$ **j** $y^3 + 3xy^2\dfrac{dy}{dx}$

2 **a** $\dfrac{dy}{dx} = -\dfrac{x}{y}$ **b** $\dfrac{dy}{dx} = -\dfrac{x}{3y}$ **c** $\dfrac{dy}{dx} = \dfrac{x}{y}$

d $\dfrac{dy}{dx} = \dfrac{2x}{3y^2}$ **e** $\dfrac{dy}{dx} = \dfrac{-2x - y}{x}$ **f** $\dfrac{dy}{dx} = \dfrac{3x^2 - 2y}{2x}$

g $\dfrac{dy}{dx} = \dfrac{2}{x^3} - \dfrac{y}{x}$ **h** $\dfrac{dy}{dx} = \dfrac{-1 - 2y}{2x + 2y}$

i $\dfrac{dy}{dx} = \dfrac{1 - 3x^2 - y^2}{2xy - 4y}$ **j** $\dfrac{dy}{dx} = \dfrac{y}{x - y^2}$

k $\dfrac{dy}{dx} = \dfrac{1}{\sin y}$ **l** $\dfrac{dy}{dx} = \dfrac{-e^y}{\cos y + xe^y - 2}$

m $\dfrac{dy}{dx} = -\dfrac{2^x \ln 2}{3^y \ln 3}$ **n** $\dfrac{dy}{dx} = \dfrac{y}{x(ye^y - 1)}$

o $\dfrac{dy}{dx} = -\dfrac{1 + \sec y\sqrt{1 - x^2}}{(1 + x\sec y\tan y)\sqrt{1 - x^2}}$

4 **a** $\dfrac{dy}{dx} = \dfrac{-(54e^{-2x} + 3x^2e^{3y} + 8xy^3)}{3x^2(xe^{3y} + 4y^2)}$

b $\dfrac{da}{db} = \dfrac{a^4b - 2ae^{2a} - a}{4abe^{2a}\ln b - 3a^3b^2 + b}$

6 **a** $\dfrac{d^2y}{dx^2} = \dfrac{-y^2 - x^2}{y^3}$ **b** $\dfrac{d^2y}{dx^2} = \dfrac{y^2 - x^2}{y^3}$

c $\dfrac{d^2y}{dx^2} = \dfrac{2y}{x^2}$

7 **a** $\dfrac{dV}{dq} = \dfrac{V - 1}{3V - q}$ **b** $\dfrac{d^2q}{dV^2} = \dfrac{2q - 3V - 3}{(1 - V)^2}$

8 **a** $\dfrac{dy}{dx} = \dfrac{3 - y^2}{2xy - 1}$ **b** **i** -3 **ii** $-\frac{1}{7}$

9 **a** $-\frac{1}{4}$ **b** -2 **c** 1 **d** $-\frac{1}{9}$ **e** $\frac{9}{8}$ **f** $\frac{1}{2}$

10 **a** $\dfrac{dy}{dx} = \dfrac{4x^3y + y^3}{2x^4 + xy^2}$

b **i** P$(1, 1)$, Q$(1, -2)$

ii At P, gradient $= \frac{5}{3}$. At Q, gradient $= -\frac{8}{3}$.

11 **a** $\dfrac{dy}{dx} = \dfrac{3y - 2x}{2y - 3x}$ **b** $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$

12 **a** $(0, 0)$, $(1, 0)$, $(-1, 0)$

b $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$

13 Q$\left(\frac{5\pi}{6}, \frac{\pi}{2}\right)$

14 $(0, 4)$, $(0, -4)$, $(-3\sqrt{5}, -1)$, and $(3\sqrt{5}, -1)$

REVIEW SET 17A

1 **a** $f'(x) = 15x^2$ **b** $f'(x) = 6x^5 - 5$

c $f'(x) = 14x + \dfrac{3}{x^2}$ **d** $f'(x) = 3 + \dfrac{8}{x^3}$

e $f'(x) = 3\sqrt{x}$ **f** $f'(x) = \dfrac{2}{\sqrt{x}} + \dfrac{1}{2x\sqrt{x}}$

2 **a** $\dfrac{dy}{dx} = 6x - 4x^3$ **b** $\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$

c $\dfrac{dy}{dx} = 2x\sqrt{x - 2} + \dfrac{x^2}{2\sqrt{x - 2}}$

3 **a** $f'(x) = (x^2 + 1)^{-\frac{3}{2}}$ **b** $(0, 0)$

4 **a** $\dfrac{dy}{dx} = 3x^2e^{x^3+2}$ **b** $\dfrac{dy}{dx} = \dfrac{1}{x + 3} - \dfrac{2}{x}$

c $\dfrac{dy}{dx} = 3x^2e^{2x} + 2x^3e^{2x}$

5 **a** $f'(x) = 5^{\sin x} \ln 5 \times \cos x$

b $f'(x) = \dfrac{2x}{(x^2 + 4)\ln 2}$ **c** $f'(x) = \dfrac{\ln 4}{\ln 3}$

6 **a** 10 **b** -15

7 **a** $5 + 3x^{-2}$ **b** $4(3x^2 + x^{\frac{1}{2}})^3(6x + \frac{1}{2}x^{-\frac{1}{2}})$

c $2x(1 - x^2)^3 - 6x(x^2 + 1)(1 - x^2)^2$

8 $(-2, 19)$ and $(1, -2)$

9 **a** $(5\cos 5x)\ln x + \dfrac{\sin 5x}{x}$ **b** $\cos x \cos 2x - 2\sin x \sin 2x$

c $-2e^{-2x}\tan x + e^{-2x}\sec^2 x$

10 $\frac{\sqrt{3}}{2}$

12 **a** $f'(x) = \dfrac{-x^2 + 6x - 3}{e^x}$ **b** $\dfrac{2}{e}$ **c** $x = 3 \pm \sqrt{6}$

13 **a** $f'(x) = 8x(x^2 + 3)^3$

b $g'(x) = \dfrac{\frac{1}{2}x(x + 5)^{-\frac{1}{2}} - 2(x + 5)^{\frac{1}{2}}}{x^3}$

c $h'(x) = \dfrac{6e^{4x} - 8xe^{4x}}{(1 - 2x)^2}$

14 **a** $\dfrac{dy}{dx} = \cos^2 x - \sin^2 x$

b **Hint:** $\cos 2x = \cos^2 x - \sin^2 x$ **c** $-\frac{1}{2}$

15 **a** $f(\frac{\pi}{2}) = 1$ **b** $f'(\frac{\pi}{2}) = 0$ **c** $f''(\frac{\pi}{2}) = 2$

16 2

18 **a** $\dfrac{dy}{dx} = -4\operatorname{cosec} 4x \cot 4x$

b $\dfrac{dy}{dx} = \sqrt{\cos x} - \dfrac{x\sin x}{2\sqrt{\cos x}}$

c $\dfrac{dy}{dx} = e^x(\cot 2x - 2\operatorname{cosec}^2 2x)$

d $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{4 - x^2}}$

e $\dfrac{dy}{dx} = 5x^4 \arcsin 3x + \dfrac{3x^5}{\sqrt{1 - 9x^2}}$

f $\dfrac{dy}{dx} = 2^x \ln 2 \times \arctan(x - \frac{\pi}{3}) + \dfrac{2^x}{1 + (x - \frac{\pi}{3})^2}$

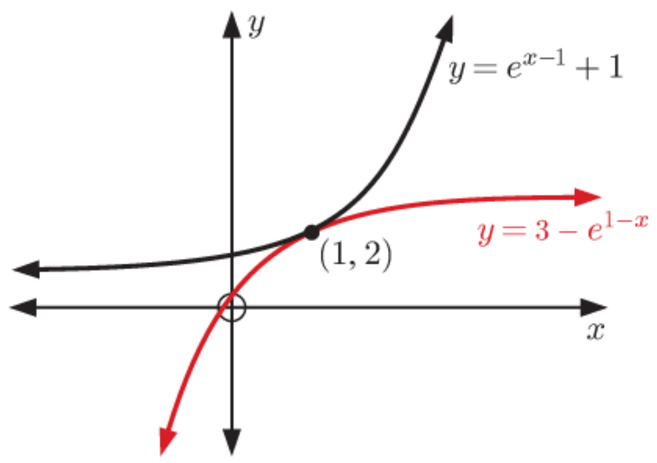
19 **a** $\dfrac{d^2y}{dx^2} = \frac{3}{2}x^2 + x - \frac{1}{2}$ **b** $\dfrac{d^2y}{dx^2} = -2e^{-x} + xe^{-x}$

20 **a** $f'(x) = \dfrac{1}{2\sqrt{x}}\cos 4x - 4\sqrt{x}\sin 4x$,

$f''(x) = -\dfrac{1}{4x\sqrt{x}}\cos 4x - \dfrac{4}{\sqrt{x}}\sin 4x - 16\sqrt{x}\cos 4x$

b **i** $\frac{1}{\sqrt{2}}(\frac{2}{\sqrt{\pi}} - \sqrt{\pi})$ **ii** $-\frac{8\sqrt{2}}{\sqrt{\pi}}$

21 **a**



b $(1, 2)$

c The gradient is $e^0 = 1$ for both. The tangents to each of the curves at this point are the same line.

22 $\dfrac{dy}{dx} = -1$, $\dfrac{d^2y}{dx^2} = 0$

25 **a** **i** $f'(x) = 3^x(1 + x\ln 3)$

ii $f''(x) = 3^x \ln 3(2 + x\ln 3)$

iii $f^{(3)}(x) = 3^x(\ln 3)^2(3 + x\ln 3)$

iv $f^{(4)}(x) = 3^x(\ln 3)^3(4 + x\ln 3)$

b $f^{(n)}(x) = 3^x(\ln 3)^{n-1}(n + x\ln 3)$

26 $-\dfrac{\pi^2}{2 + \sqrt{3}\pi}$

REVIEW SET 17B

1 **a** $f'(x) = 6x - 7$ **b** $f'(x) = 2x + 10$

c $f'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{3}{x^2}$ **d** $f'(x) = 15x\sqrt{x}$

2 **a** $\dfrac{dy}{dx} = 6x^2 - 12x + 7$ **b** $\dfrac{dy}{dx} = -\dfrac{3}{x^2} + \dfrac{15}{x^4}$

c $\dfrac{dy}{dx} = -\dfrac{5}{x^{\frac{4}{3}}}$

3 **a** $f(3) = -17$ **b** $f'(3) = -17$ **c** $f''(3) = -6$

4 **a** $\dfrac{dy}{dx} = 3x^2(1-x^2)^{\frac{1}{2}} - x^4(1-x^2)^{-\frac{1}{2}}$

b $\dfrac{dy}{dx} = \dfrac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$

5 **a** $\dfrac{dy}{dx} = e^x + xe^x$ **b** $(1, e)$

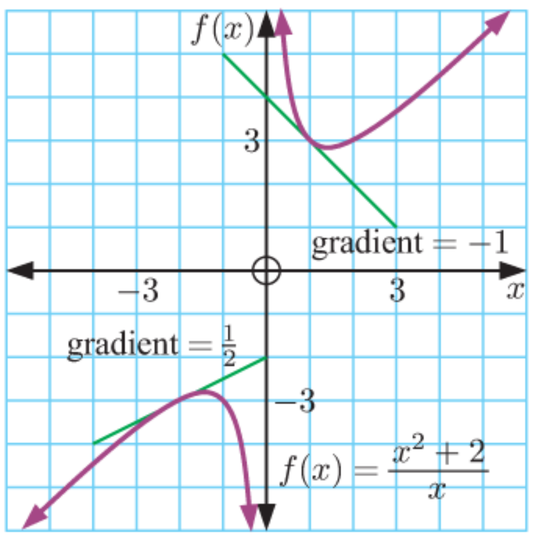
6 **a** $f'(x) = \dfrac{e^x}{e^x+3}$ **b** $f'(x) = \dfrac{3}{x+2} - \dfrac{1}{x}$

c $f'(x) = x^{x^2+1}(2\ln x + 1)$

7 **a** $f'(x) = 1 - \dfrac{2}{x^2}$

b **i** -1 **ii** $\frac{1}{2}$

c



8 when $x = 1$, $\dfrac{dy}{dx} = 0$

9 **a** $\dfrac{dy}{dx} = \dfrac{3x^2-3}{x^3-3x}$ **b** $\dfrac{dy}{dx} = \dfrac{e^x(x-2)}{x^3}$

c $\dfrac{dy}{dx} = \dfrac{e^{x+y}(y^2+1)}{2y - e^{x+y}(y^2+1)}$

10 **a** $f'(x) = 2^{x^2-5x}\ln 2 \times (2x-5)$

b $f'(x) = \dfrac{\operatorname{cosec} x \sec x}{\ln 10}$

c $f'(x) = -\dfrac{1}{\ln 3}\left(\dfrac{1}{x+5} + \dfrac{1}{x-4}\right)$

11 **a** $f''(x) = 24x^2 - 24x - 18$ **b** $x = -\frac{1}{2}$ or $\frac{3}{2}$

12 **a** $10 - 10\cos 10x$ **b** $\tan x$

c $(5\cos 5x)\ln(2x) + \dfrac{\sin 5x}{x}$

13 **a** $\frac{28}{9}$ **b** 8 **14** $a = 4,\ b = e^2$

15 **a** $-\frac{1}{4}$ **b** **Hint:** Show that $\dfrac{dy}{dx} = -\dfrac{2\sin x + 1}{(\sin x + 2)^2}$.

16 **b** **i** $x = \frac{1}{2}$ **ii** $x \leqslant 0$

18 **a** $\dfrac{dy}{dx} = -\frac{1}{4}\operatorname{cosec}^2\left(\frac{x}{4}\right)$

b $\dfrac{dy}{dx} = x\sec 3x(3x\tan 3x + 2)$

c $\dfrac{dy}{dx} = -e^{-x}\operatorname{cosec}(e^x)[e^x\cot(e^x) + 1]$

d $\dfrac{dy}{dx} = \dfrac{5}{\sqrt{1-25x^2}}$

e $\dfrac{dy}{dx} = 2e^{2x}\left(\arctan 2x + \dfrac{1}{1+4x^2}\right)$

f $\dfrac{dy}{dx} = -\dfrac{\sec^2 x}{\sqrt{1-\tan^2 x}}$

19 **a** $\dfrac{d^2y}{dx^2} = -\dfrac{10}{(1-2x)^3}$ **b** $\dfrac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$

c $\dfrac{d^2y}{dx^2} = -\dfrac{3^x(\ln 3)^2 \times 20}{(20-3^x)^2 \times \ln 2}$

21 **a** $x = -6 \pm \sqrt{33}$ **b** $x = \pm\sqrt{3}$ **c** $x = -3$, 0, or 3

22 **a** $f(x) = -5\sin 4x$ **b** $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$, or $\frac{7\pi}{8}$

23 **a** $\dfrac{dy}{dx} = \dfrac{y^2 - e^x y}{e^x - 2xy}$ **b** 0

27 **a** $f'(x) = e^{ax}(ax+1)$, $f''(x) = ae^{ax}(ax+2)$,
$f'''(x) = a^2e^{ax}(ax+3)$, $f^{(4)}(x) = a^3e^{ax}(ax+4)$

b $f^{(n)}(x) = a^{n-1}e^{ax}(ax+n)$

EXERCISE 18A

1 **a** $f'(x) = 2x - 4$ **b** $y = -2x - 1$

2 **a** $y = -7x + 11$ **b** $y = \frac{1}{4}x + 2$ **c** $y = -2x - 2$
d $y = -2x + 6$ **e** $y = -5x - 9$ **f** $y = -5x - 1$
g $y = \frac{3}{4} - \frac{1}{2}x$ **h** $y = \frac{1}{27}x + \frac{4}{9}$

3 **a** $y = 21$ and $y = -6$ **b** $y = 23$ and $y = -9$
c $y = 2$

4 **a** $k = -5$ **b** $y = 4x - 15$ **5** $y = -3x + 1$

6 $a = -4,\ b = 7$ **7** $a = 2,\ b = \frac{1}{2}$ **10** $a = 4,\ b = 3$

11 **a** $k = 1$ **b** $y = 8x - 7$ **c** $\frac{7}{8}$

12 **a** $\dfrac{dy}{dx} = -\dfrac{3x^2}{2y}$ **b** $y = 4x - 22$

13 **a** $y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$ **b** $(2, 0)$

14 **a** $y = -e^{-2}x + 3e^{-2}$ **b** $y = -\frac{1}{3}x - \frac{1}{3} + \ln 3$
c $y = 4ex - e$ **d** $y = \dfrac{e^2}{2}x - \frac{3}{2}$ **e** $y = 3ex - 5e$

15 **a** Domain is $\{x \mid x < 0 \text{ or } x > 2\}$

b $f'(x) = \dfrac{1}{x} + \dfrac{1}{x-2}$ **c** $y = \frac{4}{3}x - 4 + \ln 3$

16 **a** **i** -1 **ii** -2 **iii** -3 **b** y-intercept is $-a$

17 x-intercept $\frac{2}{3}$, y-intercept $-2e$ **18** $\dfrac{3}{4\sqrt{e}}$ units^2

19 **a** $y = x$ **b** $y = x$ **c** $y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$
d $y = 1$ **e** $y = 2$

20 **Hint:** Show that there are no tangents which have gradient $= 0$.

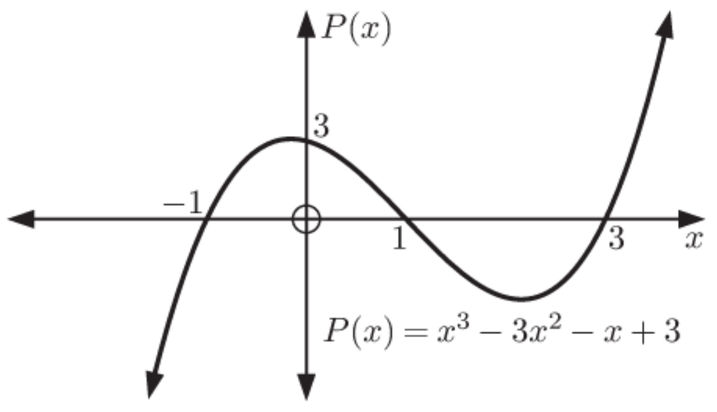
21 $\frac{1}{2}\left(\frac{\pi}{2} + 1\right)^2$ units^2

22 **a** $y = \sqrt{2}x - \sqrt{2}\left(\frac{\pi}{4} - 1\right)$ **b** $y = -2x + \frac{2\pi}{3} + \sqrt{3}$
c $y = \frac{1}{2}x + \frac{\pi}{4} - \frac{1}{2}$ **d** $y = \left(\frac{1}{\sqrt{3}} + \frac{2\pi}{3}\right)x + \frac{1}{\sqrt{3}}$

23 $(-4, -64)$ **24** $(4, -31)$ **25** $(-1, -2)$

26 **a** $P(1) = 1 - 3 - 1 + 3 = 0$
The zeros of $P(x)$ are $x = 1$, -1, and 3.

b



c $y = -x - 1$ **d** $(-1, 0)$

e **Hint:** Let $P(x) = \alpha(x-a)(x-b)(x-c)$

27 **a** $y = (2a-1)x - a^2 + 9$

b $y = 5x$, point of contact $(3, 15)$, and $y = -7x$, point of contact $(-3, 21)$

28 **a** $y = (2a+4)x - a^2$

b $y = 12x - 16$, point of contact $(4, 32)$, and $y = -4$, point of contact $(-2, -4)$

29 $y = 5x - 15$ and $y = -7x - 3$

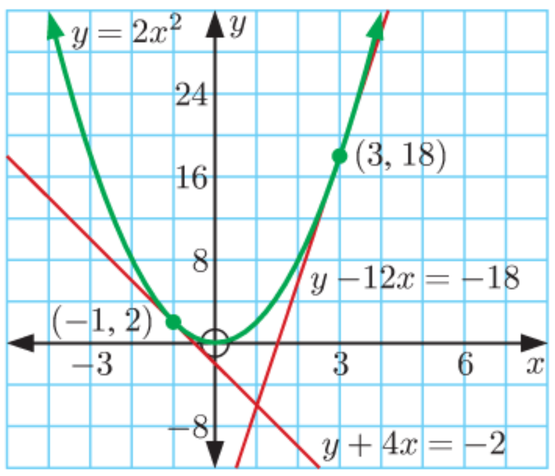
30 **a** $y = e^a x + e^a(1-a)$ **b** $y = ex$

31 **a** $y + 4x = -2$ and $y - 12x = -18$

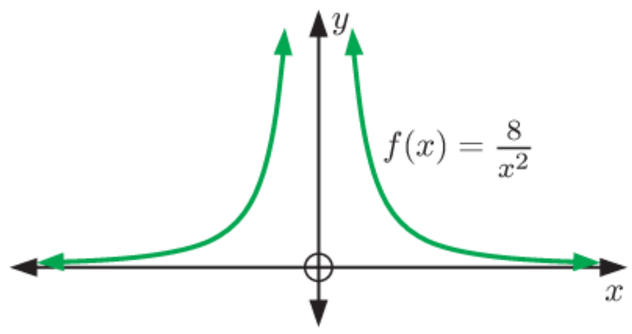
b $(-1, 2)$ for $y + 4x = -2$ and $(3, 18)$ for $y - 12x = -18$

c For a tangent to pass through $(1, 4)$, $4 = 4a - 2a^2$ must have real solutions. But $\Delta < 0$, so no real solutions.

d



32 **a**



b $16x + a^3 y = 24a$ **c** A is $(\frac{3}{2}a, 0)$, B is $\left(0, \frac{24}{a^2}\right)$

d area $= \dfrac{18}{|a|}$ units^2; as $a \to \infty$, area $\to 0$

33 $a = \frac{1}{4}$, point of intersection $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

34 $\approx 63.43°$

35 **a** **Hint:** They must have the same y-coordinate at $x = b$ and the same gradient.

c $a = \frac{1}{2e}$ **d** $y = e^{-\frac{1}{2}}x - \frac{1}{2}$

36 **a** $y = 2asx - as^2$, $y = 2atx - at^2$

EXERCISE 18B

1 **a** $x + 8y = 132$ **b** $x + 7y = 26$ **c** $x - 3y = -11$

d $x + 6y = 43$ **e** $64x + 4y = -65$ **f** $x = 2$

g $4x + 57y = 1042$ **h** $x - 2y = -1$

2 **a** $y = 4 - 2x$ **b** $y = -\frac{9}{62}x + \frac{1259}{186}$

3 **a** $\{x \mid x < 2\}$ **c** $3x - 2y = 1$ **d** $8x + 3y = -19$

4 **a** $y = x + 1$ **b** $ex + y = e^2 + 1$ **c** $x + 2ey = 1 + 2e^2$

d $2x + y = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$ **e** $y = 1 - \dfrac{1}{\ln 4}x$ **f** $y = -x$

g $2x - 2\sqrt{3}y = \pi + \sqrt{3}$ **h** $x - \sqrt{2}y = \frac{\pi}{4}$

i $(3\ln 3)x + 2y = 6\ln 3 + \dfrac{2\ln 6}{\ln 3}$

j $x - 2\sqrt{3}y = \frac{\pi}{6} - 4\sqrt{3}$ **k** $\sqrt{6}x + y = \sqrt{6}\pi + \sqrt{2}$

5 $a = 2$, $b = 4$ **6** $(-1, -2)$ and $(2, 1)$

7 $x = 0$ **8** $y = -\sqrt{14}x + 4\sqrt{14}$

9 **a** $\dfrac{dx}{dy} = e^y(y+2)$ **b** $4e^2x + y = 2 + 12e^4$ **10** $b = 3$

EXERCISE 18C

1 **a** **i** $x \geqslant 0$ **ii** never **b** **i** never **ii** $-2 < x \leqslant 3$

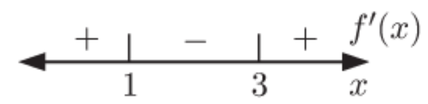
c **i** $x \leqslant 2$ **ii** $x \geqslant 2$ **d** **i** $x \in \mathbb{R}$ **ii** never

e **i** $1 \leqslant x \leqslant 5$ **ii** $x \leqslant 1$, $x \geqslant 5$

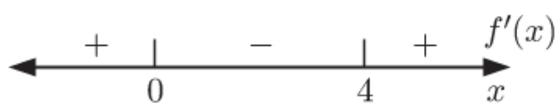
f **i** $2 \leqslant x < 4$, $x > 4$ **ii** $x < 0$, $0 < x \leqslant 2$

2 **a** **i** $x \leqslant 1$, $x \geqslant 3$ **ii** $1 \leqslant x \leqslant 3$

b $f'(x) = 3x^2 - 12x + 9$
$= 3(x-3)(x-1)$



3 **a** $f'(x) = 3x^2 - 12x$
$= 3x(x-4)$



b increasing for $x \leqslant 0$ and $x \geqslant 4$
decreasing for $0 \leqslant x \leqslant 4$

4 **a** increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

b never increasing, decreasing for all $x \in \mathbb{R}$

c increasing for $x > 0$, never decreasing

d increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$

e never increasing, decreasing for all $x \neq 0$

f increasing for $x \geqslant 1$, decreasing for $0 \leqslant x \leqslant 1$

g never increasing, decreasing for $x > 0$

h increasing for $x \leqslant 0$, and $x \geqslant 4$,
decreasing for $0 \leqslant x \leqslant 4$

i increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$, $x \geqslant \sqrt{\frac{2}{3}}$

j increasing for $-\frac{1}{2} \leqslant x \leqslant 3$,
decreasing for $x \leqslant -\frac{1}{2}$, $x \geqslant 3$

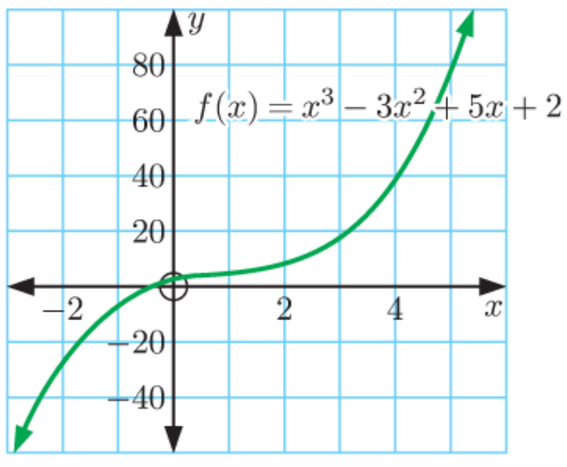
k increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

l increasing for $x \leqslant 2 - \sqrt{3}$, $x \geqslant 2 + \sqrt{3}$,
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

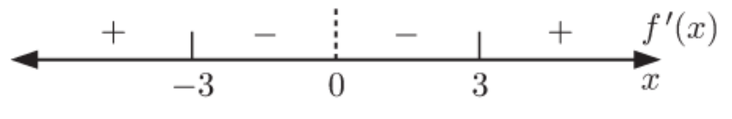
5 **a** $f'(x) = 3x^2 - 6x + 5$

b $\Delta = 36 - 60 < 0$ and $a > 0$
$\therefore$ $f'(x)$ lies entirely above x-axis.
$\therefore$ $f'(x) > 0$ for all x.
$\therefore$ $f(x)$ is increasing for all x.

c

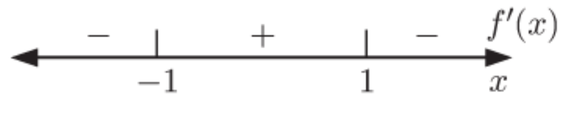


6 **a**



b increasing for $x \leqslant -3$ and $x \geqslant 3$,
decreasing for $-3 \leqslant x < 0$ and $0 < x \leqslant 3$

7 **a**



b increasing for $-1 \leqslant x \leqslant 1$,
decreasing for $x \leqslant -1$ and $x \geqslant 1$

8 **a** never increasing, decreasing for all $x \neq -1$

b increasing for $-1 \leqslant x < 1$,
decreasing for $x \leqslant -1$, $x > 1$

c increasing for $-1 \leqslant x < 1$, $1 < x \leqslant 3$,
decreasing for $x \leqslant -1$, $x \geqslant 3$

d increasing for $x \leqslant -\sqrt{3}$, $x \geqslant \sqrt{3}$,
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x < 1$,
$1 < x \leqslant \sqrt{3}$

e increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

f increasing for $x \geqslant 2$, decreasing for $x < 1$, $1 < x \leqslant 2$

9 In this case, $f(x)$ is only defined when $x > 0$.
$\therefore$ $f'(x)$ is only defined when $x > 0$.

10 **a** increasing for all $x \in \mathbb{R}$, never decreasing

b increasing for $x > -2$, never decreasing

c never increasing, decreasing for all $x \in \mathbb{R}$

d increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$

e increasing for $x \geqslant e^{-\frac{1}{3}}$, decreasing for $0 < x \leqslant e^{-\frac{1}{3}}$

f increasing for $x \leqslant 0$, decreasing for $x \geqslant 0$

g increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

h increasing for $x \leqslant -1$,
decreasing for $-1 \leqslant x < 0$, $x > 0$

i increasing for $x \geqslant \dfrac{1}{e^2}$, decreasing for $0 < x \leqslant \dfrac{1}{e^2}$

11 **a** never increasing, decreasing for $-1 \leqslant x \leqslant 1$

b increasing for $-1 \leqslant x \leqslant 1$, never decreasing

c increasing for all $x \in \mathbb{R}$, never decreasing

d increasing for $(2k+1)\pi \leqslant x \leqslant (2k+2)\pi$, $k \in \mathbb{Z}$,
decreasing for $2k\pi \leqslant x \leqslant (2k+1)\pi$, $k \in \mathbb{Z}$

e increasing for $\dfrac{(4k-1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+1)\pi}{2}$, $k \in \mathbb{Z}$,
decreasing for $\dfrac{(4k+1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+3)\pi}{2}$, $k \in \mathbb{Z}$

f increasing for $x \neq \dfrac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, never decreasing

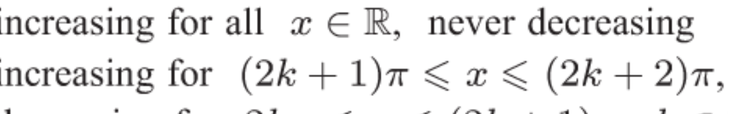
g increasing for $\dfrac{(6k+1)\pi}{3} \leqslant x \leqslant \dfrac{(6k+5)\pi}{3}$, $k \in \mathbb{Z}$,
decreasing for $\dfrac{(6k-1)\pi}{3} \leqslant x \leqslant \dfrac{(6k+1)\pi}{3}$, $k \in \mathbb{Z}$

h increasing for all $x \in \mathbb{R}$, never decreasing

EXERCISE 18D

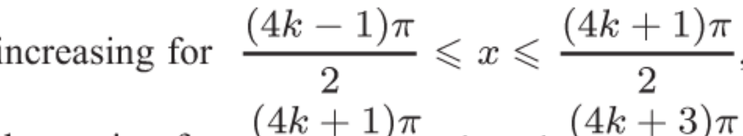
1 **a** A - local max., O - stationary inflection, B - local min.

b



c **i** $x \leqslant -2$ and $x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

d

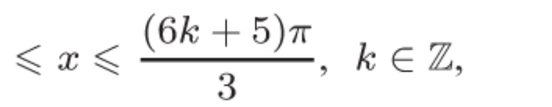


2 **a** P is a local maximum, Q is a local minimum.

b $f'(x) = 3x^2 + 12x - 15 = 3(x+5)(x-1)$

c P$(-5, 60)$, Q$(1, -48)$

3 **a** $f'(x) = x^2 - 9$
$= (x+3)(x-3)$

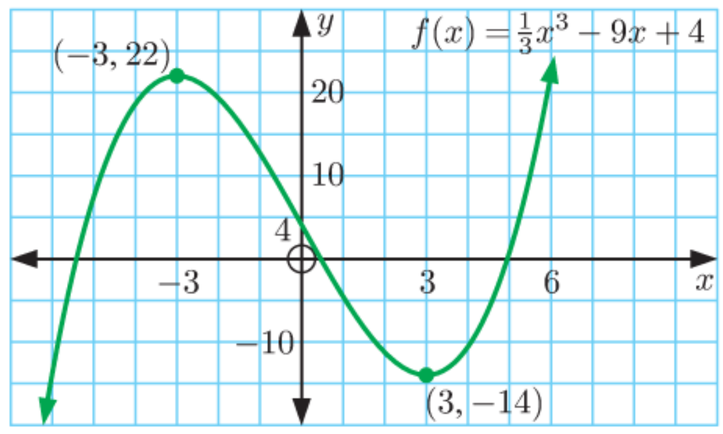


b increasing for $x \leqslant -3$ and $x \geqslant 3$,
decreasing for $-3 \leqslant x \leqslant 3$

c $(-3, 22)$ is a local maximum, $(3, -14)$ is a local minimum

d As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$.

e



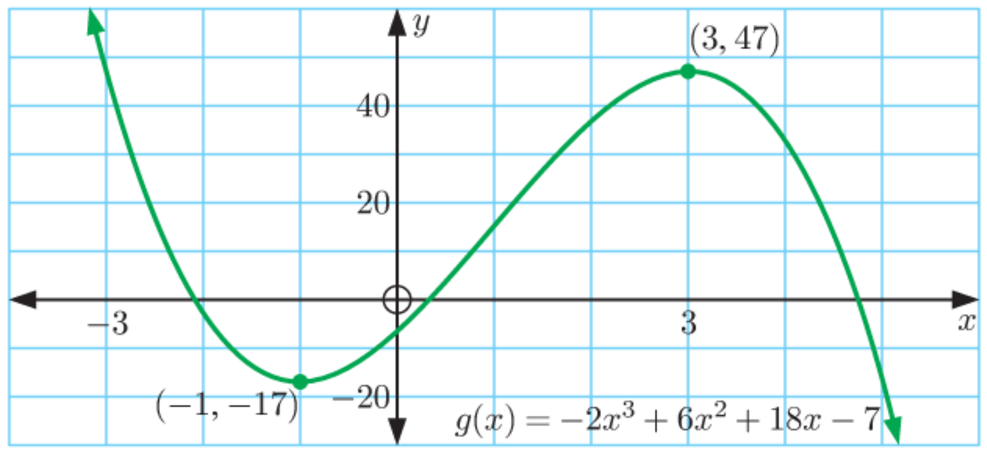
4 **a** $g'(x) = -6x^2 + 12x + 18 = -6(x-3)(x+1)$

b increasing for $-1 \leqslant x \leqslant 3$,
decreasing for $x \leqslant -1$ and $x \geqslant 3$

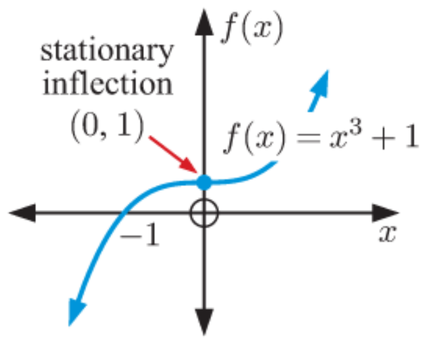
c $(3, 47)$ is a local maximum, $(-1, -17)$ is a local minimum

d As $x \to \infty$, $g(x) \to -\infty$, as $x \to -\infty$, $g(x) \to \infty$.

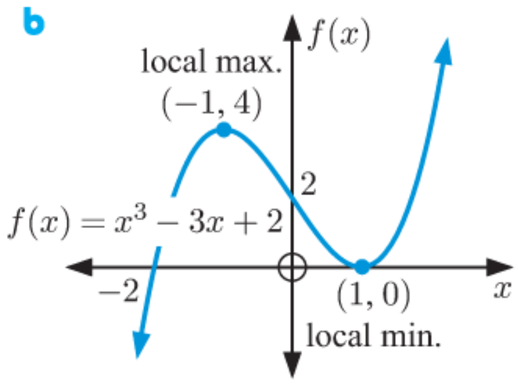
e



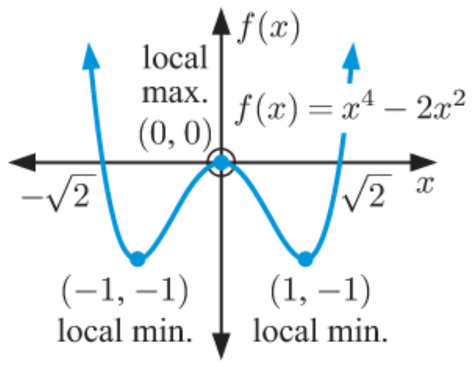
5 **a**



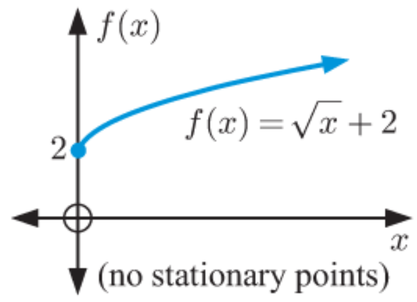
b



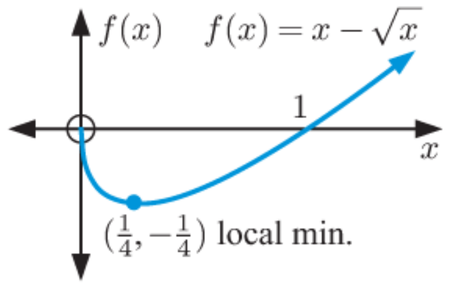
c



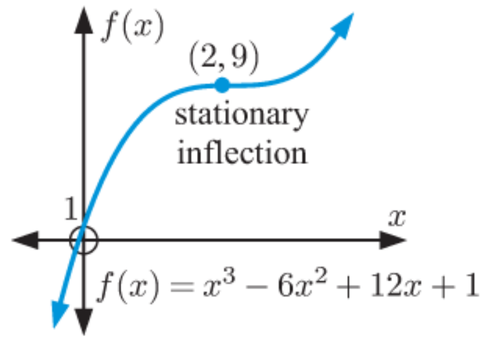
d



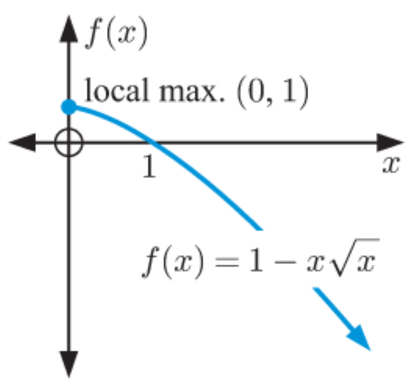
e



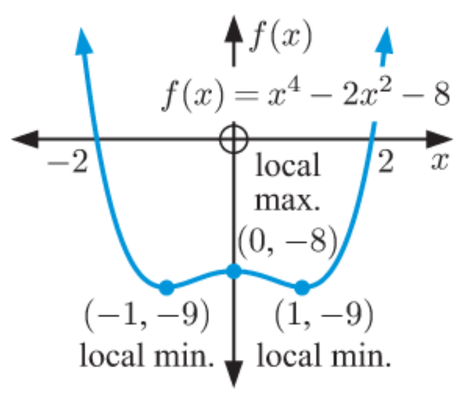
f



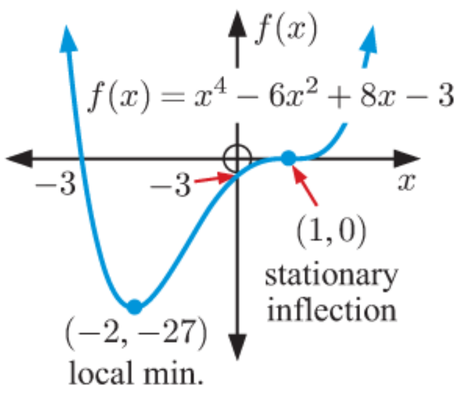
g



h

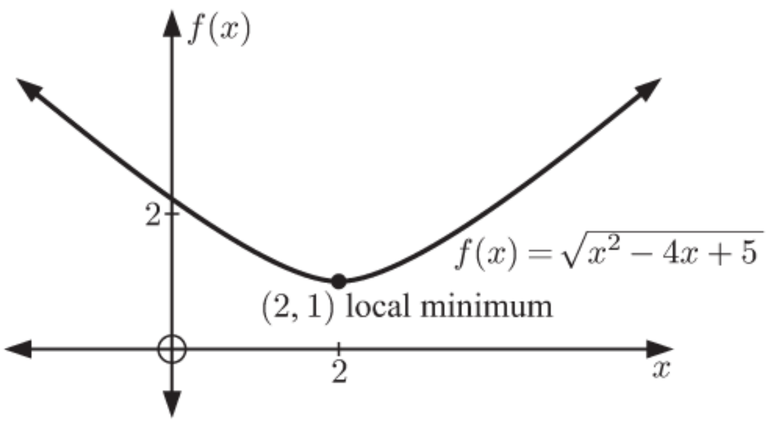


i

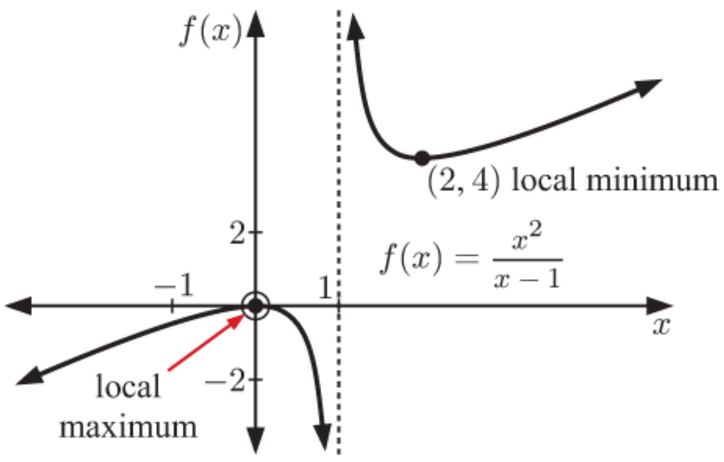


6 **a** $x = -\dfrac{b}{2a}$ **b** local min. if $a > 0$, local max. if $a < 0$

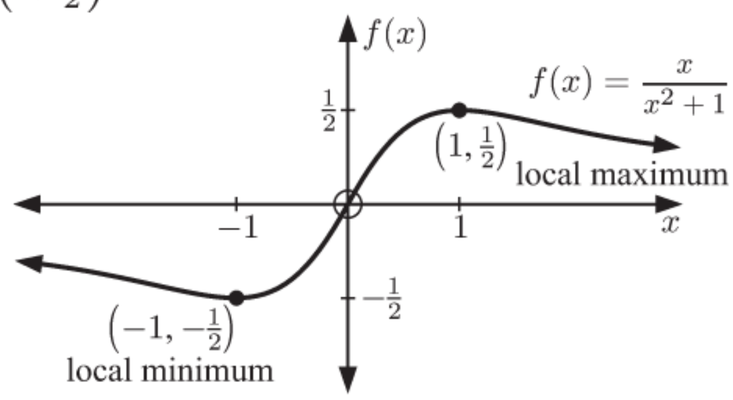
7 **a** $(2, 1)$ is a local minimum



b $(0, 0)$ is a local maximum,
$(2, 4)$ is a local minimum



c $\left(-1, -\frac{1}{2}\right)$ is a local minimum,
$\left(1, \frac{1}{2}\right)$ is a local maximum



8 **a** $\left(1, \dfrac{1}{e}\right)$ is a local maximum

b $\left(-2, \dfrac{4}{e^2}\right)$ is a local maximum, $(0, 0)$ is a local minimum

c $(1, e)$ is a local minimum **d** $(-1, e)$ is a local maximum

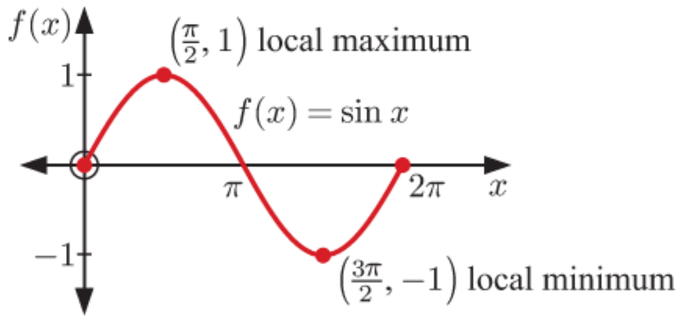
9 **a** $a = 9$ **b** $(-4, 113)$

10 **a** $a = -12$, $b = -13$

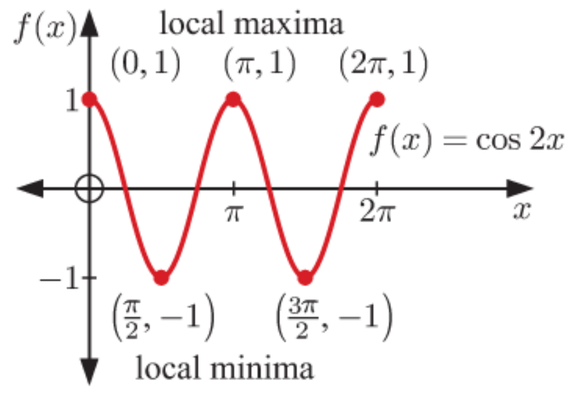
b $(-2, 3)$ is a local maximum, $(2, -29)$ is a local minimum

11 $a = 3$, $b = 6$, local minimum **12** **a** $x > 0$

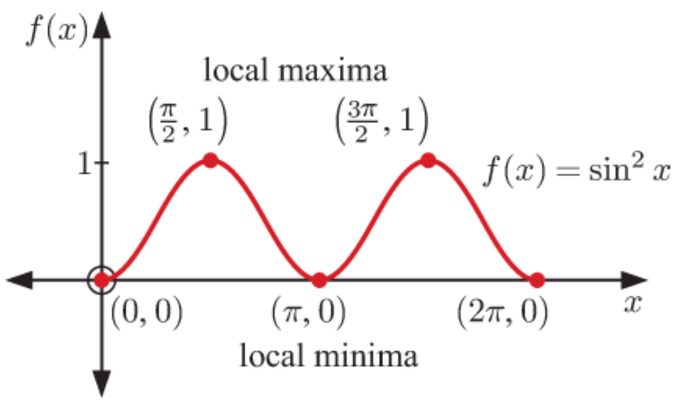
13 **a** $\left(\frac{\pi}{2}, 1\right)$ is a local maximum, $\left(\frac{3\pi}{2}, -1\right)$ is a local minimum



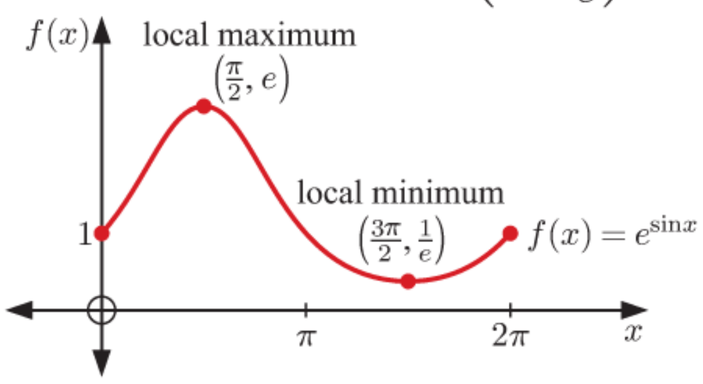
b $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$ are local maxima,
$\left(\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$ are local minima



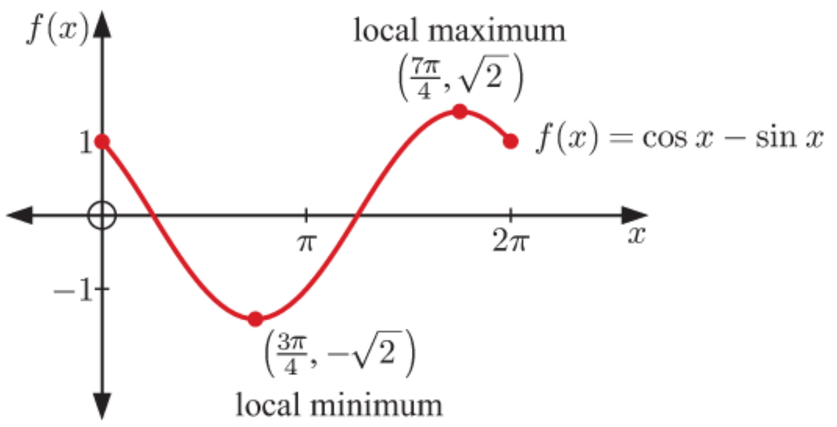
c $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$ are local minima,
$\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, 1\right)$ are local maxima



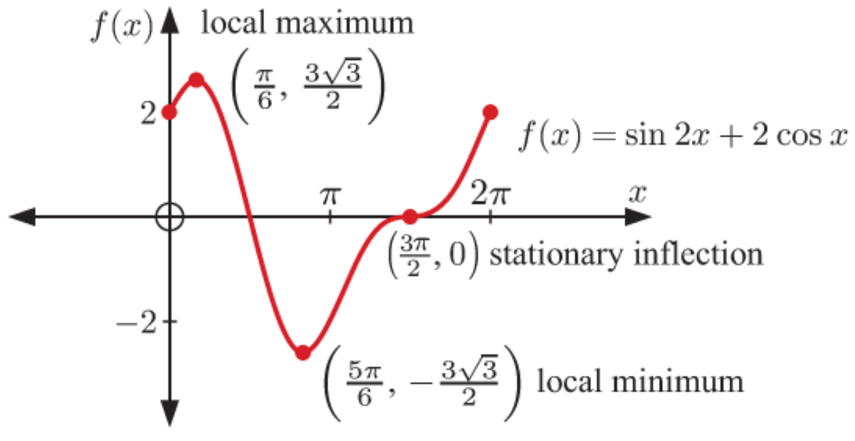
d $\left(\frac{\pi}{2}, e\right)$ is a local maximum, $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$ is a local minimum



e $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$ is a local minimum,
$\left(\frac{7\pi}{4}, \sqrt{2}\right)$ is a local maximum



f $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ is a local maximum,
$\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$ is a local minimum,
$\left(\frac{3\pi}{2}, 0\right)$ is a stationary inflection



14 $P(x) = -9x^3 - 9x^2 + 9x + 2$

15 **a** greatest value is 63 when $x = 5$, least value is -18 when $x = 2$

b greatest value is 4 when $x = 3$ and $x = 0$, least value is -16 when $x = -2$

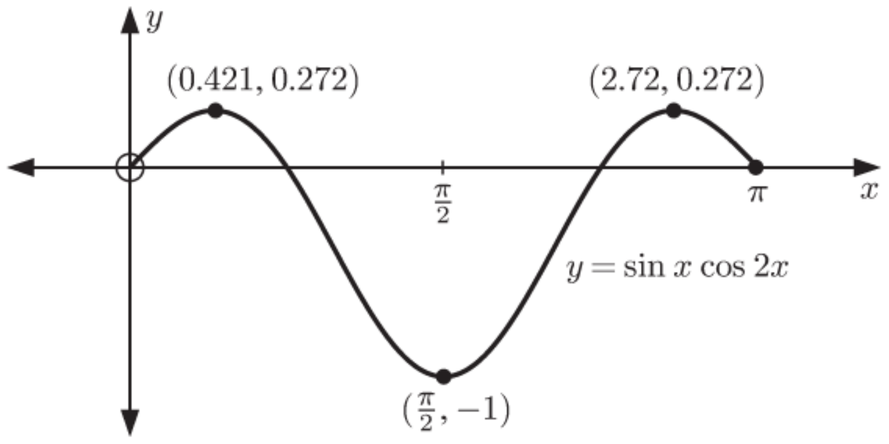
c greatest value is 20 when $x = 4$, least value is 12 when $x = 2$

d greatest value is 0 when $x = 0$, least value is -4 when $x = 4$

17 **a** $f'(x) = 6\cos^3 x - 5\cos x$

c local max. at $(0.421,\ 0.272)$, $(2.72,\ 0.272)$, local min. at $(\frac{\pi}{2},\ -1)$

d



18 $a = \dfrac{\sqrt{e}}{2}$, $b = -\frac{1}{8}$

19 **Hint:** Show that $\dfrac{\ln x}{x}$ has only one stationary point, which is a local maximum.

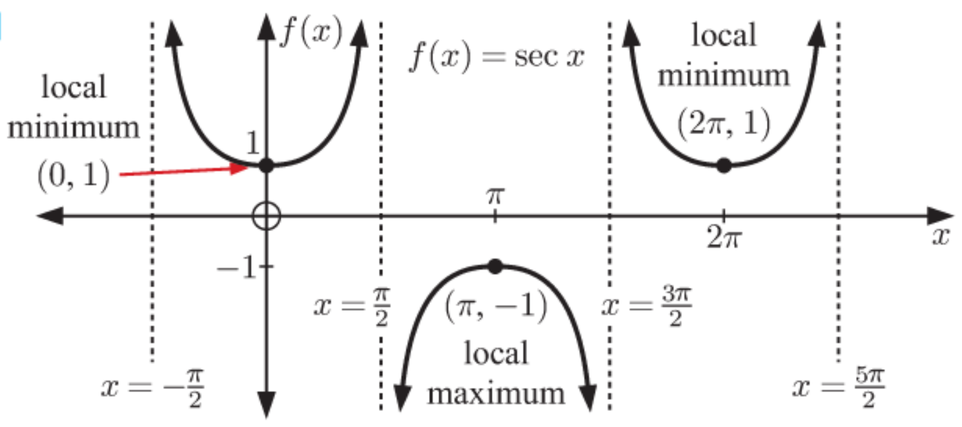
20 **Hint:** Show that $f(x) \geqslant 1$ for all $x > 0$.

21 **a** $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b $(0,\ 1)$ and $(2\pi,\ 1)$ are local minima, $(\pi,\ -1)$ is a local maximum

c $f(x)$ has period 2π

d



22 **a** $\dfrac{dy}{dx} = \dfrac{e^x(x+1)}{1-2y}$ **b** $y = x + 1$

c $\left(-1,\ \dfrac{1+\sqrt{1+\frac{4}{e}}}{2}\right)$ and $\left(-1,\ \dfrac{1-\sqrt{1+\frac{4}{e}}}{2}\right)$

EXERCISE 18E

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	+	−	+
B	−	0	+
C	+	+	0
D	+	0	−
E	0	−	−

b B is a local minimum, D is a local maximum

c C

2 **a** $f'(x) = 3x^2 + 6x - 5$, $f''(x) = 6x + 6$

b



c **i** $x \geqslant -1$ **ii** $x \leqslant -1$

3 **a** concave up **b** concave down **c** concave down **d** concave up

4 **a** **i** $x \geqslant 0$ **ii** $x \leqslant 0$ **iii** $x \in \mathbb{R}$ **iv** never

b **i** never **ii** $x \in \mathbb{R}$ **iii** $x \leqslant 0$ **iv** $x \geqslant 0$

c **i** $x \in \mathbb{R}$ **ii** never **iii** $x \in \mathbb{R}$ **iv** never

d **i** $x > 0$ **ii** never **iii** never **iv** $x > 0$

e **i** $x > 0$ **ii** never **iii** never **iv** $x > 0$

f **i** $-\sqrt{6} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{6}$

ii $x \leqslant -\sqrt{6}$ and $0 \leqslant x \leqslant \sqrt{6}$

iii $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$ **iv** $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

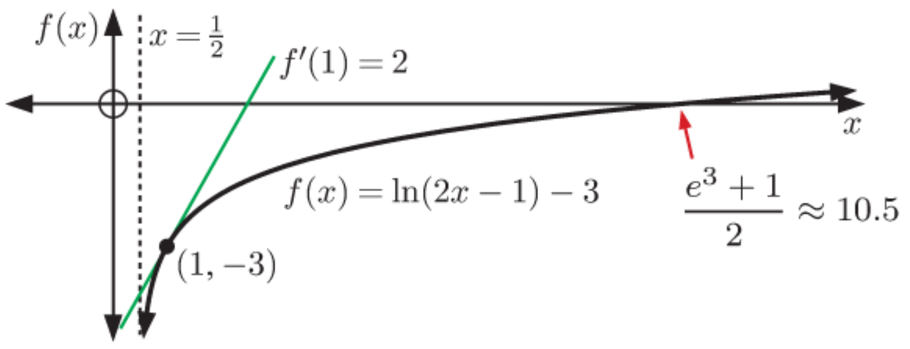
5 **a** x-intercept is $\dfrac{e^3+1}{2} \approx 10.5$

b no, $\therefore$ there is no y-intercept

c Domain is $\{x \mid x > \frac{1}{2}\}$ **d** gradient $= 2$

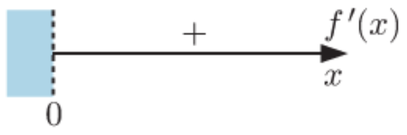
e $f''(x) = \dfrac{-4}{(2x-1)^2} < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down.

f



6 **a** $x > 0$

b $f'(x) = \dfrac{1}{x}$



$f(x)$ is increasing for $x > 0$.

$f''(x) = -\dfrac{1}{x^2}$



$f(x)$ is concave down for $x > 0$.

c $ex + y = e^2 + 1$

7 **a** concave up for $x \leqslant 0$, concave down for $x \geqslant 0$

b never concave up, concave down for all $x \neq 0$

c concave up for $0 < x \leqslant 2$, concave down for $x \geqslant 2$

d concave up for $x > -2$, concave down for $x < -2$

e concave up for $x < -1$, concave down for $x > -1$

f concave up for $0 < x \leqslant 0.753$ and $x > 2$, concave down for $0.753 \leqslant x < 2$

8 **a** $-1 \leqslant x \leqslant 1$ **b** $x \leqslant -1$ and $x \geqslant 1$

c $-\sqrt{3} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{3}$

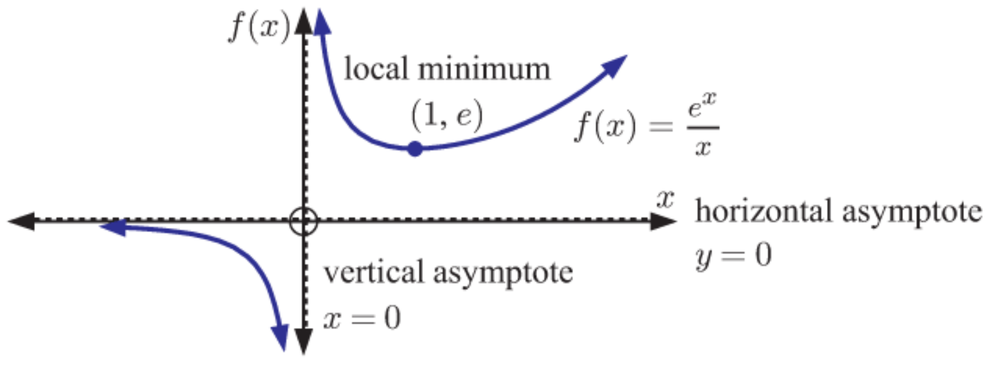
d $x \leqslant -\sqrt{3}$ and $0 \leqslant x \leqslant \sqrt{3}$

9 **a** $f(x)$ does not have any x or y-intercepts.

b as $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to 0^-$

c local minimum at $(1,\ e)$ **d** **i** $x > 0$ **ii** $x < 0$

e



f $2x + ey = -3$

EXERCISE 18F

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	0	−	+
B	−	0	+
C	0	+	0
D	+	0	0

b B is a local minimum.

c C is a non-stationary inflection point, D is a stationary inflection point.

2 **a** stationary inflection at $(0, 2)$

b non-stationary inflection at $(2, 3)$

c stationary inflection at $(0, 2)$

non-stationary inflection at $\left(-1\frac{1}{3}, 11\frac{13}{27}\right)$

d no points of inflection

e stationary inflection at $(-2, -3)$

f non-stationary inflection at $(1, 9)$

g non-stationary inflections at $(-1, 5)$ and $(1, 5)$

h non-stationary inflections at $(-\sqrt{3}, 4)$ and $(\sqrt{3}, 4)$

i non-stationary inflection at $\left(\frac{1}{e^2}, -\frac{1}{2}\right)$

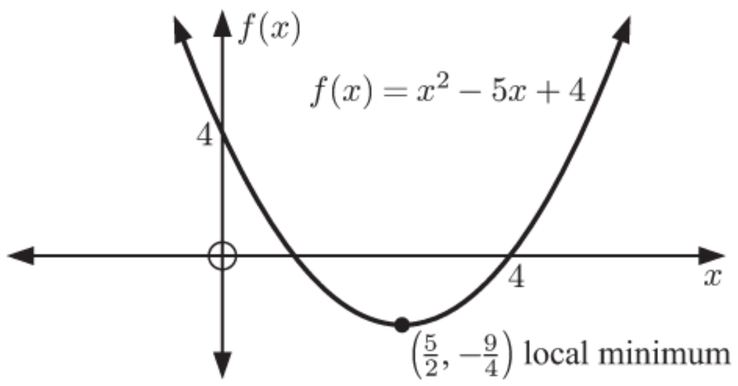
3 **a** **i** local minimum at $(\frac{5}{2}, -\frac{9}{4})$

ii no points of inflection

iii increasing for $x \geqslant \frac{5}{2}$, decreasing for $x \leqslant \frac{5}{2}$

iv concave up for all $x \in \mathbb{R}$

v



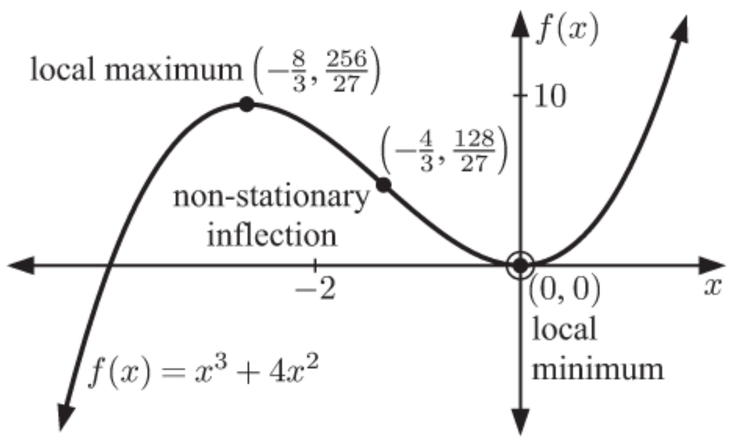
b **i** local maximum at $(-\frac{8}{3}, \frac{256}{27})$, local minimum at $(0, 0)$

ii non-stationary inflection at $(-\frac{4}{3}, \frac{128}{27})$

iii increasing for $x \leqslant -\frac{8}{3}$ and $x \geqslant 0$, decreasing for $-\frac{8}{3} \leqslant x \leqslant 0$

iv concave up for $x \geqslant -\frac{4}{3}$, concave down for $x \leqslant -\frac{4}{3}$

v

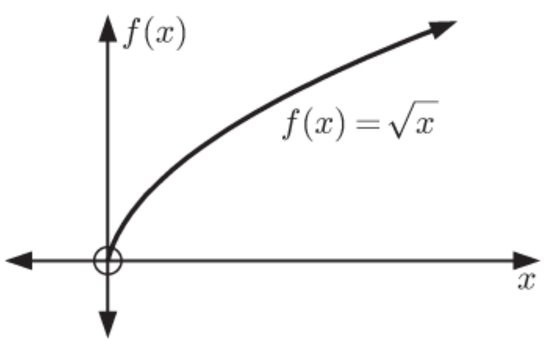


c **i** no turning points **ii** no points of inflection

iii increasing for $x > 0$, never decreasing

iv concave down for $x > 0$, never concave up

v



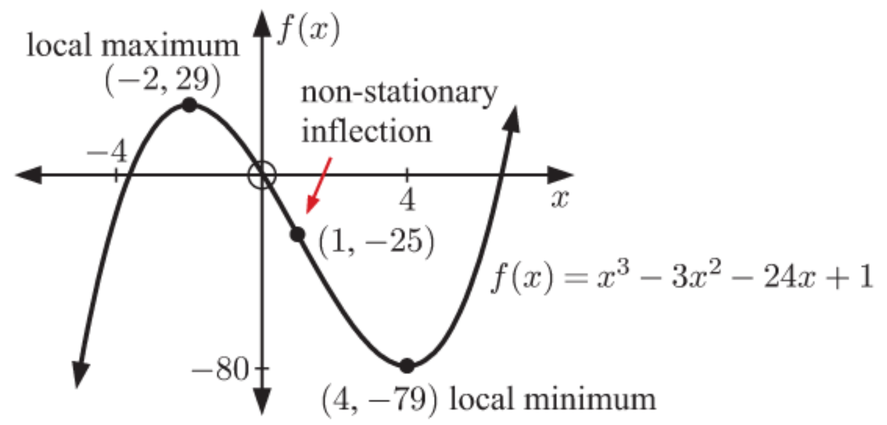
d **i** local maximum at $(-2, 29)$, local minimum at $(4, -79)$

ii non-stationary inflection at $(1, -25)$

iii increasing for $x \leqslant -2$ and $x \geqslant 4$, decreasing for $-2 \leqslant x \leqslant 4$

iv concave down for $x \leqslant 1$, concave up for $x \geqslant 1$

v



e **i** local minimum at $(-1, -3)$

ii non-stationary inflection at $\left(-\frac{2}{3}, -2\frac{16}{27}\right)$

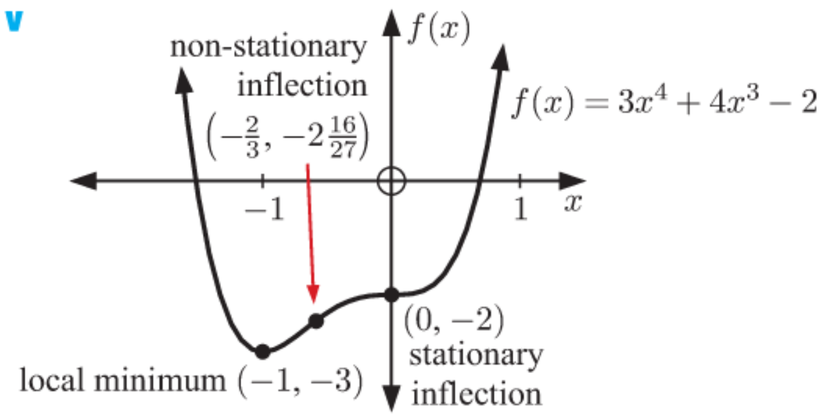
stationary inflection at $(0, -2)$

iii increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$

iv concave down for $-\frac{2}{3} \leqslant x \leqslant 0$

concave up for $x \leqslant -\frac{2}{3}$ and $x \geqslant 0$

v

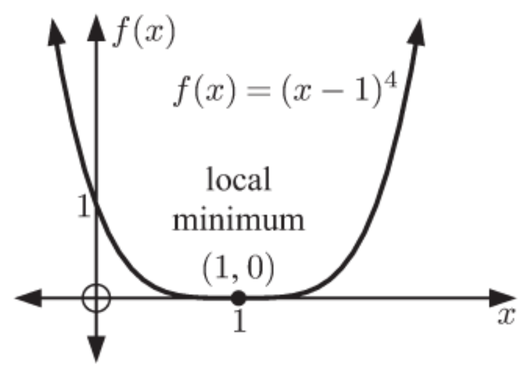


f **i** local minimum at $(1, 0)$ **ii** no points of inflection

iii increasing for $x \geqslant 1$, decreasing for $x \leqslant 1$

iv concave up for all $x \in \mathbb{R}$

v



g **i** local minima at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, -1)$, local maximum at $(0, 3)$

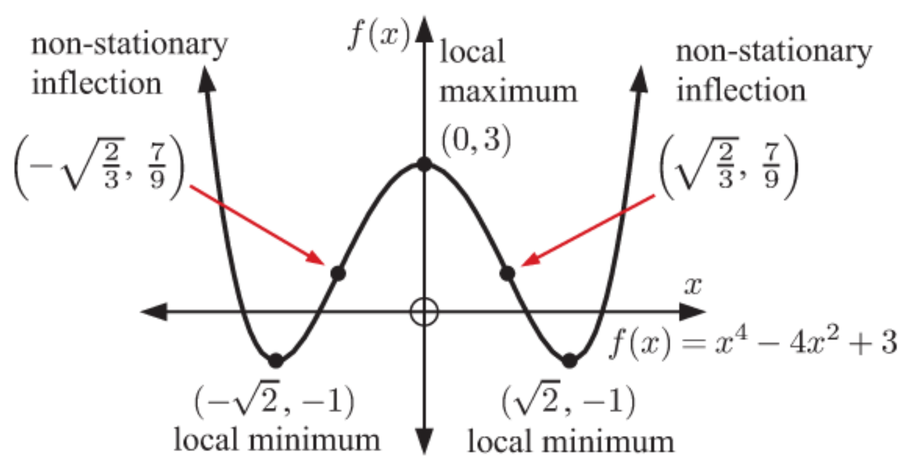
ii non-stationary inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$

iii increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, decreasing for $x \leqslant -\sqrt{2}$ and $0 \leqslant x \leqslant \sqrt{2}$

iv concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$

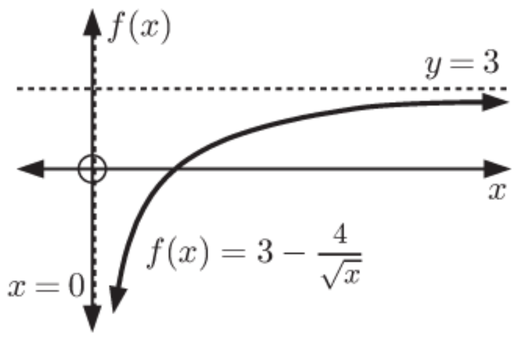
concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$

v



h **i** no turning points **ii** no points of inflection
iii increasing for $x > 0$, never decreasing
iv concave down for $x > 0$, never concave up

v



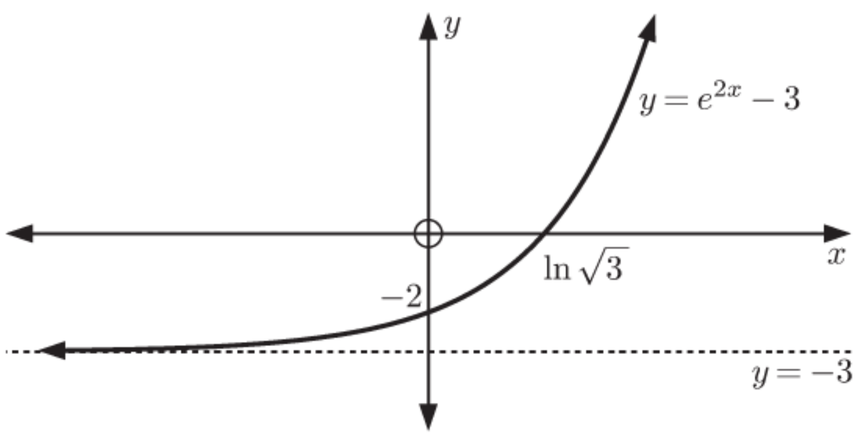
4 **a** x-intercept $\ln\sqrt{3}$, y-intercept -2

b $f'(x) = 2e^{2x} > 0$ for all $x \in \mathbb{R}$

c $f''(x) = 4e^{2x} > 0$ for all $x \in \mathbb{R}$

d as $x \to -\infty$, $e^{2x} \to 0$ $\therefore$ $e^{2x} - 3 \to -3^+$

e



5 **a** $f(x)$: x-intercept $\ln 3$, y-intercept -2
$g(x)$: x-intercept $\ln\left(\frac{5}{3}\right)$, y-intercept -2

b $f(x)$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -3^+$
$g(x)$: as $x \to \infty$, $g(x) \to 3^-$
as $x \to -\infty$, $g(x) \to -\infty$

c



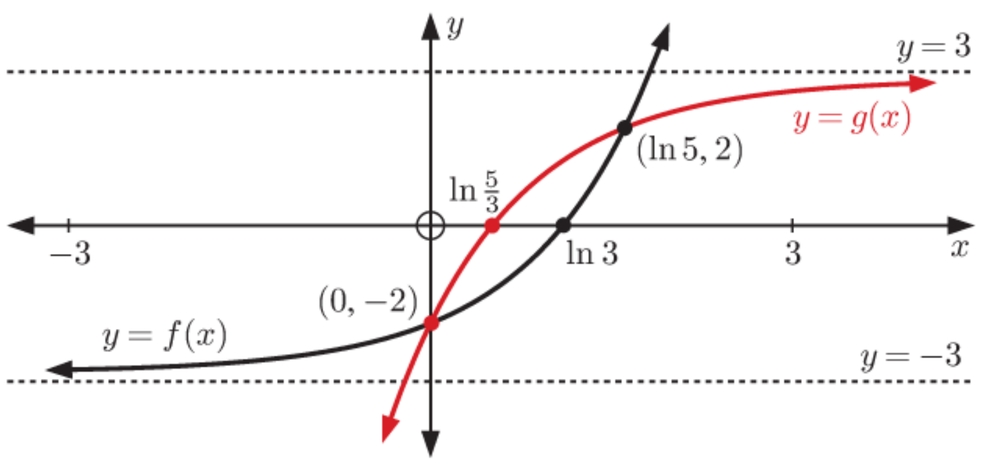
$f(x)$ is increasing and concave up for all $x \in \mathbb{R}$.



$g(x)$ is increasing and concave down for all $x \in \mathbb{R}$.

d $(0, -2)$ and $(\ln 5, 2)$

e

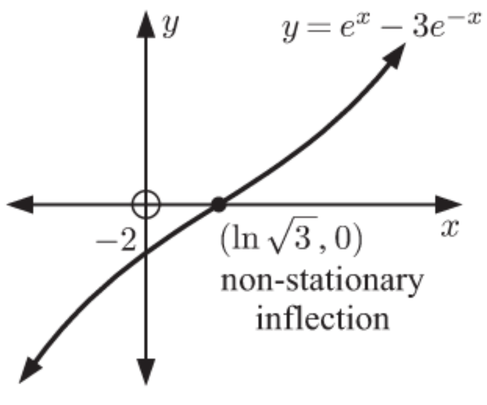


6 **a** x-intercept $\ln\sqrt{3}$, y-intercept -2

b $f'(x) = e^x + 3e^{-x} > 0$ for all $x \in \mathbb{R}$

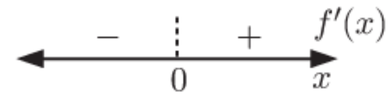
c y is concave down below the x-axis and concave up above the x-axis.

d



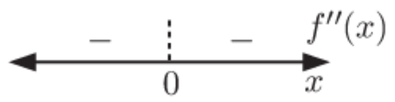
7 **a** $x \neq 0$

b $f'(x) = \frac{2}{x}$



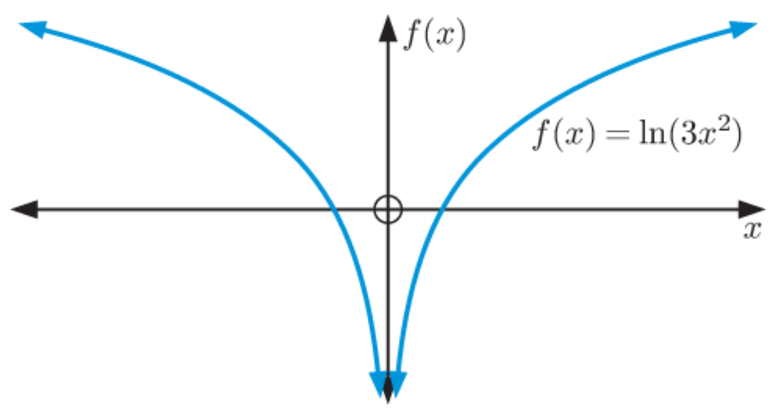
$f(x)$ is increasing for $x > 0$ and decreasing for $x < 0$.

$f''(x) = -\frac{2}{x^2}$



$f(x)$ is concave down for all $x \in \mathbb{R}$, $x \neq 0$.

c

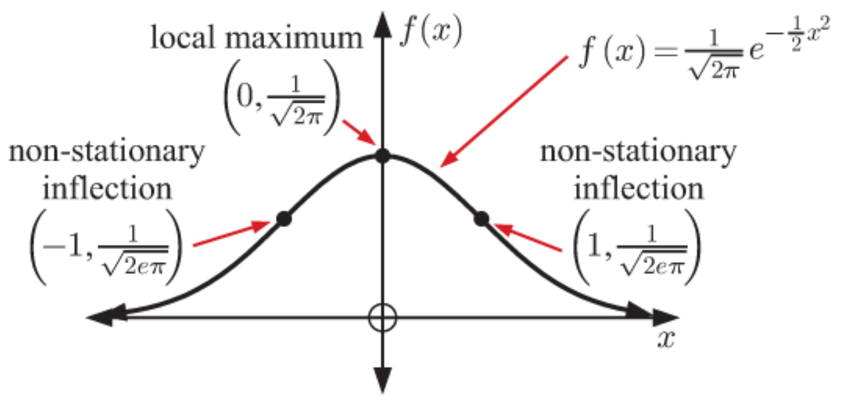


8 **a** local maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$, increasing for $x \leqslant 0$, decreasing for $x \geqslant 0$

b non-stationary inflections at $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$

c as $x \to \infty$, $f(x) \to 0^+$, as $x \to -\infty$, $f(x) \to 0^+$

d

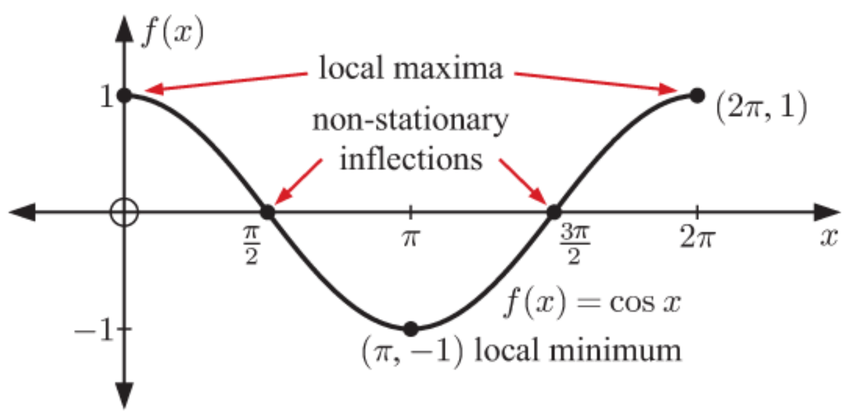


9 **a** The inflection points coincide with the x-intercepts.

b non-stationary inflection points at $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$

c **i** $\pi \leqslant x \leqslant 2\pi$ **ii** $0 \leqslant x \leqslant \pi$ **iii** $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$
iv $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

d



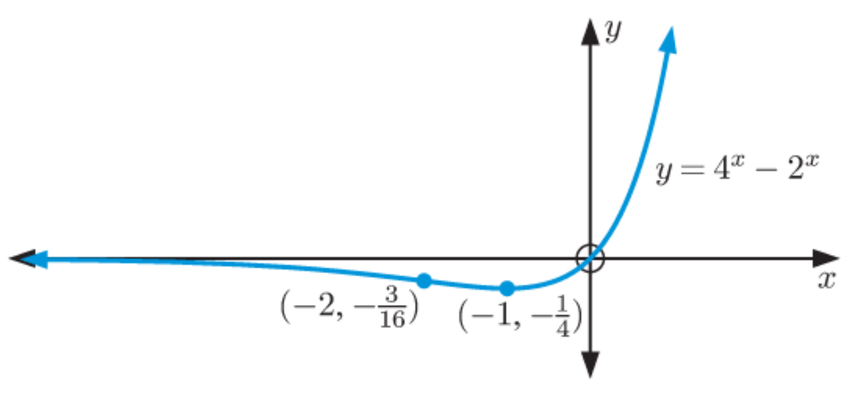
10 **a** x-intercept 0, y-intercept 0

b as $x \to \infty$, $y \to \infty$, as $x \to -\infty$, $y \to 0^-$

c local minimum at $(-1, -\frac{1}{4})$

d y is concave down for $x \leqslant -2$ and concave up for $x \geqslant -2$.

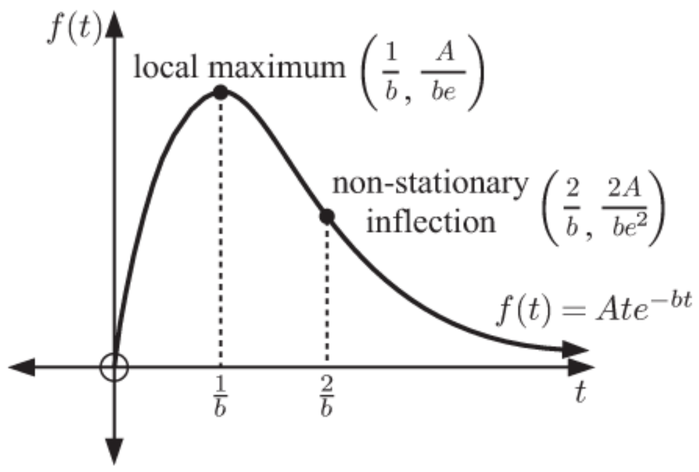
e



11 **a** **i** **Hint:** Show that $f'(t) = Ae^{-bt}(1 - bt)$.

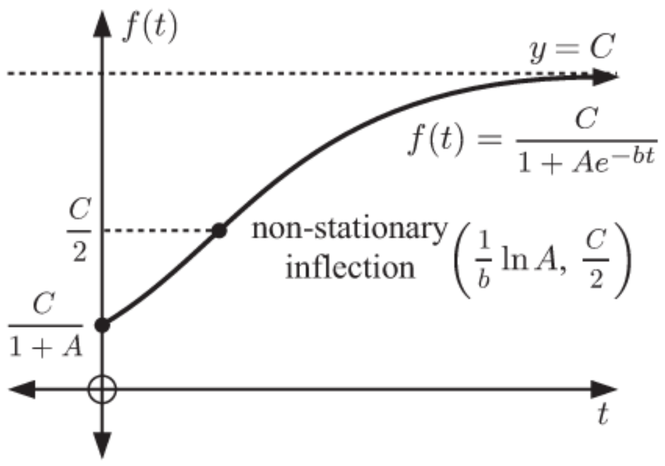
ii **Hint:** Show that $f''(t) = Abe^{-bt}(bt - 2)$.

b



12 **a** y-intercept is $\dfrac{C}{1 + A}$

d

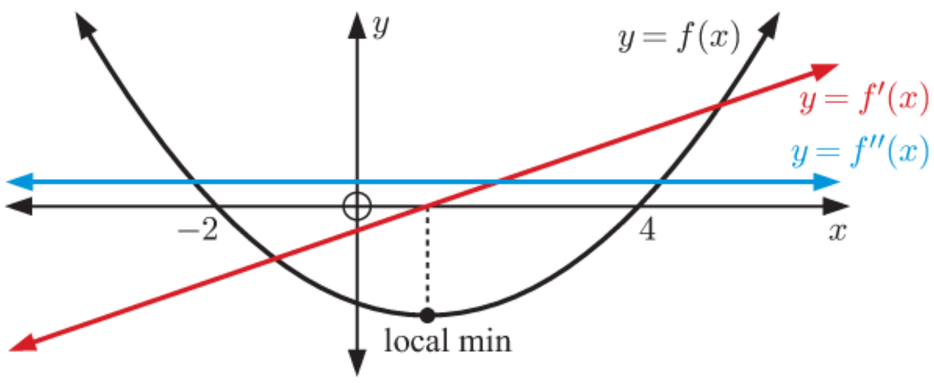


13 **a** non-stationary inflection $(0, 0)$

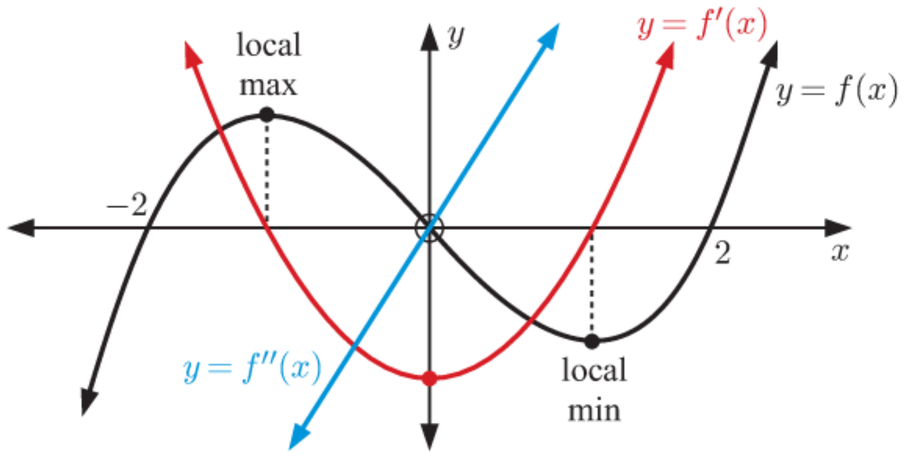
b non-stationary inflection $(\frac{1}{2}, 0)$

EXERCISE 18G

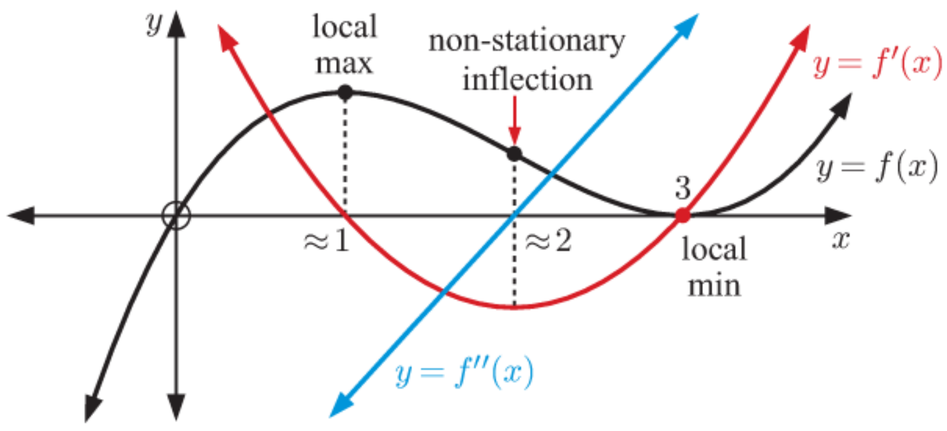
1 **a**



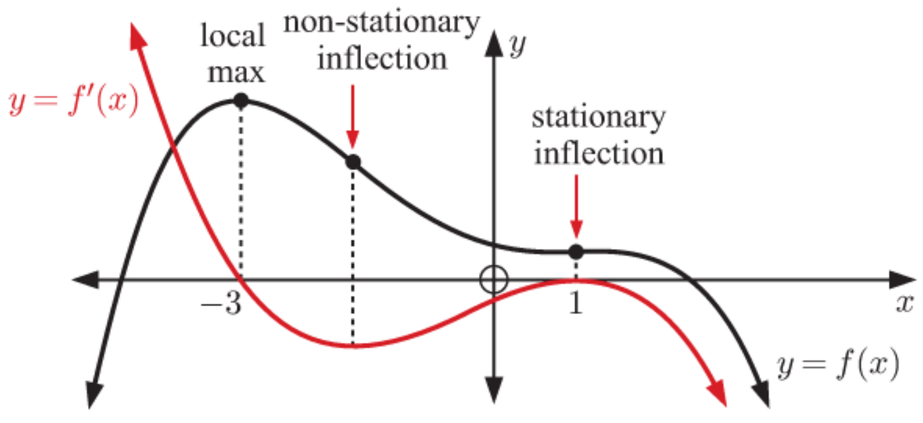
b



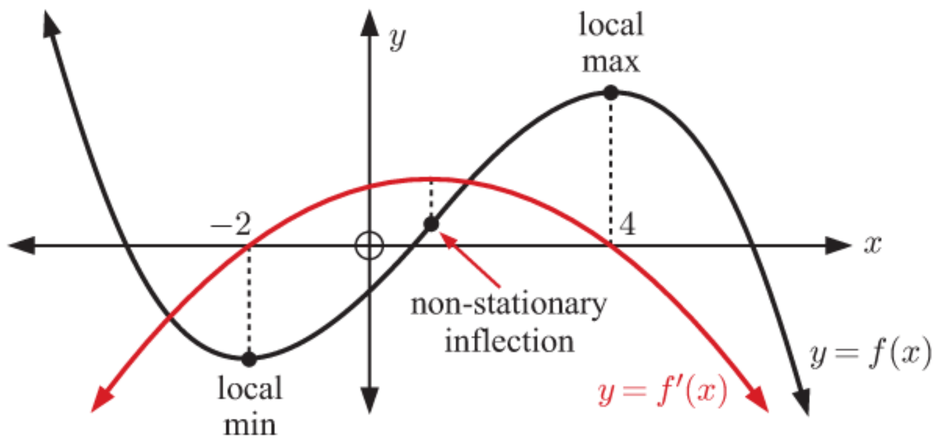
c



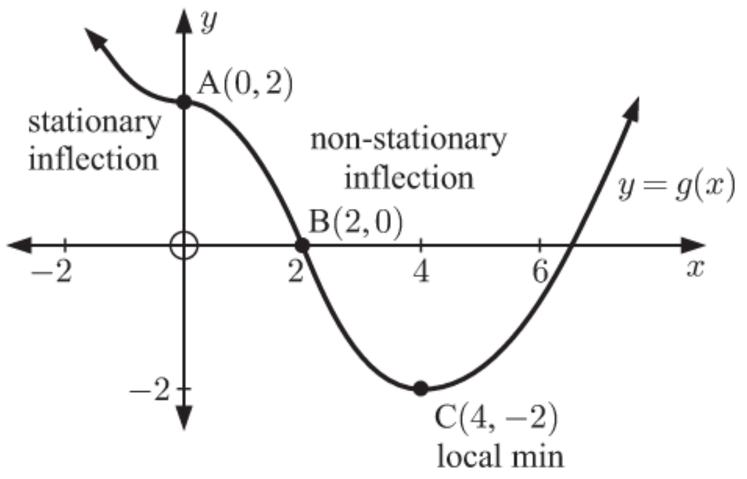
2 **a**



b



3



EXERCISE 18H

1 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** 1 **d** limit does not exist **e** 0 **f** 1

g $\frac{1}{2}$ **h** 0 **i** limit does not exist **j** 0 **k** $\ln\left(\dfrac{a}{b}\right)$

2 $f(x) = \dfrac{x^2}{e^x}$, as $x \to \infty$, $f(x) \to 0^+$ as e^x becomes larger much more quickly than x^2.

3 After applying l'Hôpital's rule once, we return to where we started. l'Hôpital's rule does not help us here.

4 No, $\lim\limits_{x \to 0} \dfrac{\frac{\pi}{2} - \arccos x - x}{x^3} = \frac{1}{6}$

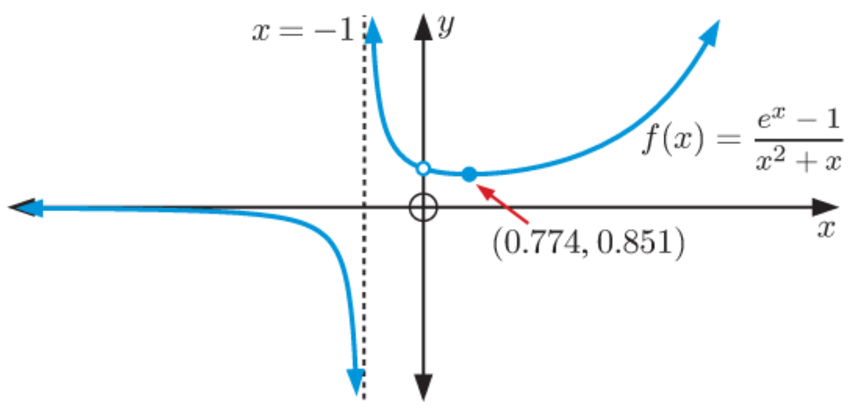
5 **a** $\{x \mid x \neq 0, -1\}$

b $f'(x) = \dfrac{e^x(x^2 - x - 1) + 2x + 1}{(x^2 + x)^2}$ **c** $(0.774, 0.851)$

domain is $\{x \mid x \neq 0, -1\}$

d **i** as $x \to -\infty$, $f(x) \to 0^-$ **ii** $\lim\limits_{x \to 0} f(x) = 1$

e



6 **a** $\frac{25}{9}$ **b** limit does not exist **7** 0

8 **a** limit does not exist **b** limit does not exist

9 **a** 0

b The result in **a** implies that for large x, e^x is greater than x^k for any fixed $k \in \mathbb{Z}^+$.

$\therefore$ e^x increases more rapidly than any fixed positive power of x.

10 **Hint:** Consider $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^k}$, $k \in \mathbb{Z}^+$.

13 $\lim\limits_{x \to \infty} x^{\frac{1}{x}} = 1$ **Hint:** Consider $x^{\frac{1}{x}} = \left(e^{\ln x}\right)^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$.

REVIEW SET 18A

1 **a** $y = 4x + 2$ **b** $y = 7x - 14$ **c** $y = -1$

d $3x - ey = -1$ **e** $y = \dfrac{2}{e}x$

2 **a** $8x + 3y = 44$ **b** $y = -\dfrac{1}{6e^2}x + \dfrac{18e^4 + 1}{6e^2}$

c $x - 5y = 11$ **d** $x + 2e^e y = e + 4e^{2e}$

3 $p = 1,\ q = -8$ **4** $(-2,\ 19)$ and $(1,\ -2)$

5 $a = 64$ **6** $a = 9,\ b = -16$ **7** $(-2,\ -25)$

8 **a** $y = -\dfrac{1}{2e^{2a}}x + \dfrac{a}{2e^{2a}} + e^{2a}$ **b** $y \approx -1.12x$

9 $\mathrm{P}(0,\ 7.5),\ \mathrm{Q}(3,\ 0)$ **10** $\frac{3267}{152} \approx 21.5$ units^2 **11** $\frac{8\sqrt{10}}{3}$ units

12 **a** $2\sqrt{3}x - y = \frac{2\sqrt{3}\pi}{3} - 2$ **b** $4x + y = 4\sqrt{3} + \frac{\pi}{3}$

14 **a** $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 3y^2 - 2}$ **b** $y = \frac{3}{2}x - \frac{5}{2}$

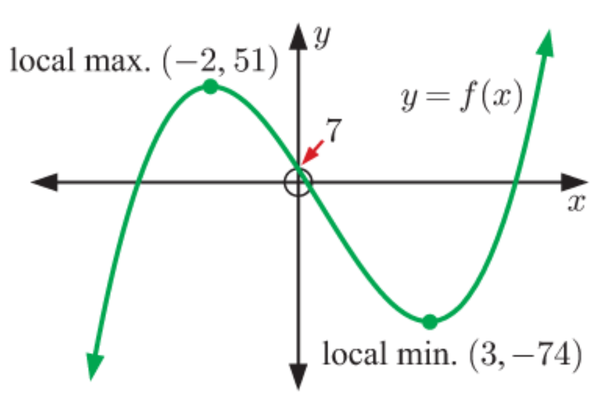
15 **a** $-6 \leqslant x \leqslant 2$ **b** $x \leqslant -6$ and $x \geqslant 2$

16 **a** local maximum at $(-2,\ 51)$, local minimum at $(3,\ -74)$

b increasing for $x \leqslant -2,\ x \geqslant 3$
decreasing for $-2 \leqslant x \leqslant 3$

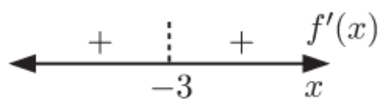
c as $x \to \infty$, $f(x) \to \infty$,
as $x \to -\infty$, $f(x) \to -\infty$

d



17 **a** $\{x \mid x \neq -3\}$ **b** x-intercept $\frac{2}{3}$, y-intercept $-\frac{2}{3}$

c $f'(x) = \dfrac{11}{(x+3)^2}$



d no stationary points

18 greatest value is ≈ 10.3 when $x = 10$,
least value is 6 when $x = 4$

19 **b** **i** $x > 0$ **ii** $x < \frac{1}{2}$ **c** local maximum at $(\frac{1}{2},\ \frac{1}{2})$

20 **a** increasing for $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$,
decreasing for $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

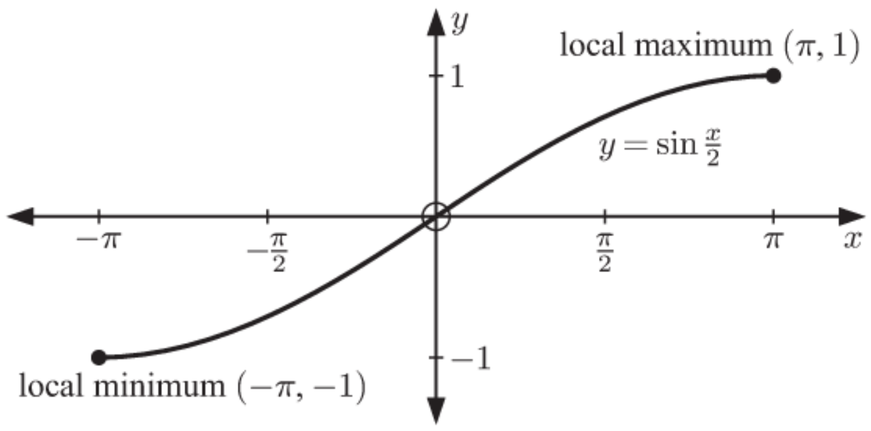
b increasing for $x \geqslant 1$, decreasing for $x \leqslant 1$

c increasing for all $x \in \mathbb{R}$, never decreasing

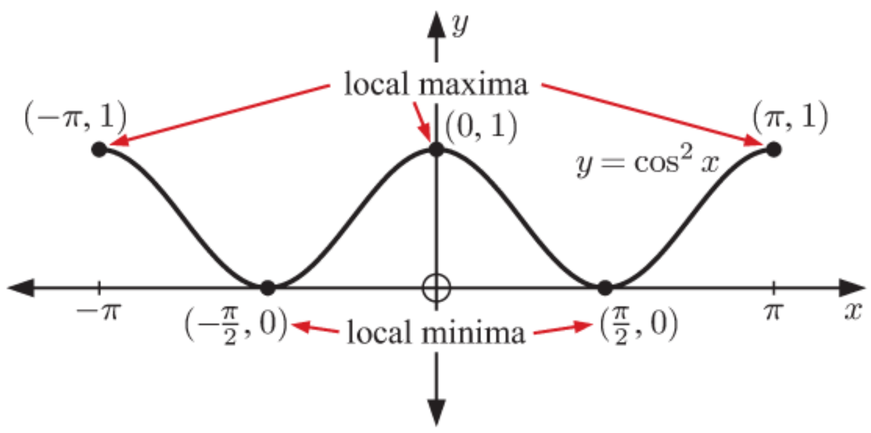
21 **a** local maximum at $(1,\ 3)$, local minimum at $(\frac{1}{3},\ \frac{77}{27})$

b local maximum at $(-6,\ -12)$, local minimum at $(0,\ 0)$

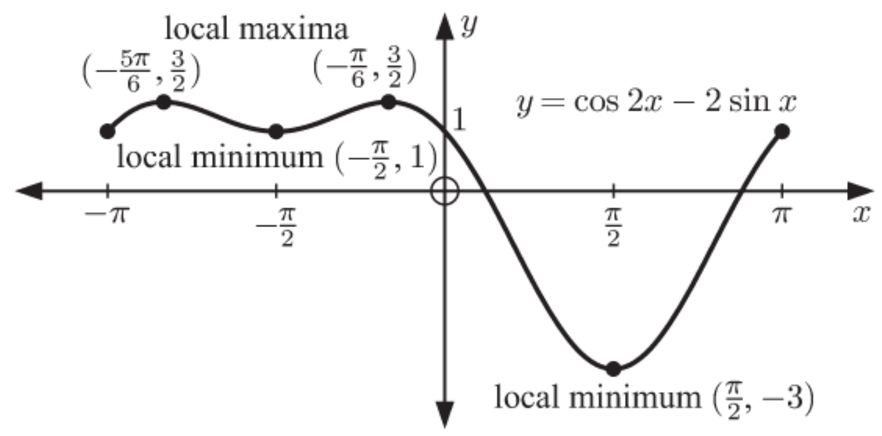
22 **a** $(-\pi,\ -1)$ is a local minimum, $(\pi,\ 1)$ is a local maximum



b $(-\pi,\ 1)$, $(0,\ 1)$, and $(\pi,\ 1)$ are local maxima,
$(-\frac{\pi}{2},\ 0)$ and $(\frac{\pi}{2},\ 0)$ are local minima

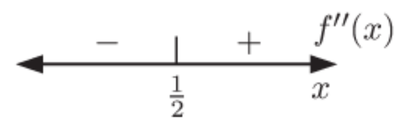


c $(-\frac{5\pi}{6},\ \frac{3}{2})$ and $(-\frac{\pi}{6},\ \frac{3}{2})$ are local maxima,
$(-\frac{\pi}{2},\ 1)$ and $(\frac{\pi}{2},\ -3)$ are local minima



23 **a** $f'(x) = 6x^2 - 6x + 1$,
$f''(x) = 12x - 6$

b



c $x \leqslant \frac{1}{2}$ **d** $(\frac{1}{2},\ -12)$

24 **a** concave up for $x \geqslant \frac{4}{3}$, concave down for $x \leqslant \frac{4}{3}$

b concave up for $x \leqslant -3$,
concave down for $-3 \leqslant x < 0$ and $x > 0$

c concave up for $-4 < x \leqslant 2$ and $x > 0$,
concave down for $x < -4$ and $-2 \leqslant x < 0$

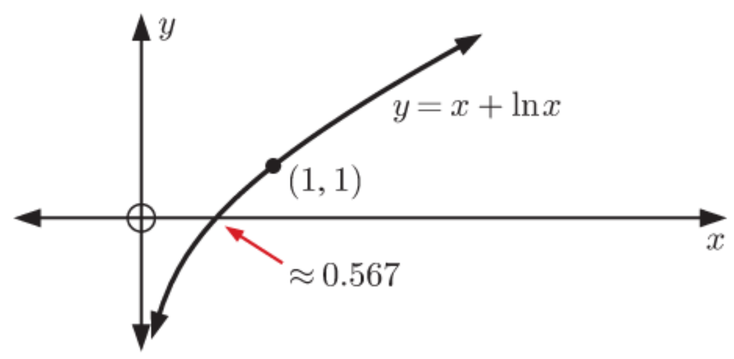
25 **a** $x > 0$

b $f'(x) = 1 + \dfrac{1}{x}$ $f''(x) = -\dfrac{1}{x^2}$



$f(x)$ is increasing and concave downwards for all $x > 0$.

c



d $x + 2y = 3$

26 **a** $f'(x) = e^{x\sqrt{3}}(\cos x + \sqrt{3}\sin x)$ **b** $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

c



d **i** $0 \leqslant x \leqslant \frac{5\pi}{6}$ and $\frac{11\pi}{6} \leqslant x \leqslant 2\pi$

ii $\frac{5\pi}{6} \leqslant x \leqslant \frac{11\pi}{6}$

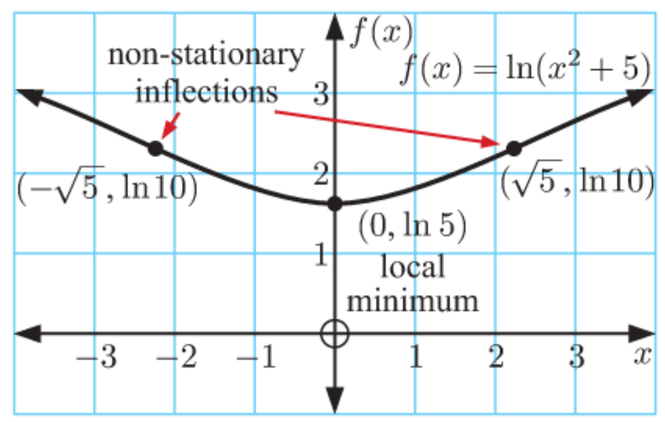
27 **a** $(0,\ \ln 5)$ is a local minimum

b $(-\sqrt{5},\ \ln 10)$ and $(\sqrt{5},\ \ln 10)$ are non-stationary inflections

c increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

d concave up for $-\sqrt{5} \leqslant x \leqslant \sqrt{5}$
concave down for $x \leqslant -\sqrt{5}$ and $x \geqslant \sqrt{5}$

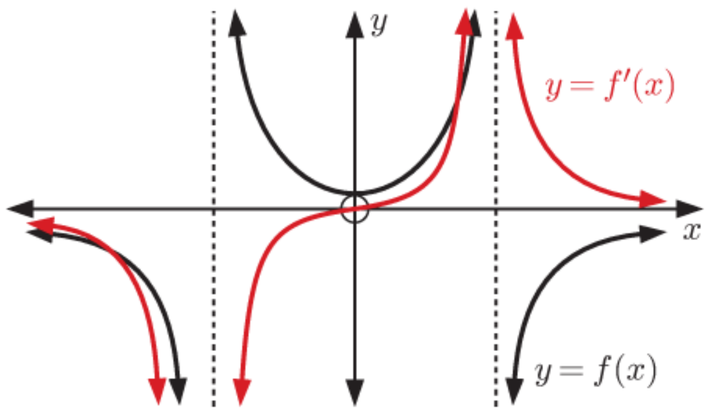
e



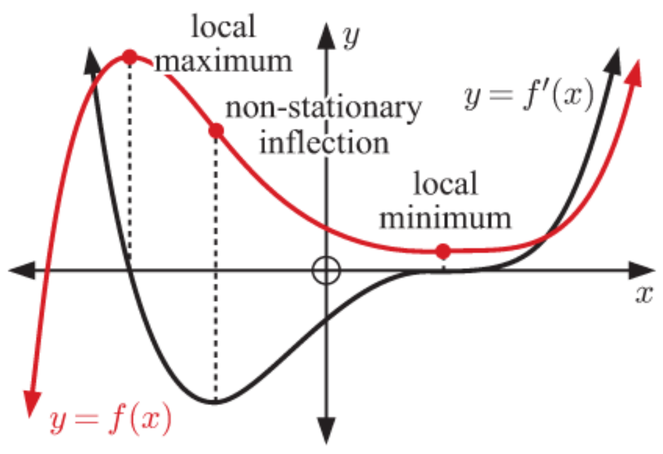
28 **a** $x = \frac{\pi}{2}$ **b** $f''(x) = e^{\sin^2 x}(\sin^2 2x + 2\cos 2x)$

c $(0.999, 2.03)$ and $(2.14, 2.03)$ are non-stationary inflections.

29



30



31 **a** 0 **b** 1 **c** $\dfrac{1}{\ln 2}$

32 **a** $f'(x) = e^{-x^2}(1 - 2x^2)$, $f''(x) = 2xe^{-x^2}(2x^2 - 3)$

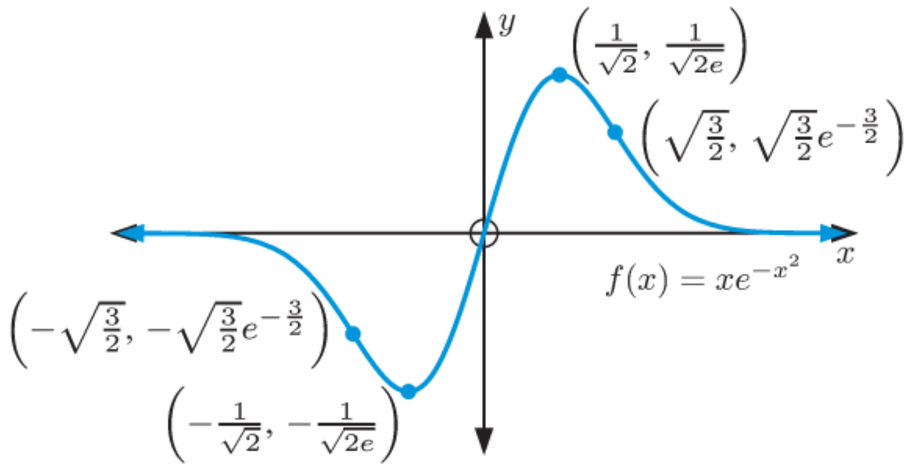
b local minimum $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$,

local maximum $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$

c as $x \to -\infty$, $f(x) \to 0^-$, as $x \to \infty$, $f(x) \to 0^+$

d non-stationary inflections at $\left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\right)$, $(0, 0)$, and $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\right)$

e



33 6

34 **a** After applying l'Hôpital's rule twice, we end up where we started, so l'Hôpital's rule does not help us here.

b 1

35 **a** $p(x) = a(x - r)^2 + (2ar + b)(x - r) + ar^2 + br + c$

c The tangent to $p(x)$ at $x = r$ is $p(x) = p(r) + p'(r)(x - r)$. We can use the form in **a** to find $p(r)$, $p'(r)$, and hence a formula for the tangent to $p(x)$ at $x = r$.

REVIEW SET 18B

1 **a** $y = 31x - 43$ **b** $x + 128y = 41$

c $y = 3x + \frac{3\sqrt{3}}{2} - \frac{\pi}{2}$ **d** $y = \frac{3}{4}x + \frac{1}{2}$ **e** $y = \ln 3$

2 **a** $x = 1$ **b** $x = 0$ **c** $y = \frac{e}{2}x + \frac{1}{e} - \frac{e}{2}$

4 $a = -14$, $b = 21$

5 **a** $f(3) = 2$, $f'(3) = -1$ **b** $f(x) = x^2 - 7x + 14$

7 **a** $a = 2$ **b** $y = 3x - 1$ **c** $(-4, -13)$

8 $(\frac{1}{2}, \frac{1}{4})$ **9** $(0, 2\ln 2 - 1)$

10 **a** $2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$ **b** $x + 2\sqrt{2}y = \frac{\pi}{2} + 2$

11 **a** $x + 4\sqrt{3}y = \frac{\pi}{3} - 8\sqrt{3}$ **b** $2x - \sqrt{2}y = \pi - 2$

12 $y = \frac{3}{4}x + \frac{5}{4}$ **13** $a = \frac{5}{2}$, $b = -\frac{3}{2}$

14 **a** $\sqrt{3}x + y = 4$, $\sqrt{3}x - y = -4$ **b** $(0, 4)$

15 **a** $\dfrac{dy}{dx} = \dfrac{7\pi\cos\frac{\pi x}{2} + 8x}{4y}$ **b** $y = \frac{3}{2}x - \frac{9}{2}$

16 **a** $-1 \leqslant x \leqslant 0$ and $x \geqslant 4$ **b** $x \leqslant -1$ and $0 \leqslant x \leqslant 4$

17 **a** $a = -9$

b local maximum at $(-1, 55)$, local minimum at $(3, 23)$

18 **a** x-intercepts 0 and 2, y-intercept 0

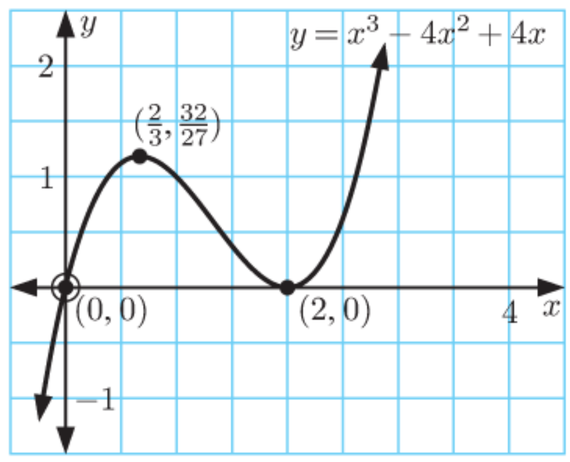
b local maximum at $(\frac{2}{3}, \frac{32}{27})$, local minimum at $(2, 0)$

c increasing for $x \leqslant \frac{2}{3}$ and $x \geqslant 2$,

decreasing for $\frac{2}{3} \leqslant x \leqslant 2$

d As $x \to \infty$, $y \to \infty$, as $x \to -\infty$, $y \to -\infty$.

e

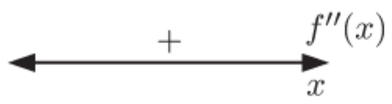


19 **a**



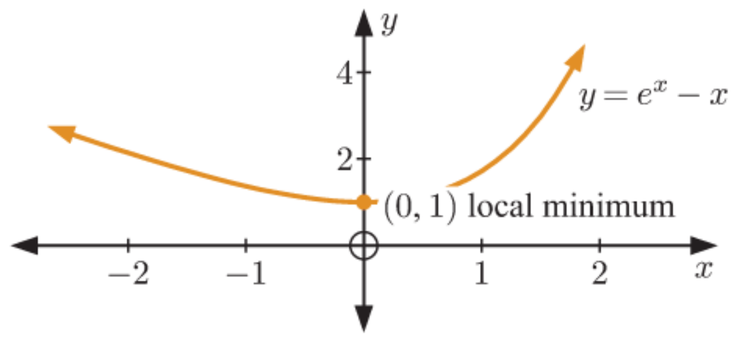
20 **a** local minimum at $(0, 1)$ **b** as $x \to \infty$, $f(x) \to \infty$

c $f''(x) = e^x$



$f(x)$ is concave up for all $x \in \mathbb{R}$.

d



21 $(1 - a,\ e^{a-1})$ is a local maximum **22** $a = 2$, $b = 3$

23 **a** $x \leqslant -1.96$ and $0.238 \leqslant x \leqslant 3.22$

b $-1.96 \leqslant x \leqslant 0.238$ and $x \geqslant 3.22$

c $-1 \leqslant x \leqslant 2$ **d** $x \leqslant -1$ and $x \geqslant 2$

24 **a** stationary inflection at $(0, 9)$,

non-stationary inflection at $(\frac{3}{2}, \frac{63}{16})$

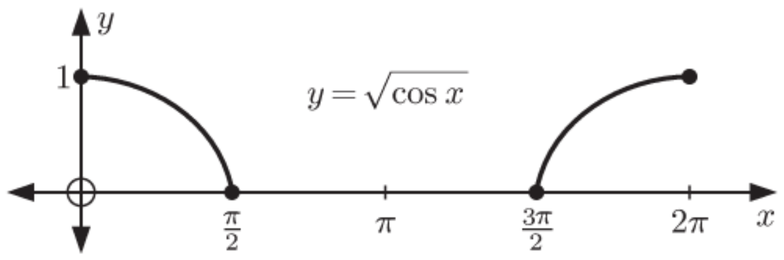
b non-stationary inflections at $(-1, 8)$ and $(\frac{3}{2}, \frac{313}{16})$

25 **a** $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b $f'(x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$, increasing for $\frac{3\pi}{2} < x \leqslant 2\pi$,

decreasing for $0 \leqslant x < \frac{\pi}{2}$

c



26 **a** $f'(x) = 4x^3 - 12x^2$



$f''(x) = 12x^2 - 24x$



b $(3, -20)$ is a local minimum

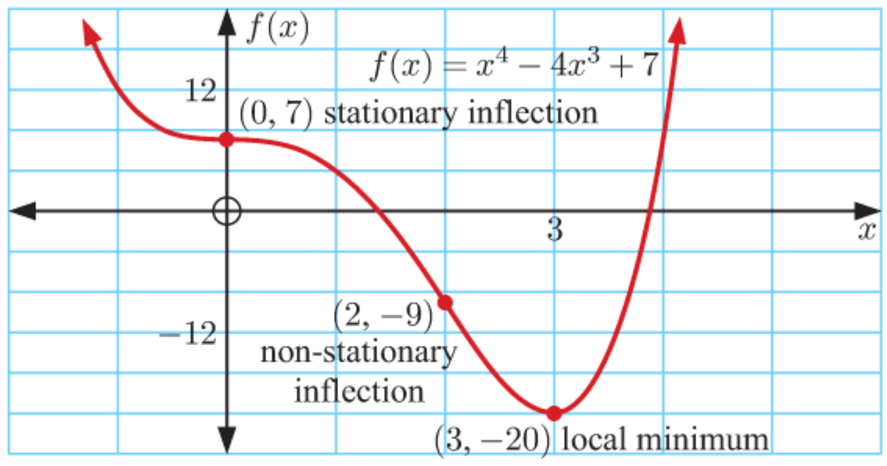
c $(0, 7)$ is a stationary inflection

$(2, -9)$ is a non-stationary inflection

d **i** $x \geqslant 3$ **ii** $x \leqslant 3$ **iii** $x \leqslant 0$ and $x \geqslant 2$

iv $0 \leqslant x \leqslant 2$

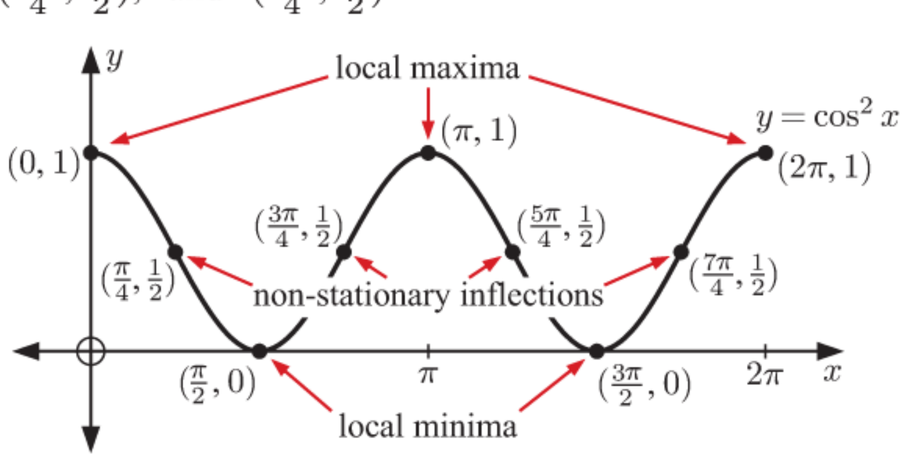
e



27 **a** local maxima at $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$,
local minima at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$

b non-stationary inflections at $(\frac{\pi}{4}, \frac{1}{2})$, $(\frac{3\pi}{4}, \frac{1}{2})$, $(\frac{5\pi}{4}, \frac{1}{2})$, and $(\frac{7\pi}{4}, \frac{1}{2})$

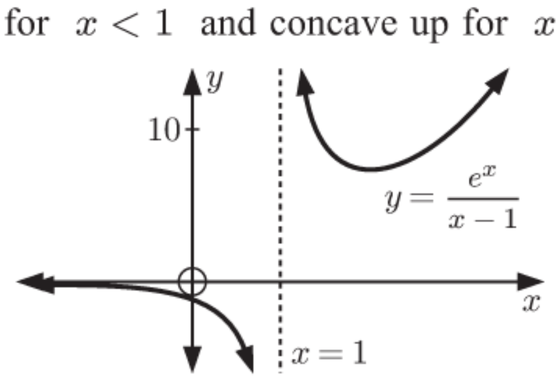
c



28 **a** -1 **b** $x \neq 1$

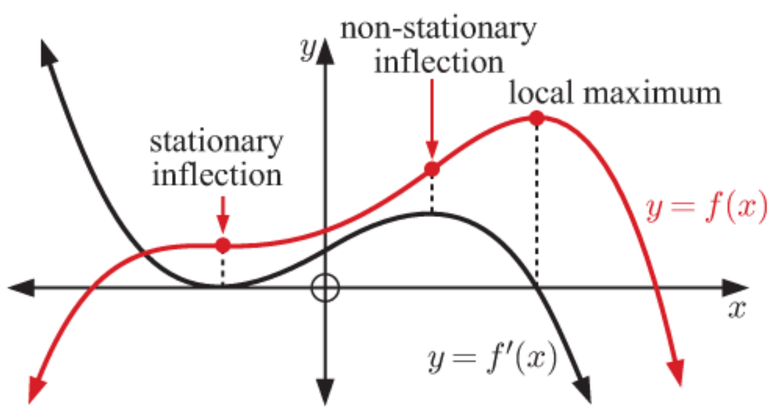
c $f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$
and $f'(x) \geqslant 0$ for $x \geqslant 2$
$f''(x) > 0$ for $x > 1$, $f''(x) < 0$ for $x < 1$
The function is decreasing for all defined values of $x \leqslant 2$, and increasing for all $x \geqslant 2$. The curve is concave down for $x < 1$ and concave up for $x > 1$.

d



e tangent is $y = e^2$

29

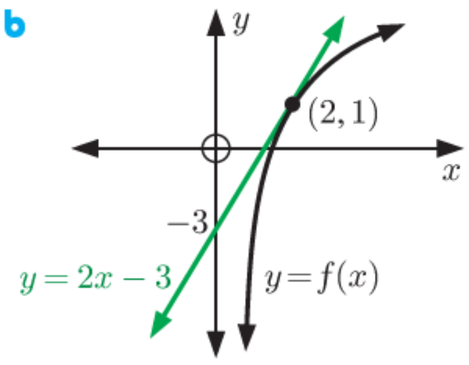


30 **a** $y = 2x - 3$

c $f'(x) > 0$ for all $x \in \mathbb{R}$
$\therefore$ $f(x)$ is increasing

d $\frac{3}{2} < x < 2$

b



31 **a** 2 **b** 1

33 To differentiate $(x + h)^n$ with respect to h, we would need to use the rule that we are trying to prove.

34 **a** $\frac{1}{6}$ **b** limit does not exist

35 **b** **i** $y = -\dfrac{2b}{a}x + b$ **ii** when $x = 0$, $y = b$

iii **Hint:** Let P″ be the point on the line $y = -b$ where the distance to P is shortest.
Show that FP = P″P.

c **i** **Hint:** Show that $\triangle$FPP′ is congruent to $\triangle$P″PP′.

ii **Hint:** Show that the tangents meet at $\left(\dfrac{a+c}{2}, \dfrac{ac}{4b}\right)$.

EXERCISE 19A

1 **a** \$118 000 **b** $\dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

c 20, which means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

2 **a** 190 m^3 per day **b** 180 m^3 per day

3 **a** **i** $Q(0) = 100$ **ii** $Q(25) = 50$ **iii** $Q(100) = 0$

b **i** decreasing by 1 unit per year

ii decreasing by $\frac{1}{\sqrt{2}}$ units per year

c $Q'(t) = -\dfrac{5}{\sqrt{t}} < 0$ for all $t > 0$

4 **a** 0.5 m **b** **i** ≈ 15.8 m **ii** ≈ 21.7 m **iii** ≈ 24.9 m

c $t = 0$: 6.9 m per year, $t = 5$: 1.725 m per year,
$t = 10$: ≈ 0.767 m per year

d $\dfrac{dH}{dt} = \dfrac{172.5}{(t+5)^2} > 0$ for all $t \geqslant 0$

The tree will continue to grow forever.

5 **a** $C'(x) = 6 + 8.4x^{-0.3}$ dollars per pair

b $C'(220) \approx$ \$7.67, this estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) \approx$ \$7.66, this is the actual cost of making the 221st pair of jeans. The answer in **b** is a very good estimate.

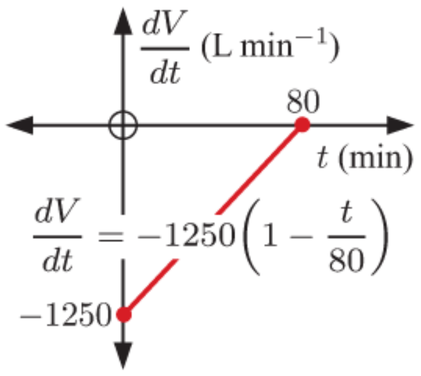
6 **a** **i** 4500 euros **ii** 4000 euros

b **i** decreasing at ≈ 210.22 euros per km h^{-1}

ii increasing at ≈ 11.31 euros per km h^{-1}

c ≈ 79.4 km h^{-1}

7 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$ L min^{-1}



b $t = 0$ when the tap was first opened

c $\dfrac{d^2V}{dt^2} = \dfrac{125}{8} > 0$

This shows that the rate of change of V is constantly increasing, so the outflow is increasing at a constant rate.

8 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}$, $\dfrac{dy}{dx} = 0.175$, the height of the hill is increasing as the gradient is positive.

When $x = 1\frac{1}{2}$, $\dfrac{dy}{dx} = -0.225$, the height of the hill is decreasing as the gradient is negative.

c ≈ 2.55 km from the sea, ≈ 63.1 m deep

9 **a** When $\dfrac{dP}{dt} = 0$, the population is not changing over time, so it is stable.

b 4000 fish **c** 8000 fish

d **Hint:** Show that $\dfrac{dP}{dt}$ is increasing for $0 \leqslant P \leqslant P_{\text{stable}}$ and decreasing for $P \geqslant P_{\text{stable}}$.

10 **a** $k = \frac{1}{50}\ln 2 \approx 0.0139$

b **i** 20 grams **ii** ≈ 14.3 grams **iii** ≈ 1.95 grams

c ≈ 216 hours or ≈ 9 days

d **i** $\approx -0.0693\ \text{g h}^{-1}$ **ii** $\approx -2.64 \times 10^{-7}\ \text{g h}^{-1}$

e **Hint:** You should find $\dfrac{dW}{dt} = -\frac{1}{50}\ln 2 \times 20e^{-\frac{t}{50}\ln 2}$

11 **a** $k = \frac{1}{15}\ln\left(\frac{19}{3}\right) \approx 0.123$ **b** 100°C

c $c = -k \approx -0.123$

d **i** decreasing at ≈ 11.7°C min^{-1}

ii decreasing at ≈ 3.42°C min^{-1}

iii decreasing at ≈ 0.998°C min^{-1}

12 **a** ≈ 43.9 cm **b** ≈ 10.4 years

c **i** growing at ≈ 5.45 cm per year

ii growing at ≈ 1.88 cm per year

13 **a** When $t = 0$, $A = 0$ litres

b **i** $k = \dfrac{\ln 2}{3}$ (≈ 0.231)

ii ≈ 0.728 litres of alcohol produced per hour

14 $\frac{21}{\sqrt{2}}$ cm^2 per radian

15 **a** $l = \sqrt{800 - 800\cos\theta}$ cm **b** 10 cm per radian

16 **a** **i** 0 volts **ii** 340 volts

b **i** $-34\,000\pi$ volts per second **ii** 0 volts per second

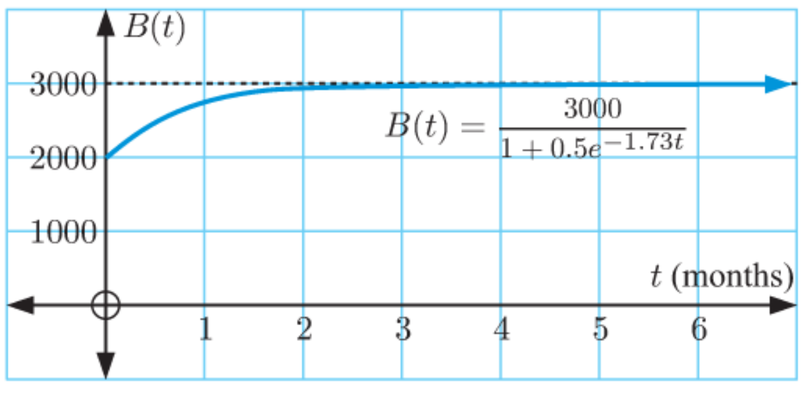
17 **a** 2000 bees **b** $\approx 37.8\%$ **c** yes, 3000 bees

d $B'(t) = \dfrac{2595}{e^{1.73t}(1 + 0.5e^{-1.73t})^2} > 0$ for all $t \geqslant 0$

$\therefore$ $B(t)$ is increasing over time.

e ≈ 0.0806 bees per month

f



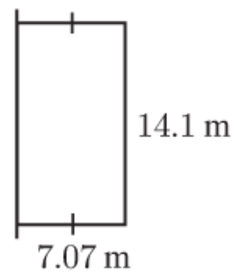
18 **b** **i** 0 metres per radian **ii** 1 metre per radian

iii ≈ 1.11 metres per radian

EXERCISE 19B

1 250 items per day **2** **b** 15 m $\times$ 30 m

3 **b** $L_{\min} \approx 28.3$ m, when $x \approx 7.07$ **c**

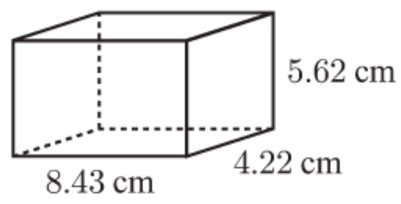


4 10 blankets per day

5 **a** **Hint:** $V = 200 = 2x \times x \times h$

b **Hint:** Show $h = \dfrac{100}{x^2}$ and substitute into the surface area equation.

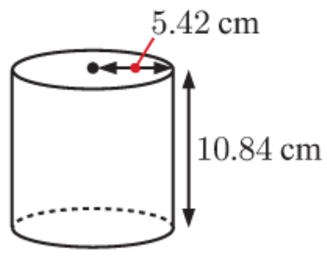
c $A_{\min} \approx 213$ cm^2, when $x \approx 4.22$ **d**



6 **a** **Hint:** Recall that $V_{\text{cylinder}} = \pi r^2 h$ and that $1\text{ L} = 1000\text{ cm}^3$.

b **Hint:** Recall that $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$.

c



7 **b** 6 cm $\times$ 6 cm

8 **a** $0 \leqslant x \leqslant \frac{200}{\pi} \approx 63.7$

b $l = 100$, $x = \frac{100}{\pi} \approx 31.8$, $A = \frac{20\,000}{\pi} \approx 6370$ m^2

9 **a** $y = 30 - x$ **b** $A(x) = x(30 - x)$ cm^2

c $A'(x) = 30 - 2x$ **d** $x = 15$, 15 cm $\times$ 15 cm

10 **a** area $= 4x\sqrt{25 - x^2}$ cm^2 **b** $5\sqrt{2}$ cm $\times$ $5\sqrt{2}$ cm

11 20 kettles **12** C$\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

13 (3.544, 8), length of pipe required ≈ 15.64 km

14 **b** $\theta \approx 1.91$, $A \approx 237$ cm^2

15 **a** $E'(t) = 750e^{-1.5t}(1 - 1.5t)$

b 40 minutes after the injection

16 **c** $\theta = \frac{\pi}{6}$, area ≈ 130 cm^2

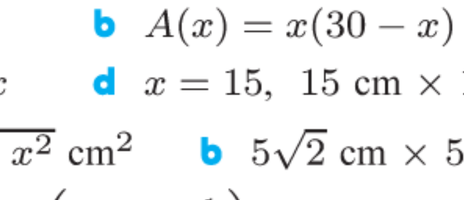
17 $3\frac{1}{3}$ km **18** after ≈ 13.8 weeks

19 **a** **Hint:** Use the cosine rule. **b** ≈ 3550

c $\approx 5{:}36$ pm

20 $\sqrt{3} : \sqrt{2}$

21 **c** $\theta = \frac{\pi}{6}$ **d**



e **i** Row from P to Q at an angle of $\frac{\pi}{6}$ to the diameter of the lake, then walk from Q to R.

ii Walk from P to R.

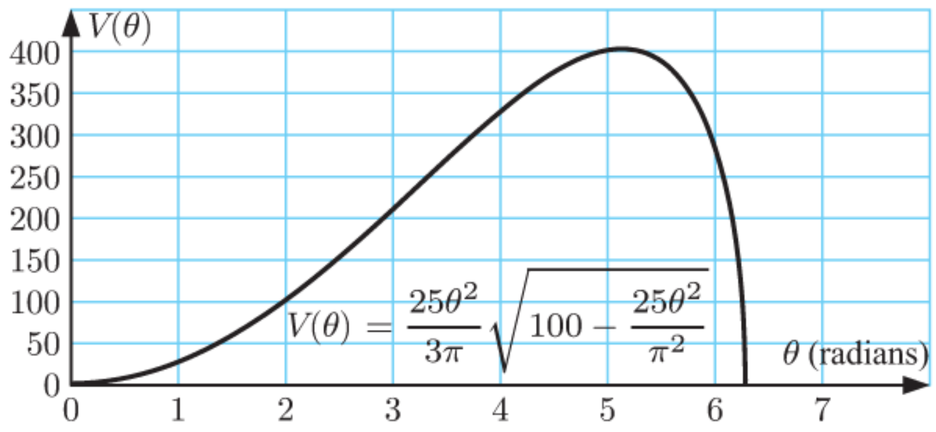
22 **d** $\approx 63.7\%$

23 **a** X is between A and C.

c $x \approx 2.67$ This is the distance in km from A to X which minimises the time taken to get from B to C.

24 **b** $V = \dfrac{25\theta^2}{3\pi}\sqrt{100 - \dfrac{25\theta^2}{\pi^2}}$

c



d $\theta \approx 5.13$

25 **b** ≈ 3.37 m **26** 4 m from the 40 cd lamp

27 **c** $L \approx 7.02$ m. This is the length of metal tube which can be carried around the corner from one corridor to the other without bending.

28 **a** $\tan\alpha = \dfrac{2}{x}$, $\tan(\alpha+\theta) = \dfrac{3}{x}$

b $\theta = \arctan\frac{3}{x} - \arctan\frac{2}{x}$ **d** $\sqrt{6}$ m

29 ≈ 1.34 m

EXERCISE 19C

1 **a** $b^3\dfrac{da}{dt} + 3ab^2\dfrac{db}{dt} = 0$

b a is decreasing at 7.5 units per second.

2 **a** $A = x^2$ **b** $\dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$ **c** 24 cm^2 per second

3 **a** $xy = 100$ **b** $y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = 0$

c **i** 0.25 cm per minute **ii** 1 cm per minute

4 **a** 4π m^2 per second **b** 8π m^2 per second

5 increasing at 0.375 m per minute

6 decreasing at 0.16 m^3 per minute **7** $\frac{20}{3}$ cm per minute

8 decreasing at $\frac{250}{13} \approx 19.2$ m s^{-1} **9** ≈ 1.35 cm per minute

10 **a** 0.2 m s^{-1} **b** $\frac{4}{45}$ m s^{-1}

11 **a** increase **b** increasing at $\frac{2\pi}{3\sqrt{3}} \approx 1.21$ cm per minute

12 $\frac{\sqrt{2}}{100}$ radians per second

13 **a** decreasing at $\frac{3}{100}$ radians per second

b decreasing at $\frac{1}{100}$ radians per second

14 increasing at 0.128 radians per second

15 increasing at 0.12 radians per minute

16 increasing at ≈ 24.3 m s^{-1}

17 **a** $\frac{\sqrt{3}}{2}\pi$ cm s^{-1} **b** 0 cm s^{-1}

18 **a** $\frac{200}{\sqrt{13}}\pi$ radians per second **b** 100π radians per second

19 **b** $\frac{\sqrt{3}}{120}$ m per minute

REVIEW SET 19A

1 **a** 60 cm **b** **i** ≈ 4.24 years **ii** ≈ 201 years

c **i** 16 cm per year **ii** ≈ 1.95 cm per year

2 **a** £20 000 **b** £146.53 per year

3 **a** $\approx$ \$4930.25

b **i** decreasing at $\approx$ \$1.39 per km h^{-1}

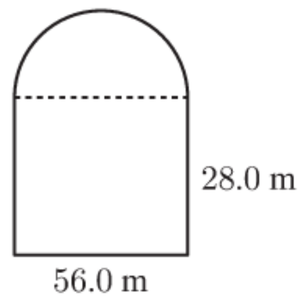
ii increasing at $\approx$ \$2.83 per km h^{-1}

c ≈ 79.4 km h^{-1}

4 **b** C$(\sqrt{3}, 6)$

5 **b** $A = 200x - 2x^2 - \frac{1}{2}\pi x^2$

c



6 **a** $y = \dfrac{1}{x^2}$, $x > 0$

c base is $\sqrt[3]{2} \approx 1.26$ m square, height is ≈ 0.630 m

7 **a** $\dfrac{dD}{dt} = -3.4476\sin(0.507t)$, this tells us the rate at which the depth of water is increasing or decreasing t hours after midnight.

b ≈ 5.15 m **c** rising, when $t = 8$, $\dfrac{dD}{dt} \approx 2.73 > 0$

d midnight and $\approx$ 12:24 pm, maximum depth of 16.1 m

8 **a** **i** 5 km **ii** $2\sqrt{10}$ km

9 **a** $A = \frac{\sqrt{3}}{4}x^2$ **b** $\frac{45\sqrt{3}}{2}$ cm^2 per minute

10 **b** $\dfrac{d}{dx}(A^2) = 5000x - 4x^3$ **c** $x = 25\sqrt{2}$

11 **a** $V(r) = \frac{8}{9}\pi r^3$ m^3

b decreasing at $\frac{8}{375\pi} \approx 0.006\,79$ m min^{-1}

12 **a** $a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$

REVIEW SET 19B

1 **a** $C'(x) = 2.805x^{-0.15} + 1.4x^{-0.5}$ euros per item

b $C'(1000) \approx$ €1.04, this estimates the cost of making the 1001st item each day.

c $C(1001) - C(1000) \approx$ €1.04, this is the actual cost of making the 1001st item each day. The answer in **b** is a very good estimate.

2 **a** $P(0) = 20\,000$ **b** $P'(t) = \dfrac{30\,000e^{-\frac{t}{4}}}{(1+2e^{-\frac{t}{4}})^2}$

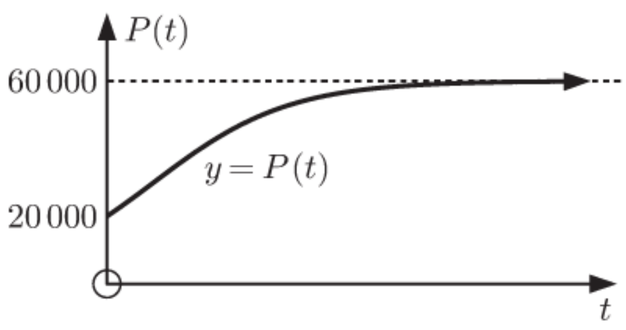
c **Hint:** Use the fact that $e^{-\frac{t}{4}}$ is never negative.
$P(t)$ is increasing for all $t \geqslant 0$.

d $P''(t) = \dfrac{7500e^{-\frac{t}{4}}(2e^{-\frac{t}{4}} - 1)}{(1+2e^{-\frac{t}{4}})^3}$

e 3750 per year when $t = 4\ln 2$ years

f as $t \to \infty$, $P(t) \to 60\,000^-$

g



3 **a** $A = 242\pi(1 - \cos\theta)$ cm^2

b $121\sqrt{2}\pi \approx 538$ cm^2 per radian

4 6 cm from each end **5** x-coordinate of P is $\ln a$.

6 $x \approx 2.11$ **7** **b** $\frac{1}{\sqrt{2}}$ metres

8 **a** LQ $= \dfrac{8}{x}$

b **Hint:** Show that
(length of pipe)$^2 = (LQ+1)^2 + (8+x)^2$
then simplify.

c $\dfrac{d}{dx}(L^2) = 2\left(\dfrac{x+8}{x}\right)\left(\dfrac{x^3-8}{x^2}\right)$ **d** ≈ 11.2 km

9 **a** **Hint:** Use Pythagoras to find h as a function of x and then substitute into the equation for the volume of a cylinder.

b radius ≈ 4.08 cm, height ≈ 5.77 cm

10 $\approx 3.60\ \text{m s}^{-1}$ **11** $\frac{20\sqrt{10}}{3} \approx 21.1$ m per minute

12 **a** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $\dfrac{dy}{dx} = -\dfrac{2\cos\theta}{3\sin\theta}$

c 6 units^2, when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

EXERCISE 20A

1 **a** **i** 0.4 units^2 **ii** 0.6 units^2 **b** 0.5 units^2

2 **a** ≈ 0.653 units^2 **b** ≈ 0.737 units^2

3

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3}$.

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{3}$ **iv** $\frac{3}{4}$ **c** area $= \dfrac{1}{a+1}$

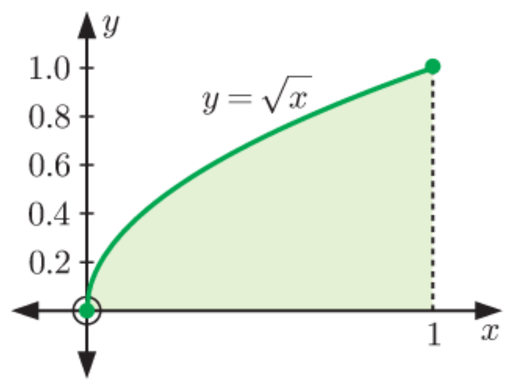
5 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $n = 10\,000$

EXERCISE 20B

1 **a**



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\displaystyle\int_0^1 \sqrt{x}\,dx \approx 0.67$

2 **a** $A_L = \dfrac{2}{n}\sum_{i=0}^{n-1}\sqrt{1+x_i^3}$, $A_U = \dfrac{2}{n}\sum_{i=1}^{n}\sqrt{1+x_i^3}$,

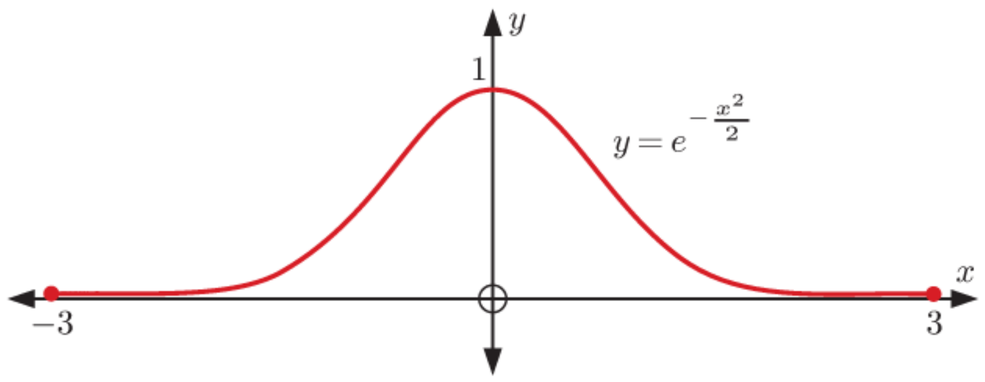
where $x_i = \dfrac{2i}{n}$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

3 **a**



b lower ≈ 1.2493, upper ≈ 1.2506

c lower ≈ 1.2493, upper ≈ 1.2506

d $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx \approx 2.4999$, $\sqrt{2\pi} \approx 2.5066$

4 **a** 18 **b** 4.5 **c** 2π **5** **b** **i** 0 **ii** 22

EXERCISE 20C

1 **a** **i** $\dfrac{x^2}{2}$ **ii** $\dfrac{x^3}{3}$ **iii** $\dfrac{x^6}{6}$ **iv** $-\dfrac{1}{x}$

v $-\dfrac{1}{3x^3}$ **vi** $\frac{3}{4}x^{\frac{4}{3}}$ **vii** $\frac{3}{2}x^{\frac{2}{3}}$ **viii** $\frac{3}{5}x^{\frac{5}{3}}$

b The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

2 **a** **i** $\frac{1}{3}e^{3x}$ **ii** $\frac{1}{5}e^{5x}$ **iii** $2e^{\frac{1}{2}x}$ **iv** $100e^{0.01x}$

v $\frac{1}{\pi}e^{\pi x}$ **vi** $3e^{\frac{x}{3}}$

b The antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$, where $k \neq 0$ is a constant.

3 **a** **i** $\frac{2^x}{\ln 2}$ **ii** $\frac{2^{3x}}{3\ln 2}$ **iii** $\frac{2^{-x}}{-\ln 2}$ **iv** $\frac{2^{\frac{x}{2}}}{\frac{1}{2}\ln 2}$

b The antiderivative of 2^{kx} is $\frac{2^{kx}}{k\ln 2}$, where k is a constant.

4 **a** $\frac{d}{dx}(x^3+x^2) = 3x^2+2x$

$\therefore$ the antiderivative of $6x^2+4x$ is $2x^3+2x^2$.

b $\frac{d}{dx}(x\sqrt{x}) = \frac{3}{2}\sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$.

c $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$

$\therefore$ the antiderivative of $\frac{1}{x\sqrt{x}}$ is $-\frac{2}{\sqrt{x}}$.

d $\frac{d}{dx}(\cos x) = -\sin x$

$\therefore$ the antiderivative of $\sin x$ is $-\cos x$.

e $\frac{d}{dx}(4x+1)^4 = 16(4x+1)^3$

$\therefore$ the antiderivative of $(4x+1)^3$ is $\frac{1}{16}(4x+1)^4$.

f $\frac{d}{dx}\left(\tan\frac{x}{3}\right) = \frac{1}{3}\sec^2\left(\frac{x}{3}\right)$

$\therefore$ the antiderivative of $\sec^2\left(\frac{x}{3}\right)$ is $3\tan\frac{x}{3}$.

g $\frac{d}{dx}(\sin(3x-1)) = 3\cos(3x-1)$

$\therefore$ the antiderivative of $\cos(3x-1)$ is $\frac{1}{3}\sin(3x-1)$.

EXERCISE 20D

1 **a** $\frac{d}{dx}(x^2) = 2x$

$\therefore$ the antiderivative of $2x$ is x^2.

b 8 units^2

2 **a** $\frac{2}{3}x^{\frac{3}{2}}$ **b** $\frac{2}{3}$ units^2

c $\frac{2}{3} \approx 0.67$ $\therefore$ the answers are the same.

3 **a** **i** 4 units^2 **ii** $16\frac{1}{4}$ units^2 **iii** $20\frac{1}{4}$ units^2

b $\int_0^3 x^3\,dx = \int_0^2 x^3\,dx + \int_2^3 x^3\,dx$

4 **a** $3\frac{3}{4}$ units^2 **b** $8\frac{2}{3}$ units^2 **c** $\frac{4\sqrt{2}-2}{3}$ units^2

d $(e^{\frac{3}{2}}-1)$ units^2 **e** 2 units^2

6 $\frac{9\pi}{4}$ units^2

7 **c** **i** $\int_0^1 (-x^2)\,dx = -\frac{1}{3}$, the area between $y=-x^2$ and the x-axis from $x=0$ to $x=1$ is $\frac{1}{3}$ units^2.

ii $\int_0^1 (x^2-x)\,dx = -\frac{1}{6}$, the area between $y=x^2-x$ and the x-axis from $x=0$ to $x=1$ is $\frac{1}{6}$ units^2.

iii $\int_{-2}^0 3x\,dx = -6$, the area between $y=3x$ and the x-axis from $x=-2$ to $x=0$ is 6 units^2.

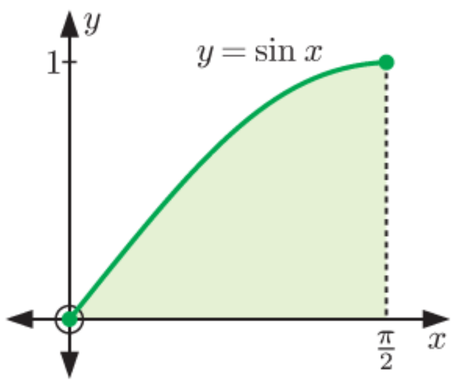
d $-\pi$

REVIEW SET 20A

1 **a** $A=\frac{17}{4}$, $B=\frac{25}{4}$ **b** $\int_0^2 (4-x^2)\,dx \approx \frac{21}{4}$

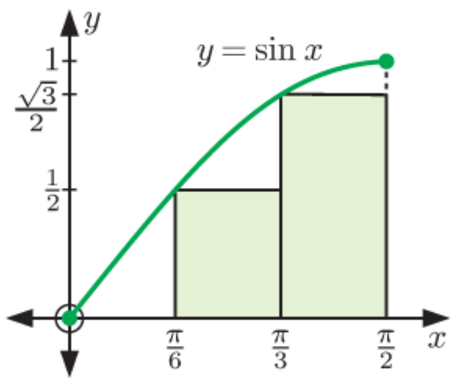
2 **a** $\frac{x^5}{5}$ **b** $-\frac{1}{2x}$ **c** $-2e^{-\frac{1}{2}x}$ **d** $\sin x$ **e** $\frac{3^x}{\ln 3}$

3 **a**

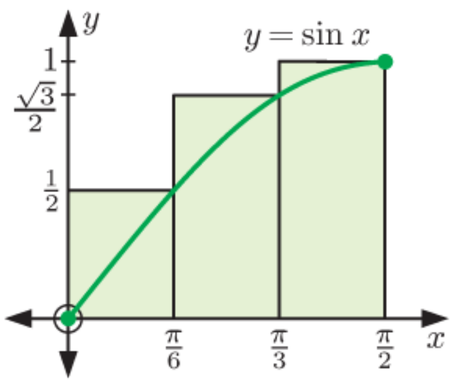


b

lower rectangles



upper rectangles



$$\frac{\pi(1+\sqrt{3})}{12} < \int_0^{\frac{\pi}{2}} \sin x\,dx < \frac{\pi(3+\sqrt{3})}{12}$$

or $0.715 < \int_0^{\frac{\pi}{2}} \sin x\,dx < 1.24$

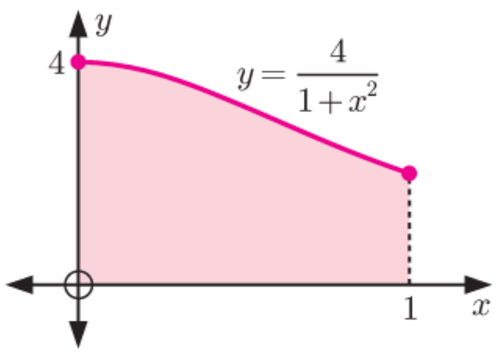
4 **a** 2π **b** 4

5 **a** **i** $\frac{1}{3}$ units^2 **ii** $2\frac{1}{3}$ units^2 **iii** $2\frac{2}{3}$ units^2

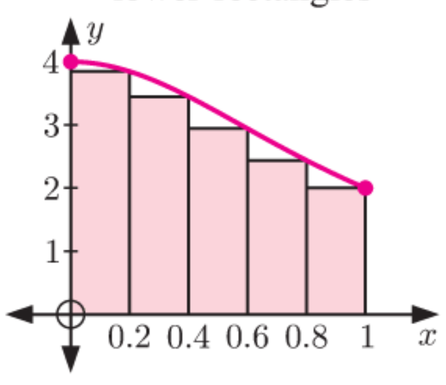
b $\int_0^2 x^2\,dx = \int_0^1 x^2\,dx + \int_1^2 x^2\,dx$

REVIEW SET 20B

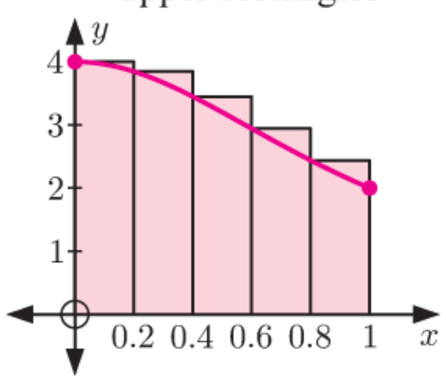
1 **a**



lower rectangles



upper rectangles



b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\int_0^1 \frac{4}{1+x^2}\,dx$

≈ 3.1416

$\approx \pi$

2 **a** 10 **b** $\frac{\pi}{2}$

3 **a** $\dfrac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

$\therefore$ the antiderivative of $3x^2 - 2$ is $x^3 - 2x$.

b $\dfrac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$

$\therefore$ the antiderivative of $\sqrt[3]{x}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

c $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$

$\therefore$ the antiderivative of $\dfrac{1}{\sqrt{1-x^2}}$ is $-\arccos x$.

4 **a** 3 **b** $\frac{15}{2}$ **5** **a** 18 units^2 **b** 18 units^2

6 **b** Area of top half $= \displaystyle\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx$ units^2

$\therefore$ total area $= 2 \times \displaystyle\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx$ units^2

$= \dfrac{b}{a}\displaystyle\int_{-a}^{a} 2\sqrt{a^2 - x^2}\, dx$ units^2

c $2\sqrt{a^2 - x^2}$

EXERCISE 21A

1 $\dfrac{d}{dx}(kx) = k$

$\therefore \displaystyle\int k\, dx = kx + c$

2 **a** $\dfrac{d}{dx}\left(x^7\right) = 7x^6$

$\therefore \displaystyle\int x^6\, dx = \frac{1}{7}x^7 + c$

b $\dfrac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \displaystyle\int \sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}} + c$

c $\dfrac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}}$

$\therefore \displaystyle\int x^{-\frac{3}{2}}\, dx = -2x^{-\frac{1}{2}} + c$

d $\dfrac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n$

$\therefore \displaystyle\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

3 **a** $\dfrac{d}{dx}\left(e^{4x}\right) = 4e^{4x}$

$\therefore \displaystyle\int e^{4x}\, dx = \frac{1}{4}e^{4x} + c$

b $\dfrac{d}{dx}\left(e^{-\frac{x}{2}}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$

$\therefore \displaystyle\int e^{-\frac{x}{2}}\, dx = -2e^{-\frac{x}{2}} + c$

c $\dfrac{d}{dx}\left(e^{kx}\right) = ke^{kx}$

$\therefore \displaystyle\int e^{kx}\, dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$

4 **a** $\dfrac{d}{dx}(\sin x) = \cos x$

$\therefore \displaystyle\int \cos x\, dx = \sin x + c$

b $\dfrac{d}{dx}(\cos x) = -\sin x$

$\therefore \displaystyle\int (-\sin x)\, dx = \cos x + c$

c $\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\therefore \displaystyle\int \sec^2 x\, dx = \tan x + c$

5 **a** $\dfrac{d}{dx}\left(x^3 + x^2\right) = 3x^2 + 2x$

$\therefore \displaystyle\int (3x^2 + 2x)\, dx = x^3 + x^2 + c$

b $\dfrac{d}{dx}\left(3x^4 - 2x^2\right) = 12x^3 - 4x$

$\therefore \displaystyle\int (3x^3 - x)\, dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 + c$

6 **a** $f(x) + g(x)$

7 **a** $\dfrac{d}{dx}(3^x) = 3^x \ln 3$

$\therefore \displaystyle\int 3^x \ln 3\, dx = 3^x$

$\therefore \displaystyle\int 3^x\, dx = \frac{3^x}{\ln 3} + c$

b $\dfrac{d}{dx}(a^x) = a^x \ln a$

$\therefore \displaystyle\int a^x \ln a\, dx = a^x$

$\therefore \displaystyle\int a^x\, dx = \frac{a^x}{\ln a}, \quad a > 0, \quad a \neq 1$

8 **a** $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\therefore \displaystyle\int (-\operatorname{cosec} x \cot x)\, dx = \operatorname{cosec} x + c$

b $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\therefore \displaystyle\int \sec x \tan x\, dx = \sec x + c$

c $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\therefore \displaystyle\int \left(-\operatorname{cosec}^2 x\right) dx = \cot x + c$

9 **a** $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$

$\therefore \displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c$

b $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$

$\therefore \displaystyle\int \left(-\frac{1}{\sqrt{1-x^2}}\right) dx = \arccos x + c$

c $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

$\therefore \displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + c$

10 **a** $\dfrac{1}{x}$ **b** $\dfrac{-1}{-x} = \dfrac{1}{x}$

c $\displaystyle\int \frac{1}{x}\,dx = \begin{cases} \ln x + c, & x > 0 \\ \ln(-x) + c, & x < 0 \end{cases}$

11 **a** $\dfrac{1}{x \ln a}$ **b** $\dfrac{-1}{-x \ln a} = \dfrac{1}{x \ln a}$

c $\displaystyle\int \frac{1}{x \ln a}\,dx = \begin{cases} \log_a x + c, & x > 0 \\ \log_a(-x) + c, & x < 0 \end{cases}$

12 **a** $\dfrac{d}{dx}(\sin 3x) = 3\cos 3x$

$\therefore \displaystyle\int \cos 3x\,dx = \tfrac{1}{3}\sin 3x + c$

b $\dfrac{d}{dx}\left(\cos\left(\tfrac{\pi}{3} - x\right)\right) = \sin\left(\tfrac{\pi}{3} - x\right)$

$\therefore \displaystyle\int \sin\left(\tfrac{\pi}{3} - x\right)\,dx = \cos\left(\tfrac{\pi}{3} - x\right) + c$

c $\dfrac{d}{dx}(\sin(1-5x)) = -5\cos(1-5x)$

$\therefore \displaystyle\int \cos(1-5x)\,dx = -\tfrac{1}{5}\sin(1-5x) + c$

d $\dfrac{d}{dx}\left(e^{3x+1}\right) = 3e^{3x+1}$

$\therefore \displaystyle\int e^{3x+1}\,dx = \tfrac{1}{3}e^{3x+1} + c$

e $\dfrac{d}{dx}\left(\sqrt{5x-1}\right) = \tfrac{5}{2}(5x-1)^{-\frac{1}{2}} = \dfrac{5}{2\sqrt{5x-1}}$

$\therefore \displaystyle\int \frac{1}{\sqrt{5x-1}}\,dx = \tfrac{2}{5}\sqrt{5x-1} + c$

f $\dfrac{d}{dx}\left((2x+1)^4\right) = 8(2x+1)^3$

$\therefore \displaystyle\int (2x+1)^3\,dx = \tfrac{1}{8}(2x+1)^4 + c$

g $\dfrac{d}{dx}\left((x^2-x)^3\right) = 3(2x-1)(x^2-x)^2$

$\therefore \displaystyle\int (2x-1)(x^2-x)^2\,dx = \tfrac{1}{3}(x^2-x)^3 + c$

h $\dfrac{d}{dx}\left(2^{3x-5}\right) = 3\ln 2 \times 2^{3x-5}$

$\therefore \displaystyle\int 2^{3x-5}\,dx = \frac{2^{3x-5}}{3\ln 2} + c$

i $\dfrac{d}{dx}\left(\arctan\tfrac{x}{2}\right) = \dfrac{2}{x^2+4}$

$\therefore \displaystyle\int \frac{-3}{x^2+4}\,dx = -\tfrac{3}{2}\arctan\tfrac{x}{2} + c$

EXERCISE 21B

1 **a** $\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c$ **b** $\frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c$

c $-\frac{1}{4}x^4 + \frac{4}{3}x^3 - 3x + c$ **d** $\frac{1}{4}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + c$

e $\frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c$ **f** $\frac{4}{3}x^3 + \ln|x| + c$

g $\frac{4}{3}x\sqrt{x} - 6\sqrt{x} + c$ **h** $\frac{1}{3}\ln|x| + \dfrac{2}{x} + c$

i $\frac{2}{5}x^2\sqrt{x} - 9x + c$ **j** $-6x^{-\frac{1}{2}} + \frac{4}{5}x^{\frac{5}{4}} + c$

2 **a** $2e^x - \frac{3}{2}x^2 + c$ **b** $4\ln|x| + \frac{1}{3}x^3 - e^x + c$

c $5e^x + \frac{1}{6}x^3 + c$ **d** $\frac{2}{3}x\sqrt{x} + e^x + c$

e $3e^x + \dfrac{2}{\sqrt{x}} + c$ **f** $\frac{1}{2}\ln|x| + \frac{1}{3}x^3 - \frac{1}{2}e^x + c$

3 **a** $-3\cos x - 2x + c$ **b** $2x^2 - 2\sin x + c$

c $-\cos x - 2\sin x + e^x + c$ **d** $\frac{2}{7}x^3\sqrt{x} + 10\cos x + c$

e $\frac{1}{9}x^3 - \frac{1}{6}x^2 + \tan x + c$ **f** $\cos x + \frac{4}{3}x\sqrt{x} + c$

4 **a** $y = 6x + c$ **b** $y = \frac{4}{3}x^3 + c$ **c** $y = -\dfrac{1}{x} + c$

d $y = 3x^{\frac{2}{3}} + c$ **e** $y = \frac{1}{2}x^4 - 4x + c$

f $y = x^4 + x^3 + c$ **g** $y = 2x - \ln|x| + c$

h $y = -\cos x + 2\sin x + c$ **i** $y = 2e^x - 5x + \frac{1}{2}x^2 + c$

5 We cannot divide by zero.

6 **a** $\dfrac{4^x}{\ln 4} + c$ **b** $3\ln|x| + \log_2|x| + c$

c $\dfrac{5^x}{\ln 5} - \dfrac{2 \times 7^x}{\ln 7} + c$ **d** $-\operatorname{cosec} x + c$

e $\sin x - \cot x + c$ **f** $-\dfrac{8}{3x\sqrt{x}} - \frac{1}{9}\sec x + c$

7 **a** $2e^t + 4\cos t + c$ **b** $3\sin t - \ln|t| + c$

c $-5\cos t - \frac{2}{3}t\sqrt{t} + c$ **d** $\tan x - 2\cos x + c$

e $\frac{1}{2}\theta^2 + \cos\theta + c$ **f** $2\ln|\theta| - \tan\theta + c$

g $-2\cot\theta - 3\operatorname{cosec}\theta + c$ **h** $4\sec\theta + c$

i $\tan\theta - \theta + c$

8 **a** $4\arcsin x + c$ **b** $5\arccos x + c$

c $7\arctan x + c$ **d** $3\ln|x| + 5\arcsin x + c$

e $\arccos x + \frac{3}{2}x^2 + c$ **f** $6x - 2\arctan x + c$

9 Both are correct. Notice that:

$$\begin{aligned} &\cos(-\arcsin x + \tfrac{\pi}{2}) \\ &= \cos(-\arcsin x)\cos\tfrac{\pi}{2} - \sin(-\arcsin x)\sin\tfrac{\pi}{2} \\ &= \sin(\arcsin x) \\ &= x \end{aligned}$$

$\therefore \; -\arcsin x + \frac{\pi}{2} = \arccos x$

So the answers are equivalent.

10 **a** $\frac{4}{3}x^3 + 2x^2 + x + c$ **b** $\frac{1}{3}x^3 + 2x - \dfrac{1}{x} + c$

c $-\dfrac{2}{\sqrt{x}} - 8\sqrt{x} + c$ **d** $\frac{4}{3}x\sqrt{x} - 2\sqrt{x} + c$

e $\frac{1}{2}x^2 - 2x + \ln|x| + c$ **f** $\ln|x| - \frac{1}{2}x^2 + c$

g $-\dfrac{4}{x} + 4\ln|x| + x + c$ **h** $\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

i $\frac{1}{5}x^5 - x^4 + 2x^3 - 2x^2 + x + c$

j $\frac{2}{3}x\sqrt{x} + \dfrac{8}{\sqrt{x}} + c$ **k** $\frac{1}{2}x^2 - 4x + 10\ln|x| + c$

l $\frac{3}{2}x^2 - 2x - \dfrac{5}{x} + c$

11 **a** $f(x) = x - 2x^2 + \frac{4}{3}x^3 + c$

b $f(x) = \frac{2}{3}x\sqrt{x} - 4\sqrt{x} + c$ **c** $f(x) = x + \dfrac{5}{x} + c$

d $f(x) = 2x + 3\arctan x$

e $f(x) = -\frac{1}{2}\operatorname{cosec} x + \frac{1}{3}\cot x + c$

12 **a** $\frac{1}{5}x^5 - x^2 - \frac{1}{x} + c$ **b** $\frac{2}{5}x^2\sqrt{x} - \frac{8}{3}x\sqrt{x} + 4\sqrt{x} + c$

c $\frac{18}{7}x^3\sqrt{x} - \frac{12}{5}x^2\sqrt{x} + \frac{2}{3}x\sqrt{x} + c$

13 $\frac{d}{dx}(x\cos x) = \cos x - x\sin x$

$\therefore \int x\sin x\,dx = \sin x - x\cos x + c$

14 $\frac{d}{dx}(xe^x) = e^x + xe^x$

$\therefore \int xe^x\,dx = xe^x - e^x + c$

15 $\frac{d}{dx}(x\ln x) = \ln x + 1$

$\therefore \int \ln x\,dx = x\ln x - x + c$

16 **a** $\frac{d}{dx}(2x\sin x) = 2\sin x + 2x\cos x$

$\frac{d}{dx}\left(-x^2\cos x\right) = -2x\cos x + x^2\sin x$

b $\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + c$

EXERCISE 21C

1 **a** $f(x) = x^2 - x + 3$ **b** $f(x) = x^3 + x^2 - 7$

c $f(x) = e^x + 2\sqrt{x} - 1 - e$ **d** $f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

e $f(x) = \frac{2}{3}x\sqrt{x} - 2x + \frac{8}{3}$ **f** $f(x) = \ln|x| + 1$

2 $y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{22}{3}$ **3** $y = x - e^x + 2e^3 - 3$

4 **a** $f(x) = \frac{1}{3}x^3 - 4\sin x + 3$

b $f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$

c $f(x) = \frac{2}{3}x\sqrt{x} - 2\tan x - \frac{2}{3}\pi\sqrt{\pi}$

d $f(x) = e^x + 3\sin x - e^\pi$ **e** $f(x) = 2\sec x + 3$

f $f(x) = \frac{10}{\pi}\arcsin x - \frac{3}{\pi}\arctan x - \frac{1}{4}$

5 $f(x) = -x^2 + x + 3$ **6** $f(x) = 3x^3 + 1$

7 **a** $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f(x) = 4x^2\sqrt{x} + 4x\sqrt{x} - 4x + 5$

c $f(x) = -\cos x - x + 4$

d $f(x) = \frac{3^x}{(\ln 3)^2} - \frac{x}{\ln 3} - \frac{3}{(\ln 3)^2}$

e $f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

8 $f(x) = \cos x - \sin x + \frac{6}{\pi}x - 3$

EXERCISE 21D

1 **a** $\frac{1}{8}(2x+5)^4 + c$ **b** $\frac{1}{32}(4x-3)^8 + c$

c $-\frac{3}{5}(1-x)^5 + c$ **d** $\frac{1}{2(3-2x)} + c$

e $\frac{-2}{3(2x-1)^3} + c$ **f** $\frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

g $-4\sqrt{1-5x} + c$ **h** $-2\sqrt{3-4x} + c$

i $-\frac{5}{6(3x-2)^2} + c$

2 $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$ **3** $(-8, -19)$

4 **a** $\frac{1}{2}(2x-1)^3 + c$ **b** $\frac{1}{12}(4x-5)^3 + c$

c $-\frac{1}{12}(1-3x)^4 + c$ **d** $-\frac{1}{15}(2-5x)^3 + c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}} + c$ **f** $\frac{1}{35}(7x+1)^5 + c$

5 $y = \frac{1}{2}x^2 - \frac{5}{1-x} - 7$

6 **a** $-\frac{1}{3}\cos 3x + c$ **b** $-\frac{1}{2}\sin(-4x) + x + c$

c $\frac{1}{2}\tan 2x + c$ **d** $6\sin\frac{x}{2} + c$

e $-\frac{3}{2}\cos 2x + e^{-x} + c$ **f** $\frac{1}{2}e^{2x} - 4\tan\frac{x}{2} + c$

g $-\cos\left(2x + \frac{\pi}{6}\right) + c$ **h** $3\sin\left(\frac{\pi}{4} - x\right) + c$

i $-2\tan\left(\frac{\pi}{3} - 2x\right) + c$ **j** $\frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x + c$

k $-\frac{2}{3}\cos 3x + \frac{5}{4}\sin 4x + c$ **l** $\frac{1}{16}\sin 8x + 3\cos x + c$

7 **a** $2e^x + \frac{5}{2}e^{2x} + c$ **b** $\frac{3}{5}e^{5x-2} + c$

c $-\frac{1}{3}e^{7-3x} + c$ **d** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

e $-\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

f $\frac{1}{3}e^{3x} - 10e^x - 25e^{-x} + c$

8 $y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$

9 **b** **i** $\frac{3^{2x-1}}{2\ln 3} + c$ **ii** $-\frac{5^{-x}}{\ln 5} + c$

iii $\frac{2^{5x}}{5\ln 2} - \frac{7^{1-2x}}{2\ln 7} + c$

10 $f(x) = -e^{-2x} + 4$ **11** $p = -\frac{1}{4}$, $f(x) = \frac{1}{2}\cos\frac{x}{2} + \frac{1}{2}$

13 $y = \frac{2}{3}x\sqrt{x} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

14 $\int(\sin x + \cos x)^2\,dx = x - \frac{1}{2}\cos 2x + c$

15 **a** $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

b **i** $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ **ii** $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$

16 **a** $\frac{5}{2}x + \frac{1}{4}\sin 2x + c$ **b** $\frac{1}{2}x - \frac{1}{4}\sin 2x + 2x^2 + c$

c $\frac{3}{2}x + \frac{1}{8}\sin 4x + c$ **d** $\frac{5}{2}x + \frac{1}{12}\sin 6x + c$

e $\frac{1}{4}x + \frac{1}{32}\sin 8x + c$ **f** $\frac{3}{2}x + 2\sin x + \frac{1}{4}\sin 2x + c$

g $x - \frac{1}{2}\sin 2x + \cos x + c$

h $\frac{11}{2}x + 6\cos x - \frac{9}{4}\sin 2x + c$

17 **b** $\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$

18 **a** $6\ln|x+4| + c$ **b** $\frac{1}{2}\ln|2x-1| + c$

c $-\frac{5}{3}\ln|1-3x| + c$ **d** $4x + \frac{1}{5}\ln|5x-2| + c$

e $x - x^2 + 4\ln|x-3| + c$

f $-\frac{1}{2}\cos 2x + \frac{3}{2}\ln|1-2x| + c$

g $-e^{-x} - 2\ln|2x+1| + c$

h $\ln|x+2| + 2\ln|x-3| + c$

i $5\ln|x-6| - \frac{2}{3}\ln|3x-1| + c$

19 Both are correct. Recall that:

$\frac{d}{dx}(\ln(Ax)) = \frac{d}{dx}(\ln A + \ln x) = \frac{1}{x}$, $A, x > 0$

20 $\int\frac{3x-1}{x+2}\,dx = 3x - 7\ln|x+2| + c$

21 $\frac{1}{3}x^3 + 4x + \ln|x-1| + c$

22 $f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

23 **b** **i** $\frac{1}{2}\log_3|2x+5| + c$ **ii** $-3\log_2|1-x| + c$

24 **b** **i** $\int \sec(ax+b)\tan(ax+b)\,dx$

$= \frac{1}{a}\sec(ax+b) + c, \quad a \neq 0$

ii $\int \left(-\operatorname{cosec}^2(ax+b)\right)dx$

$= \frac{1}{a}\cot(ax+b) + c, \quad a \neq 0$

c **i** $-\frac{1}{2}\operatorname{cosec}\left(2x - \frac{\pi}{6}\right) + c$ **ii** $-3\sec\left(\frac{\pi}{3} - \theta\right) + c$

iii $-\frac{1}{20}\cot(4x - \pi) + c$

25 **b** **i** $\int \frac{-1}{\sqrt{1-(ax+b)^2}}\,dx$

$= \frac{1}{a}\arccos(ax+b) + c, \quad a \neq 0$

ii $\int \frac{1}{1+(ax+b)^2}\,dx$

$= \frac{1}{a}\arctan(ax+b) + c, \quad a \neq 0$

c **i** $\arcsin(x+2) + c$ **ii** $2\arccos(x-5) + c$

iii $\frac{1}{2}\arctan(2x+3) + c$

26 **b** **i** $\arcsin\frac{x}{5} + c$ **ii** $4\arccos\frac{x}{3} + c$

iii $6\arctan\frac{x}{3} + c$ **iv** $\arcsin 2x + c$

v $5\arccos 3x + c$ **vi** $4\arctan 4x + c$

EXERCISE 21E

1 **b** $3\ln|x+2| - \ln|x-2| + c$

2 **b** $\frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|2x+1| + c$

3 **a** $\frac{8}{2x-3} - \frac{4}{x+1}$ **b** $4\ln|2x-3| - 4\ln|x+1| + c$

4 **a** $\frac{5}{2(x+1)} - \frac{3}{2(x-3)}$

b $\frac{5}{2}\ln|x+1| - \frac{3}{2}\ln|x-3| + c$

5 **a** $\ln|x-2| - \ln|x+1| + c$

b $3\ln|x+3| + \ln|x-1| + c$

c $\frac{1}{2}\ln|x| - \frac{3}{2}\ln|x+2| + c$

d $\frac{7}{3}\ln|x+1| - \frac{1}{3}\ln|x-2| + c$

e $-\frac{5}{14}\ln|2x+1| - \frac{1}{7}\ln|x-3| + c$

f $\frac{10}{57}\ln|3x-5| + \frac{3}{19}\ln|2x+3| + c$

6 **a** $\frac{2}{x+3} + \frac{4}{x-1} - \frac{3}{(x-1)^2}$

b $2\ln|x+3| + 4\ln|x-1| + \frac{3}{x-1} + c$

EXERCISE 21F

1 $\frac{d}{dx}\left((2x^2-5x)^3\right) = 3(4x-5)(2x^2-5x)^2$

$\therefore \int (4x-5)(2x^2-5x)^2\,dx = \frac{1}{3}(2x^2-5x)^3 + c$

2 $\frac{d}{dx}\left(\sin(x^2)\right) = 2x\cos(x^2)$

$\therefore \int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + c$

3 $\frac{d}{dx}\left(\ln(5-3x+x^2)\right) = \frac{2x-3}{5-3x+x^2}$

$\therefore \int \frac{4x-6}{5-3x+x^2}\,dx = 2\ln\left|5-3x+x^2\right| + c$

4 **a** $\frac{1}{5}(x^3+1)^5 + c$ **b** $\frac{1}{3}e^{x^3+1} + c$ **c** $\frac{1}{5}\sin^5 x + c$

d $\sin(x^2-3) + c$ **e** $e^{\frac{x-1}{x}} + c$

5 **a** $\frac{1}{4}(2+x^4)^4 + c$ **b** $2\sqrt{x^2+3} + c$

c $-\frac{1}{3(2x^3-1)^3} + c$ **d** $\frac{2}{3}(x^3+x)^{\frac{3}{2}} + c$

e $\frac{1}{5}(x^3+2x+1)^5 + c$ **f** $\frac{1}{8(1-x^2)^4} + c$

g $-\frac{1}{2(x^2+4x-3)} + c$ **h** $-\frac{1}{27(3x^3-1)^3} + c$

i $\frac{1}{5}(x^2+x)^5 + c$

6 **a** $e^{1-2x} + c$ **b** $e^{x^2} + c$ **c** $2e^{\sqrt{x}} + c$

d $-e^{x-x^2} + c$

7 **a** $\ln\left|x^2+1\right| + c$ **b** $-\frac{1}{2}\ln\left|2-x^2\right| + c$

c $-2\ln\left|5x-x^2\right| + c$ **d** $-\frac{1}{3}\ln\left|x^3-3x\right| + c$

e $\ln\left|e^x - e^{-x}\right| + c$ **f** $\frac{1}{2}x^2 - \frac{1}{2}\ln\left|1+x^2\right| + c$

8 **a** $-\frac{1}{9}(3-x^3)^3 + c$ **b** $-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$

c $-\frac{1}{2}e^{1-x^2} + c$ **d** $\frac{1}{4}(\ln x)^4 + c$

e $4\ln|\ln x| + c$ **f** $\frac{2^{x^3}}{3\ln 2} + c$

g $-\sqrt{5-e^{2x}} + c$ **h** $\frac{3^{\sin x}}{\ln 3} + c$

9 **a** $\frac{1}{8}\sin^8 x + c$ **b** $-\frac{1}{6}\cos^6 x + c$

c $-2\sqrt{\cos x} + c$ **d** $-\ln|\cos x| + c$

e $\frac{2}{3}(\sin x)^{\frac{3}{2}} + c$ **f** $-\frac{1}{2+\sin x} + c$

g $\frac{1}{2\cos^2 x} + c$ **h** $\ln|1-\cos x| + c$

i $\ln|\sin x - \cos x| + c$ **j** $\frac{1}{2}\ln|\sin 2x - 3| + c$

k $-\frac{1}{2}\cos(x^2) + c$ **l** $\frac{1}{4}\tan^4 x + c$

m $\frac{1}{2}\arctan^2 x + c$ **n** $\ln|\cos x| + \frac{1}{2\cos^2 x} + c$

o $\sin x - \frac{1}{3}\sin^3 x + c$

p $\frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + c$

10 **a** $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

b $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$ **c** $\frac{1}{8}\sin^4 2x + c$

11 **a** $-e^{\cos x} + c$ **b** $e^{\tan x} + c$ **c** $-e^{\cot x} + c$

12 **a** $\frac{1}{3}\ln|\sin 3x| + c$ **b** $\frac{1}{2}\sec^2 x + c$

c $-\frac{1}{3}\operatorname{cosec}^3 x + c$ **d** $\frac{2}{3}(\tan x)^{\frac{3}{2}} + c$

e $-\frac{1}{2}\cot(x^2) + c$ **f** $\frac{3}{2}(\sec x)^{\frac{2}{3}} + c$

g $-2\sqrt{\cot x} + c$ **h** $-\frac{1}{6}\operatorname{cosec}^3(2x) + c$

i $\frac{1}{5}\sec^5 x + c$

13 **a** $\frac{1}{4}[\ln(x^2+7)]^2 + c$ **b** $8\arcsin(\sqrt{x}) + c$

14 $\frac{2}{7}(x-16)^{\frac{7}{2}} + \frac{64}{5}(x-16)^{\frac{5}{2}} + \frac{512}{3}(x-16)^{\frac{3}{2}} + c$

15 **a** $\frac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c$

b $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

c $\frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

d $\frac{2}{3}(x-3)^{\frac{3}{2}} + 6\sqrt{x-3} + c$

e $-(3-x^2)^{\frac{3}{2}} + \frac{1}{5}(3-x^2)^{\frac{5}{2}} + c$

f $\frac{1}{5}(t^2+2)^{\frac{5}{2}} - \frac{2}{3}(t^2+2)^{\frac{3}{2}} + c$

16 $2\sqrt{x-1} - 2\arctan\sqrt{x-1} + c$

17 **a** $\frac{1}{12}\arctan\frac{x}{3} + c$ **b** $\frac{1}{5}\sqrt{4x^2-1} - \frac{1}{5}\operatorname{arccos}\frac{1}{2x} + c$

18 **a** $x - 3\arctan\frac{x}{3} + c$ **b** $\frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + c$

c $\frac{9}{2}\arcsin\frac{x}{3} + \frac{1}{2}x\sqrt{9-x^2} + c$ **d** $2\ln\left(1+(\ln x)^2\right) + c$

e $\frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2} + \frac{1}{4}x^3\sqrt{1-x^2} + c$

f $\sqrt{x^2-4} - 2\arccos\frac{2}{x} + c$ **g** $\frac{1}{2}\arcsin\frac{2x}{3} + c$

h $\frac{1}{6}\arctan\left(\frac{2\ln x}{3}\right) + c$ **i** $\arcsin\frac{x}{2} + 2\sqrt{4-x^2} + c$

j $\frac{1}{2}\ln(x^2+4) + 2\arctan\frac{x}{2} + c$

k $\frac{1}{16}\ln\left|\frac{x}{\sqrt{x^2+16}}\right| + c$ **l** $\frac{3}{2}\operatorname{arccos}\frac{2}{x} + c$

m $-\frac{\sqrt{16-x^2}}{16x} + c$ **n** $\frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c$

o $\ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$

p $2\arcsin\frac{x}{2} - \frac{1}{4}x\sqrt{4-x^2}(2-x^2) + c$

19 **a** $\frac{1}{4\xi} - \frac{x}{(x+\xi)^2} + c$ **b** $\frac{1}{3\xi}\left(\frac{x-\xi}{x+\xi}\right)^{\frac{3}{2}} + c$

20 $2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln(x^{\frac{1}{6}}+1) + c$

21 $\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + d$ if $b^2 < 4ac$,

$\frac{1}{b^2-4ac}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + d$ if $b^2 > 4ac$,

$-\frac{2}{2ax+b} + d$ if $b^2 = 4ac$.

EXERCISE 21G

1 **a** $xe^x - e^x + c$ **b** $-x\cos x + \sin x + c$

c $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$ **d** $\frac{(\ln x)^2}{2} + c$

e $\frac{x\,3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2} + c$ **f** $-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + c$

g $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$ **h** $x\tan x + \ln|\cos x| + c$

2 **a** $x\ln x - x + c$ **b** $x(\ln x)^2 - 2x\ln x + 2x + c$

3 $x\arctan x - \frac{1}{2}\ln(1+x^2) + c$

4 **a** $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$

b $\frac{1}{2}e^x(\sin x + \cos x) + c$ **c** $-\frac{1}{2}e^{-x}(\cos x + \sin x) + c$

d $-x^2\cos x + 2x\sin x + 2\cos x + c$

e $-\frac{(\ln x)^2 + 2\ln x + 2}{x} + c$

5 **a** $u^2e^u - 2ue^u + 2e^u + c$

b $x(\ln x)^2 - 2x\ln x + 2x + c$

6 **a** $-\frac{1}{4}\cos 2x + c$ **b, c** $\frac{1}{2}\sin^2 x + c$

7 $-\frac{1}{15}(\sin 4x\sin x + 4\cos 4x\cos x) + c$

8 **a** $-u\cos u + \sin u + c$

b $-\sqrt{2x}\cos\sqrt{2x} + \sin\sqrt{2x} + c$

9 $\frac{2}{3}\sqrt{3x}\sin\sqrt{3x} + \frac{2}{3}\cos\sqrt{3x} + c$

REVIEW SET 21A

1 $\frac{d}{dx}\left(x^4 - x^2\right) = 4x^3 - 2x$

$\therefore \int (2x^3 - x)\,dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 + c$

2 $\frac{d}{dx}\left(\sin(\frac{\pi}{3} - 2x)\right) = -2\cos(\frac{\pi}{3} - 2x)$

$\therefore \int \cos(\frac{\pi}{3} - 2x)\,dx = -\frac{1}{2}\sin(\frac{\pi}{3} - 2x) + c$

3 **a** $\frac{2}{3}x\sqrt{x} + \frac{2}{x} + c$ **b** $x^2 - \frac{9}{2}x^{\frac{2}{3}} + c$

c $4x\sqrt{x} + 10\sqrt{x} + c$

4 **a** $8\sqrt{x} + c$ **b** $-\frac{3}{2}\ln|1-2x| + c$

c $\frac{1}{12}x^4 + x^2 + c$ **d** $-\frac{1}{2x^2} + \frac{2}{x} + c$

5 **a** $-\frac{3}{5}x^5 + 2x^3 + c$ **b** $\frac{3}{2}x^2 - x + \frac{1}{x} + c$

c $\frac{4}{3}x^3 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$ **d** $4e^x - 3\ln|x| + c$

e $-\frac{1}{4}\cos(4x-5) + c$ **f** $-\frac{1}{3}e^{4-3x} + c$

g $-\frac{5^{2-x}}{\ln 5} + c$ **h** $2\tan\frac{x}{2} + c$

i $-\cot(x - \frac{\pi}{4}) + c$

6 **a** $y = -3e^{-x} - 2\cos(\frac{\pi}{2} - x) + c$

b $y = \frac{1}{4}\sin 4x - \frac{1}{6}x^3 + c$

7 $f(x) = x^3 - 2x^2 + x + 2$ **8** $y = -2x^2 + 3x + 2$

9 $f(x) = \frac{3}{2}e^{2x} + \frac{1}{2}$ **10** $a = 6\sqrt{2}$, $f(x) = 2\sqrt{2}\sin 3x - 1$

11 **a** $\frac{1}{2}x^2 - 7\ln|x| + c$ **b** $\frac{1}{2}e^{2x-3} - \frac{2}{3}\ln|3x-1| + c$

c $-\frac{1}{12}(4-3x)^4 + \frac{1}{2}\cos(-2x) + c$

d $3\arctan x + 2\ln|5-x| + c$

12 **a** $\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x + c$ **b** $\sqrt{x^2-4} + c$

c $-\frac{1}{2}\cos(x^2 + \frac{\pi}{3}) + c$

13 **b** $4\ln|x-2| + 3\ln|x+1| + c$

14 **a** $\frac{1}{3}\ln|x+2| + \frac{2}{3}\ln|x-4| + c$

b $\frac{1}{2}\ln|2x-5| + \ln|x+1| + c$

15 $\frac{1}{3}\tan^3 x + \tan x + c$

16 **a** $\frac{1}{2}\ln\left|x^2+4x\right| + c$ **b** $e^{x^2-1} + c$

c $\frac{1}{10}\sin^{10} x + c$ **d** $-\frac{1}{2}\ln|\cos 2x| + c$

e $e^{\sin x} + c$ **f** $\frac{1}{2}(\arcsin x)^2$

17 **a** $-\frac{32}{3}(4-x)^{\frac{3}{2}} + \frac{16}{5}(4-x)^{\frac{5}{2}} - \frac{2}{7}(4-x)^{\frac{7}{2}} + c$

b $-x + 6\ln|2-x| + \frac{12}{2-x} - \frac{4}{(2-x)^2} + c$

c $\frac{2}{3}(x+2)^{\frac{3}{2}} - (x+2) - 2\sqrt{x+2} + 2\ln(\sqrt{x+2}+1) + c$

18 **a** $\frac{1}{3}x^3(\ln(x^3)-1)+c$ **b** $\frac{1}{5}e^{2x}(2\sin x-\cos x)+c$

19 **a** $\frac{1}{2}\ln\left|x^2-9\right|+c$ **b** $\frac{1}{2}\ln|x+3|+\frac{1}{2}\ln|x-3|+c$

c $\ln\left|\cos\left(\arcsin\frac{x}{3}\right)\right|+c$

20 **a** $\frac{1}{2}e^u(\sin u-\cos u)+c$ **b** $\frac{1}{2}x(\sin(\ln x)-\cos(\ln x))+c$

21 $\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\tan x\right)+c$

REVIEW SET 21B

1 $\dfrac{d}{dx}\left(e^{-2x}\right)=-2e^{-2x}$

$\therefore \int e^{-2x}\,dx=-\frac{1}{2}e^{-2x}+c$

2 $\dfrac{d}{dx}\left(\ln(2x+1)\right)=\dfrac{2}{2x+1}$

$\therefore \int \dfrac{1}{2x+1}\,dx=\frac{1}{2}\ln|2x+1|+c$

3 **a** $x+\dfrac{2}{x}+c$ **b** $3x^3-12x^2+16x+c$ **c** $4x-\frac{2}{3}x^3+c$

4 **a** $\frac{3}{4}x^{\frac{4}{3}}+3x+c$ **b** x^3-2x+c

c $9x+6x^2+\frac{4}{3}x^3+c$ **d** $\frac{1}{2}x^2-2x+\ln|x|+c$

e $5\log_2|x|+c$ **f** $3\arccos x+c$

5 $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x+\frac{13}{6}$

6 **a** $y=\frac{1}{5}x^5-\frac{2}{3}x^3+x+c$ **b** $y=400x-40x^{\frac{1}{2}}+c$

c $y=-\frac{5}{3}\ln|7-3x|+\dfrac{7^x}{\ln 7}+c$

7 **a** $\frac{1}{2}x^4-\frac{5}{2}x^2+7x+c$ **b** $\frac{3}{2}x^2-\ln|x|+c$

c $x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7+c$ **d** $-2e^{-x}+3x+c$

e $2\sin 2x+c$ **f** $9x+3e^{2x-1}+\frac{1}{4}e^{4x-2}+c$

8 $f(x)=2\ln|x|-x+e+2-2\ln 2$

9 $y=-\frac{3}{68}x^3+\frac{21}{68}(x-1)^{\frac{3}{2}}+\frac{275}{68}$

10 $f(x)=-2\sqrt{4-3x}+8$

11 **a** $-\frac{1}{2}\ln|3-2x|+c$ **b** $\frac{4}{5}\ln|5x+1|+c$

c $x+\frac{1}{2}\cos 2x+c$

12 **a** $\dfrac{2}{x-6}-\dfrac{3}{2x+1}$ **b** $2\ln|x-6|-\frac{3}{2}\ln|2x+1|+c$

13 $-\ln|x+2|+2\ln|x-1|+c$

15 **a** $\dfrac{x^3-3x^2+5x+4}{x-3}=x^2+5+\dfrac{19}{x-3}$

b $\displaystyle\int \frac{x^3-3x^2+5x+4}{x-3}\,dx$

$=\frac{1}{3}x^3+5x+19\ln|x-3|+c$

16 **a** $\frac{1}{3}(3x^2+x)^3+c$ **b** $2\sqrt{x^2-5}+c$

c $\dfrac{1}{3\cos^3 x}+c$ **d** $-2e^{-x^2}+c$

e $-\cos x+\frac{1}{3}\cos^3 x+c$ **f** $\cos x-\frac{2}{3}\cos^3 x+c$

g $\frac{1}{4}\arccos\frac{4}{x}+c$ **h** $-9\sqrt{9-x^2}+\frac{1}{3}(9-x^2)^{\frac{3}{2}}+c$

17 $\dfrac{2}{n-2}(\cos x)^{\frac{2-n}{2}}+c$ for $n\neq 2$, $-\ln|\cos x|+c$ for $n=2$

18 $-3\ln\left|1-\tan^2 x\right|+c$

19 **a** $\frac{1}{2}e^{-x}(\sin x-\cos x)+c$

b $\dfrac{2^x(\ln 2\times\sin x-\cos x)}{1+(\ln 2)^2}+c$

c $\sqrt{x^2-4}-2\arccos\frac{2}{x}+c$

20 **a** $5\arcsin\frac{x}{3}+c$ **b** $\frac{1}{6}\arctan\frac{2x}{3}+c$

c $\frac{3}{2}\ln\left|x^2+1\right|+5\arctan x+c$

21 The argument has not accounted for the constant of integration c.

EXERCISE 22A

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx=\frac{14}{3}$, $\displaystyle\int_1^4 (-\sqrt{x})\,dx=-\frac{14}{3}$

b $\displaystyle\int_0^1 x^7\,dx=\frac{1}{8}$, $\displaystyle\int_0^1 (-x^7)\,dx=-\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1

3 **a** -4 **b** $\frac{25}{4}=6\frac{1}{4}$ **c** $\frac{9}{4}=2\frac{1}{4}$

4 **a** $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** 1

5 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** 18 **d** $\frac{3}{2}=1\frac{1}{2}$ **e** $\frac{101}{6}=16\frac{5}{6}$

f $\frac{20}{3}=6\frac{2}{3}$ **g** $\frac{2}{3}$ **h** $\frac{61}{3}=20\frac{1}{3}$ **i** $21+\ln 4$

6 **a** $\frac{341}{5}=68\frac{1}{5}$ **b** 2 **c** $\frac{14}{3}=4\frac{2}{3}$

7 **a** $e-1$ **b** $2e^3-11$ **c** $\frac{1}{3}(e^6-1)$

d $e-1$ **e** $\frac{3}{2}$ **f** $-\dfrac{1}{2e^4}-\dfrac{3}{2e^2}+\dfrac{2}{e}+1$

8 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\sqrt{3}-1$ **d** $\frac{1}{3}$ **e** 1

f $\frac{1}{\sqrt{2}}$ **g** $\frac{\pi}{8}+\frac{1}{4}$ **h** $\frac{\pi}{4}$ **i** $-\frac{1}{6}$

9 **a** $-\ln 3$ **b** $\ln 3$ **c** $\frac{4}{3}\ln 2$ **d** $2\ln\left(\frac{13}{5}\right)$

11 **a** $m=-1$ or $\frac{4}{3}$ **b** $m=-5$

12 **a** $\frac{12}{\ln 2}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi}{6}$ **d** $\frac{1}{2}\ln\left(\frac{3}{2}\right)$

e $\frac{5}{3}+\frac{3\sqrt{3}}{8}+\frac{\pi}{3}$ **f** -3π

13 **a** $13\ln 2-7\ln 3$ **b** $2\ln 3-2\ln 2$ **c** $\frac{5}{3}\ln 10-\ln 2$

14 **a** $\dfrac{2}{e}-1$ **b** 1 **c** $3\ln 3-2$ **d** $\dfrac{e^2+1}{2}$

15 **a** ≈ 1.49 **b** ≈ -0.189 **c** ≈ 0.783

16 **a** $\dfrac{\pi}{4}-\dfrac{\ln 2}{2}$ **b** **ii** $a\approx 0.822$ **c** $a\approx 0.937$

18 **a** $\displaystyle\int_2^7 f(x)\,dx$ **b** $\displaystyle\int_4^6 f(x)\,dx$ **c** $\displaystyle\int_1^9 g(x)\,dx$

19 **a** -5 **b** 4

21 **a** 4 **b** 0 **c** -8 **d** $k=-\frac{7}{4}$ **22** 0

EXERCISE 22B

1 **a** $\frac{81}{4}=20\frac{1}{4}$ **b** $\frac{1}{12}$ **c** $\frac{1}{3}(e^2-e)\approx 1.56$

d $\frac{61}{3}=20\frac{1}{3}$ **e** $\frac{1}{4}(e^{-2}-e^{-8})\approx 0.0337$

f $\frac{1}{2}\ln\left(\frac{2}{7}\right)\approx -0.626$ **g** $\frac{1}{2}(\ln 2)^2\approx 0.240$

h 0 **i** $2\ln 7\approx 3.89$

2 **a** $2-\sqrt{2}$ **b** $\frac{1}{24}$ **c** $\frac{1}{2}\ln 2$ **d** $\ln 2$ **e** $\ln 2$ **f** $\frac{1}{4}$

3 $\dfrac{3^{n+1}}{2n+2}$, $n\neq -1$, undefined for $n=-1$

4 **a** $\frac{28\sqrt{3}}{5}-\frac{44\sqrt{2}}{15}$ **b** $\frac{48\sqrt{6}-54}{5}$ **c** $\frac{1054\sqrt{3}}{35}$

EXERCISE 22C

1 **a** 6 units^2 **b** 6 units^2

2 **a** 30 units^2 **b** $4\frac{1}{2}$ units^2 **c** $13\frac{1}{2}$ units^2

3 **a** $5\frac{1}{3}$ units^2 **b** $12\frac{2}{3}$ units^2

4 **a** $\frac{1}{3}$ units^2 **b** $63\frac{3}{4}$ units^2 **c** $2\frac{1}{6}$ units^2
d $6\frac{2}{3}$ units^2

5 **a** A$(-2, 0)$, B$(3, 0)$ **b** $20\frac{5}{6}$ units^2

6 **a** $4\frac{1}{2}$ units^2 **b** 9 units^2 **c** $4\sqrt{3}$ units^2

8 1 unit^2 **9** $\frac{2}{3}$ units^2

10 **a** region A
b A: $\frac{1}{2}$ units^2, B: $1 - \frac{1}{\sqrt{2}} \approx 0.293$ units^2
$\therefore$ region A is larger.

11 **a** $\frac{\pi}{4}$ **b** $-\ln|\cos x| + c$ **c** $\ln\sqrt{2}$ units^2

12 **a** $\frac{1}{2}$ units^2 **b** $(e-1)$ units^2 **c** $4\frac{1}{2}$ units^2
d 18 units^2 **e** $\left(2e - \frac{2}{e}\right)$ units^2
f $\frac{3\ln 3 - 1}{\ln 3}$ units^2 **g** $\frac{\pi}{6}$ units^2 **h** $4\arctan\frac{1}{2}$ units^2

13 **b** $\frac{\sin 2}{1 + \cos 2} \approx 1.56$ units^2

14 **b** $(8\ln 2 - 3) \approx 2.55$ units^2

15 **a** $-x\cos x + \sin x + c$ **b** $(1 + \cos 1 - \sin 1)$ units^2

16 **a** $b = \left(\frac{3}{2}\right)^{\frac{2}{3}} \approx 1.3104$ **b** $a = \sqrt{3}$
c $k = \dfrac{3e^{0.4} - 1}{2} \approx 1.7377$ **d** $k = 2\sqrt{5}$

EXERCISE 22D

1 **a** $4\frac{1}{2}$ units^2 **b** $(1 + e^{-2})$ units^2
c 2 units^2 **d** $2\frac{1}{4}$ units^2

2 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2 **c** 8 units^2

3 **a** $\displaystyle\int_3^5 f(x)\,dx = -(\text{area between } x = 3 \text{ and } x = 5)$

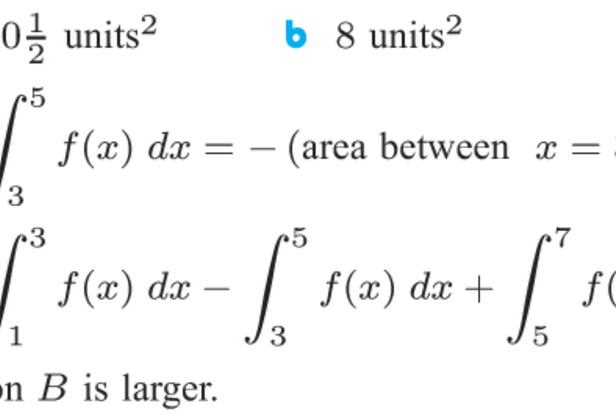
b $\displaystyle\int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

4 Region B is larger.

$\displaystyle\int_{-2}^4 f(x)\,dx = \text{area of region } A + (-\text{area of region } B)$
{region B is below the x-axis}
$= \text{area of region } A - \text{area of region } B$
$= -6$
$\therefore$ area of region $B >$ area of region A

5 $k = \frac{3}{2}$ or 6

6 **a**



b $\left(2\sqrt{3} - \frac{2\pi}{3}\right) \approx 1.37$ units^2

7 **a** $33\frac{1}{3}$ cm^2 **b** $66\frac{2}{3} \approx 66.7$ L

EXERCISE 22E

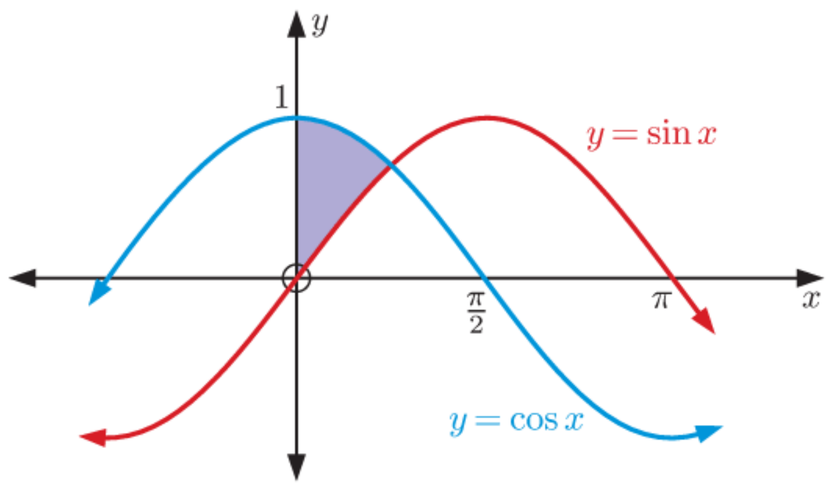
1 **a** $10\frac{2}{3}$ units^2 **b** $1\frac{1}{3}$ units^2 **c** $\frac{1}{12}$ units^2 **d** $\frac{1}{3}$ units^2

2 $\frac{1}{2}$ units^2 **3** **a** 36 units^2 **b** $\frac{82\sqrt{41}}{75} \approx 7.00$ units^2

4 **a** A$(\frac{1}{2}, \frac{5}{2})$, B$(2, 1)$ **b** $\left(\frac{21}{8} - \frac{5}{2}\ln\left(\frac{5}{2}\right)\right)$ units^2

5 $k \approx 2.3489$

6

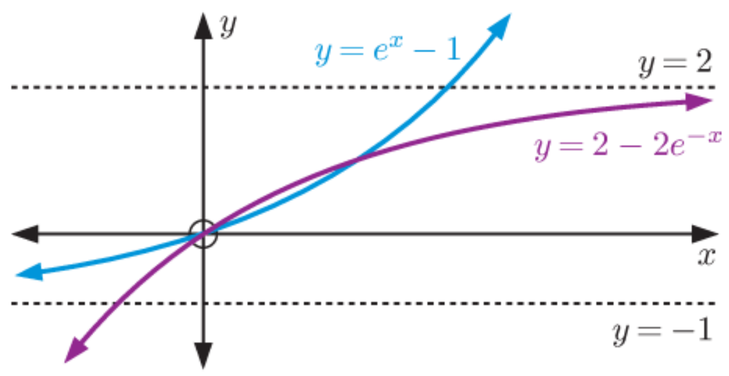


enclosed area $= (\sqrt{2} - 1)$ units^2

7 **a** C_1 is $y = 4\sin x$, C_2 is $y = \sin x$ **b** 6 units^2

8 **a** C_1 is $y = \cos^2 x$, C_2 is $y = \cos 2x$
b A$(0, 1)$, B$(\frac{\pi}{4}, 0)$, C$(\frac{\pi}{2}, 0)$, D$(\frac{3\pi}{4}, 0)$, E$(\pi, 1)$

9 **a**

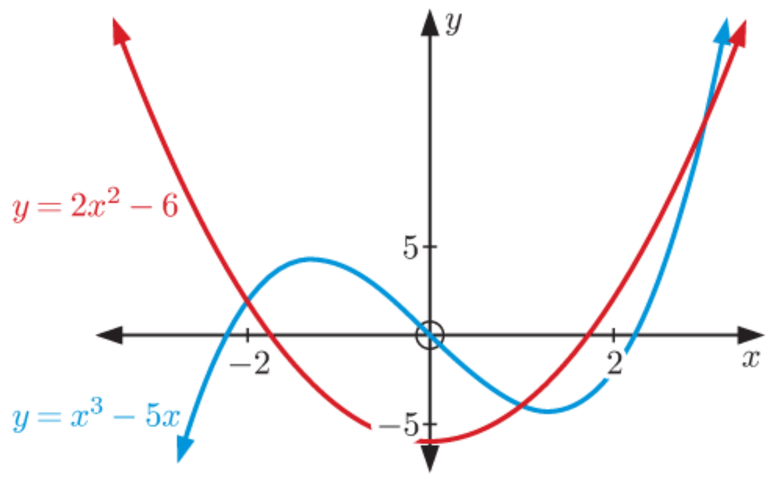


b $(0, 0)$ and $(\ln 2, 1)$ **c** $(3\ln 2 - 2)$ units^2

10 **a** $\displaystyle A = \int_{-2}^{-1} (x^3 - 7x - 6)\,dx + \int_{-1}^{3} (-x^3 + 7x + 6)\,dx$

b $32\frac{3}{4}$ units^2

11 **a**

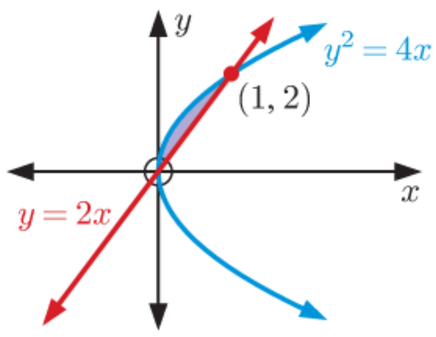


b -2, 1, and 3 **c** $21\frac{1}{12}$ units^2

12 **a** 8 units^2 **b** $101\frac{3}{4}$ units^2

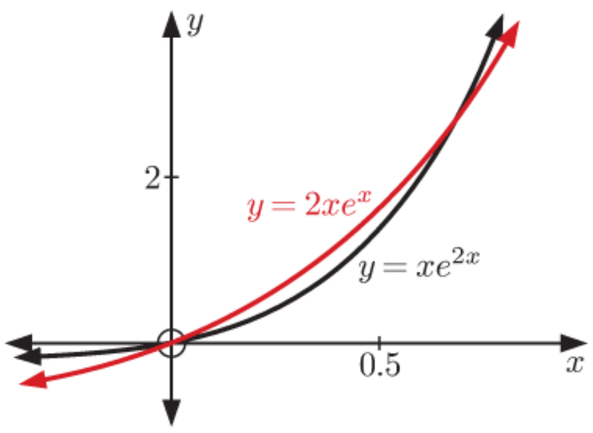
13 **a** C_1 is $y = \sin 2x$, C_2 is $y = \sin x$
b A$(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ **c** $2\frac{1}{2}$ units^2

14



enclosed area $= \frac{1}{3}$ units^2

15 **a**



b 0 and $\ln 2$ **c** $\left(2\ln 2 - \frac{5}{4}\right)$ units^2

EXERCISE 22F

1 **a** $f^{-1}(y) = \dfrac{1}{y^2}$ **b** $\frac{1}{2}$ units^2

2 **a** $\dfrac{45}{2 \times \sqrt[3]{2}}$ units^2 **b** 12 units^2 **c** $\dfrac{e^3 - 1}{e}$ units^2

d $2\ln 3$ units^2 **e** $\left[6 - 3\ln\left(\frac{5}{2}\right)\right]$ units^2 **f** $\frac{\sqrt{3}}{2}$ units^2

3 **a** $3\frac{1}{3}$ units^2 **b** $12\frac{2}{3}$ units^2

4 **a** $\left[\ln\left(\frac{27}{4}\right) - 1\right]$ units^2 **b** $\frac{1}{2}\ln 3$ units^2 **c** $\frac{\pi}{6}$ units^2

EXERCISE 22G.1

1 **a** 36π units^3 **b** 8π units^3 **c** $\frac{127\pi}{7}$ units^3

d $\frac{255\pi}{4}$ units^3 **e** $\frac{992\pi}{5}$ units^3 **f** $\frac{250\pi}{3}$ units^3

g $\frac{\pi}{2}$ units^3 **h** $\frac{40\pi}{3}$ units^3

2 **a** 186π units^3 **b** $\frac{146\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^8 - 1)$ units^3

3 **a** a cone of base radius r and height h

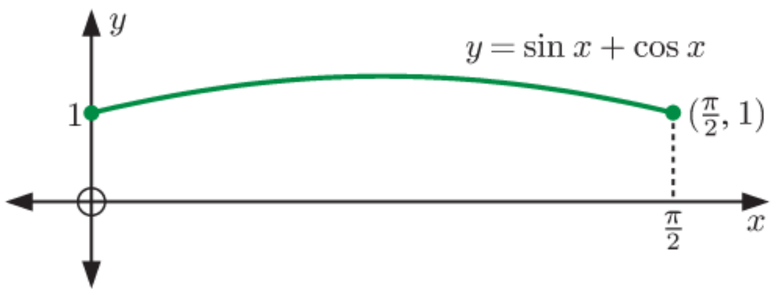
b $y = -\left(\dfrac{r}{h}\right)x + r$ **c** $V = \frac{1}{3}\pi r^2 h$

4 **a** a sphere of radius r

5 **a** $\frac{\pi^2}{4}$ units^3 **b** $\frac{\pi^2}{8}$ units^3 **c** 2π units^3

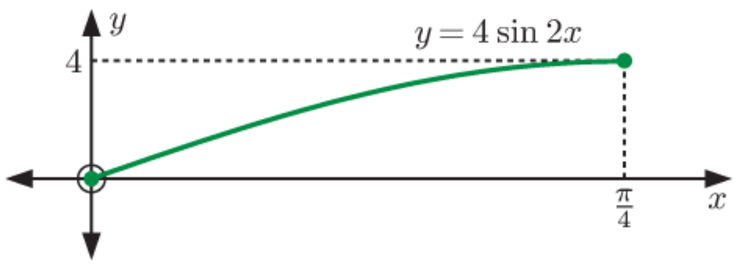
d $\pi\sqrt{3}$ units^3 **e** $\frac{\pi}{3}$ units^3 **f** $\pi(2 - \frac{\pi}{2})$ units^3

6 **a**



b $\pi(\frac{\pi}{4} + \frac{1}{2})$ units^3

7 **a**



b $2\pi^2$ units^3

8 **a** ≈ 18.6 units^3 **b** ≈ 30.2 units^3

EXERCISE 22G.2

1 **a** 8π units^3 **b** $\frac{1023\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^4 - 1)$ units^3

d $\frac{483\pi}{5}$ units^3 **e** $\frac{256\pi}{5}$ units^3

2 48π units^3 **3** 32π units^3

4 **a** 63π units^3 **b** $63\pi \approx 198$ cm^3

EXERCISE 22G.3

1 **a** A$(-1, 3)$, B$(1, 3)$ **b** $\frac{136\pi}{15}$ units^3

2 **a** A$(2, e)$ **b** $\pi(e^2 + 1)$ units^3

3 **a** A$(1, 1)$ **b** $\frac{11\pi}{6}$ units^3 **4** $\frac{162\pi}{5}$ units^3

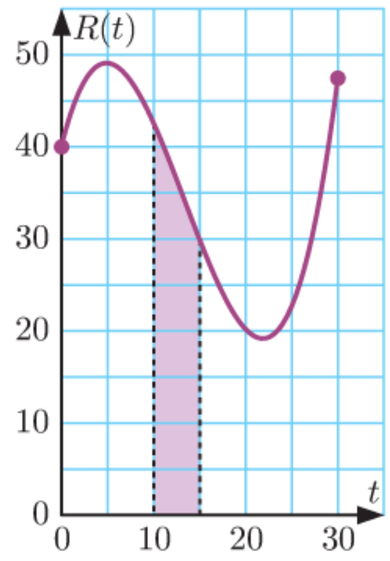
5 **a** A$(5, 1)$ **b** $\frac{9\pi}{2}$ units^3 **6** $24\pi^2 \approx 237$ units^3

7 **b** $V = 36\pi$ units^3, which is independent of r.

EXERCISE 22H

1 **a** 20 cars per minute **b** $\approx$ 8:05 am

c



$\int_{10}^{15} R(t)\,dt$ represents the total number of cars going past the pedestrian crossing from 8:10 am to 8:15 am.

d 1031 cars

2 **a** **i** ≈ 1.65 L per minute **ii** ≈ 1.09 L per minute

b The rate of water leaking into the kayak is greater than the rate of water being bailed from the kayak after 2 minutes. So, the amount of water in the kayak is increasing after 2 minutes.

c **i** $\int_0^3 R_1(t)\,dt \approx 3.72$

About 3.72 litres of water have leaked into the kayak in the first 3 minutes.

ii $\int_2^5 R_2(t)\,dt \approx 5.27$

About 5.27 litres of water have been bailed out of the kayak from $t = 2$ minutes to $t = 5$ minutes.

iii $\int_0^8 [R_1(t) - R_2(t)]\,dt \approx 5.09$

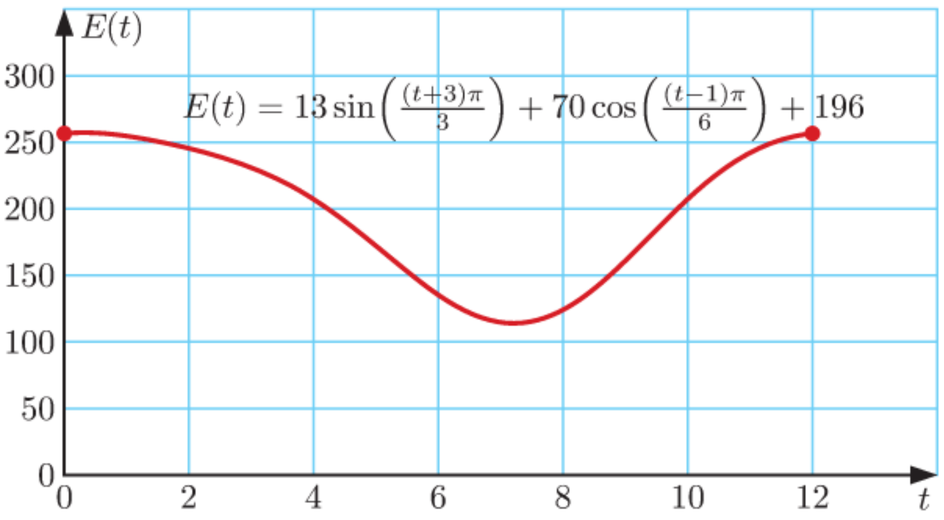
There are about 5.09 litres of water in the kayak 8 minutes after striking the rock.

d ≈ 6.31 litres

3 **a** C_1 is $y = 3\sin\frac{\pi t}{10}$, C_2 is $y = \sin\frac{\pi t}{10}$ **b** $\frac{40}{\pi}$ units

c The area in **b** represents the difference in the total amount of energy that enters the greenhouse in the first 10 hours on a sunny day versus an overcast day.

4 **a**



b **i** $\int_3^4 E(t)\,dt \approx 220.12$

The power consumption of the United Kingdom in April is about 220.12 TWh.

ii $\int_5^8 E(t)\,dt \approx 392.96$

The power consumption of the United Kingdom from June 1st to September 1st is about 392.96 TWh.

iii $\int_0^{12} E(t)\,dt = 2352$

The yearly power consumption of the United Kingdom is 2352 TWh.

EXERCISE 22I

1 **a** $\frac{1}{24}$ **b** does not exist **c** does not exist **d** $\frac{5}{18}$ **e** $\frac{1}{e}$ **f** $\frac{1}{2}$

2 **a** $\frac{1}{4}$ units^2 **b** $\frac{\pi}{2}$ units^2 **c** $\ln 3$ units^2

3 **a** 2 **b** 2 **c** $\dfrac{1}{\ln 2}$ **4** $\frac{\pi}{6}$

5 **a** $\int_0^\infty e^{-x} = 1$, $\int_0^\infty xe^{-x} = 1$, $\int_0^\infty x^2e^{-x} = 2$, $\int_0^\infty x^3e^{-x} = 6$

b $\int_0^\infty x^n e^{-x}\,dx = n!$

6 $\frac{2}{5}$ units^2

REVIEW SET 22A

1 **a** 8 **b** $\frac{1}{8} - \frac{1}{3\sqrt{2}}$ **c** $\frac{178}{15} = 11\frac{13}{15}$

2 **a** $b = 3$ **b** $b \approx 1.86$ **c** $b = \frac{1}{2}\ln 2$

3 **a** $\frac{112}{9} = 12\frac{4}{9}$ **b** $\sqrt{2}$ **c** $2\ln 3$ **d** $\frac{\pi}{12} - \frac{1}{4}$ **e** $\frac{8}{\sqrt{3}}$

4 $\dfrac{d}{dx}\left(e^{-2x}\sin x\right) = e^{-2x}(\cos x - 2\sin x)$

$\therefore \int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx = e^{-\pi}$

5 **a** $3 - \frac{2\sqrt{7}}{3}$ **b** $e(e-2)$ **c** $\ln 2$ **d** $\frac{1}{2}\ln 3$ **e** $\frac{5}{18}$ **f** $-15\ln 3$

6 **a** 6 **b** 3 **c** $k = -\frac{5}{3}$

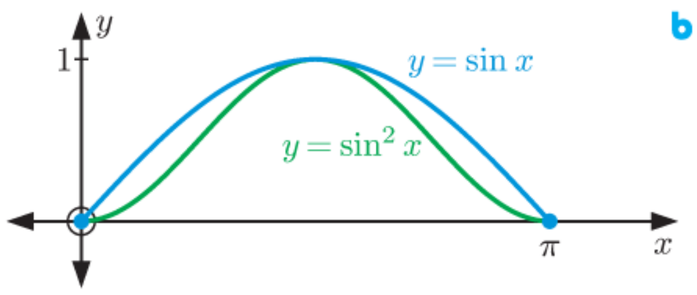
7 **a** $(3\pi + 1)$ units^2 **b** 36 units^2 **c** $(3\ln 3 - 2)$ units^2

8 **a** 39 units^2 **b** $\left(\frac{16}{3} - 2\sqrt{3}\right)$ units^2 **c** 3 units^2

9 $4\frac{1}{2}$ units^2 **10** $k = \sqrt[3]{16}$

11 **a** $a = \ln 3$ **b** $b = \ln 5$ **12** $40\frac{1}{2}$ units^2

13 **a**

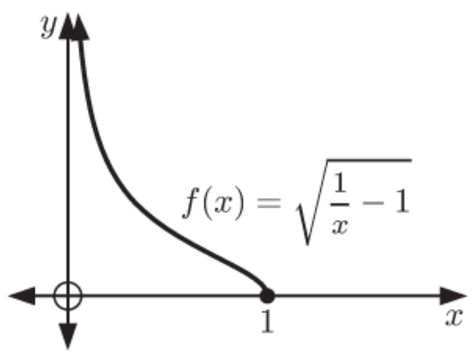


b $(1 - \frac{\pi}{4})$ units^2

15 **a** -2.3566, -0.7292, and 0 **b** 0.2009 units^2

16 **a** $2\frac{2}{3}$ units^2 **b** $\left(\frac{\sqrt{3}-1}{2}\right)$ units^2

17 **a**



b $\frac{\pi}{12}$ units^2

18 **a** 312π units^3 **b** 402π units^3 **c** 2π units^3 **d** $\pi\left(\frac{3\pi-8}{4}\right)$ units^3

19 **a** $\pi\left(\frac{3\pi}{32} - \frac{1}{8\sqrt{2}}\right)$ units^3 **b** ≈ 124 units^3

20 **a** $\frac{31\pi}{5}$ units^3 **b** $\frac{65\pi}{4}$ units^3 **21** $\frac{\pi}{2}$ units^3

22 **a** **i** $\int_0^{\frac{1}{2}} R_2(t)\,dt \approx 0.655$

About 655 millilitres of water leak from the watering can in the first 30 seconds.

ii $\int_0^1 [R_1(t) - R_2(t)]\,dt \approx 5.03$

There are about 5.03 litres of water in the watering can after 1 minute.

b ≈ 199 seconds

23 **a** does not exist **b** does not exist **c** $\dfrac{2}{\ln 2}$

24 $\frac{\pi}{3}$ units^3

25 **a** $f(x)$ is even, so $V = \pi\int_{-\infty}^{\infty} [f(x)]^2\,dx$

$= 2\pi\int_0^{\infty} [f(x)]^2\,dx$

b 1 unit^3

REVIEW SET 22B

1 **a** $2 - \frac{2\sqrt{2}}{\sqrt{3}}$ **b** -3 **c** $\frac{8}{15}$

2 **a** $a = \frac{3}{2}$ **b** $a = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

3 **a** $\frac{2}{3}(\sqrt{5} - \sqrt{2})$ **b** $4\ln\left(\frac{4}{3}\right)$ **c** $\frac{\pi}{4} + \sqrt{2}$ **d** $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$ **e** $\frac{1}{2}\ln 2$ **f** $\frac{1}{3}e^3 + 3e^2 + 12e - \frac{22}{3}$

4 **a** ≈ 3.528 **b** ≈ 2.963

5 **b** $\frac{\pi}{4} - \frac{1}{2}$ **6** **b** $3\ln 5 - 2\ln 2$

7 **a** $\frac{5}{54}$ **b** $\frac{19}{384}$ **c** $\frac{14\sqrt{7}}{3}$ **d** $-\frac{609}{8}$ **e** $\dfrac{e-1}{3}$

8 **a** $\ln 3$ units^2 **b** $\left(\dfrac{e^4 - 1}{2}\right)$ units^2 **c** $\frac{4\pi}{3}$ units^2

9 $4\frac{1}{2}$ units^2

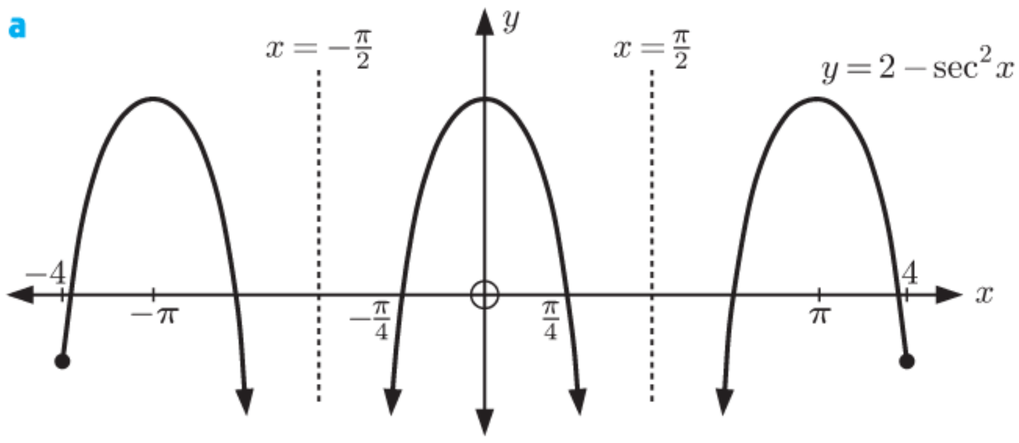
10 **Hint:** Show that the areas represented by the integrals can be arranged to form a $1 \times e$ unit rectangle.

11 **a** $(3 - \ln 4)$ units^2 **b** $\frac{92\sqrt{3}}{5}$ units^2 **12** $k = \frac{4}{3}$

13 $m = \frac{\pi}{3}$ **14** **a** $a = -3$ **b** A has x-coordinate $\sqrt[3]{4}$.

15 **a** $21\frac{1}{12}$ units^2 **b** $\left(\frac{\pi^2}{2} - 2\right)$ units^2 **c** $\left(2 - \frac{\pi}{2}\right)$ units^2 **d** $\ln 2$ units^2

16 **a**



b $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

c x-intercepts are $-\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

y-intercept is 1

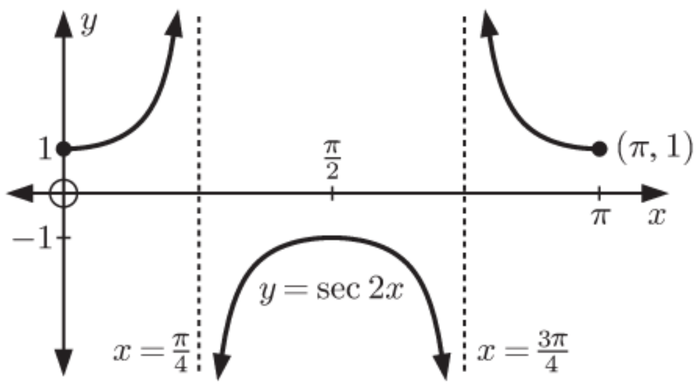
d $(\pi - 2)$ units^2

17 **a** $(6 \ln 3 - 4)$ units^2 **b** $\left(3\sqrt[3]{4} - \frac{3}{4}\right)$ units^2

18 **a** $\dfrac{d}{dx}\left[\ln(\tan x + \sec x)\right] = \sec x$

$$\therefore \int \sec x \, dx = \ln|\tan x + \sec x| + c$$

b **i**



ii ≈ 0.965 units^2

19 **a** $\frac{\pi^2}{2}$ units^3 **b** 18π units^3

20 **a** $-\operatorname{cosec}^2 x$ **b** 2π units^3

21 **a** 16 cm **b** $\frac{1280\pi}{9}$ cm^3 **c** ≈ 192 mL

22 **a** $\displaystyle\int_5^{12} E(t)\, dt \approx 10.7$

The solar energy transferred into Callum's solar panels from 5 am to 12 pm is about 10.7 kWh.

b $\displaystyle\int_{12}^{20} E(t)\, dt \approx 12.9$

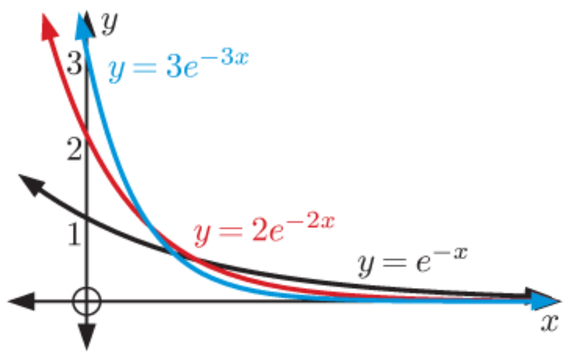
The solar energy transferred into Callum's solar panels from 12 pm to 8 pm is about 12.9 kWh.

c $\displaystyle\int_5^{20} E(t)\, dt \approx 23.5$

The solar energy transferred into Callum's solar panels from 5 am to 8 pm is about 23.5 kWh.

23 **a** 1 for all $a > 0$

b



The area between the curves and the x-axis appears to remain constant.

24 **b** **i** $\Gamma(1) = 1$

EXERCISE 23A

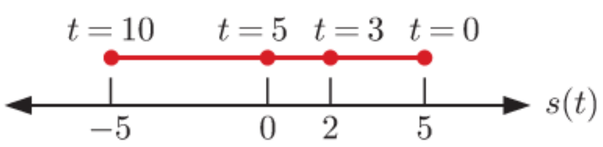
1 **a** 5 cm to the right of the origin

b **i** 2 cm to the right of the origin **ii** 5 cm to the left of the origin

c at $t = 5$ s

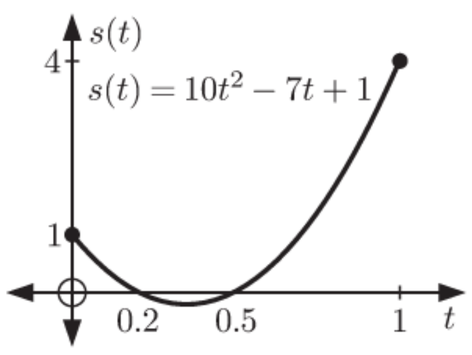
d No, the displacement function $s(t)$ is linear, so it has no turning points.

e



2 **a** 1 m to the right of the origin

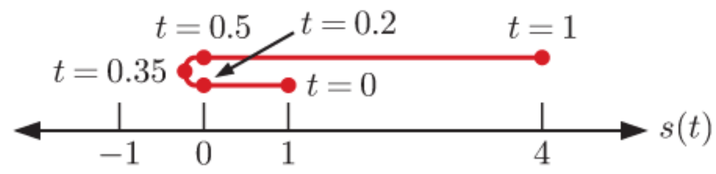
b



c $t = 0.35$ s, 0.225 m to the left of the origin

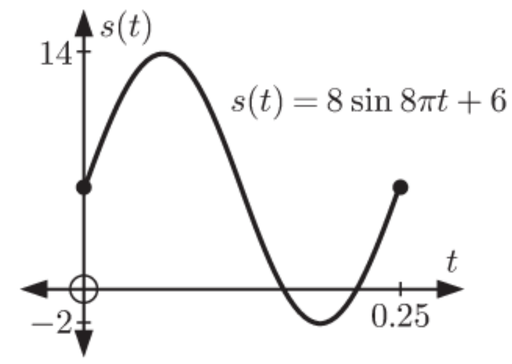
d at $0 \leqslant t < 0.2$ s and $0.5 < t \leqslant 1$ s

e



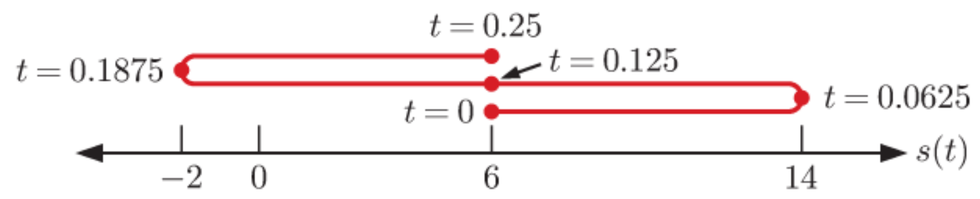
3 **a** at $t = 0$ s, $t = 0.125$ s, and $t = 0.25$ s

b



c 14 cm to the right of the origin, at $t = 0.0625$ s, and 2 cm to the left of the origin, at $t = 0.1875$ s

d



EXERCISE 23B.1

1 **a** **i** 3 m **ii** -3 m **b** -2 m s^{-1} **c** $-\frac{2}{3}$ m s^{-1}

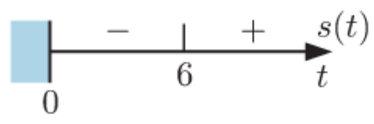
2 **a** -2 m s^{-1} **b** $v(t) = 2t - 6$ m s^{-1}

c **i** -4 m s^{-1} **ii** 4 m s^{-1}

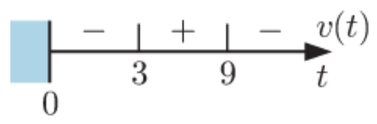
3 **a** 2 cm to the left of the origin **b** $t = 6$ s

c to the right **d** $t = 3$ s and $t = 9$ s

e **i**



ii



f 1 cm s^{-1}

4 **a** $\frac{2}{3}$ cm s^{-1} **b** 3 cm to the right of the origin

c $v(t) = \dfrac{1}{\sqrt{t}}$ cm s^{-1} **d** **i** $\frac{1}{2}$ cm s^{-1} **ii** $\frac{1}{4}$ cm s^{-1}

5 **a** $v(t) = 3t^2 - 22t + 24$ m s^{-1}

b The object is initially at the origin, moving to the right at 24 m s^{-1}, with decreasing speed.

c

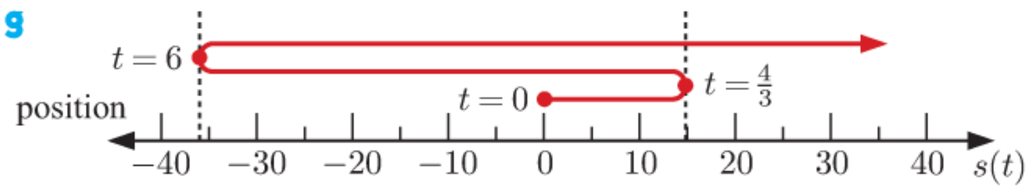


d at 0 s, 3 s, and 8 s

e at $\frac{4}{3}$ s and 6 s; $s(\frac{4}{3}) \approx 14.8$ m, $s(6) = -36$ m

f The object starts at O, and moves towards the right at 24 m s^{-1}. Its velocity is decreasing. After $\frac{4}{3}$ seconds, when it is 14.8 m to the right of O, it changes direction and moves to the left, passing O after 3 seconds. After 6 seconds, when it is 36 m to the left of O, it changes direction again and moves towards the right, passing O once more after 8 seconds.

g



6 **b** **i** 69.58 m s^{-1} **ii** ≈ 247 m

EXERCISE 23B.2

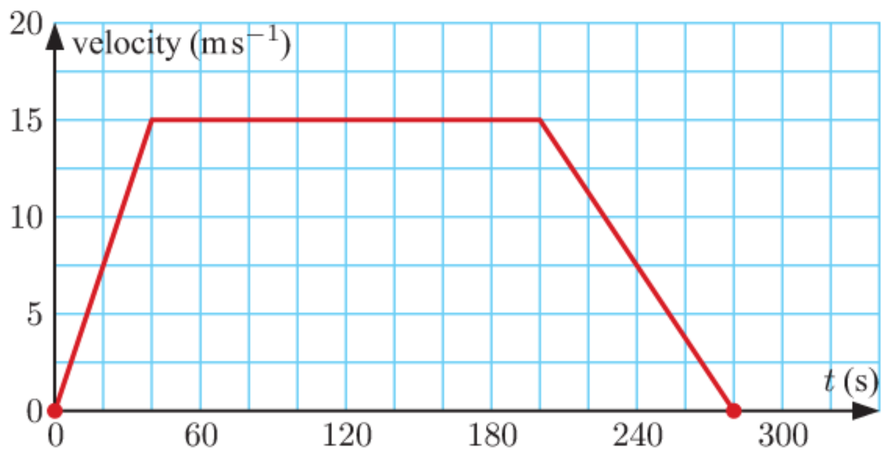
1 110 m

2 **a** **i** travelling forwards

ii travelling backwards (opposite direction)

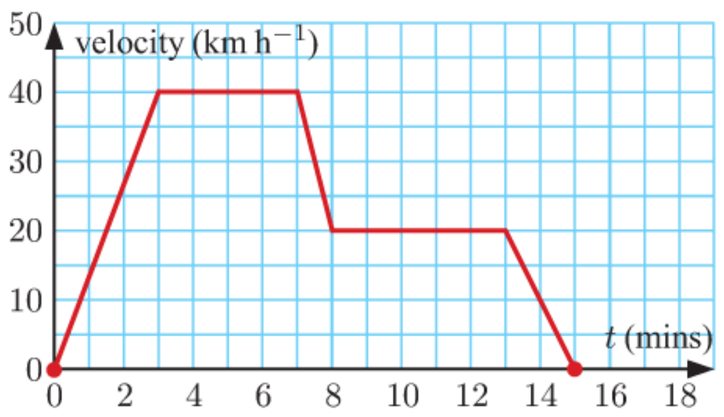
b 16 km **c** 8 km from starting point (on positive side)

3 **a**



b 3.3 km

4 **a**



b $6\frac{1}{6}$ km

5 **a** at $t = \frac{1}{2}$ second **b** $\frac{1}{2}$ cm **c** 0 cm

6 **a** $s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t$ cm **b** $5\frac{1}{6}$ cm

c $1\frac{1}{2}$ cm left of its starting point

7 **a** $s(t) = 29.4t - 4.9t^2 + 1$ m **b** 45.1 m

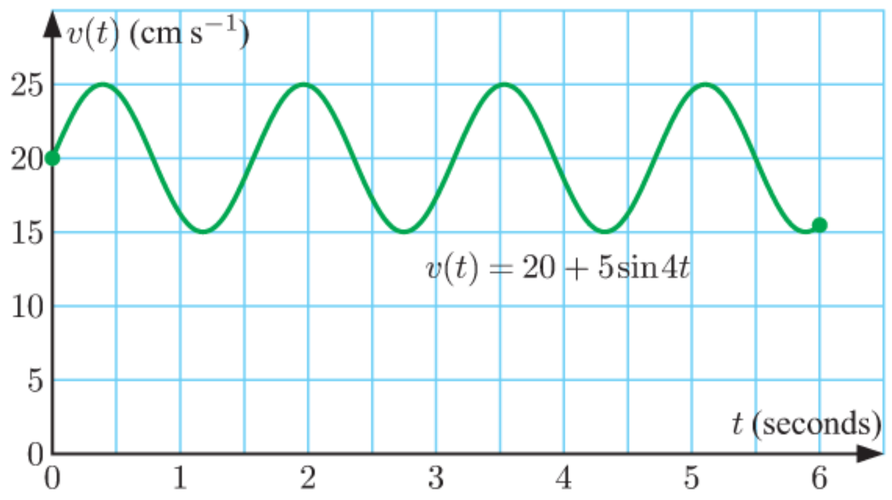
8 **a** $s(t) = 32t + 2t^2 + 16$ m

b There is no change in direction,
so displacement $= s(\tau) - s(0) = \int_0^\tau (32 + 4t)\,dt$

c 160 m

9 **a** 1 m **b** $s(\frac{\pi}{3}) = \frac{\sqrt{3}+2}{4}$ m

10 **a**



b $\approx 16.2\ \text{cm s}^{-1}$ **c** ≈ 41.4 cm

11 **a** $s(t) = -4t + \frac{2}{3}t^{\frac{3}{2}}$ m **b** $t = 16$ s

c ≈ 10.5 m left of the origin **d** ≈ 32.2 m

12 **a** **i** $10\ \text{m s}^{-1}$ **ii** $10\sqrt{2}\ \text{m s}^{-1}$ **b** $s(t) = \frac{20}{3}t^{\frac{3}{2}}$ m

c $\int_0^2 v(t)\,dt \approx 18.9$ The motorcyclist travels about 18.9 m in the first 2 seconds.

d **i** 4 seconds

ii Yes, he only needs $53\frac{1}{3}$ m to reach the required speed.

13 **a** ≈ 513 m **b** ≈ 0.637 cm

EXERCISE 23C

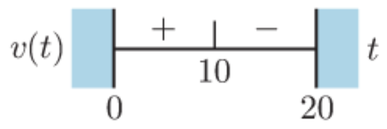
1 **a** $16\ \text{cm s}^{-1}$ **b** $6\ \text{cm s}^{-2}$

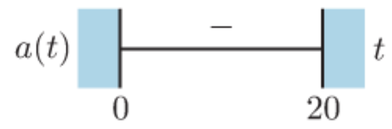
c $a(t) = 10 - 2t\ \text{cm s}^{-2}$ **d** $4\ \text{cm s}^{-2}$

2 **a** $s(2) = -1$ m, $v(2) = 8\ \text{m s}^{-1}$, $a(2) = 10\ \text{m s}^{-2}$

b $t = \frac{1}{3}$ s

3 **a** $v(t) = 98 - 9.8t\ \text{m s}^{-1}$ $a(t) = -9.8\ \text{m s}^{-2}$





b $s(0) = 0$ m above the ground, $v(0) = 98\ \text{m s}^{-1}$ upward

c At $t = 5$ s, the stone is 367.5 m above the ground and moving upward at $49\ \text{m s}^{-1}$. Its speed is decreasing.
At $t = 12$ s, the stone is 470.4 m above the ground and moving downward at $19.6\ \text{m s}^{-1}$. Its speed is increasing.

d 490 m **e** 20 seconds

4 **a** $v(t) = 100 - 40e^{-\frac{t}{5}}\ \text{cm s}^{-1}$, $a(t) = 8e^{-\frac{t}{5}}\ \text{cm s}^{-2}$

b $s(0) = 200$ cm to the right of the origin
$v(0) = 60\ \text{cm s}^{-1}$, $a(0) = 8\ \text{cm s}^{-2}$

c as $t \to \infty$, $v(t) \to 100\ \text{cm s}^{-1}$ (below)

d as $t \to \infty$, $a(t) \to 0\ \text{cm s}^{-2}$

5 **b** $v(t) = 1 - \dfrac{2}{2t+1}\ \text{cm s}^{-1}$

c **i** $t \geqslant \frac{1}{2}$ **ii** $0 \leqslant t \leqslant \frac{1}{2}$ **e** $\frac{4}{25}\ \text{cm s}^{-2}$

f $1 + \ln\left(\frac{4}{5}\right) \approx 0.777$ cm

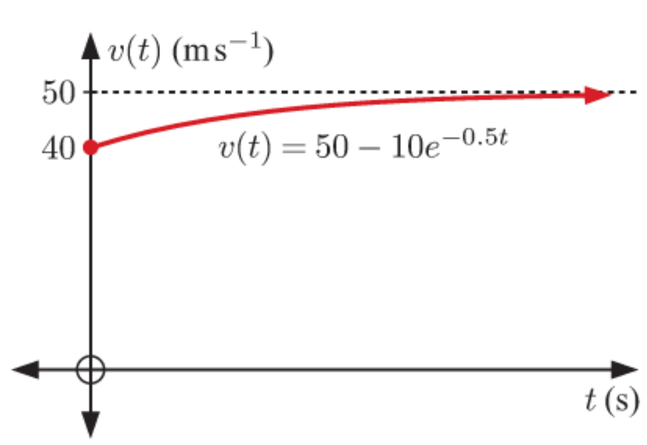
6 **a** $40\ \text{m s}^{-1}$ **b** $\approx 47.8\ \text{m s}^{-1}$ **c** $2\ln 2 \approx 1.39$ seconds

d as $t \to \infty$, $v(t) \to 50$ from below

e $a(t) = 5e^{-0.5t}$ and as $e^x > 0$ for all x, then $a(t) > 0$ for all t.

f $t = 2\ln\left(\frac{5}{2}\right)$ seconds

g



h ≈ 134 m

7 **a** $v(t) = \dfrac{t^2}{20} - 3t + 45\ \text{m s}^{-1}$

b $\int_0^{60} v(t)\,dt = 900$ The train travels a total of 900 m in the first 60 seconds.

8 **a** **Hint:** Show that $v(t) = 100 - 80e^{-\frac{1}{20}t}\ \text{m s}^{-1}$ and as $t \to \infty$, $v(t) \to 100\ \text{m s}^{-1}$.

b ≈ 370 m

9 **a** $v(t) = -\dfrac{1}{(t+1)^2} + 1\ \text{m s}^{-1}$

b $s(t) = \dfrac{1}{t+1} + t - 1$ m

c The particle is $\frac{4}{3}$ m to the right of the origin, moving to the right at $\frac{8}{9}\ \text{m s}^{-1}$, and accelerating at $\frac{2}{27}\ \text{m s}^{-2}$.

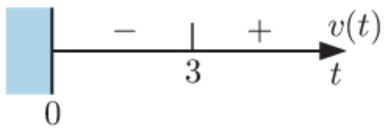
10 **a** $\approx 5.28\ \text{cm s}^{-1}$ **b** $\approx 5.07\ \text{cm s}^{-2}$

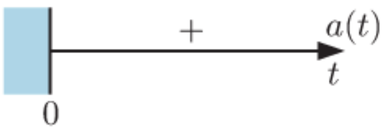
c $s(t) = -\frac{5}{29}e^{-2t}(9\sin 5t + 8\cos 5t) + \frac{330}{29}$ cm

d ≈ 0.221 cm

EXERCISE 23D

1 **a** $v(t) = 2t - 6\ \text{m s}^{-1}$ $a(t) = 2\ \text{m s}^{-2}$

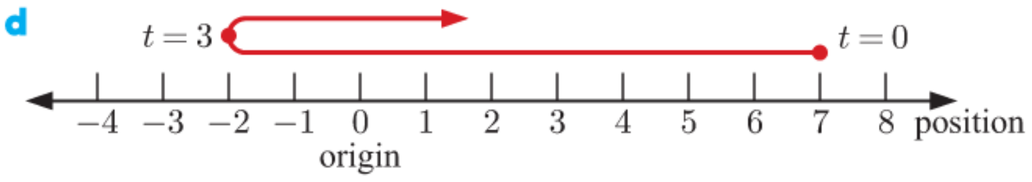




b $s(0) = 7$ m, $v(0) = -6$ m s^{-1}, $a(0) = 2$ m s^{-2}
Initially, the object is 7 m to the right of O, moving to the left at 6 m s^{-1}, with acceleration 2 m s^{-2}.

c 2 m to the left of O

d



e $0 \leqslant t \leqslant 3$

2 **a** 1.2 m

b $s'(t) = 28.1 - 9.8t$ represents the instantaneous velocity of the ball.

c ≈ 41.5 m

d **i** 28.1 m s^{-1} **ii** 8.5 m s^{-1} **iii** 20.9 m s^{-1}

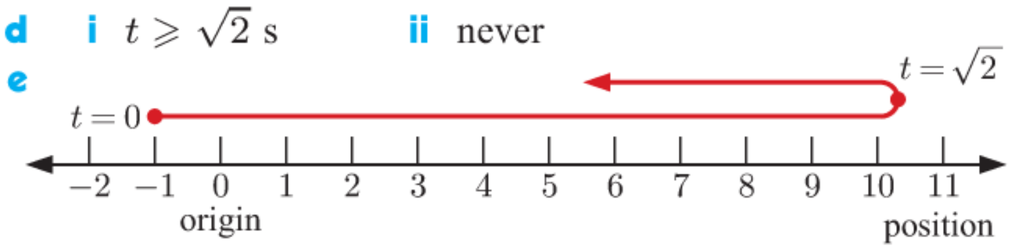
3 **a** $v(t) = 12 - 6t^2$ cm s^{-1}, $a(t) = -12t$ cm s^{-2}

b $s(0) = -1$ cm, $v(0) = 12$ cm s^{-1}, $a(0) = 0$ cm s^{-2}
The particle started 1 cm to the left of the origin and was travelling to the right at a constant speed of 12 cm s^{-1}.

c $t = \sqrt{2}$ s, $s(\sqrt{2}) = 8\sqrt{2} - 1$ cm

d **i** $t \geqslant \sqrt{2}$ s **ii** never

e



4 **a** $v(t) = -\dfrac{1}{2\sqrt{t+1}}$ m s^{-1} $a(t) = \dfrac{1}{4(t+1)^{\frac{3}{2}}}$ m s^{-2}



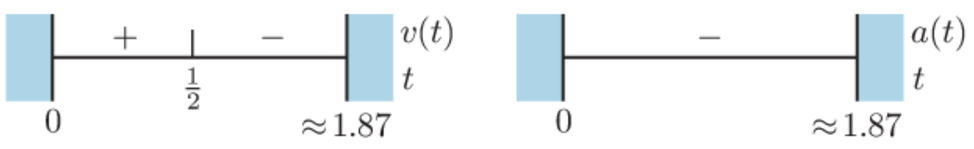
b $s(0) = 3$ m, $v(0) = -\frac{1}{2}$ m s^{-1}, $a(0) = \frac{1}{4}$ m s^{-2}
Initially, the particle is 3 m to the right of O, moving to the left at $\frac{1}{2}$ m s^{-1} with acceleration $\frac{1}{4}$ m s^{-2}.

c After 3 seconds, the particle is 2 m to the right of O, moving to the left at $\frac{1}{4}$ m s^{-1}, with acceleration $\frac{1}{32}$ m s^{-2}.

d The particle's speed is continuously decreasing.

5 **a** $k \approx 1.87$

b $v(t) = -9.8t + 4.9$ m s^{-1} $a(t) = -9.8$ m s^{-2}



c **i** decreasing **ii** increasing

6 **a** $x(0) = -1$ cm, $v(0) = 0$ cm s^{-1}, $a(0) = 2$ cm s^{-2}

b At $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1)$ cm to the left of O, moving to the right at $\sqrt{2}$ cm s^{-1}, with acceleration $\sqrt{2}$ cm s^{-2}.

c changes direction when $t = \pi$ s, $x(\pi) = 3$ cm

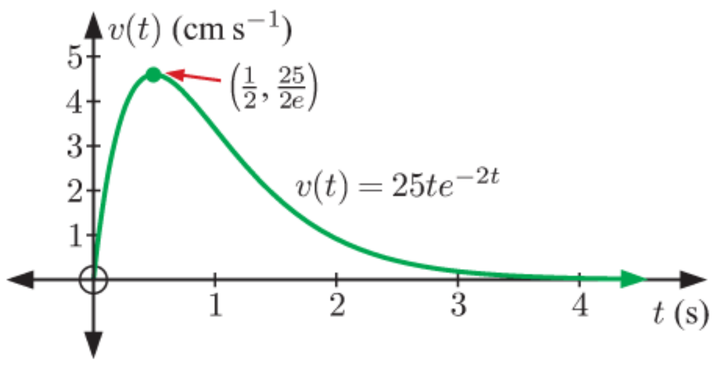
d increasing for $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

7 **a** **i** right **ii** left **b** $v(t) = 4\cos\frac{t}{2}$ m s^{-1}

c **i** left **ii** right **d** $a(t) = -2\sin\frac{t}{2}$ m s^{-2}

e decreasing

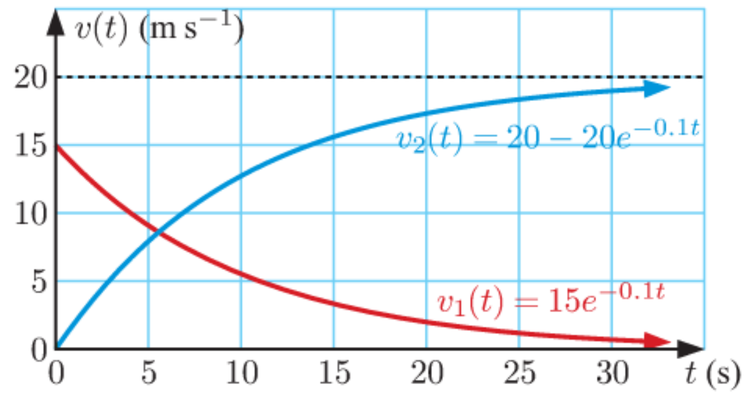
8 **a**



c $0 \leqslant t \leqslant \frac{1}{2}$ **d** $t \geqslant \frac{1}{2}$

9 **a** lion: ≈ 13.6 m s^{-1}, zebra: ≈ 1.90 m s^{-1}

b



The lion's speed $v_1(t)$ decreases over time whereas the zebra's speed $v_2(t)$ increases over time.

c $\displaystyle\int_0^3 v_1(t)\,dt = 150 - 150e^{-0.3} \approx 38.9$

The lion has travelled about 38.9 m in the first 3 seconds.

d $\displaystyle\int_0^3 [v_1(t) - v_2(t)]\,dt = 290 - 350e^{-0.3} \approx 30.7$

In the first 3 seconds, the lion has gained about 30.7 m on the zebra.

e At the time when $v_1(t) = v_2(t)$, the lion and the zebra will be moving at the same speed. Since the lion's speed decreases over time and the zebra's speed increases over time, the zebra will be faster than the lion after that time. So, they will be closest at the point when their speeds are equal.

f $-10\ln\left(\frac{4}{7}\right) \approx 5.60$ s

g No, the lion was about 1.92 m from the zebra at their closest point.

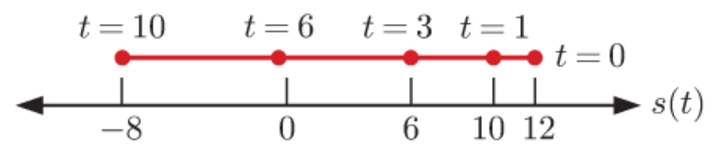
REVIEW SET 23A

1 **a** 12 m to the right of the origin

b **i** 10 m to the right of the origin **c** $t = 6$ s
ii 6 m to the right of the origin

d No, the displacement function is linear, so it has no turning points.

e



2 **a** 13 cm s^{-1} **b** **i** 9 cm s^{-1} **ii** 17 cm s^{-1}

c $a(t) = 4$ cm s^{-2}

3 **a** $v(t) = 6t^2 - 18t + 12$ cm s^{-1}, $a(t) = 12t - 18$ cm s^{-2}

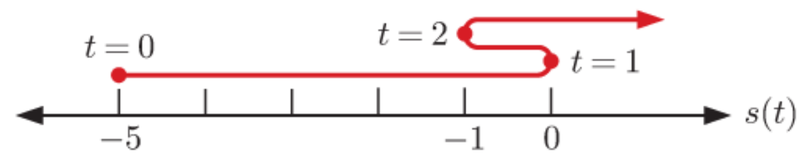


b $s(0) = 5$ cm to left of origin
$v(0) = 12$ cm s^{-1} towards origin
$a(0) = -18$ cm s^{-2} (decreasing speed)

c At $t = 2$, the particle is 1 cm to the left of the origin, is instantaneously stationary, and is beginning to accelerate.

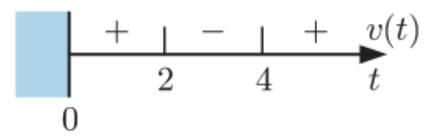
d at $t = 1$ s, $s = 0$ cm, and at $t = 2$ s, $s = -1$ cm

e



f Speed is increasing for $1 \leqslant t \leqslant 1\frac{1}{2}$ and $t \geqslant 2$.

4 **a**



b The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point again.

c $6\frac{2}{3}$ m **d** $9\frac{1}{3}$ m

5 **a** **i** 2.75 m s^{-1} **ii** $\approx 1.62 \text{ m s}^{-1}$

b **Hint:** Show that $a(t)$ and $v(t)$ are opposite in sign for all $0 \leqslant t \leqslant 6$.

c $\int_0^2 v(t)\,dt \approx 4.54$ The kayak travels approximately 4.54 m in the first 2 seconds after the kayaker stops paddling.

6 **a** $v(t) = 15 + \dfrac{120}{(t+1)^3} \text{ cm s}^{-1}$, $a(t) = -\dfrac{360}{(t+1)^4} \text{ cm s}^{-2}$

b At $t = 3$, the particle is 41.25 cm to the right of O, moving to the right at $\approx 16.9 \text{ cm s}^{-1}$, with decreasing speed ($a(3) \approx -1.41 \text{ cm s}^{-2}$).

c The particle's speed is never increasing.

7 **a** $x(0) = 3$ m, $x'(0) = 2\pi \text{ m s}^{-1}$, $x''(0) = 0 \text{ m s}^{-2}$

b $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}$ s **c** 20 m

8 **a** 1 m **b** 6 m **c** 4 m

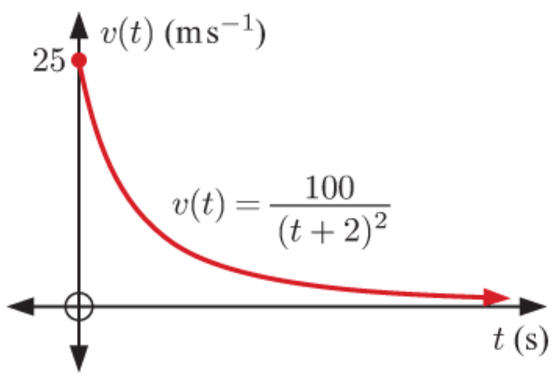
9 **a** $v(t) = -2t + 65 \text{ m s}^{-1}$ **b** $s(t) = -t^2 + 65t$ m

c **i** 31 s **ii** 1054 m

10 **a** $v(0) = 25 \text{ m s}^{-1}$, $v(3) = 4 \text{ m s}^{-1}$

b as $t \to \infty$, $v(t) \to 0$ from above

c



d $\int_0^2 v(t)\,dt = 25$ The boat travels a total distance of 25 m in the first 2 seconds after its engine is turned off.

e 3 seconds

REVIEW SET 23B

1 **a** 1 m to the right of the origin **b** 8 m s^{-1}

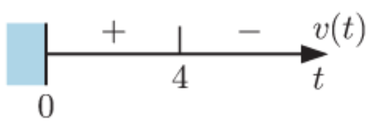
c $v(t) = 2t + 4 \text{ m s}^{-1}$ **d** 6 m s^{-1}

2 200 m

3 **a** $v(t) = 3t^2 - 30t + 27 \text{ cm s}^{-1}$

b -162 cm (162 cm to the left of the origin)

4 **a** $v(t) = 3 - \frac{3}{2}\sqrt{t} \text{ cm s}^{-1}$ $a(t) = -\dfrac{3}{4\sqrt{t}} \text{ cm s}^{-2}$





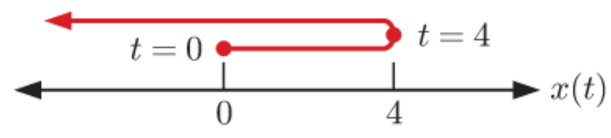
b $x(0) = 0$, $v(0) = 3$

The particle is initially at the origin, moving to the right at 3 cm s^{-1}.

c The particle is ≈ 3.17 cm to the right of the origin, travelling to the right at $\approx 0.879 \text{ cm s}^{-1}$, with decreasing speed ($a(2) \approx -0.530 \text{ cm s}^{-2}$).

d at $t = 4$ s, 4 cm to the right of O **e** $0 \leqslant t \leqslant 4$

f



g ≈ 4.70 cm

5 **a** **i** 7.2 m s^{-2} **ii** 9.6 m s^{-2} **iii** 0 m s^{-2} **iv** -12 m s^{-2}

b $\int_0^3 v(t)\,dt = 27$ The human cannonball travels 27 m in the first 3 seconds.

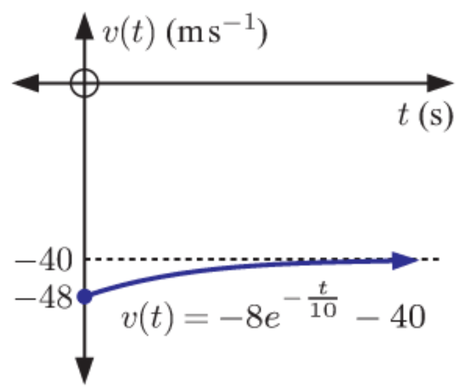
c ≈ 3.14 s

6 **a** $v(t) = -8e^{-\frac{t}{10}} - 40 \text{ m s}^{-1}$

$a(t) = \frac{4}{5}e^{-\frac{t}{10}} \text{ m s}^{-2}$ $\{t \geqslant 0\}$

b $s(0) = 80$ m, $v(0) = -48 \text{ m s}^{-1}$, $a(0) = 0.8 \text{ m s}^{-2}$

c



d $t = 10 \ln 2$ seconds

7 **a** $v(0) = 0 \text{ cm s}^{-1}$, $v(\frac{1}{2}) = -\pi \text{ cm s}^{-1}$, $v(1) = 0 \text{ cm s}^{-1}$, $v(1\frac{1}{2}) = \pi \text{ cm s}^{-1}$, $v(2) = 0 \text{ cm s}^{-1}$

b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, and so on

So, for $2n \leqslant t \leqslant 2n + 1$, $n \in \{0, 1, 2, 3, ...\}$

8 **a** $\approx 6.76 \text{ m s}^{-1}$

b $a(t) = 0.15(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3) \text{ m s}^{-2}$

c $\approx 1.79 \text{ m s}^{-2}$ **d** ≈ 109 m

9 **a** $s(t) = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2$ m **b** ≈ 3.46 s

10 **a** Tyson

b $\int_0^5 v_1(t)\,dt \approx 42.0$ Tyson has travelled about 42.0 m in the first 5 seconds of the race.

c $s_1(t) = 10t + 8e^{-1.25t} - 8$ m **d** Tyson

$s_2(t) = 10.5t + 10.5e^{-t} - 10.5$ m

e **Hint:** Find t such that $s_1(t) = 100$. **f** Maurice

EXERCISE 24A

1 **a** $f^{(n)}(x) = e^x$

$$\therefore \quad e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + ...$$

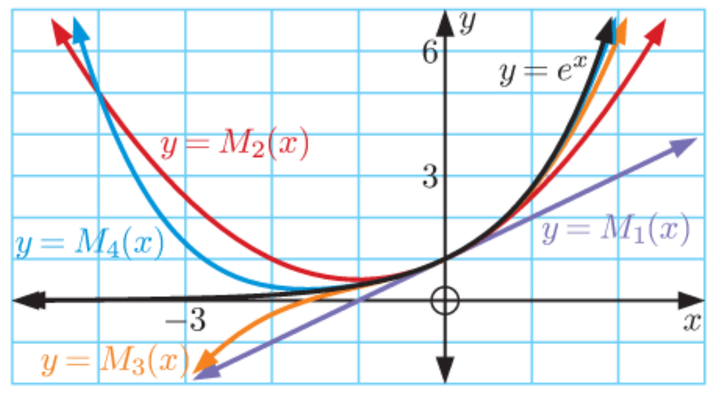
b $M_3(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

$M_4(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$M_5(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!}$

$M_6(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!}$

c



The accuracy of the approximation is improved by the addition of extra terms.

d

n	1	2	3	4	5	6
$M_n(1)$	2	2.5	2.667	2.708	2.717	2.718

As n increases, $M_n(1)$ gets closer to $f(1) = e \approx 2.718\,282$.

3 $\ln x$ and all of its derivatives are undefined for $x = 0$.

4 **a** $\dfrac{1}{1+x} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k$ **b** $\dfrac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$

5 **b** $\cos x$ is even, so its Maclaurin series representation will only contain even powers of x.

c $M_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$M_4(\frac{\pi}{3}) \approx 0.502, \quad \cos\frac{\pi}{3} = 0.5$

6 **b** $\sin x$ is odd, so its Maclaurin series representation will only contain odd powers of x.

c $M_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

$M_5(\frac{\pi}{6}) \approx 0.500, \quad \sin\frac{\pi}{6} = 0.5$

EXERCISE 24B

1 **a** For any x, we can find a value of n for which $n!$ grows faster than x^n, and $\dfrac{x^n}{n!}$ will reduce in size.

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ converges for all $x \in \mathbb{R}$.

b For any x, we can find a value of n for which $n!$ grows faster than x^n, and $\dfrac{x^n}{n!}$ will reduce in size.

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$ converges for all $x \in \mathbb{R}$.

c For any $|x| > 1$, the terms *increase* in size.
For any $|x| \leqslant 1$, the terms *decrease* in size.
$\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ converges for $|x| \leqslant 1$.

d For any $|x| > 1$, the terms *increase* in size.
For any $|x| < 1$, the terms *decrease* in size.
The series does not converge for $x = \pm 1$ where $\ln\left(\dfrac{1+x}{1-x}\right)$ is undefined.

$\ln\left(\dfrac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \ldots$ converges for $|x| < 1$.

2 **a** yes

b In the coefficients of x^k, the product of the k terms in the numerator counteracts the effect of the $k!$ in the denominator, so we require $|x| < 1$ for the series to converge.

3 **b** The interval of convergence needs to include 0, and *not* include $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ where $\tan x$ is undefined.

4 **b** **i**

n	1	2	3	4	5	6
$M_n(1)$	3	7	15	31	63	127

ii no

c **i**

n	1	2	3	4	5	6
$M_n(0.1)$	1.2	1.24	1.248	1.2496	1.249 92	1.249 98

ii yes

d $|x| < \frac{1}{2}$

5 **a** $|x| \leqslant 1$

b

n	1	2	3	4	5
M_n	1.5	1.375	1.438	1.398	1.426

The series appears to be convergent for $x = 1$, as the values approach $\sqrt{2} \approx 1.414$.

c $(1.8)^{\frac{1}{2}} + \dfrac{1}{2(1.8)^{\frac{1}{2}}}x - \dfrac{1}{8(1.8)^{\frac{3}{2}}}x^2 + \dfrac{1}{16(1.8)^{\frac{5}{2}}}x^3 - \dfrac{5}{128(1.8)^{\frac{7}{2}}}x^4 + \dfrac{7}{256(1.8)^{\frac{9}{2}}}x^5$

n	1	2	3	4	5
M_n	1.416 18	1.414 11	1.414 22	1.414 21	1.414 21

The Maclaurin series for $\sqrt{x+1.8}$ approximated $\sqrt{2}$ much more quickly and accurately than the Maclaurin series for $\sqrt{x+1}$ (0.2 is closer to 0 than 1 is to 0).

EXERCISE 24C

1 **a** $\dfrac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + \ldots.$

b $\dfrac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \ldots.$

c $\dfrac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots.$

d $\dfrac{2}{1+x} = 2\sum_{k=0}^{\infty} (-1)^k x^k$
$= 2 - 2x + 2x^2 - 2x^3 + 2x^4 - \ldots.$

2 **a** $\cos 2x = \sum_{k=0}^{\infty} \dfrac{(-4)^k x^{2k}}{(2k)!} = 1 - 2x^2 + \dfrac{2x^4}{3} - \dfrac{4x^6}{45} + \ldots.$

b $\cos(-x) = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k}}{(2k)!}$
$= 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \ldots.$

c $\cos(x^2) = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{4k}}{(2k)!} = 1 - \dfrac{x^4}{2} + \ldots.$

d $x\cos x = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k)!} = x - \dfrac{x^3}{2} + \dfrac{x^5}{24} - \ldots.$

3 **a** $\sin(-x) = \sum_{k=0}^{\infty} \dfrac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}$

b $\sin 3x = \sum_{k=0}^{\infty} \dfrac{(-1)^k (3x)^{2k+1}}{(2k+1)!}$

c $\sin(x^2) = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{4k+2}}{(2k+1)!}$

d $x\sin x = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+2}}{(2k+1)!}$

4 **a** $\sum_{k=0}^{\infty} \dfrac{3^k x^k}{k!} = 1 + 3x + \dfrac{9x^2}{2} + \dfrac{9x^3}{2} + \dfrac{27x^4}{8} + \ldots.$

b $e^{x^2} = \sum_{k=0}^{\infty} \dfrac{x^{2k}}{k!} = 1 + x^2 + \dfrac{x^4}{2} + \ldots.$

c 2^x
$= \sum_{k=0}^{\infty} \dfrac{x^k (\ln 2)^k}{k!}$
$= 1 + x\ln 2 + \dfrac{x^2(\ln 2)^2}{2} + \dfrac{x^3(\ln 2)^3}{6} + \dfrac{x^4(\ln 2)^4}{24} + \ldots.$

5 **a** $\sin(x+\frac{\pi}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+\frac{\pi}{2})^{2k+1}}{(2k+1)!}$

b $\sin(x+\frac{\pi}{2}) = \left(\frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840} - \frac{\pi^7}{645\,120} + \ldots.\right)$
$+ x\left(1 - \frac{\pi^2}{8} + \frac{\pi^4}{384} - \frac{\pi^6}{46\,080} + \ldots.\right)$
$+ x^2\left(-\frac{\pi}{4} + \frac{\pi^3}{96} - \frac{\pi^5}{7680} + \ldots.\right)$
$+ x^3\left(-\frac{1}{6} + \frac{\pi^2}{48} - \frac{\pi^4}{2304} + \ldots.\right)$
$+ x^4\left(\frac{\pi}{48} - \frac{\pi^3}{1152} + \frac{\pi^5}{92\,160} - \ldots.\right)$

Since $\sin(x+\frac{\pi}{2}) = \cos x$, the coefficients of the odd powers must be zero.

So, $1 - \frac{\pi^2}{8} + \frac{\pi^4}{384} - \frac{\pi^6}{46\,080} + \ldots. = 0$ and
$-\frac{1}{6} + \frac{\pi^2}{48} - \frac{\pi^4}{2304} + \ldots. = -\frac{1}{6}\left(1 - \frac{\pi^2}{8} + \frac{\pi^4}{384} - \ldots.\right) = 0$

Also, $\frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840} - \frac{\pi^7}{645\,120} + \ldots = 1$,

$-\frac{\pi}{4} + \frac{\pi^3}{96} - \frac{\pi^5}{7680} + \ldots.$
$= -\frac{1}{2!}\left(\frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840} - \ldots.\right) = -\frac{1}{2!}$,

and $\frac{\pi}{48} - \frac{\pi^3}{1152} + \frac{\pi^5}{92\,160} - \ldots.$
$= \frac{1}{4!}\left(\frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840} - \ldots.\right) = \frac{1}{4!}$

6 **a** **i** $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots.$

ii $e^{-3x} = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^k}{k!}$
$= 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \ldots.$

iii $e^{-(2n+1)x}$
$= \sum_{k=0}^{\infty} \frac{(-1)^k ((2n+1)x)^k}{k!}$
$= 1 - (2n+1)x + \frac{(2n+1)^2 x^2}{2!} - \frac{(2n+1)^3 x^3}{3!} + \ldots.$

b $\sin(e^{-x}) \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x$

c $\sin(e^{-x}) \approx \sin(1) - \cos(1)x$

d $M_2(x) = \sin(1) - \cos(1)x + (\cos(1) - \sin(1))\frac{x^2}{2!}$

EXERCISE 24D

1 **a** $\sin x + \cos x$
$= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots.$

b $\sin x + \sin 2x = 3x - \frac{3x^3}{2} + \frac{11x^5}{40} - \frac{43x^7}{1680} + \ldots.$

c $\sin(x^2) - \cos 3x = -1 + \frac{11x^2}{2} - \frac{27x^4}{8} + \frac{203x^6}{240} + \ldots.$

2 **a** $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots.$

c $\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots.$

EXERCISE 24E

2 $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

3 **a** $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots.$

b $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots.$

c **Hint:** $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

4 **a** $\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{6} + \ldots.$

b $\int_0^1 \sin(x^2)\,dx \approx 0.3103$

5 **a** $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!}$

b **i** ≈ 0.6827 **ii** ≈ 0.9545 **iii** ≈ 0.9973

6 **a** $\frac{1}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k x^{3k} = 1 - x^3 + x^6 - \ldots.$

We expect the series to converge for $|x| < 1$.

b $\int_0^{\frac{1}{3}} \frac{1}{1+x^3}\,dx \approx 0.3303$

7 **a** $(1+x^2)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - \ldots.$

b $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$
which is valid for $|x| < 1$.

c $\arctan(x^2) \approx x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7}$
$\int_0^1 \arctan(x^2)\,dx \approx 0.2944$

8 **a** $\arcsin x = x + \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} x^{2k+1}$
$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots.$
which is valid for $|x| < 1$.

EXERCISE 24F

1 $e^x \sin x = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \ldots.$

2 **a** $2\sin x \cos x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \ldots.$

b $\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \ldots.$

3 **a** $e^x \arccos x = \frac{\pi}{2} + (\frac{\pi}{2} - 1)x + (\frac{\pi}{4} - 1)x^2 + (\frac{\pi}{12} - \frac{2}{3})x^3 + \ldots.$

b $|x| < 1$, $\arccos x$ only converges for $|x| < 1$.

4 **a** $e^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!} = 1 - x^3 + \frac{x^6}{2} - \ldots.$

b $e^{-x^3}\cos x = 1 - \frac{1}{2}x^2 - x^3 + \frac{1}{24}x^4 + \frac{1}{2}x^5 + \frac{359}{720}x^6$
$- \frac{1}{24}x^7 - \frac{10\,079}{40\,320}x^8 - \frac{119}{720}x^9 + \ldots.$

EXERCISE 24G

1 $\frac{x}{\cos x} = x + \frac{1}{2}x^3 + \frac{5}{24}x^5 + \ldots.$

2 $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots.$

3 **a** **i** $e^{2x} - 1 = \sum_{k=1}^{\infty} \frac{2^k x^k}{k!} = 2x + 2x^2 + \frac{4}{3}x^3 + \ldots.$

ii $\sin 4x = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1} x^{2k+1}}{(2k+1)!}$
$= 4x - \frac{32}{3}x^3 + \frac{1}{5}x^5 - \ldots.$

b $\frac{1}{2}$

4 **a** $\dfrac{\sin x}{\ln(1+x)} = \dfrac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots.}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots.}$ **b** 1

6 **a** $e^{\cos x} = e - \frac{e}{2}x^2 + \frac{e}{6}x^4 + \ldots.$

b $\sin(e^{2x} - 1) = 2x + 2x^2 - \frac{10}{3}x^4 + \ldots.$ **c** 0

7 **a** $\sin x$ has zeros $n\pi$, $n \in \mathbb{Z}$.

$\dfrac{\sin x}{x}$ has zeros $n\pi$, $n \in \mathbb{Z}$, $n \neq 0$.

b $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \ldots.$

c $x = n\pi$, $n \in \mathbb{Z}$, $n \neq 0$

d This evidence supports Euler's claim. **f** $\frac{\pi^2}{8}$

REVIEW SET 24A

1 **a** $\dfrac{1}{3-x} = \sum_{k=0}^{\infty} \dfrac{x^k}{3^{k+1}}$

b $M_3(x) = \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3$

$M_3(1) = \frac{40}{81}$, $f(1) = \frac{1}{2}$

2 $\approx 0.999\,39$

3 For any $|x| \leqslant 1$, the terms *decrease* in size.
The series converges for $|x| \leqslant 1$.

4 **a** $-2 \leqslant x < 2$

5 **a** $\ln(1+3x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1} 3^k x^k}{k}$

b

n	1	2	3	4	5
$M_n(1)$	3	−1.5	7.5	−12.75	35.85
$M_n(-\frac{1}{4})$	−0.75	−1.0313	−1.1719	−1.2510	−1.2984

i not convergent **ii** convergent

c $-\frac{1}{3} < x \leqslant \frac{1}{3}$

6 ≈ 1.463 **7** $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots.$

8 **a** $\dfrac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots.$

b $\dfrac{x^2}{(1+x^3)^2} = x^2 - 2x^5 + 3x^8 - \ldots.$

9 **a** **i** $\tan 2x = 2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \ldots.$ **b** 2

ii $e^{\tan x} = 1 + x + \dfrac{x^2}{2} + \ldots.$

REVIEW SET 24B

1 $xe^x = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \ldots.$

2 **a**

n	2	4	6
$M_n(0.1)$	−0.2506	−0.2506	−0.2506
$M_n(1)$	−0.3125	−0.3281	−0.3320
$M_n(2)$	−0.5	−0.75	−1

b $|x| < 2$

3 **a** $\arctan\frac{x}{2} = \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{160}x^5 - \frac{1}{896}x^7 + \ldots.$

b $\arctan(x^2) = x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \ldots.$

c $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots.$

4 **a** $(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \ldots.$

b $(1-x)e^{1-x} = e + \sum_{k=1}^{\infty} (-1)^k \left(\dfrac{1}{(k-1)!} + \dfrac{1}{k!}\right) ex^k$
$= e - 2ex + \frac{3}{2}ex^2 - \frac{2}{3}ex^3 + \ldots.$

c $\dfrac{1}{(1+e^{2x})^2} = \frac{1}{4} - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{6}x^3 - \ldots.$

5 ≈ 0.708 **6** **a** $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots.$

7 **a** $a_0 = 1$, $a_1 = -1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{5}{6}$

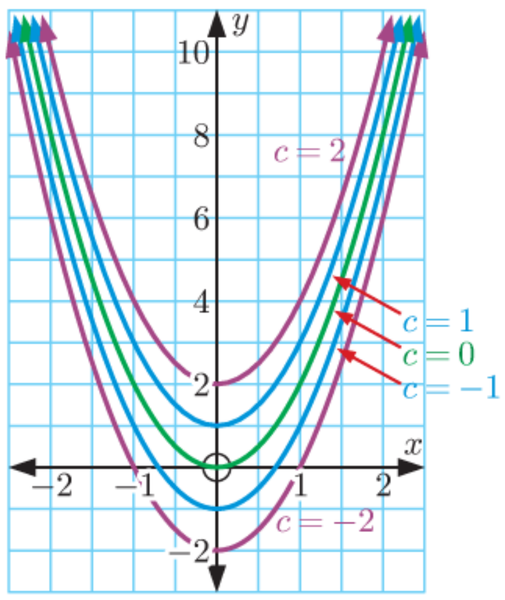
8 **a** **i** $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6$

ii $\sin^2 x = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \ldots.$

EXERCISE 25A

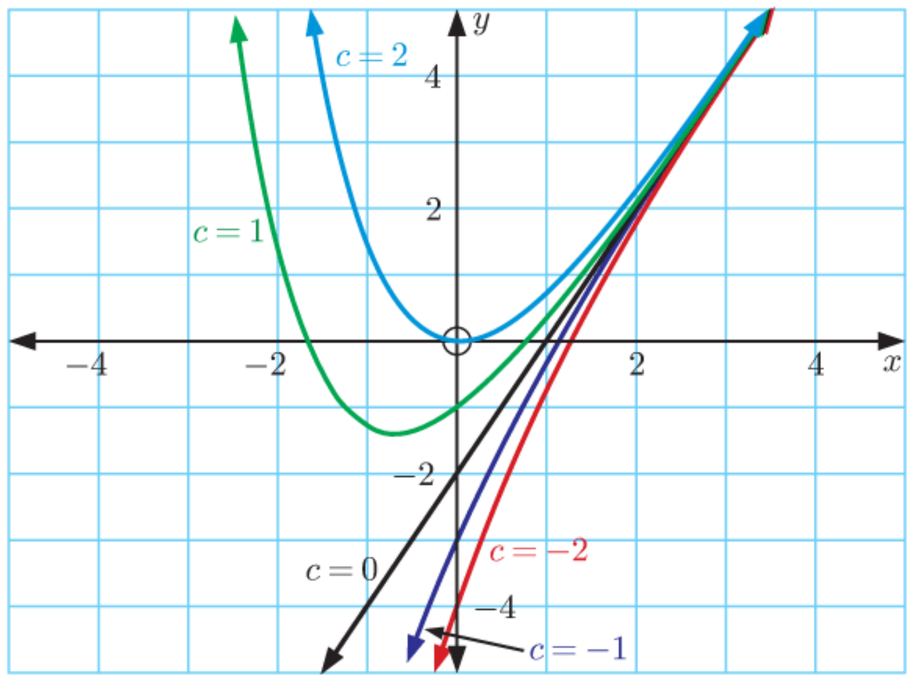
2 **a** B **b** C **c** A

4 **b**



c $y = 2x^2 - \frac{3}{2}$ **d** $y = 4x - \frac{7}{2}$

5 **b**



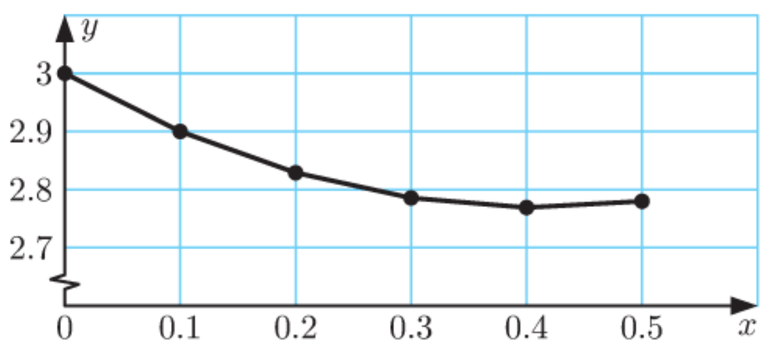
c $y = 2x - 2 + 3e^{-x}$ **d** $y = -x + 1$

EXERCISE 25B

1 **a**

Iteration	x_{n-1}	y_{n-1}	$\frac{dy}{dx}$	x_n	y_n
1	0	3	−1	0.1	2.9
2	0.1	2.9	−0.71	0.2	2.829
3	0.2	2.829	−0.4342	0.3	2.7856
4	0.3	2.7856	−0.1643	0.4	2.7691
5	0.4	2.7691	0.1077	0.5	2.7799

b



2 **a** **i** ≈ 3.4865 **ii** ≈ 3.8283

b $y(0.5) = \frac{3}{2}e \approx 4.0774$

The accuracy of Euler's method was improved by decreasing the step size.

3 $y(1) \approx 0.7869$

4 **a** $y(0.5) \approx -0.4851$ **b** $y(0.5) \approx -0.4795$

c $y(0.5) = -\sin(0.5) \approx -0.4794$

The accuracy of Euler's method was improved by decreasing the step size.

5 **a** ≈ 1 **b** ≈ 1.3021 **c** ≈ 1.3018

EXERCISE 25C

1 **a** $y = x^4 + c$ **b** $y = \frac{1}{3}x^3 + 3x^2 + c$

c $y = \frac{1}{3}e^{3x} + 4x + c$ **d** $y = \sin x - \frac{1}{2}\cos 2x + c$

e $x = \frac{1}{2}t + \frac{1}{4}\sin 2t + c$

f $y = \ln|x+4| - \frac{2}{3}\ln|3x-5| + c$

g $M = \ln\left|t^3 - 4\right| + c$ **h** $y = -\sqrt{25 - x^2} + c$

i $f(t) = -\frac{1}{2}e^{-t^2+1} + 2t + c$ **j** $S = \frac{2}{3}(\ln t)^{\frac{3}{2}} + c$

k $y = x - \frac{1}{3}\sin 3x + c$ **l** $y = \frac{2}{3}x^{\frac{3}{2}} - 2\ln|x| + c$

2 **a** $y = \frac{3}{2}x^2 - 2x + 5$ **b** $y = \frac{1}{3}e^{3x} + x - \frac{1}{3}$

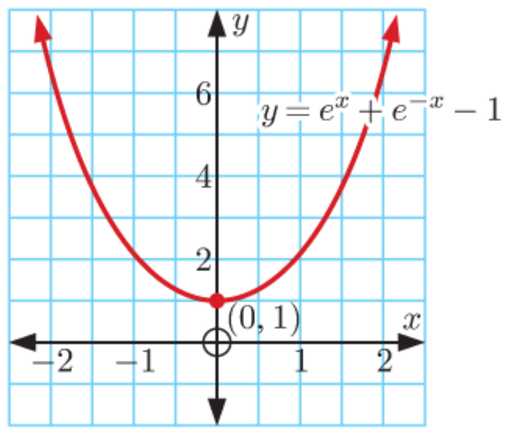
c $y = \ln|x| + \ln 6$

3 **a** $y = e^{2t} + e^{-t} - 2$ **b** $M = \frac{1}{2}\sin 2\alpha + 3\cos\alpha + 2$

c $P = 2x\sin x + 2\cos x - \frac{\pi}{6}$

4 $f(-2) = 2$

5 **a** $y = e^x + e^{-x} - 1$ **b**



c $3x - 2y = 3\ln 2 - 3$

6 $f(x) = \frac{1}{2}x^2 + \frac{1}{x} - \frac{3}{2}$ **7** $y = \ln(x^2+2) + \ln 3$

8 **a** $f(x) = x^3 - 2x^2 + 4x - 1$ **b** $y = -\frac{1}{4}\sin 2x + 4x$

9 £4250

10 **a** $P(x) = 15x - 0.015x^2 - 650$ pounds

b maximum profit is £3100, when 500 plates are made

c $46 \leqslant x \leqslant 954$ (you cannot produce part of a plate)

11 $\approx 225°\text{C}$

12 **a** $\approx 400°\text{C}$ **b** ≈ 0.387 m **c** ≈ 0.187 m

13 **a** $y = \left(\frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 - \frac{0.08}{3}x\right)$ metres

b ≈ 3.33 cm; yes, it seems reasonable that the maximum sag occurs when $x = 2$.

c 2.375 cm **d** $\approx 1.05°$

EXERCISE 25D

1 **a** $y = \sqrt[3]{\frac{3}{2}x^2 + c}$ **b** $y = \ln(x^2 + c)$ **c** $y = Ae^{\frac{3}{2}x^2}$

d $y = \left(\frac{x^2}{2} + c\right)^2$ **e** $y = Ae^{-\cos x}$

f $y = \left(-\frac{1}{4}x^2 + c\right)^2 - 1$ **g** $y = Ax$

h $y = -\ln(c - x^3)$ **i** $y = A(x-1) - 2$

2 **a** $y = Ae^x$ **b** $y = \pm\sqrt{2x + c}$ **c** $y = Ae^t + 4$

d $P = \left(\frac{3}{2}t + c\right)^2$ **e** $Q = Ae^t - \frac{3}{2}$ **f** $t = Q^2 + 3Q + c$

3 **a** $y = A\sqrt[3]{3x+1}$ **b** $y = Ae^{2x} + 2$

c $y = \sqrt[3]{3\ln(x^2+5) + c}$ **d** $y = Ae^{2\sqrt{4-x}} + 1$

e $y = \dfrac{1}{-\frac{1}{2}x^2 + 2x + c}$ **f** $y = \left(\frac{9}{2}\ln(x^2+5) + c\right)^{\frac{2}{3}}$

4 $y = \dfrac{1}{c - x}$, defined for $x \neq c$

5 **a** $y = \sqrt[3]{\frac{9}{2}x^2 + 1}$ **b** $y = \frac{1}{36}(x-26)^2$

c $y = e^{x + \frac{1}{3}x^3}$ **d** $y = \arcsin\left(\frac{3}{2}x^2 - \frac{3}{2}\right)$

e $y = \left(\frac{9}{2}\sin 2x + 3\sqrt{3}\right)^{\frac{2}{3}}$

f $y = \ln\left[\sqrt[4]{2x^2 + 4x + 1}\left(e^2 + 3\right) - 3\right]$

6 $y = (x+3)^2$, defined for $x \geqslant -3$ **7** **b** $y = \dfrac{1-x}{(x+1)^2}$

8 **a** $\dfrac{5}{x-4} + \dfrac{3}{x+3}$

b $y = \frac{1}{4}(5\ln|x-4| + 3\ln|x+3| + c)^2$

9 $y = -\dfrac{1}{\ln\left|\frac{(x+2)^2(x-1)^3}{4}\right| + 2}$

$= \dfrac{-1}{2\ln\left|\frac{x+2}{2}\right| + 3\ln|x-1| + 2}$

10 $y = Ae^x\left(\dfrac{x-1}{x+1}\right)$

11 **a** $P = 40e^{\frac{t}{2}}$ **b** ≈ 803 rabbits

12 **a** $I = 350e^{-0.4t}$ **b** ≈ 47.4 milliamps

c ≈ 7.16 milliseconds

13 ≈ 5.19 minutes **14** 0.8%

15 **a** $(V_0 - V)$ is the amount of water remaining

so $\dfrac{dV}{dt} \propto (V_0 - V)$

$\therefore \dfrac{dV}{dt} = k(V_0 - V)$ for some constant k

b $\approx 17.7\%$

16 **a** ≈ 12.5 minutes **b** midnight

17 **a** $I(t) = Ae^{-\frac{100}{3}t} + 2$ **b** $I(t) = 2 - 2e^{-\frac{100}{3}t}$

c 2 amps **d** ≈ 0.138 seconds

18 4 hours

19 **a** $\dfrac{dh}{dt} = \dfrac{r^2}{\pi h^2 - 2\pi rh}$

b **ii** ≈ 14.4 hours **iii** ≈ 20.9 hours

20 **b** ≈ 35.6 minutes **21** $x^2 + y^2 = 5$

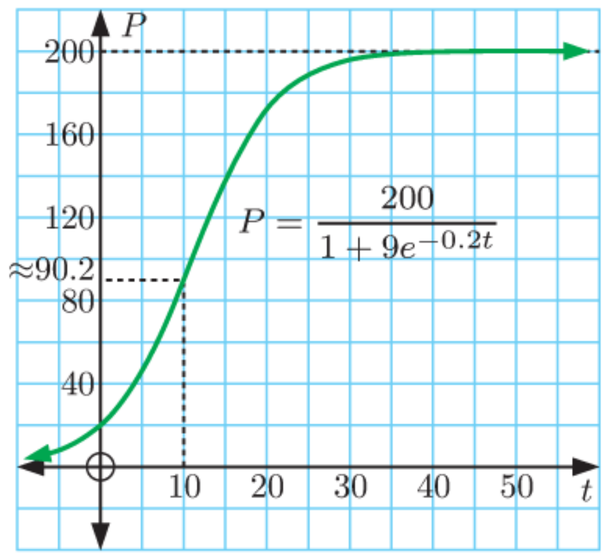
22 $y = \dfrac{1}{\sqrt{x}}$ or $y = \sqrt{x}$

EXERCISE 25E

1 **a** $P = \dfrac{200}{1 + 9e^{-0.2t}}$ **b** $P \approx 90.2$

c as $t \to \infty$, $P \to 200$

d



2 **a** The population P grows logistically over time t.

b The graph would increase exponentially at first, but then level off to approach the maximum value.

c $P = \dfrac{3000}{1 + 5e^{-0.1t}}$

d **i** ≈ 924 rodents **ii** ≈ 23 years **iii** 3000 rodents

3 **a** $A = \dfrac{500}{1 + 24e^{-0.1t}}$ **b** ≈ 118 cm^2 **c** 500 cm^2

4 **a** $N = \dfrac{600}{1 + 299e^{-0.8t}}$ **b** ≈ 21 people **c** 600 people

d $\approx$ 5:08 pm

5 **a** $N \approx \dfrac{10^{30}}{1 + (5 \times 10^{27} - 1)e^{-kt}}$ **b** $k \approx 1.12 \times 10^6$

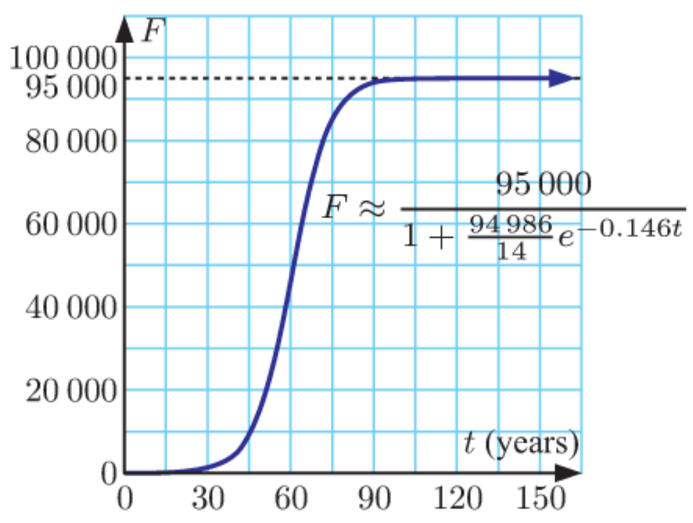
c after $\approx 6.09 \times 10^{-5}$ seconds

6 **a** The population of foxes increased quickly at first, but later levelled off to approach a maximum.

b **i** $A = 95\,000$ **ii** $F \approx \dfrac{95\,000}{1 + \frac{94\,986}{14}e^{-0.146t}}$

c $\approx 85\,100$ foxes **d** **i** ≈ 1894 **ii** ≈ 1911

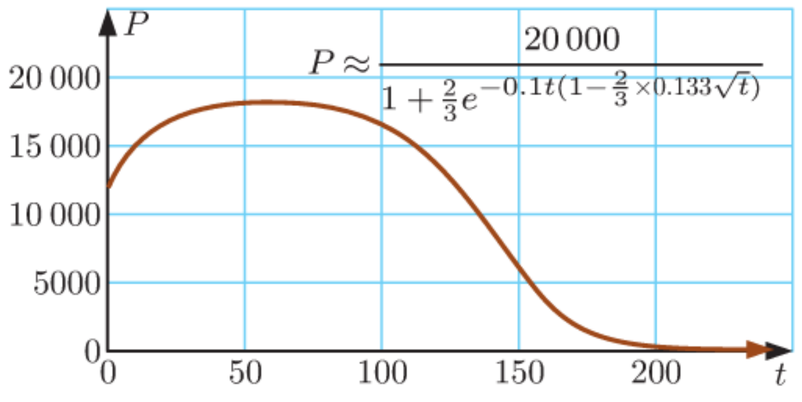
e



f In 1905, it appears as an inflection on the graph.

7 **a** $P = \dfrac{20\,000}{1 + be^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})}}$ **b** $b = \frac{2}{3}, \quad \alpha \approx 0.133$

c



d $t \approx 56.9$, maximum population $\approx 18\,200$

e $P(220) \approx 32$, so the model predicts that, if nothing had been done, 32 numbats would still exist today.

f The introduced species will drive the numbat population to extinction, that is, $P \to 0$ as $t \to \infty$.
Realistically, it is possible for the numbat population to become extinct in a finite time.

EXERCISE 25F

1 **a** $y = \dfrac{x}{2} + \dfrac{A}{x}$ **b** $y = \dfrac{x^2 + 3}{2x}$ **2** $y = \dfrac{x + 3\sqrt{x}}{2}$

3 **a** $y = -\dfrac{x}{\ln|Ax|}$ **b** $y = x \arctan(\ln|Ax|)$

c $y = \pm\sqrt{Ax - x^2}$

4 $x = \sqrt{2}e^{-\frac{\pi}{4}}$ **5** **b** $y = -x \ln\left(\dfrac{1}{x} + c\right)$

EXERCISE 25G

1 **a** $y = 3 + ce^{-4x}$ **b** $y = -\frac{1}{2}e^x + \frac{5}{2}e^{3x}$

2 $y = \dfrac{3 - \cos 2x}{2\cos x}$

3 **a** $y = x - 1 + \frac{1}{2}e^x + e^{1-x} - \frac{1}{2}e^{2-x}$

b $y = \sin x + \dfrac{\cos x}{x} + \dfrac{c}{x}$ **c** $y = 3 - x - \dfrac{3}{x} + \dfrac{c}{xe^x}$

EXERCISE 25H

1 **a** $y = -\frac{2}{3}x + \frac{1}{9}x^2 + \frac{1}{81}x^3 + \ldots.$

2 **a** $y = -3 + x + 2x^2 + \ldots.$ **b** $y = 2x^2 + x - 3$

3 **a** $y = 2 + x - x^2 + \ldots.$ **b** $y = \sin x + 2\cos x$

4 **b** $y = A(1 + x)^p$

REVIEW SET 25A

2 $a = 2, \ b = -1$ **3** $y \approx 1.0088$

4 **a** $y = \frac{3}{4}\sin 2x - \frac{1}{2}x + c$ **b** $y = 3x + \frac{1}{2}e^{-2x} + c$

c $y = \frac{1}{2}\ln|2x + 1| + 2$ **d** $y = \frac{1}{2}(t^2 + e^{t^2} + 3e - 1)$

5 **a** $P(x) = 20x - \frac{1}{8}x^2 - 250$ pounds

b £550 per day when 80 vases are made.

c between 14 and 146 vases per day inclusive

6 **a** **i** 160 spectators per minute

ii ≈ 249 spectators per minute

b $\dfrac{d}{dt}\left(\dfrac{1}{1 + 4e^{-0.05t}}\right) = \dfrac{0.2e^{-0.05t}}{(1 + 4e^{-0.05t})^2}$

$\therefore \quad \displaystyle\int S'(t)\,dt = \frac{20\,000}{1 + 4e^{-0.05t}} + c$

c $\displaystyle\int_0^{60} S'(t)\,dt \approx 12\,700$

About 12 700 spectators entered the stadium between noon and 1 pm.

d $\approx 19\,500$ spectators

7 **a** $y = Ae^{\frac{5}{3}x^3}$ **b** $y = \dfrac{-1}{x^2 - x + c}$

c $x = \left(-\frac{9}{16}\cos 2t + c\right)^{\frac{2}{3}}$

8 **a** $\dfrac{dM}{dt} = kM, \quad M(0) = M_0$ **b** ≈ 93.2 days

9 $y = \ln(e - 3x)$

10 **a** $t = -100\ln(10C) + \dfrac{0.1 - C}{2.66 \times 10^{-4}}$

b ≈ 509 seconds (≈ 8 minutes 29 seconds)

11 $y^2 = \dfrac{2x^2}{3x^4 - 40}$

12 **a** $N = \dfrac{694}{1 + 693e^{-t}}$ **b** ≈ 51 people **c** ≈ 7.49 weeks

13 $y^2 = 2x^2(\ln|x| + c)$

14 **a** $y = 2x\sqrt{x} - 4x$ **b** $y = -\frac{1}{2}\cot x \cos x$

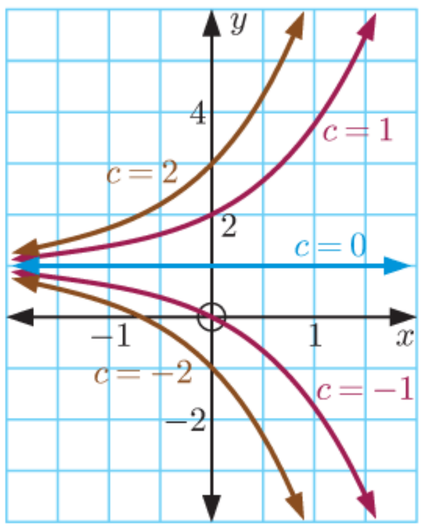
15 **a** $y = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + \ldots.$ **b** $y = \ln(x^2 + 1)$

16 **a** $x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (mt)^{2k}}{(2k)!}$ **b** $x(t) = \cos mt$

REVIEW SET 25B

2 **a** $\frac{dy}{dx} = ce^x = (ce^x + 1) - 1 = y - 1$ ✓

b



c $y = 3e^x + 1$

d $y = 3x + 4$

3 $y(0.5) \approx 0.8555$

4 **a** $y = \ln|e^x - 2| + c$ **b** $y = -\frac{1}{4}\sin\left(\frac{\pi}{3} - 2x\right) - \frac{\sqrt{3}}{8}$

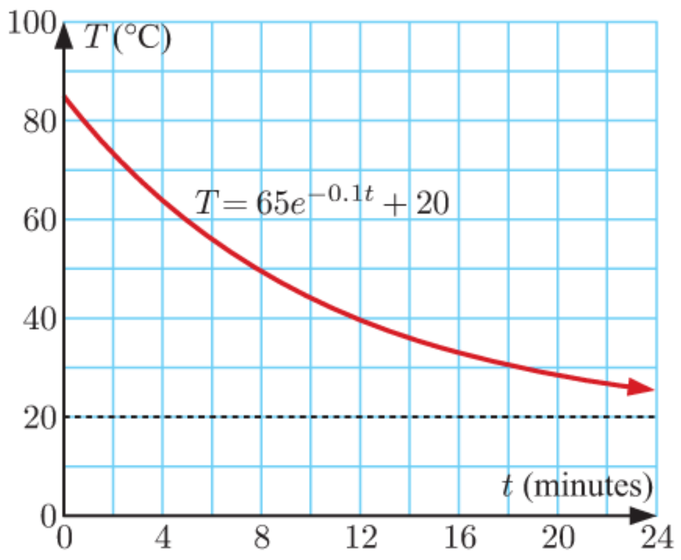
6 **a** $y = \frac{1}{\sqrt[3]{c - 6x}}$ **b** $P = A\sqrt{t^2 + 1}$

7 **a** $y = (\frac{1}{2}x \pm 2)^2$ **b** $y = e^{\sin x - 3}$ **8** $y = \frac{(x+2)^2}{25(x-2)}$

9 **a** $T = 65e^{-0.1t} + 20$ **b** $\approx 63.6°\text{C}$

c as $t \to \infty$, $T \to 20$

d



e **i** ≈ 3.68 minutes **ii** ≈ 8.11 minutes

10 $y = \frac{5}{\sqrt{x}}$

11 **a** $\frac{dV}{dt} = -k\sqrt{h}$ **b** $V = 2 \times 2 \times h$, $\frac{dh}{dt} = -\frac{k}{4}\sqrt{h}$

c 20 minutes

12 **a** $P(t) \approx \frac{200\,000}{1 + 7e^{-0.129t}}$ **b** $\approx 42\,800$ ostriches

c after ≈ 15.1 years

13 $y^2 = 3x^2 - 2x - 1$ **14** $y = x^5 - \frac{1}{x^3}$

15 **a** $y = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots.$ **b** $y = (x+1)^{x+1}e^{-x}$

16 **d** $y^2 = 2cx + c^2$

e The mirror is a parabola (x is a quadratic in y).

EXERCISE 26A

1 **a** weak, positive, linear correlation, with no outliers

b strong, negative, linear correlation, with one outlier

c no correlation

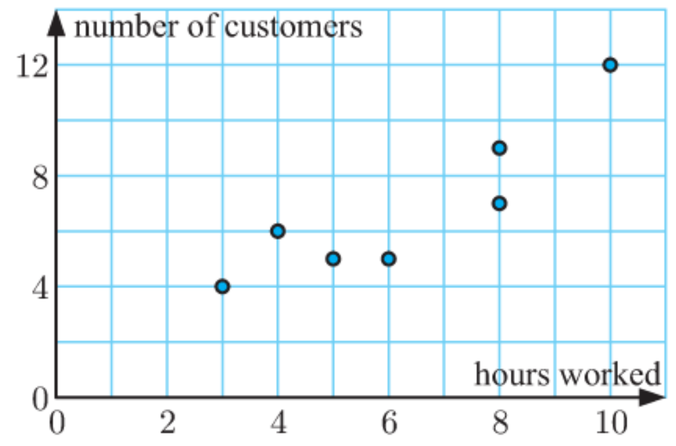
d strong, negative, non-linear correlation, with one outlier

e moderate, positive, linear correlation, with no outliers

f weak, positive, non-linear correlation, with no outliers

2 **a** *Hours worked* is the explanatory variable.
Number of customers is the response variable.

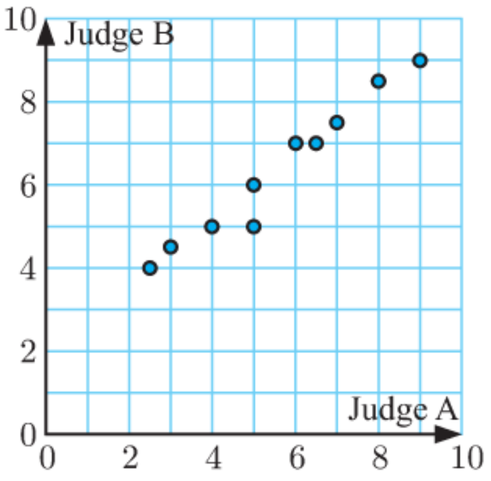
b



c **i** Monday and Friday **ii** Wednesday and Sunday

d The more hours that Tiffany works, the more customers she is likely to have.

3 **a**

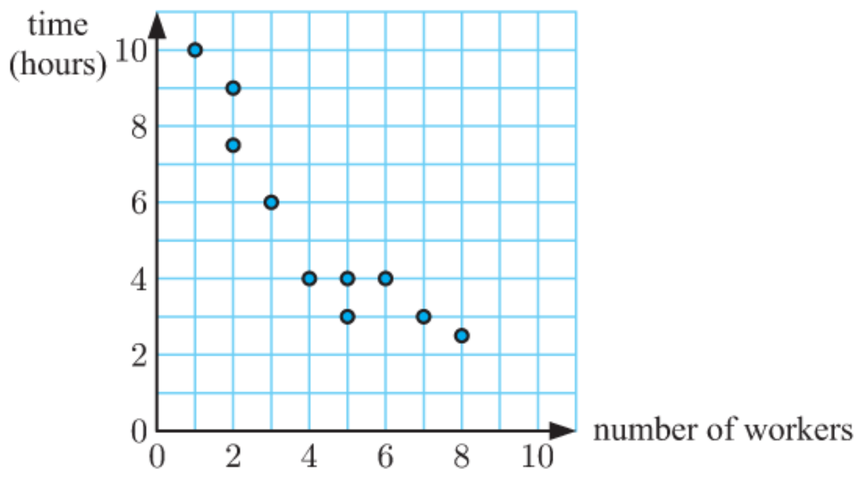


b There appears to be **strong, positive, linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.

c No, the scores are related to the quality of the ice skaters' performances.

4 **a** **i** job G **ii** job C

b



c There is a strong, negative, non-linear correlation between *number of workers* and *time*.

5 **a** **D** **b** **A** **c** **B** **d** **C**

6 **a** There is a moderate, positive, linear correlation between *hours of study* and *marks obtained*.

b The test is out of 50 marks, so the outlier (> 50) appears to be an error. It should be discarded.

c Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.

7 **a** Not causal, dependent on genetics and/or age.

b Not causal, dependent on the size of the fire.

c Causal, an increase in advertising is likely to cause an increase in sales.

d Causal, the childrens' adult height is determined by the genetics they receive from their parents to a great extent.

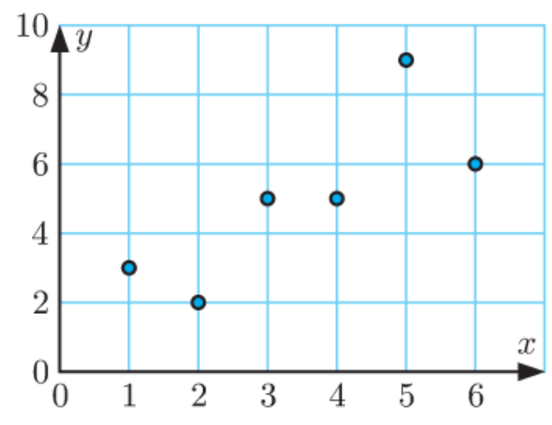
e Not causal, dependent on population of town.

EXERCISE 26B

1 weak, positive correlation

2 **a** B **b** A **c** D **d** C **e** E

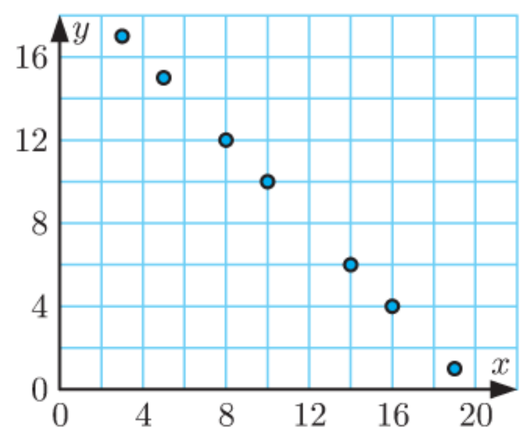
3 **a** **i**



ii $r \approx 0.786$

iii moderate, positive correlation

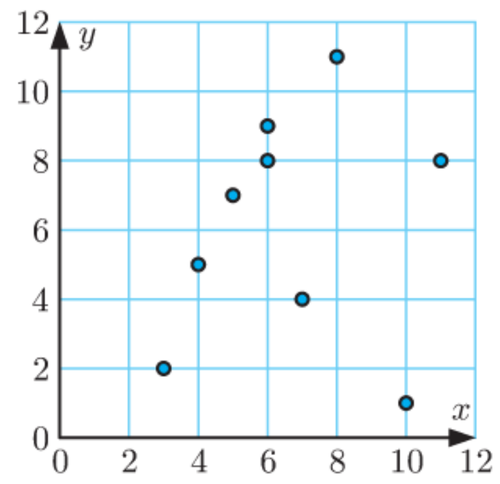
b **i**



ii $r = -1$

iii perfect, negative correlation

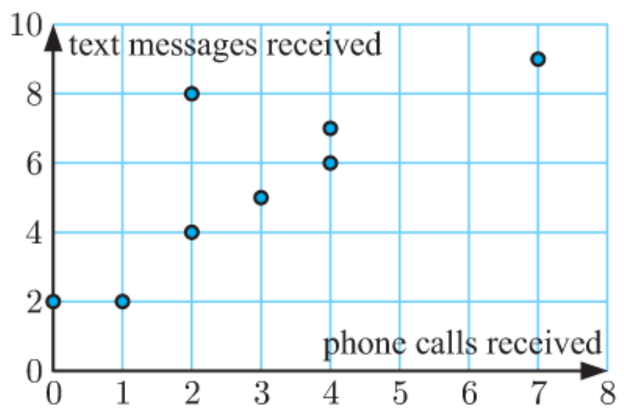
c **i**



ii $r \approx 0.146$

iii very weak, positive correlation

4 **a**



b $r \approx 0.816$

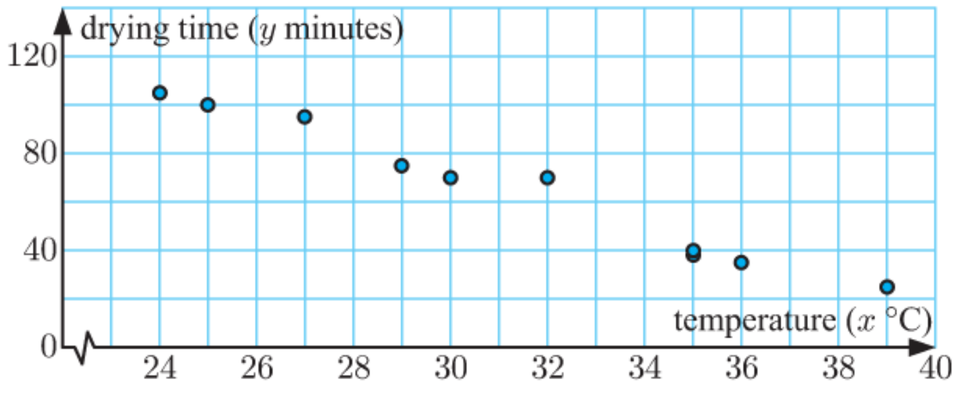
c moderate, positive correlation

d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

5 **a** $r \approx 0.917$

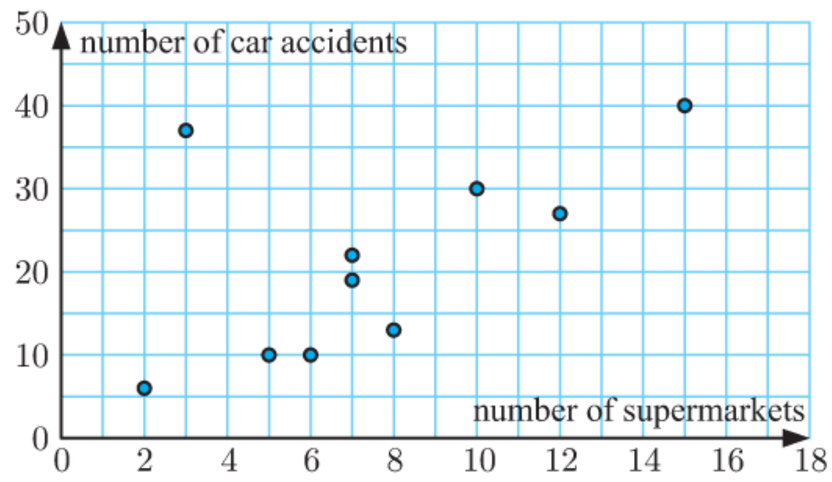
b strong, positive correlation
In general, the higher the young athlete's age, the further they can throw a discus.

6 **a**



b $r \approx -0.987$ **c** very strong, negative correlation

7 **a**



b $r \approx 0.572$

c The point (3, 37), which represents 37 car accidents in a town with 3 supermarkets, is an outlier.

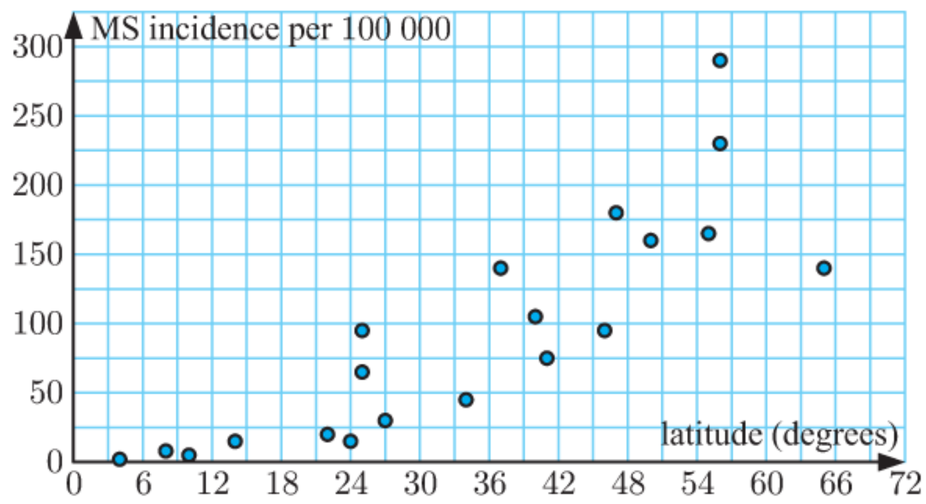
d **i** $r \approx 0.928$

ii strong, positive correlation

iii Removing the outlier had a very significant effect on the value of r.

e No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

8 **a**

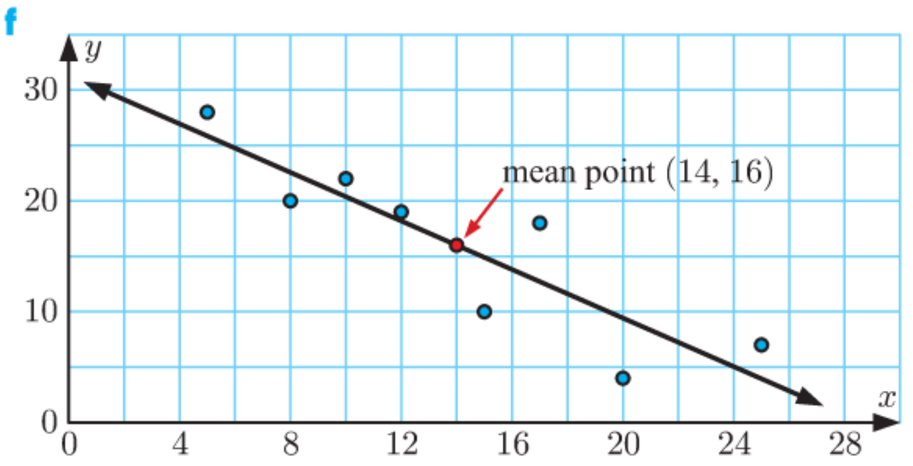


b $r \approx 0.849$ **c** moderate, positive correlation

d The incidence of MS is higher near the poles.

EXERCISE 26C

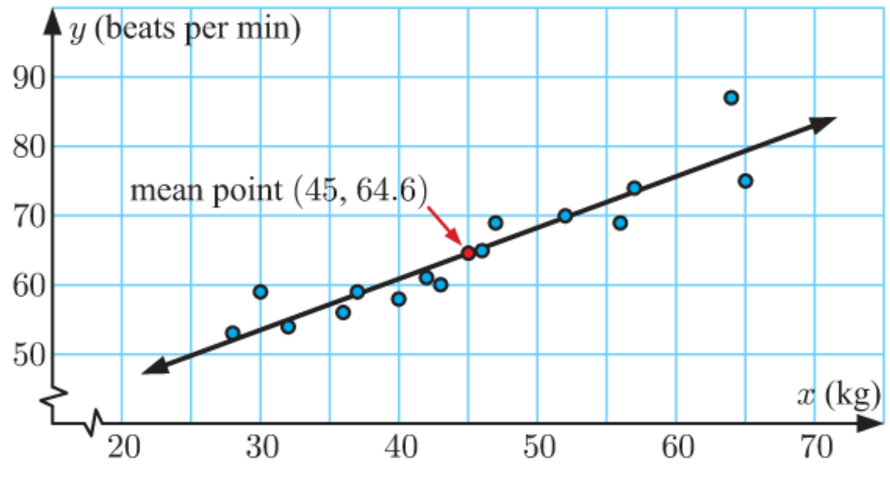
1 **a**, **f**



b negatively correlated **c** $r \approx -0.881$

d strong, negative correlation **e** (14, 16) **g** $y \approx 7$

2 **a**, **e**



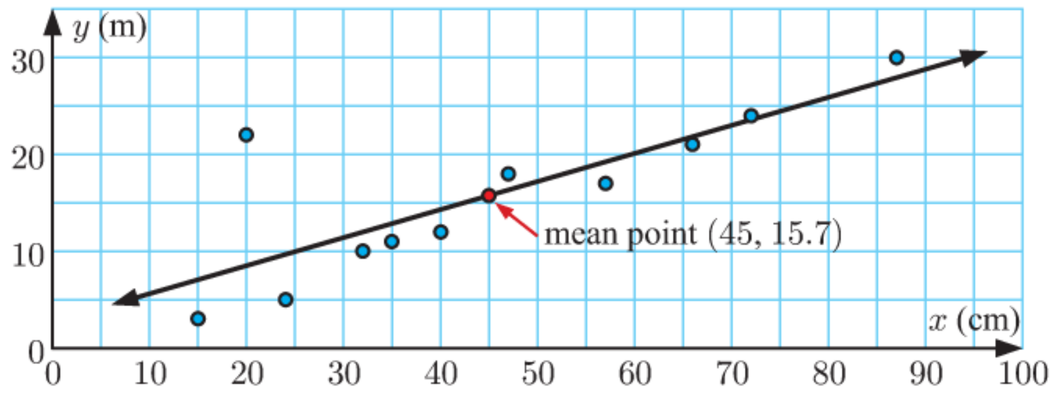
b $r \approx 0.929$

c There is a strong, positive correlation between *weight* and *pulse rate*.

d (45, 64.6)

f 68 beats per minute. This is an interpolation, so the estimate is reliable.

3 **a, e**



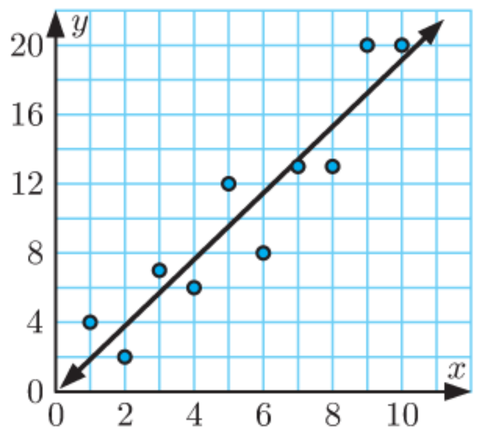
b (20, 22) **c** very tall and thin **d** (45, 15.7)

f ≈ 37 m. This is an extrapolation, so the prediction may not be reliable.

g ≈ 25 cm. This is an interpolation, so the estimate is reliable.

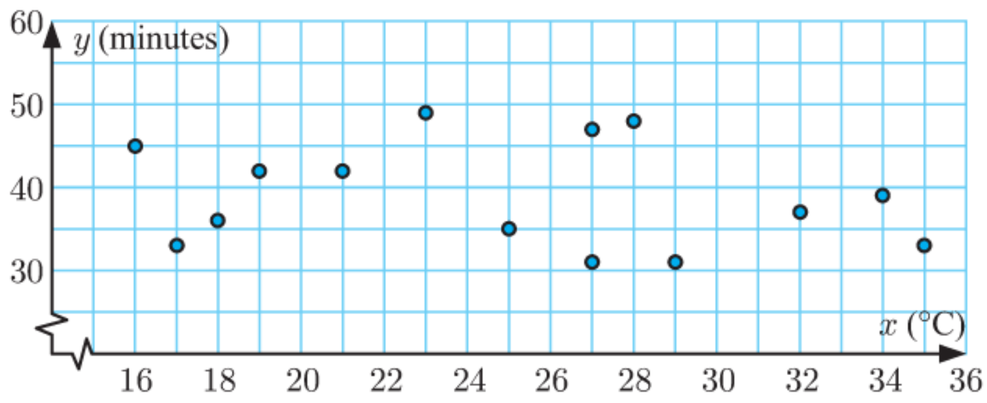
EXERCISE 26D

1 **a, c**



b $y \approx 1.92x - 0.0667$

2 **a**



b $r \approx -0.219$

c There is a very weak, negative correlation between *temperature* and *time*.

d No, as there is almost no correlation.

3 **a** $r \approx -0.924$

b There is a strong, negative, linear correlation between the *petrol price* and the *number of customers*.

c $y \approx -4.27x + 489$

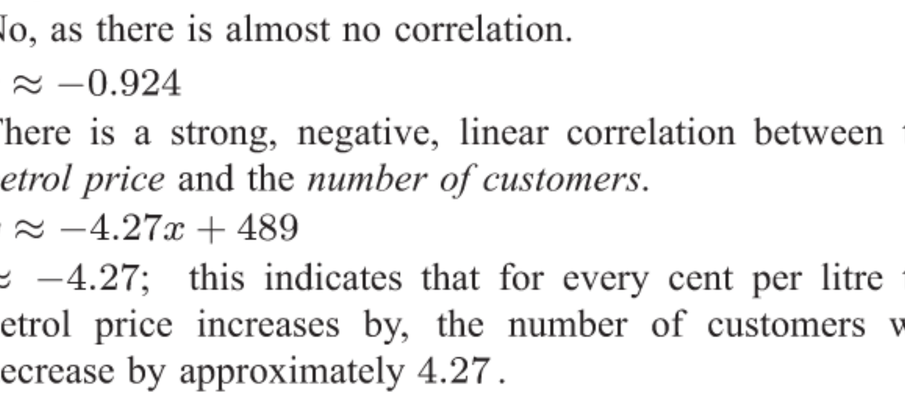
d ≈ -4.27; this indicates that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

e ≈ -5.10 customers **f** ≈ 105.3 cents per litre

g In **e**, it is impossible to have a negative number of customers. This extrapolation is not valid.

In **f**, this is an interpolation, so this estimate is likely to be reliable.

4 **a**



b $r \approx -0.878$

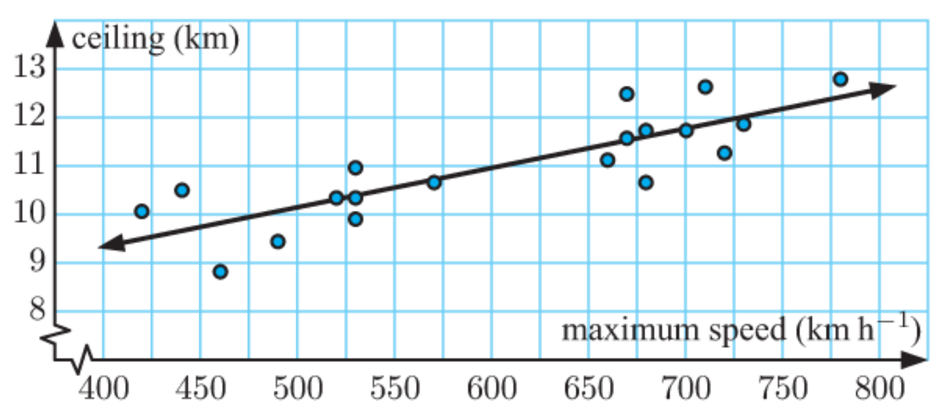
c There is a strong, negative correlation between *number of speed cameras* and *number of car accidents*.

d $y \approx -1.06x + 52.0$

e gradient: ≈ -1.06; this indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.
y-intercept: ≈ 52.0; this indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

f ≈ 41.4 car accidents

5 **a, d**



b $r \approx 0.840$

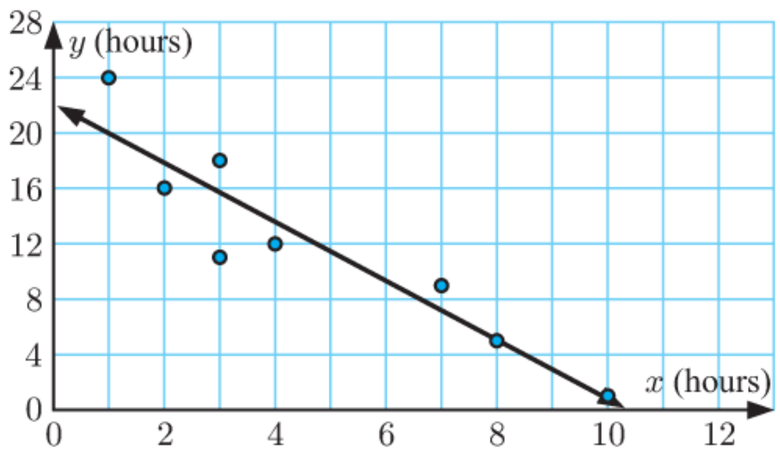
c moderate, positive, linear correlation

d $y \approx 0.008\,12x + 6.09$

e $\approx 0.008\,12$; this indicates that for each additional km h^{-1}, the ceiling increases by an average of 0.008 12 km or 8.12 m.

f ≈ 11.0 km **g** ≈ 605 km h^{-1}

6 **a, d**



b $r \approx -0.927$

c There is a strong, negative, linear correlation between *time exercising* and *time watching television*.

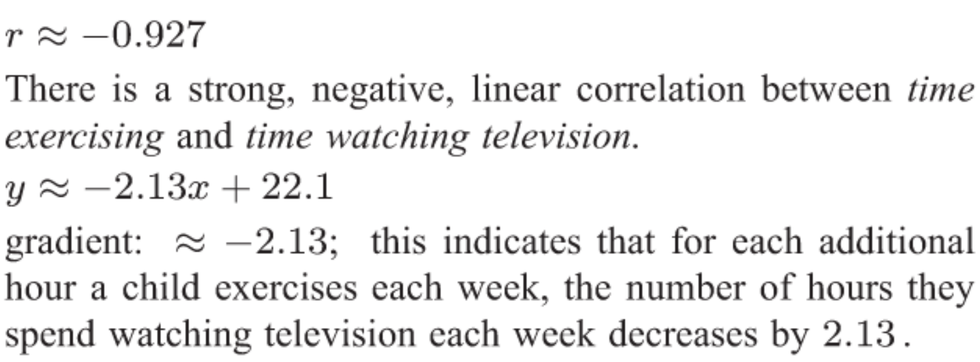
d $y \approx -2.13x + 22.1$

e gradient: ≈ -2.13; this indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.
y-intercept: ≈ 22.1; this indicates that for children who do not spend time exercising, they would watch television for an average of 22.1 hours per week.

f **i** 9 hours per week **ii** ≈ 7.22 hours per week

iii This particular child spent more time watching television than predicted.

7 **a**



(50, 4.4) is the outlier.

b **i** reduces the strength of the correlation

ii decreases the gradient of the regression line

c **i** $r \approx 0.798$ **ii** $r \approx 0.993$

d **i** $y \approx 0.0672x + 2.22$ **ii** $y \approx 0.119x + 1.32$

e The one which excludes the outlier, as this will be more accurate for an interpolation.

f Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

8 **b** $\beta = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^{\,2}}$

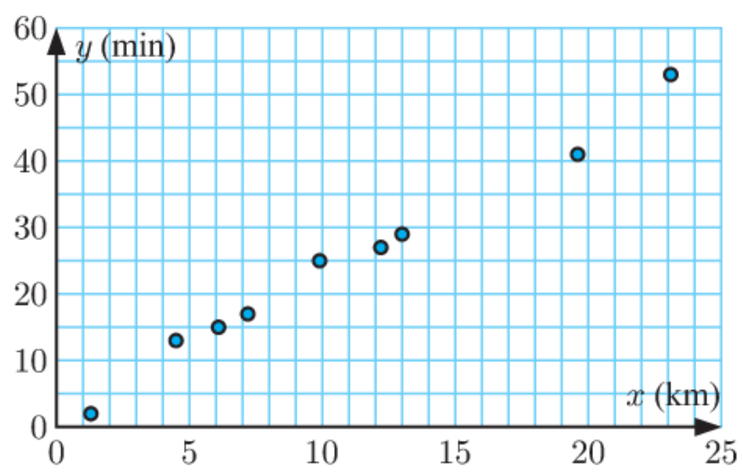
EXERCISE 26E

1 **a** The y variable, money spent on fast food, can be measured exactly. The x variable, time spent on homemade meals, will not be measured exactly.

b $x \approx -0.0576y + 8.29$

c **i** ≈ 5.70 hours **ii** $\approx \$57.13$

2 **a**

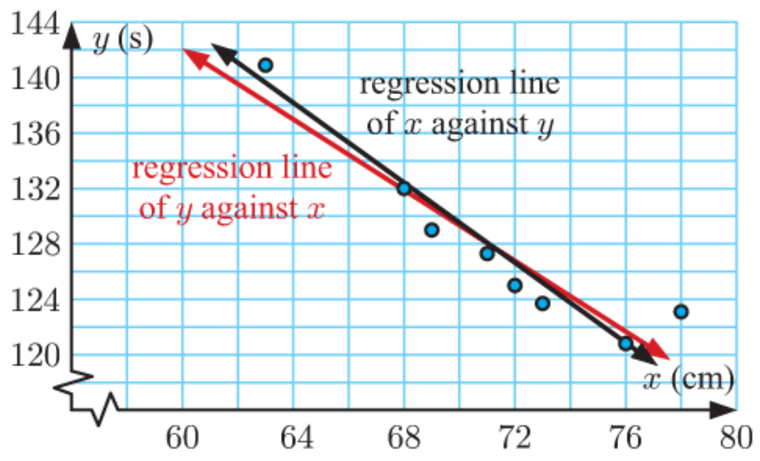


b x against y, since a student's time taken to travel to school can be more precisely measured than their distance from school.

c ≈ 33.9 min

d This is an interpolation, so this estimate is likely to be reliable.

3 **a, c**



b **i** $y \approx -1.28x + 219$ **ii** $x \approx -0.693y + 160$

c The two regression lines are very similar. The regression line of x against y is slightly steeper.

4 **b** The regression lines are the same if $r^2 = 1$.

REVIEW SET 26A

1 **a** strong, positive, linear correlation, with no outliers

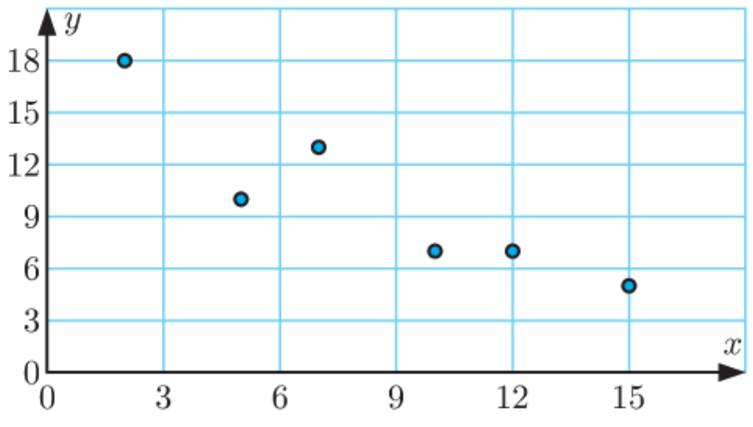
b weak, negative, linear correlation, with one outlier

c strong, negative, non-linear correlation, with no outliers

2 **a** The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.

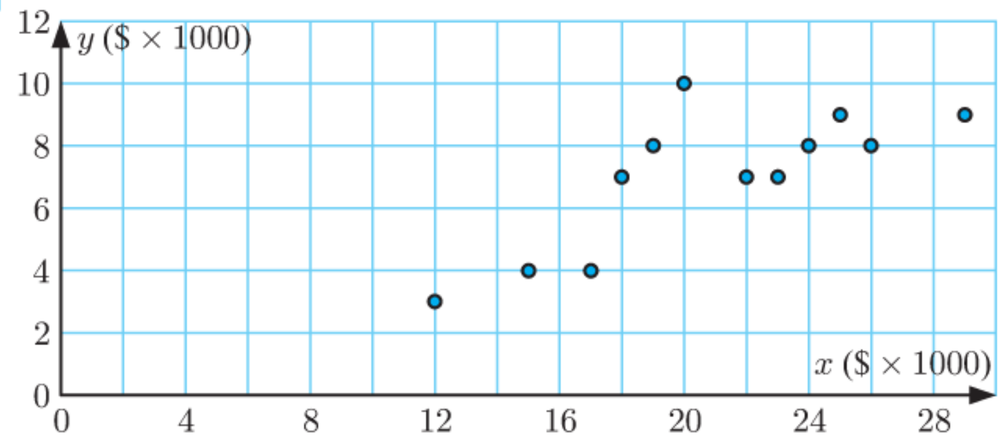
b No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

3 **a**



b negative **c** $r \approx -0.906$

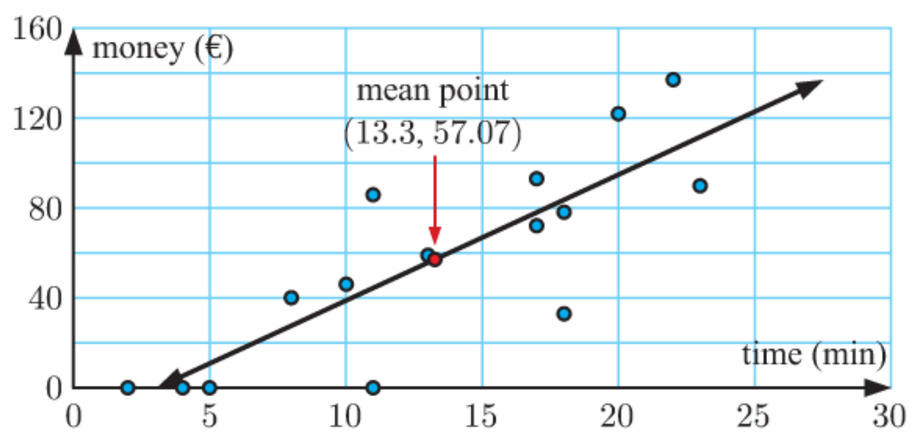
4 **a**



b $r \approx 0.776$ **c** moderate, positive correlation

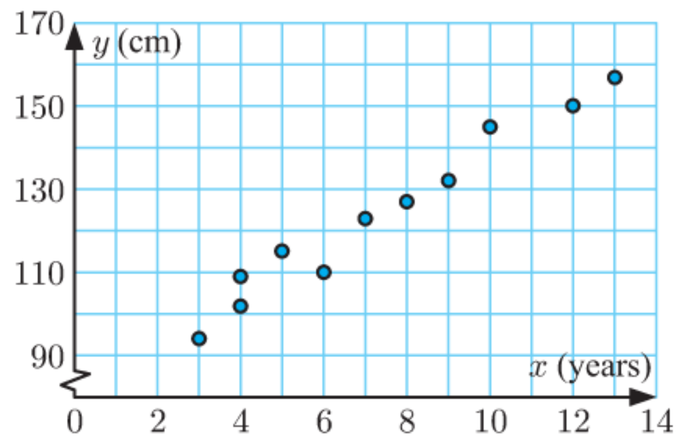
5 **a** mean time ≈ 13.3 min, mean spending $\approx €57.07$

b



c There is a moderate positive linear correlation between *time in the store* and *money spent*.

6 **a**

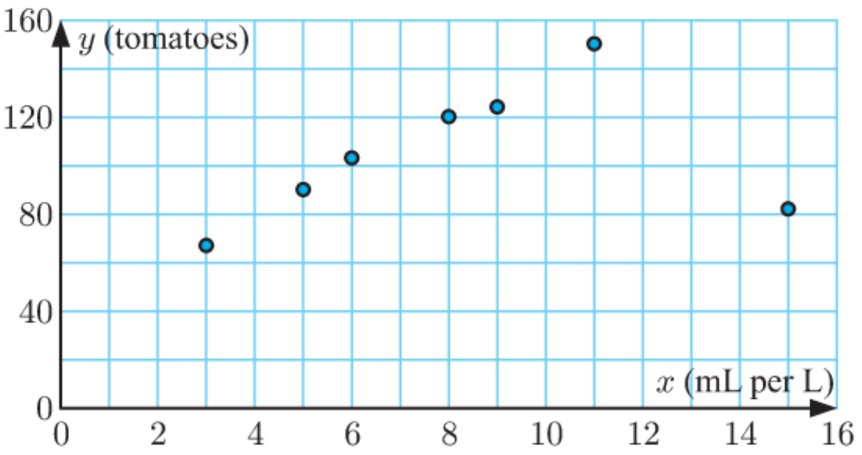


b $y \approx 5.98x + 80.0$

c ≈ 5.98; this indicates that each year, a child grows taller by an average of 5.98 cm.

d ≈ 110 cm **e** 10 years old

7 **a**



b $r \approx 0.340$. There is a very weak, positive, linear correlation between spray concentrations and yield.

c Yes, $(15, 82)$ is an outlier.

d $r \approx 0.994$. Yes it is now reasonable to draw a regression line.

e $y \approx 9.93x + 39.5$

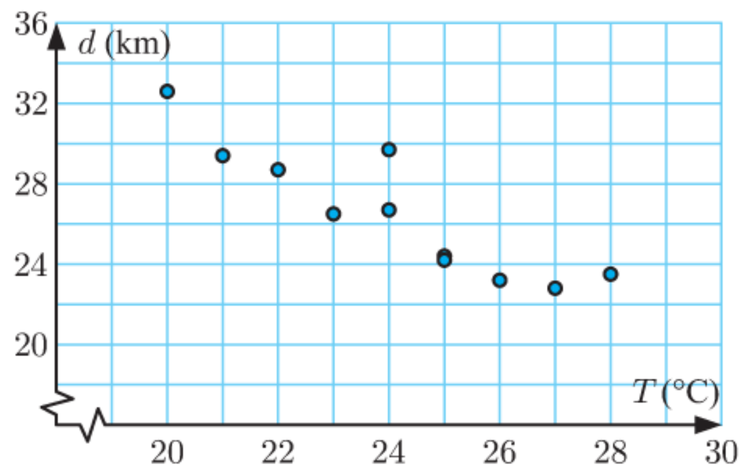
f gradient: ≈ 9.93; this indicates that for every additional mL per L the spray concentration increases, the yield of tomatoes per bush increases on average by 9.93.

y-intercept: ≈ 39.5; this indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g **i** ≈ 109 tomatoes per bush **ii** ≈ 16.2 mL per L

h In **g i**, this is an interpolation, so this estimate is likely to be reliable.
In **g ii**, this is an extrapolation, so this estimate may not be reliable.

8 **a**



b The values for the distance travelled d are more precisely measured than the daily temperature which Thomas is just estimating.

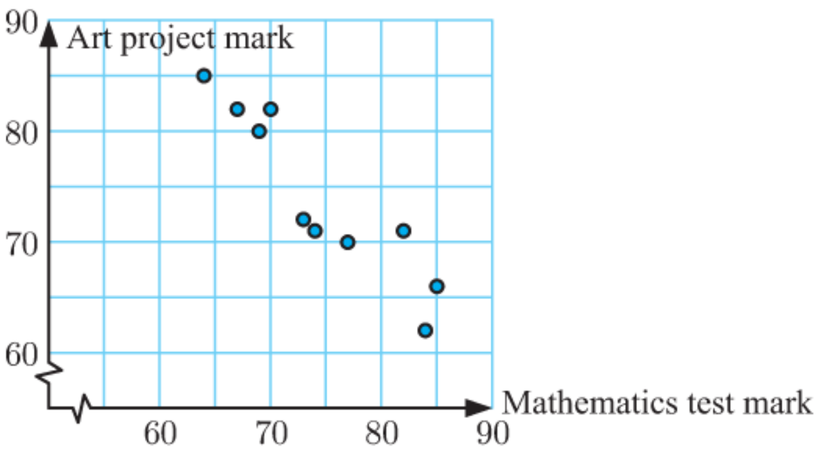
c $T \approx -0.689d + 42.3$ **d** ≈ 17.9 km

REVIEW SET 26B

1 **a** Negative correlation. As prices increase, the number of tickets sold is likely to decrease.
Causal. Less people will be able to afford tickets as the prices increase.

b Positive correlation. As ice cream sales increase, the number of shark attacks is likely to increase.
Not causal. Both of these variables are dependent on the number of people at the beach.

2 **a**

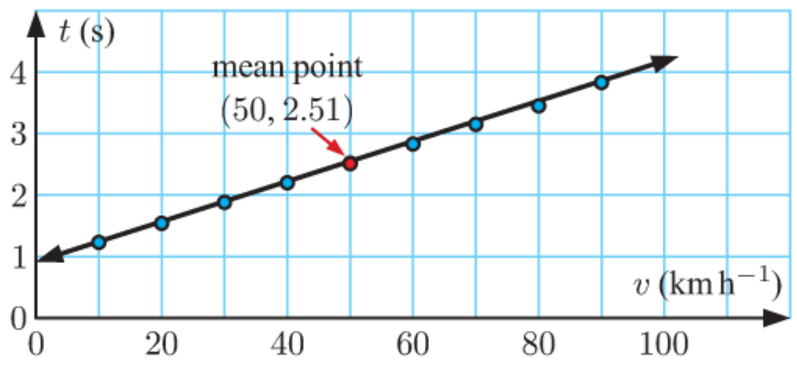


b There is a strong, negative, linear correlation between the Mathematics and Art marks.

c $r \approx -0.930$

3 **a** (50, 2.51)

b

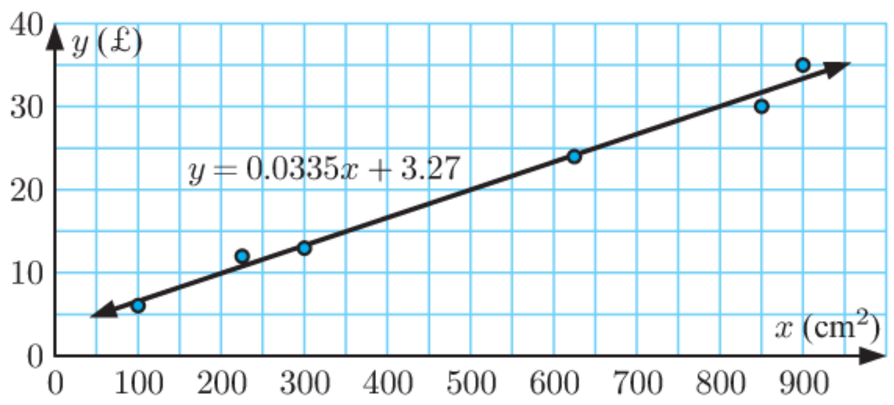


c **i** ≈ 2.68 seconds **ii** ≈ 4.44 seconds

d The estimate in **c i**, since it is an interpolation.

4 **a** $r \approx 0.983$ **b** $y \approx 3.36x + 8.64$ **c** ≈ 42.2

5 **a**, **d**

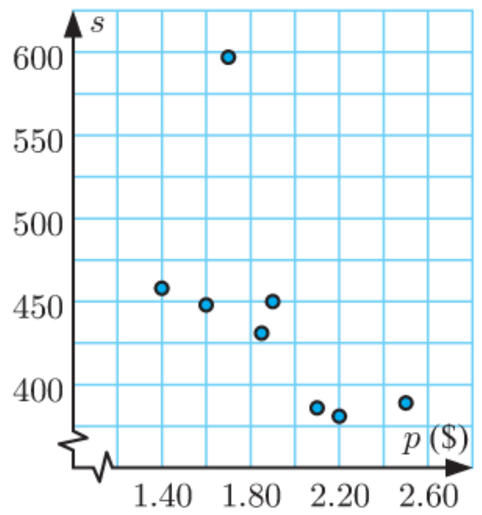


b $r \approx 0.994$

c There is a very strong, positive correlation between *area* and *price*.

e $\approx$ £43.42, this is an extrapolation, so it may be unreliable.

6 **a**



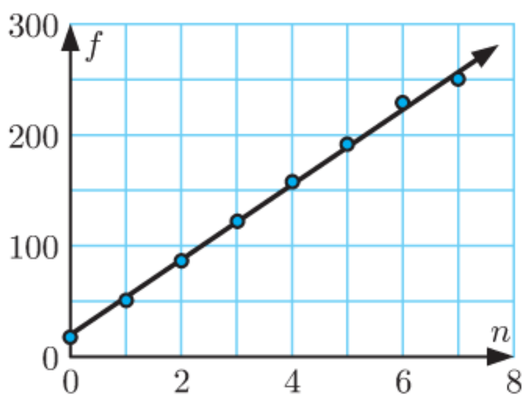
b Yes, the point $(1.7, 597)$ is an outlier. It should not be deleted as there is no evidence that it is a mistake.

c $s \approx -116p + 665$

d ≈ -116; this indicates that with every additional dollar the price increases by, the number of sales decreases by 116.

e No, the prediction would not be accurate, as it is an extrapolation.

7 **a**, **d**



There is a very strong, positive correlation between number of waterings and flowers produced.

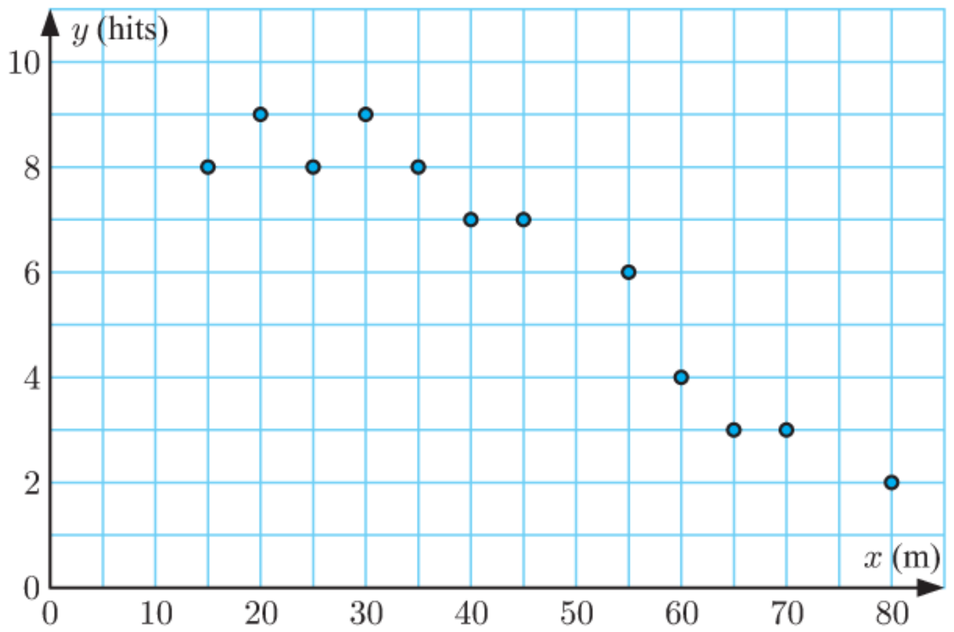
b $f \approx 34.0n + 19.3$

c Yes, plants need water to grow, so it is expected that an increase in watering will result in an increase in flowers.

e **i** 104 flowers $(n = 2.5)$, 359 flowers $(n = 10)$

ii $n = 2.5$ is reliable, as it is an interpolation.
$n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

8 **a**



b The number of hits can be measured exactly, while the distance from the target might not be exact.

c $x \approx -7.89y + 93.7$

d ≈ -0.8 shots, but it is impossible to make a negative number of shots. This extrapolation is not valid.

EXERCISE 27A

1 **a** continuous **b** discrete **c** continuous **d** continuous
e discrete **f** discrete **g** continuous **h** continuous

2 **a** **i** X = the height of water in the rain gauge

ii continuous **iii** $0 \leqslant X \leqslant 400$ mm

b **i** X = stopping distance **ii** continuous

iii $0 \leqslant X \leqslant 50$ m

c **i** number of switches until failure **ii** discrete

iii any integer $\geqslant 1$

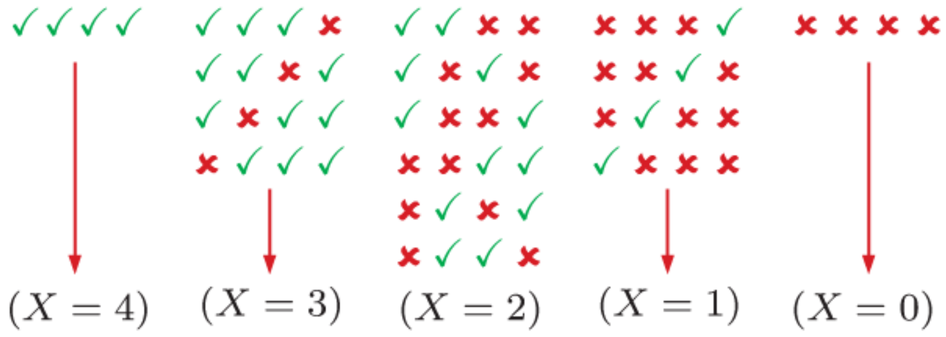
3 **a** X has a set of distinct possible values.

b $X = 2, 3, 4, 5, 6, 7, 8, 9$, or 10

4 **a** $X = 4, 5, 6$, or 7 **b** **i** $X = 5$ **ii** $X = 6$ or 7

5 **a** $X = 0, 1, 2, 3$, or 4

b



c **i** $X = 2$ **ii** $X = 2, 3$, or 4

6 **a** $X = 0, 1, 2$, or 3

b

$X = 3$	$X = 2$	$X = 1$	$X = 0$
HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	

c No, for example there is probability $\frac{1}{8}$ that $X = 3$, and probability $\frac{3}{8}$ that $X = 2$.

EXERCISE 27B

1 **a** **i** yes **ii** no **iii** yes **iv** no

b For **a iii**, X is a uniform random variable.

2 **a** $k = 0.2$ **b** $k = \frac{1}{7}$

3 **a** $a = 0.2$

b No, as the probabilities of each outcome are not all equal.

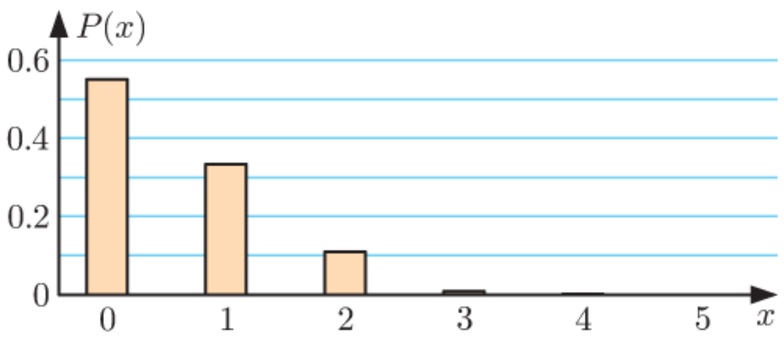
c 2 **d** $\mathrm{P}(X \geqslant 2) = 0.65$

4 **a** $P(2) = 0.1088$

b $a = 0.5488$ is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.

d



e mode = 0 home runs, median = 0 home runs

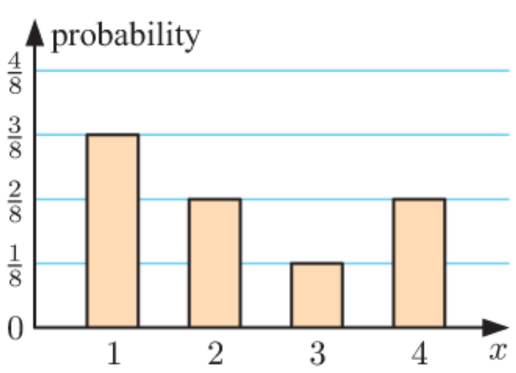
5 **a** $k = 0.04$ **b** 0 tyres

c $\mathrm{P}(X > 1) = 0.12$ which is the probability that more than 1 tyre will need replacing on a car being inspected.

6 **a**

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

b



c mode = 1, median = 2 **d** $\mathrm{P}(X \leqslant 3) = \frac{3}{4}$

7 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$\mathrm{P}(X = x)$	0.24	0.35	0.27	0.14

c mode = 2 bedrooms, median = 2 bedrooms

8 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$\mathrm{P}(X = x)$	0.48	0.28	0.08	0.16

c mode = 1 shot, median = 2 shots

9 **a** $P(0) = \frac{1}{10}$, $P(1) = \frac{2}{10}$, $P(2) = \frac{3}{10}$, $P(3) = \frac{4}{10}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and

$\sum_{i=1}^{n} P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$

$\therefore$ $P(x)$ is a valid probability function.

b $P(1) = \frac{6}{11}$, $P(2) = \frac{3}{11}$, $P(3) = \frac{2}{11}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and

$\sum_{i=1}^{n} P(x_i) = \frac{6}{11} + \frac{3}{11} + \frac{2}{11} = 1$

$\therefore$ $P(x)$ is a valid probability function.

10 **a** $k = \frac{1}{12}$ **b** $k = \frac{12}{25}$

11 **a** $a = 10$ **b** $\mathrm{P}(X = 1) = \frac{3}{10}$ **c** 2

12 **a** $\frac{3}{2}$ **b** $a = \frac{2}{3}$ **13** $a = \frac{3}{5}$

14

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

15 **a** $X = 1, 2$, or 3 **b**

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

16 **a**

Die 1 \ Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b

s	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(S = s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c 7 **d** $\mathrm{P}(S \geqslant 10) = \frac{1}{6}$

17 **a** $\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!} = 1$

b **i** $\sum P(x_i) = 1$ **ii** $\mathrm{P}(X = 0) = e^{-0.2} \approx 0.819$,

iii $\mathrm{P}(X \geqslant 3) \approx 0.001\,15$ $\mathrm{P}(X = 1) = 0.2e^{-0.2} \approx 0.164$,

$\mathrm{P}(X = 2) = 0.02e^{-0.2} \approx 0.0164$

EXERCISE 27C.1

1 **a** $\mathrm{E}(X) = 1.7$ **b** $\mathrm{E}(X) = 2.5$ **c** $\mathrm{E}(X) = 3.85$

d $\mathrm{E}(X) = 30$

2 **a** $a = \frac{1}{2}$ **b** 3 **c** $\mu = 2\frac{2}{5}$ **3** ≈ 11.7 points

4 1.57 fish **5** **a** $a = 0.25$ **b** 4 books **c** 3.13 books

6 5.25 lollies **7** **a** $\frac{4}{15}$ **b** ≈ 8.93 pins

8 $a = 0.1,\ b = 0.4$

9 **a** offensive strategy: $\text{P(draw)} = 0.15$
defensive strategy: $\text{P(draw)} = 0.5$
b offensive strategy: 1.05 points per game
defensive strategy: 1.1 points per game
c defensive strategy
d Yes, an offensive strategy would then be better.

10 **a** **i** car park B **ii** car park A **iii** car park B
b Zoe should choose car park A as the expected cost for car park A is \$14.80 whereas the expected cost for car park B is \$15.25.

11 \$390

EXERCISE 27C.2

1 fair **2** **a** \$3.50 **b** −\$0.50 **c** no

3 **a** $\approx -\$0.05$ **b** lose $\approx$ \$5.41 **4** −\$0.75

5 **a** **i** 0.3 **ii** 0.1 **b** $\text{E}(X) = 2.5$ tokens
c No, as the player can expect to lose half a token on average per game.

6 **a** Expected gain $\approx -\$0.67 \neq \$0$ **b** \$30

7 **a** $\text{P}(X \leqslant 3) = \frac{1}{12},\ \text{P}(4 \leqslant X \leqslant 6) = \frac{1}{3},\ \text{P}(7 \leqslant X \leqslant 9) = \frac{5}{12},\ \text{P}(X \geqslant 10) = \frac{1}{6}$ **c** $a = 7$
d The organisers would lose \$2.50 per game. **e** \$4010

8 \$4.75

EXERCISE 27D

1 **a** **i** $\mu = 2$ **ii** $\sigma^2 = 0.6$ **iii** $\sigma \approx 0.775$
b **i** $\mu = 1.5$ **ii** $\sigma^2 = 1.25$ **iii** $\sigma \approx 1.12$

2 **a** $k = 0.15$ **b** 20 **c** $\mu = 13$ **d** $\sigma \approx 6.88$

3 **b** Michelle: $\sigma = 1$ ace, $\sigma^2 = 1$ ace^2
Amanda: $\sigma \approx 1.30$ aces, $\sigma^2 = 1.7$ aces^2 **c** Amanda

4 **a** $k = 0.03$ **b** $\mu = 0.74$ **c** $\sigma \approx 0.996$

5 **a**

x	1	2	3
$\text{P}(X = x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

b $\mu \approx 2.17$ **c** $\sigma \approx 0.898$

6 **a** mode = 3 **b** median = 3 **c** $\mu = 2.5$ **d** $\sigma \approx 0.671$

8 **a**

x_i	1	2	3	4	5
p_i	0.1	0.2	0.4	0.2	0.1

b $\mu = 3,\ \sigma \approx 1.10$ **c** **i** $\text{P}(\mu - \sigma < x < \mu + \sigma) = 0.8$ **ii** $\text{P}(\mu - 2\sigma < x < \mu + 2\sigma) = 1$

9 **a** **i** Y would have the greater mean as the possible values for Y will be greater overall than the possible values for X.
ii X would have the greater standard deviation as the probabilities of obtaining each outcome are more spread out.

b

x	1	2	3	4
$\text{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

y	1	2	3	4
$\text{P}(Y = y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

c X **d** X: $\mu = 2.5,\ \sigma \approx 1.12$
Y: $\mu = 3.125,\ \sigma \approx 0.927$

10 **Hint:** Use $\sigma^2 = \sum x_i^{\,2} p_i - \mu^2$

EXERCISE 27E

1 $\text{E}(aX + b) = \text{E}(aX) + \text{E}(b) = a\,\text{E}(X) + b$

2 **a** 13 **b** −5 **c** $3\frac{1}{3}$

3 mean = 17, standard deviation = 4

4 **a** $\text{E}(Y) = 13,\ \text{Var}(Y) = 16$
b $\text{E}(Y) = -22,\ \text{Var}(Y) = 100$
c $\text{E}(Y) = 0,\ \text{Var}(Y) = 1$

5 **a** $\text{E}(X) = 2$ **b** $\text{Var}(X) = 1$ **c** $\sigma(X) = 1$
d $\text{E}(X + 1) = 3$ **e** $\text{Var}(3X + 1) = 9$ **f** $\sigma(5 - X) = 1$
g $\text{E}\left(\frac{2X + 5}{3}\right) = 3$ **h** $\text{Var}(20 - 4X) = 16$

6 **a** $a = \frac{1}{4}$
b **i** $\text{E}(X) = 2\frac{11}{12},\ \text{Var}(X) \approx 1.91,\ \sigma(X) \approx 1.38$
ii $\text{E}(X + 4) = 6\frac{11}{12},\ \text{Var}(X + 4) \approx 1.91,\ \sigma(X + 4) \approx 1.38$
iii $\text{E}(3X - 1) = 7\frac{3}{4},\ \text{Var}(3X - 1) \approx 17.2,\ \sigma(3X - 1) \approx 4.15$

7 **a** **i** $\text{E}(X) = 1.85$ **ii** $\text{Var}(X) \approx 1.73$ **iii** $\sigma(X) \approx 1.31$
b $Y = 25X + 100$
c **i** $\text{E}(Y) = 146.25$ **ii** $\text{Var}(Y) \approx 1080$
iii $\sigma(Y) \approx 32.9$

EXERCISE 27F

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (H or T) and each toss is independent of every other toss.
b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

2 **a** $(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$
b **i** $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ **ii** $4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}$
iii $6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$

3 **a** $(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
b **i** $5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{5}{32}$ **ii** $10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{16}$
iii $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$

4 **a** $\left(\frac{2}{3} + \frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$
b **i** $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ **ii** $6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$ **iii** $\frac{8}{9}$

5 **a** $\left(\frac{3}{4} + \frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
b **i** $10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{135}{512}$ **ii** $\frac{53}{512}$ **iii** $\frac{47}{128}$

6 **b** $\binom{n}{x},\ p^x,\ (1 - p)^{n-x} \geqslant 0 \quad \therefore \quad P(x) \geqslant 0$
Now $\sum_{x=0}^{n} P(x) = 1 \quad \therefore \quad P(x) \leqslant 1$
$\therefore \quad 0 \leqslant P(x) \leqslant 1$ for all $x = 0, 1, \ldots, n$.

c $P(x)$ is a valid probability distribution.

EXERCISE 27G

1 **a** ≈ 0.0305 **b** ≈ 0.265

2 **a** ≈ 0.476 **b** ≈ 0.840 **c** ≈ 0.160 **d** ≈ 0.996

3 **a** ≈ 0.0280 **b** $\approx 0.002\,46$ **c** ≈ 0.131 **d** ≈ 0.710

4 $\approx 0.000\,864$ **5** **a** ≈ 0.998 **b** ≈ 0.807

6 **a** ≈ 0.0388 **b** ≈ 0.405 **c** ≈ 0.573 **7** ≈ 0.0341

8 **a** ≈ 0.863 **b** ≈ 0.475 **9** **a** $\frac{1}{36}$ **b** ≈ 0.846

10 **a** ≈ 0.0905 **b** ≈ 0.622

c Yes, the probability that Shelley is on time for work each day of a 5 day week is now $\approx 87.2\%$.

11 **a** ≈ 0.0388 **b** 25 solar components

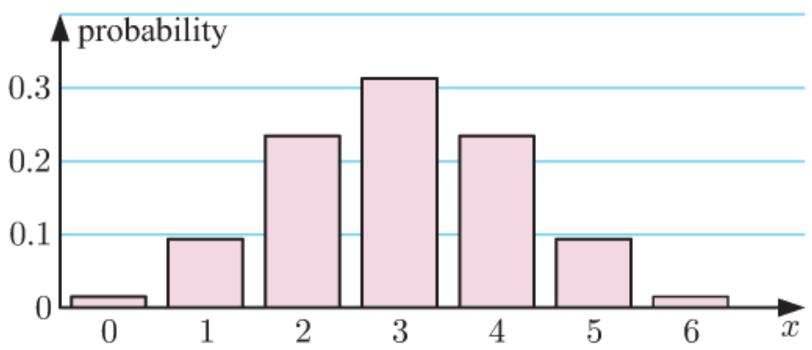
EXERCISE 27H

1 **a** **i** $\mu = 3, \quad \sigma \approx 1.22$

ii

x_i	0	1	2	3
$P(x_i)$	0.0156	0.0938	0.2344	0.3125

x_i	4	5	6
$P(x_i)$	0.2344	0.0938	0.0156



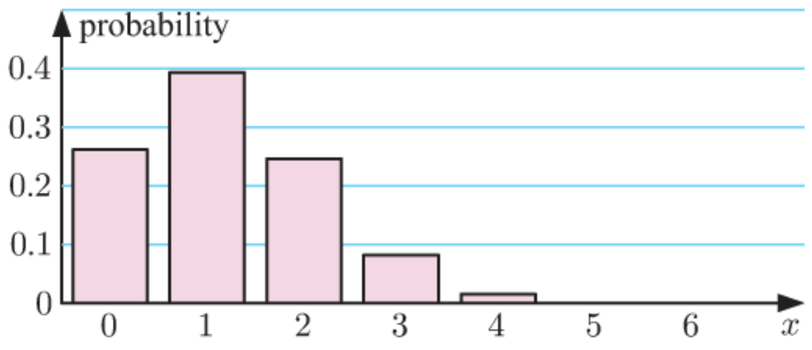
iii The distribution is symmetric.

b **i** $\mu = 1.2, \quad \sigma \approx 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.2621	0.3932	0.2458	0.0819

x_i	4	5	6
$P(x_i)$	0.0154	0.0015	0.0001



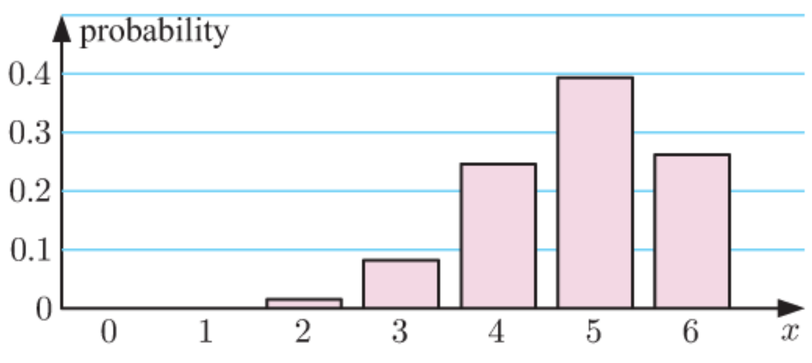
iii The distribution is positively skewed.

c **i** $\mu = 4.8, \quad \sigma = 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.0001	0.0015	0.0154	0.0819

x_i	4	5	6
$P(x_i)$	0.2458	0.3932	0.2621



iii The distribution is symmetric and is the exact reflection of **b**.

2 $\mu = 5, \quad \sigma^2 = 2.5$

3 **a** $\mu = 1.2, \quad \sigma \approx 1.07$ **b** $\mu = 28.8, \quad \sigma \approx 1.07$

4 $\mu = 3.9, \quad \sigma \approx 1.84$

5 **a** $\mu = 28.5, \quad \sigma \approx 2.67$ **b** ≈ 0.740

6 **a** $\mu_X = np = 100 \times \frac{1}{2} = 50$ $\quad \mu_Y = np = 300 \times \frac{1}{6} = 50$

b $\sigma_X = 5, \quad \sigma_Y \approx 6.45$

c X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y, which means there are more values of X which lie close to the mean.

d **i** ≈ 0.729 **ii** ≈ 0.606

REVIEW SET 27A

1 **a** discrete **b** continuous **c** discrete

2 **a** **i** yes **ii** no **iii** no **iv** yes **v** yes **vi** yes

b the distribution in **a iv**

3 **a** $a = \frac{5}{9}$ **b** $\frac{4}{9}$

4 **a** $k = 0.05$ **b** 0.15 **c** 2 **d** $\mathrm{E}(X) = 1.7$

5 **a** X has a set of distinct possible values.

b $X = 0$, 1, or 2

c

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d 1.2 green balls

6 ≈ 3.83

7 **a** \$7 **b** No, she would lose \$1 per game in the long run.

8 **a** $\mu = 3$ **b** $\sigma^2 = 1.1$ **c** $\sigma \approx 1.05$

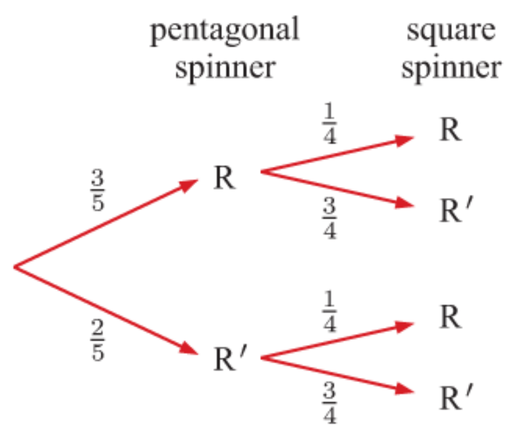
9 **a** $a = -\frac{1}{84}$ **b** 4 marsupials **c** 3

10 mean $= 27$, standard deviation $= 8$

11 **a** $\left(\frac{4}{5} + \frac{1}{5}\right)^5 = \left(\frac{4}{5}\right)^5 + 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) + 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 + 10\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 + 5\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5$

b **i** $\frac{64}{3125} = 0.020\,48$ **ii** $\frac{128}{625} = 0.2048$

12 **a**



b $\frac{11}{20}$

c **i** $X \sim \mathrm{B}(10, \frac{11}{20})$

ii $\mathrm{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)\left(\frac{9}{20}\right)^9 \approx 0.004\,16$

$\mathrm{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right) \approx 0.0207$

It is more likely that exactly one red will occur 9 times.

iii $\mu = 5.5, \quad \sigma \approx 1.57$

13 **a** **i** ≈ 0.0751 **ii** ≈ 0.166 **b** ≈ 4.97 games

REVIEW SET 27B

1 **a** X is the number of hits that Sally has in each match.

$X = 0$, 1, 2, 3, 4, or 5

b **i** $k = 0.23$ **ii** $\mathrm{P}(X \geqslant 2) = 0.79$

iii $\mathrm{P}(1 \leqslant X \leqslant 3) = 0.83$

c mode = 3 hits, median = 3 hits

3 **a** 2 **b** 3 **c** 2.7

4 **a** **i** Naomi **ii** Rosslyn **b** Rosslyn

5 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{1}{10}$ **b** \$2.70 per game

6 **a** $a = 0.15,\ b = 0.35$ **b** $\text{Var}(X) = 1.26$

7 **a** $\frac{4}{5}$ **b** **i** ≈ 0.0881 **ii** ≈ 0.967

8 **a** The probability of rolling a two is not the same for each die. So X is not a binomial random variable.

b

x	0	1	2
$\text{P}(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c **i** $\frac{5}{12}$ **ii** ≈ 0.571

9 **a** **i** $\text{E}(X) = 2.1$ **ii** $\text{Var}(X) = 1.29$

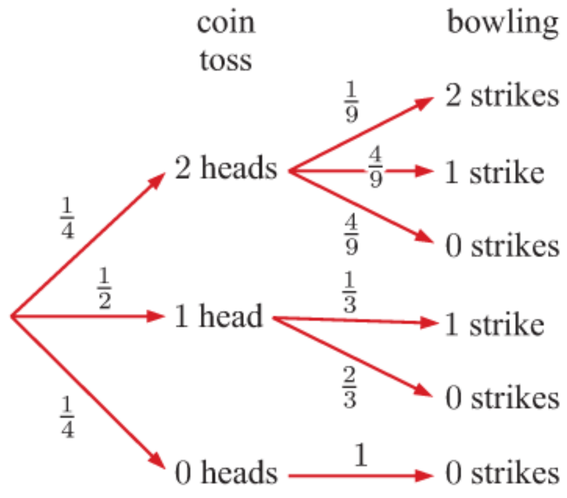
b **i** $Y = 4 - X$ **ii** $\text{E}(Y) = 1.9,\ \text{Var}(Y) = 1.29$

10 **a** The probability of spinning a 3 is the same for each spin.

b $\mu = 4,\ \sigma \approx 1.79$

11 **a** 42 donations **b** **i** ≈ 0.334 **ii** ≈ 0.0931

12 **a**



b

x	0	1	2
$\text{P}(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c $\approx$ \$3.33

d $\approx -$ \$1.67, Suvi should not play the game many times.

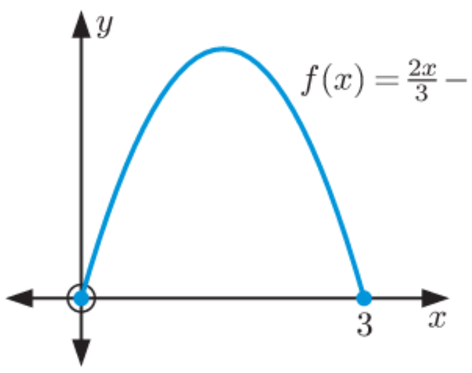
13 **a** **i** ≈ 0.544 **ii** ≈ 0.456

b **i** $\text{P}(X = 0) = \binom{n}{0}\left(\frac{3}{100}\right)^0\left(\frac{97}{100}\right)^n = (0.97)^n$ **ii** $n = 12$

EXERCISE 28A

1 **b** $\frac{5}{8}$

2 **a**



c $\frac{7}{27}$

4 **a** $a = \frac{3}{16}$ **b** $\frac{1}{8}$

5 **a** $f(x) \not\geqslant 0$ for all x **b** $k = e$ **c** $\ln 2 \approx 0.693$

6 **a** $k = \dfrac{e}{e-1}$ **b** ≈ 0.472 **7** $a = \frac{5}{32},\ k = 2$

8 **a** $a = \frac{1}{972}$ **b** **i** ≈ 0.374 **ii** ≈ 0.126

9 **a** For $f(x)$ to be a valid probability density function, we only require $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$ and $\int_a^b f(x)\,dx = 1$.

b $b - a \geqslant 1$

EXERCISE 28B

1 **b** **i** 4 **ii** $2\sqrt{2}$ **iii** $2\frac{2}{3}$ **c** $\text{Var}(X) = \frac{8}{9},\ \sigma = \frac{2\sqrt{2}}{3}$

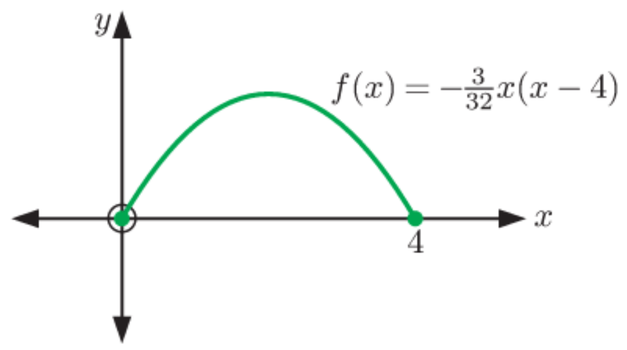
2 **b** $\frac{2}{3}$

c **i** $2\ln 2$ **ii** $2 - 4(\ln 2)^2$ **iii** $\sqrt{2 - 4(\ln 2)^2}$

d **i** $4\ln 2 + 8$ **ii** $8 - 16(\ln 2)^2$ **iii** $2\sqrt{2 - 4(\ln 2)^2}$

3 **a** $a = -\frac{3}{32}$

b



c **i** 2 **ii** 2 **iii** $\frac{4}{5}$ **iv** $\frac{2}{\sqrt{5}}$

4 **a** $k = e^2$ **b** **i** $\dfrac{e^2-1}{2}$ **ii** $\dfrac{e^2-1}{2}$ **iii** $\sqrt{\dfrac{e^2-1}{2}}$

c $\frac{1}{3}\sqrt{\dfrac{e^2-1}{2}}$

5 **a** $f(x) \not\geqslant 0$ for all $0 \leqslant x \leqslant \frac{1}{2}$ **b** $k = \frac{1}{3}$

c $\frac{11}{25}$ **d** mean $= \frac{7}{54}$, median $= \frac{5-\sqrt{13}}{12}$

6 **a** $k = \dfrac{1}{b-a}$ **b** mean $= \dfrac{a+b}{2}$, median $= \dfrac{a+b}{2}$

c $\text{Var}(X) = \dfrac{(b-a)^2}{12},\ \sigma = \dfrac{b-a}{\sqrt{12}}$

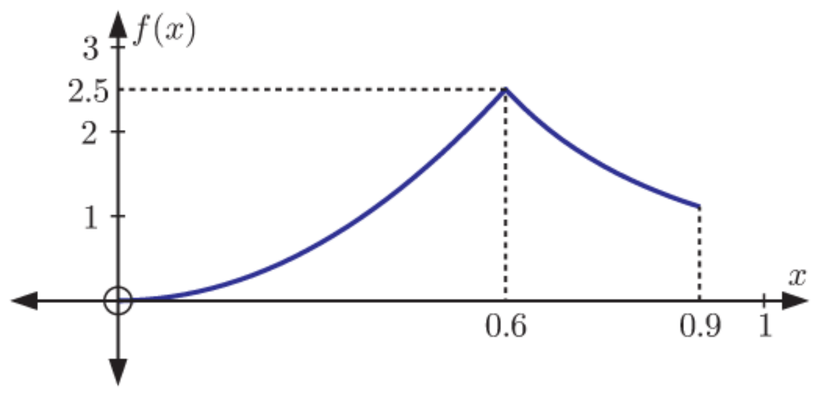
7 **a** $k \approx 1.0524$ **b** ≈ 0.843

8 **a** $k = -\frac{4}{375}$ **b** ≈ 3.46 **c** $3\frac{1}{3}$ **d** $1\frac{1}{9}$

9 **a** $\frac{8}{9}$ **b** $\sqrt{2}$ **c** $\mu = \int_1^\infty x\,f(x)\,dx = \int_1^\infty \frac{2}{x^2}\,dx$

d 2

10 **a**



c mean ≈ 0.590, median $= 0.6$, mode $= 0.6$

d $\text{Var}(X) \approx 0.0300,\ \sigma \approx 0.173$

e ≈ 0.652; the task can be performed between 18 minutes and 42 minutes about 65.2% of the time.

11 **a** $\frac{1}{2}$ **b** $k = -\frac{1}{4}$ **c** $\frac{\pi}{2}$ **d** $\frac{5}{6} + \frac{\pi}{4} \approx 1.62$

12 **a** $\lambda > 0$ and $e^{-\lambda t} > 0$ for all $t \geqslant 0$.

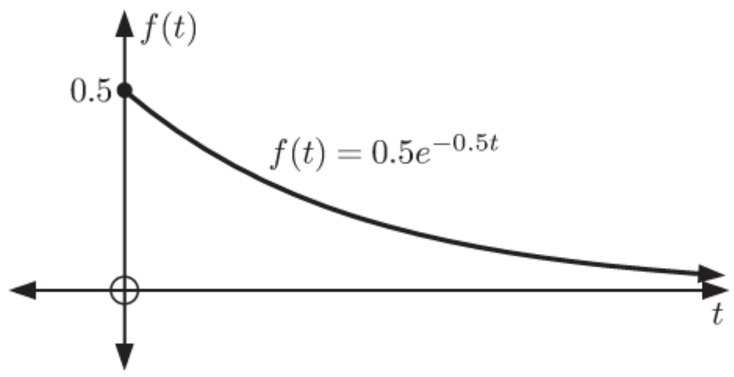
c **i** $\text{E}(T) = \int_0^\infty t\,f(t)\,dt = \int_0^\infty \lambda t e^{-\lambda t}\,dt$

ii $\int_0^k \lambda t e^{-\lambda t}\,dt = -e^{-\lambda k}\left(k + \frac{1}{\lambda}\right) + \frac{1}{\lambda}$

d **i** $\text{Var}(T) = \int_0^\infty t^2 f(t)\,dt - [\text{E}(T)]^2 = \int_0^\infty \lambda t^2 e^{-\lambda t}\,dt - \frac{1}{\lambda^2}$

ii $\int_0^k \lambda t^2 e^{-\lambda t}\,dt = -e^{-\lambda k}\left(k^2 + \frac{2k}{\lambda} + \frac{2}{\lambda^2}\right) + \frac{2}{\lambda^2}$

e **i**



ii $\mathrm{E}(T) = 2, \ \mathrm{Var}(T) = 4$ **iii** $1 - e^{-0.5} \approx 0.393$

iv $2 \ln 2$ minutes

15 **a** **i** $2a + 1 \leqslant y \leqslant 2b + 1$ **ii** $F_Y(y) = F_X\left(\frac{y-1}{2}\right)$

iii $f_Y(y) = \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$

iv $\mathrm{E}(Y) = 2\mathrm{E}(X) + 1, \ \mathrm{Var}(Y) = 4\mathrm{Var}(X)$

EXERCISE 28C.1

1 **B**, **D**, and **F**

2 **a** The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

b The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

c The times may be affected by:

- the distance that the students live from their school
- walking speed
- physical fitness
- the terrain.

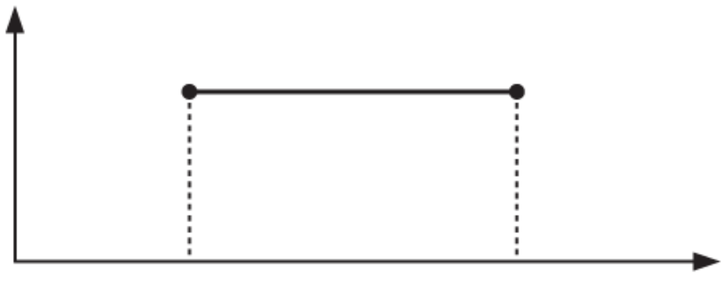
3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



b The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



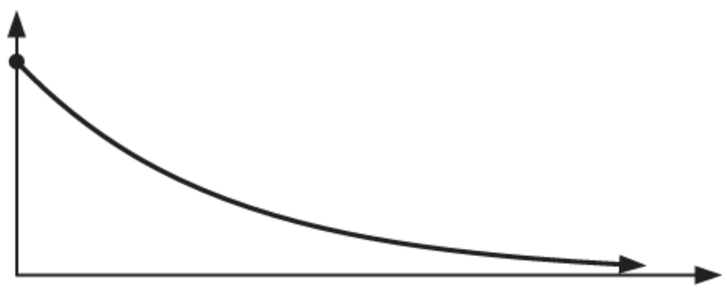
c The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



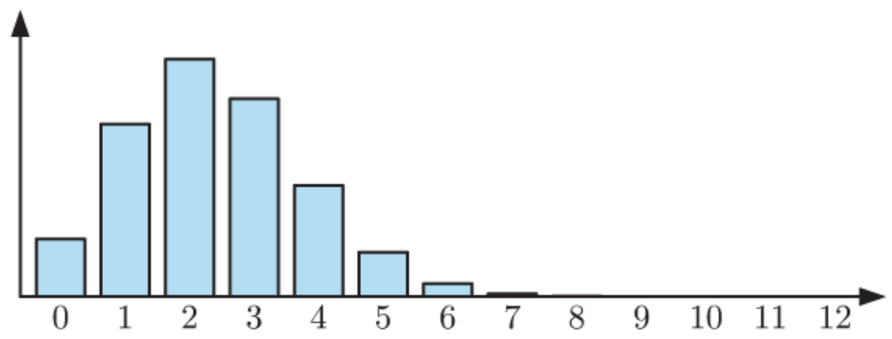
d The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



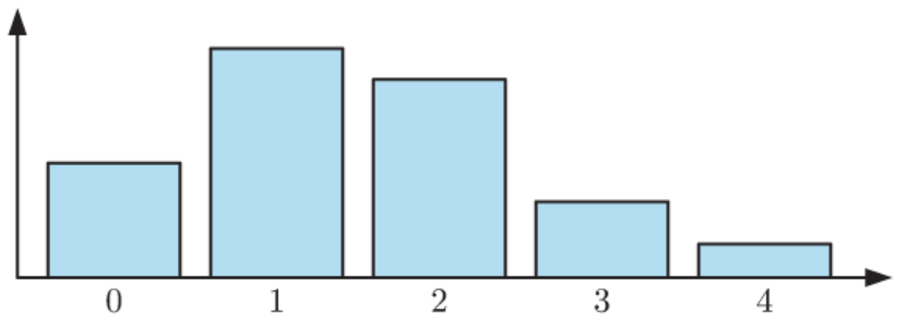
e The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



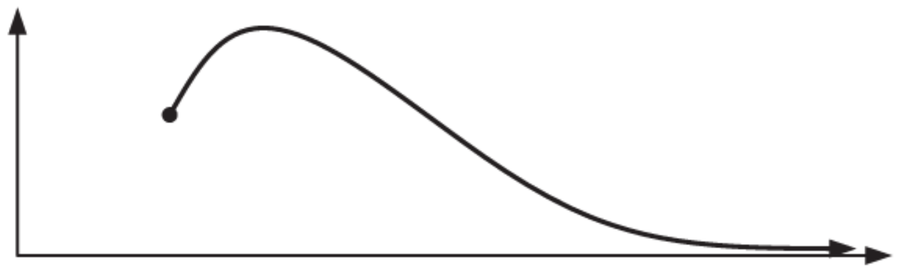
f The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



g The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



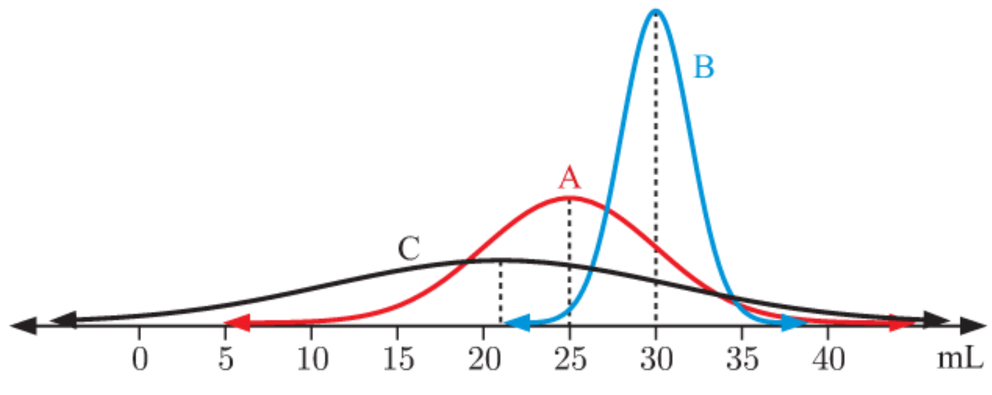
h The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



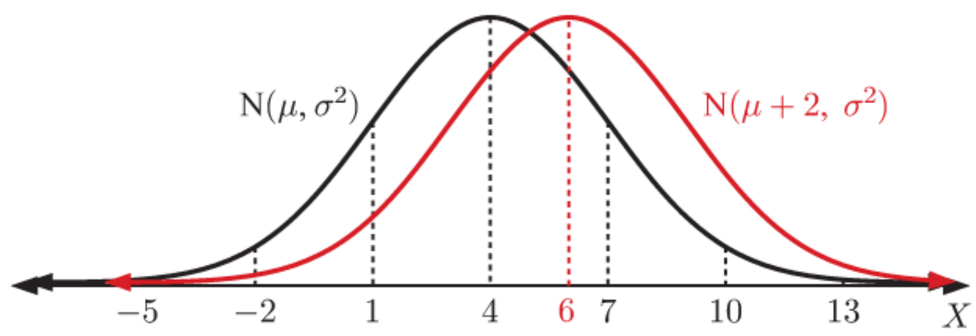
EXERCISE 28C.2

1 **a** B **b** D **c** A **d** C

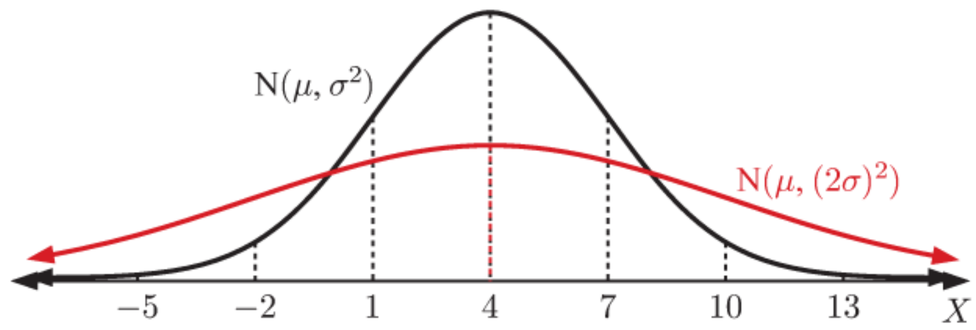
2



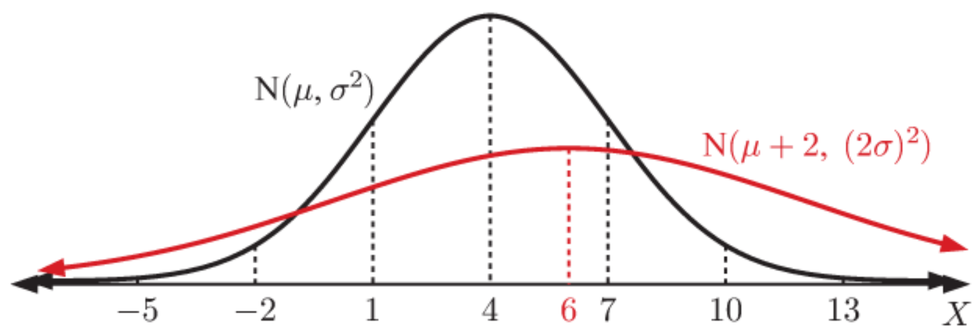
3 **a**



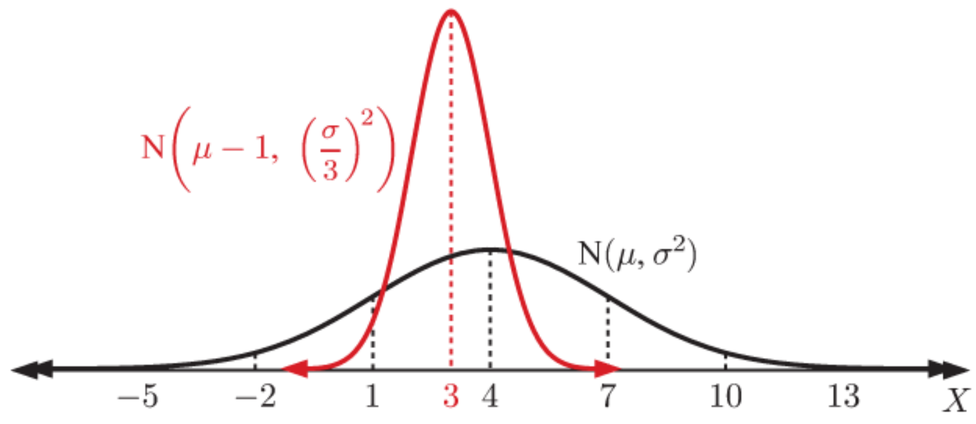
b



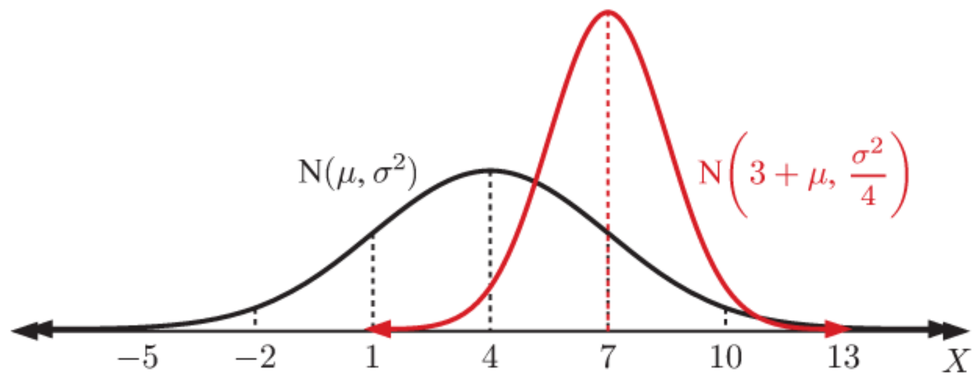
c



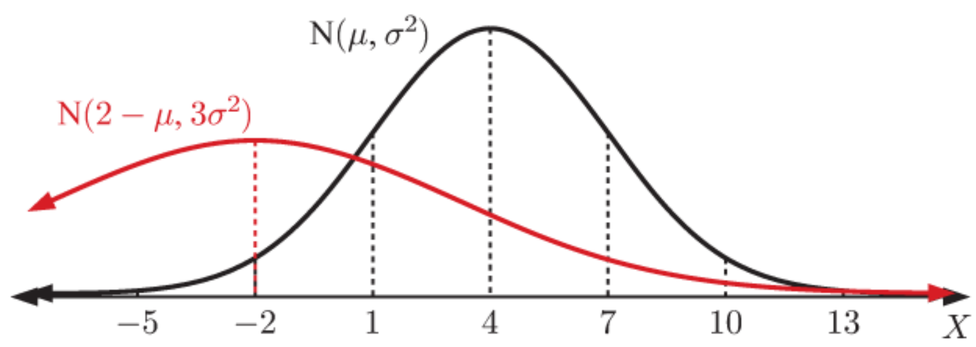
d



e



f



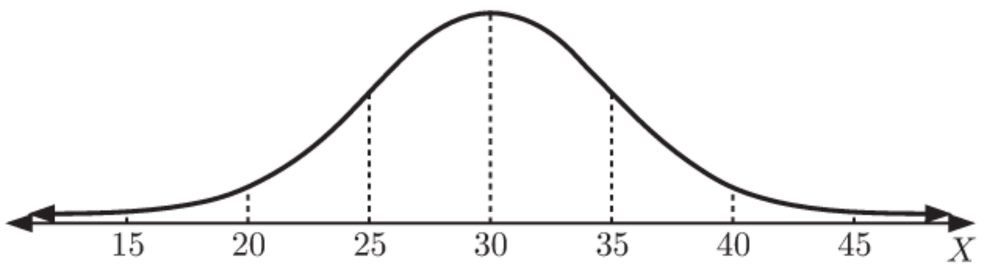
5 **d** **i** **Another hint:** Complete the square in the exponent of e.

EXERCISE 28D.1

1 **a** **i** 40 **ii** 25

b **i** 1 standard deviation above the mean
ii 2 standard deviations below the mean
iii 3 standard deviations above the mean

c

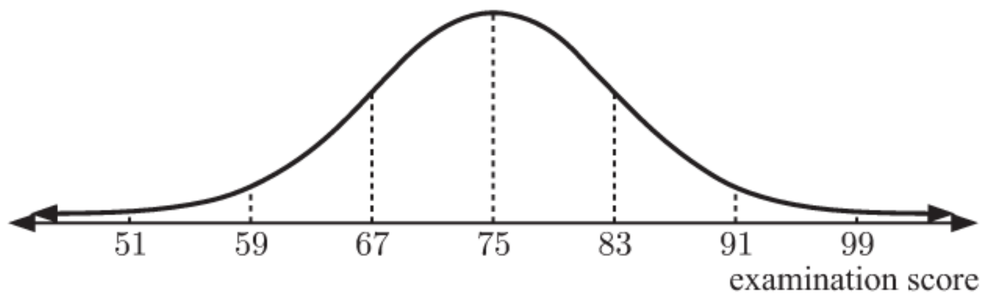


d **i** ≈ 34.13% **e** ≈ 0.1359

2 **a** $\mu = 20, \quad \sigma = 4$

b **i** ≈ 34.13% **ii** ≈ 13.59% **iii** ≈ 2.28%

3 **a**



b **i** ≈ 15.87% **ii** ≈ 2.28% **iii** ≈ 81.85%

4 **a** ≈ 0.6826 **b** ≈ 0.0228

5 **a** **i** ≈ 34.13% **ii** ≈ 47.72%

b **i** ≈ 0.0228 **ii** ≈ 0.8413

c ≈ 68 students **d** $k \approx 178$

6 **a** ≈ 459 babies **b** ≈ 446 babies

7 **a** ≈ 41 days **b** ≈ 254 days **c** ≈ 213 days

8 **a** ≈ 5 competitors **b** ≈ 32 competitors

c ≈ 137 competitors

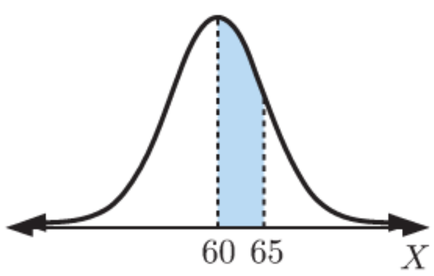
9 **a** $\mu = 176$ g, $\sigma = 24$ g **b** ≈ 81.85%

10 **a** **i** ≈ 84.13% **ii** ≈ 2.28%

b **i** ≈ 0.0215 **ii** ≈ 0.9544 **c** ≈ 0.0223

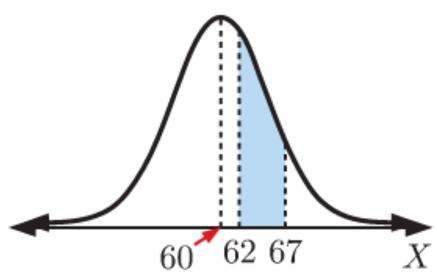
EXERCISE 28D.2

1 **a**



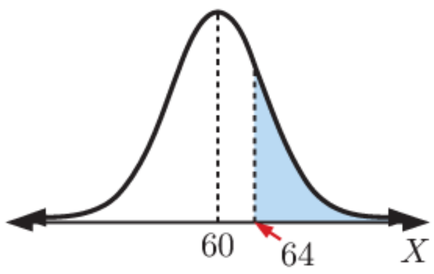
$P(60 \leqslant X \leqslant 65) \approx 0.341$

b



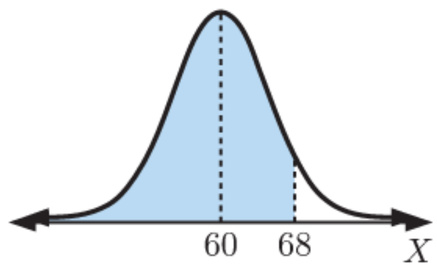
$P(62 \leqslant X \leqslant 67) \approx 0.264$

c



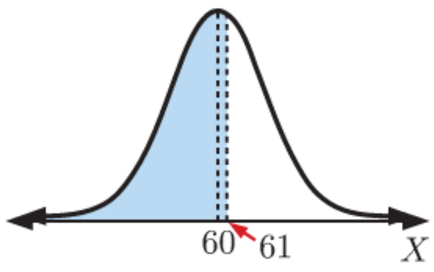
$P(X \geqslant 64) \approx 0.212$

d



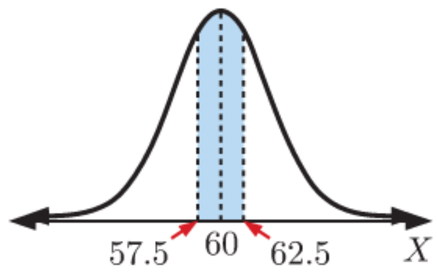
$P(X \leqslant 68) \approx 0.945$

e



$P(X \leqslant 61) \approx 0.579$

f



$P(57.5 \leqslant X \leqslant 62.5) \approx 0.383$

2 **a** ≈ 0.334 **b** ≈ 0.166 **3** ≈ 0.378

4 **a** ≈ 0.303 **b** ≈ 0.968 **c** ≈ 0.309

5 **a** ≈ 0.0509 **b** ≈ 52.1% **c** ≈ 47 eels

6 **a** **i** ≈ 90.4% **ii** ≈ 4.78% **b** $4160

7 **a** **i** ≈ 12.7% **ii** ≈ 52.0%

b **i** 21.6 kL **ii** ≈ 76 customers

8 **a** **i** ≈ 21.5% **ii** ≈ 95.2%

b **i** Enrique **ii** Damien

9 **a** ≈ 10.3% **b** ≈ 0.456

10 **a** ≈ 84.1% **b** ≈ 0.880

EXERCISE 28E.1

1 **a** Emma's z-scores:
English ≈ 1.82, Mandarin ≈ 2.33, Geography ≈ 1.61,
Biology $= 0.9$, Mathematics ≈ 2.27

b Mandarin, Mathematics, English, Geography, Biology

c The scores in each of Emma's classes are normally distributed.

2 **a** Sergio's z-scores:
Physics ≈ -0.463, Chemistry ≈ 0.431,
Mathematics ≈ 0.198, German ≈ 0.521,
Biology ≈ -0.769

b German, Chemistry, Mathematics, Physics, Biology

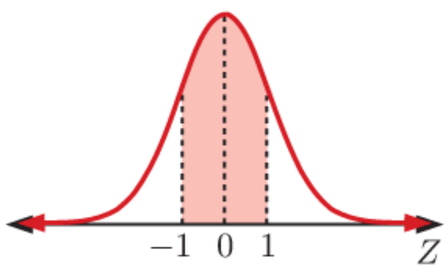
3 **a** Frederick's z-scores:
50 m freestyle ≈ 1.95, 100 m backstroke ≈ -1.07,
200 m breaststroke ≈ -0.578, 100 m butterfly ≈ 0.345

b Lower times are better as they indicate that the person swims faster.

c 100 m backstroke, 200 m breaststroke, 100 m butterfly, 50 m freestyle

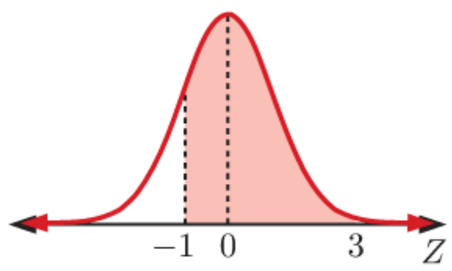
EXERCISE 28E.2

1 **a**



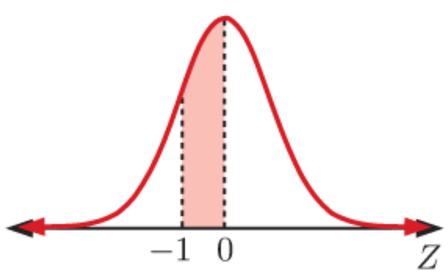
$P(-1 < Z < 1) \approx 0.683$

b



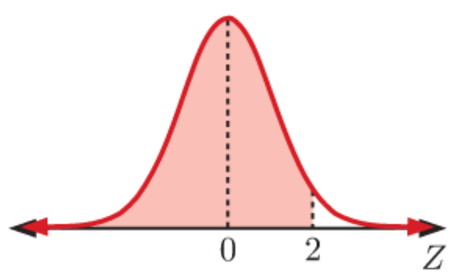
$P(-1 \leqslant Z \leqslant 3) \approx 0.840$

c



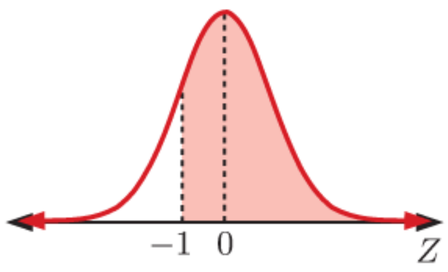
$P(-1 < Z < 0) \approx 0.341$

d



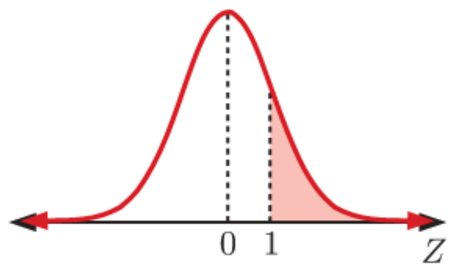
$P(Z < 2) \approx 0.977$

e



$P(-1 < Z) \approx 0.841$

f

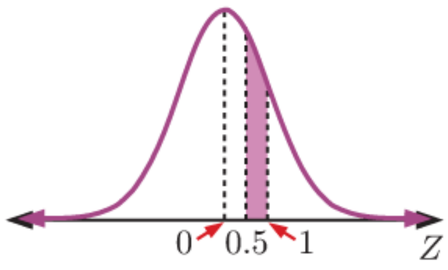


$P(Z \geqslant 1) \approx 0.159$

2 **a** 0 **b** 1

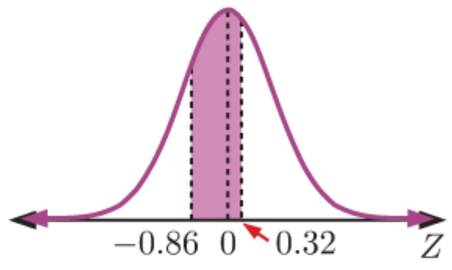
3 **a** $a = -1,\ b = 2$ **b** $a = -0.5,\ b = 0$
c $a = 0,\ b = 3$

4 **a**



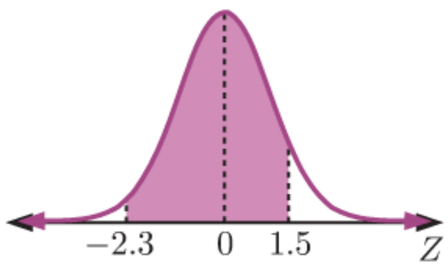
$P(0.5 \leqslant Z \leqslant 1) \approx 0.150$

b



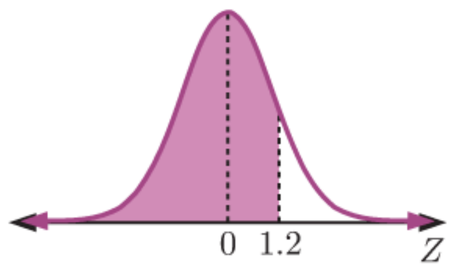
$P(-0.86 \leqslant Z \leqslant 0.32) \approx 0.431$

c



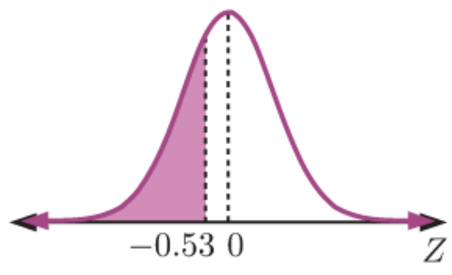
$P(-2.3 \leqslant Z \leqslant 1.5) \approx 0.922$

d



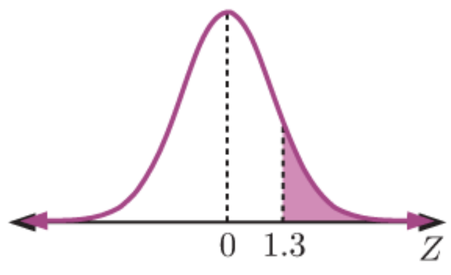
$P(Z \leqslant 1.2) \approx 0.885$

e



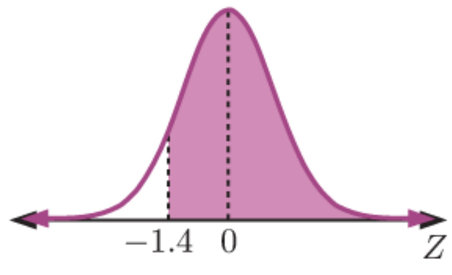
$P(Z \leqslant -0.53) \approx 0.298$

f



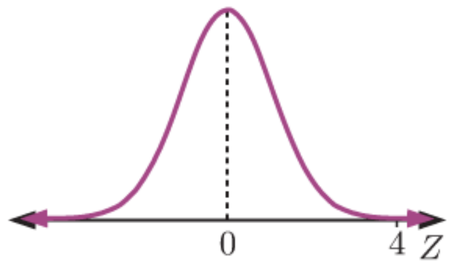
$P(Z \geqslant 1.3) \approx 0.0968$

g



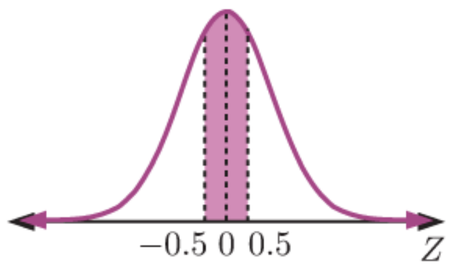
$P(Z \geqslant -1.4) \approx 0.919$

h



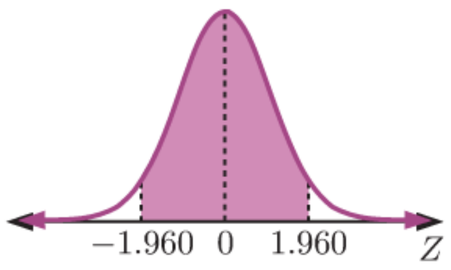
$P(Z > 4) \approx 3.17 \times 10^{-5}$

i



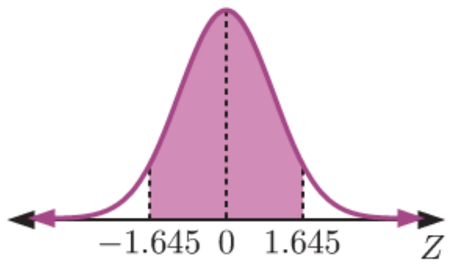
$P(-0.5 < Z < 0.5) \approx 0.383$

j



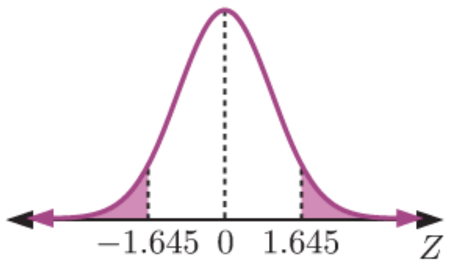
$P(-1.960 \leqslant Z \leqslant 1.960) \approx 0.950$

k



$P(-1.645 \leqslant Z \leqslant 1.645) \approx 0.900$

l



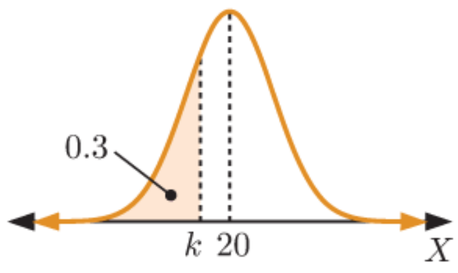
$P(|Z| > 1.645) \approx 0.100$

5 **a** **ii** ≈ 0.976 **b** **i** ≈ 0.910 **ii** ≈ 0.302

6 **a** **i** $z_1 \approx -0.859,\ z_2 \approx 1.18$ **ii** ≈ 0.687

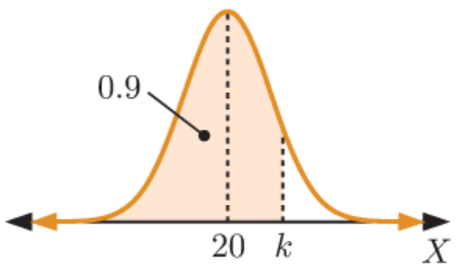
EXERCISE 28F.1

1 **a**



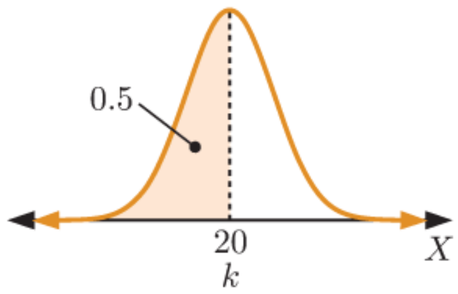
$k \approx 18.4$

b



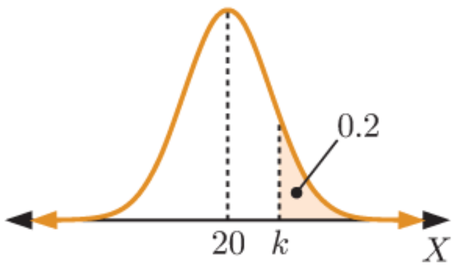
$k \approx 23.8$

c



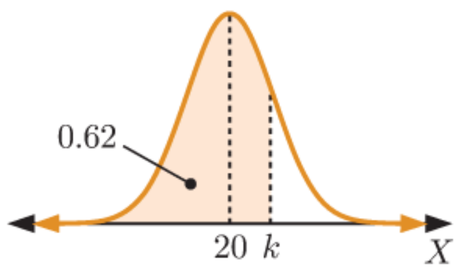
$k = 20$

d



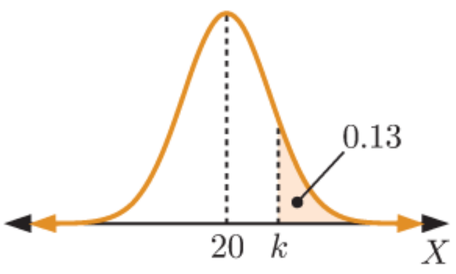
$k \approx 22.5$

e



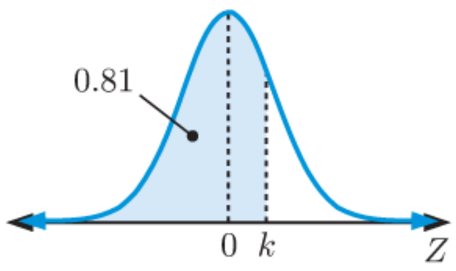
$k \approx 20.9$

f



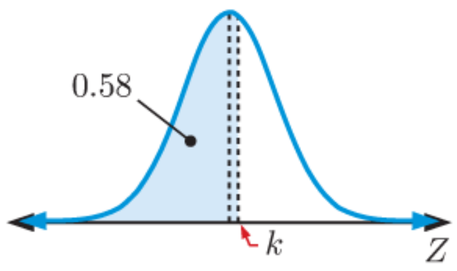
$k \approx 23.4$

2 **a**



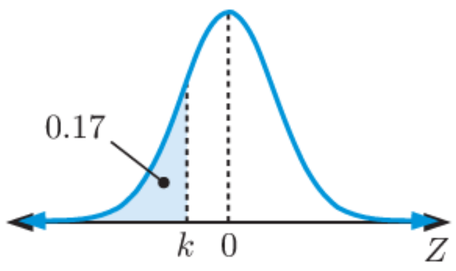
$k \approx 0.878$

b



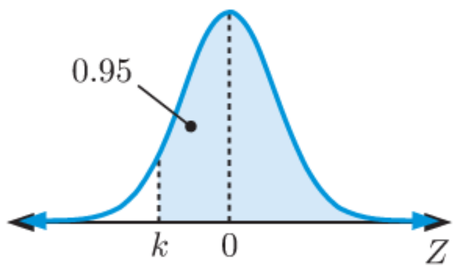
$k \approx 0.202$

c



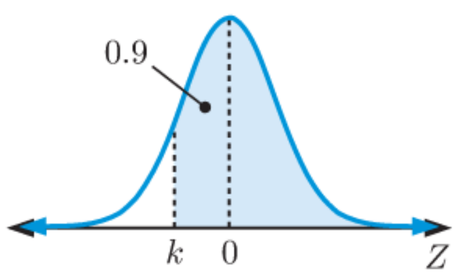
$k \approx -0.954$

d



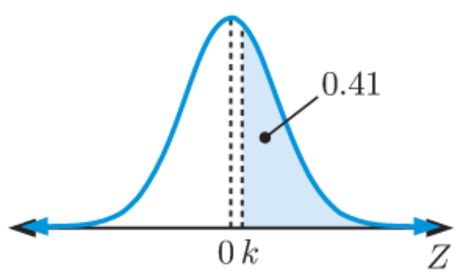
$k \approx -1.64$

e



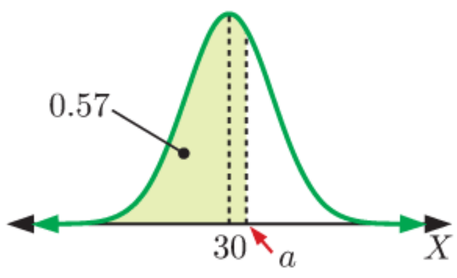
$k \approx -1.28$

f



$k \approx 0.228$

3 **a**



$\therefore \quad a > 30$

b $a \approx 30.9$

c **i** 0.43

ii 0.07

4 **a** $k \approx 12.5$ **b** $k \approx 18.8$ **c** $k \approx 4.93$

5 **a** ≈ 0.212 **b** $k \approx 75.1$

6 **a** $a \approx 42.0$ **b** $a \approx 46.7$ **c** $a \approx 40.1$

7 ≈ 24.7 cm **8** ≈ 75.2 mm

9 ≈ 501.8 mL to 504.0 mL **10** $\approx 31.0°$C

EXERCISE 28F.2

1 **a** Greater. Data values less than 40 make up only 20% of all values.

b $\mu \approx 45.0$

2 $\sigma \approx 3.90$ **3** ≈ 112 **4** ≈ 0.193 m

5 $\approx$ €96.48 **6** $\approx$ 4:01:24 pm **7** $\mu \approx 23.6$, $\sigma \approx 24.3$

8 **a** $\mu \approx 52.4$, $\sigma \approx 21.6$ **b** $\approx 54.3\%$

9 **a** $\mu \approx 4.00$ cm, $\sigma \approx 0.003\,53$ cm **b** ≈ 0.603

10 **a** $\mu \approx 2.00$ cm, $\sigma \approx 0.0305$ cm **b** ≈ 0.736

c ≈ 0.153

REVIEW SET 28A

1 **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\pi}{6}$ **2** **b** ≈ 0.0998 **c** $\frac{1}{2}$ **d** $\frac{19}{4}$

3 **a** $a = \frac{1}{4}$ **b** $\frac{7}{16}$ **c** $\frac{16}{15}$ **d** $\frac{\sqrt{44}}{15}$

4 **a** $\approx 2.28\%$ **b** $\approx 68.26\%$ **c** $\approx 95.44\%$

5 **a** $\approx 50.2\%$ **b** ≈ 7 oysters

6 **a** Harri's test score is 2 standard deviations below the mean.

b $\approx 97.7\%$ **c** 7

7 **a** ≈ 0.364 **b** ≈ 0.356 **c** $k \approx 18.2$

8 **a** $\approx 6.68\%$ **b** ≈ 0.854

9 **a** $k \approx 28.1$ **b** $k \approx 26.5$ **c** $k \approx 25.0$

10 **a** ≈ 0.260 **b** ≈ 29.3 weeks **11** ≈ 0.842

12 **a** $\mu = 29$, $\sigma \approx 10.7$ **b** **i** ≈ 0.713 **ii** ≈ 0.250

13 **a** **i** ≈ 0.0736 **ii** ≈ 0.0406 **b** ≈ 0.644

14 **a** ≈ 0.330 **b** ≈ 0.796

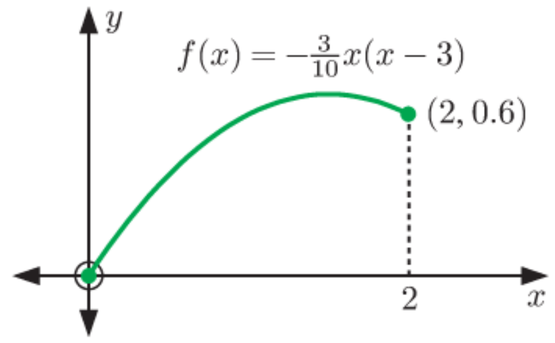
REVIEW SET 28B

1 **b** ≈ 0.414 **c** greater **d** $\frac{9}{4}$

2 **a** **i** $\approx 2.28\%$ **ii** $\approx 84.0\%$ **b** ≈ 0.3413

3 **a** $a = -\frac{3}{10}$

b



c **i** $\frac{6}{5}$

ii $\frac{6}{25}$

iii $\frac{\sqrt{6}}{5}$

d $\frac{13}{20}$

4 **a** **i** $\approx 76.1\%$ **ii** $\approx 96.0\%$ **b** ≈ 0.598

c $x \approx 61.9$

5 $k \approx 1.96$ **6** **a** ≈ 0.479 **b** ≈ 0.0766 **c** $k \approx 55.2$

7 ≈ 162 seconds

8 **a** $a \approx 9.05$ **b** $a \approx 13.7$ **c** $a \approx 10.4$ **9** 8.97

10 ≈ 0.0708

11 **a** **i** ≈ 0.0362 **ii** ≈ 0.610 **iii** ≈ 0.566

b $k \approx 74.4$ **c** $a \approx 81.0$, $b \approx 102$ **d** ≈ 0.506

12 **a** $\approx 68.3\%$ **b** ≈ 0.0884

13 **a** **i** ≈ 0.722 **ii** ≈ 0.798 **b** ≈ 0.0563

14 **a** $k = 8$ **b** $2\frac{2}{7}$ **c** $\mathrm{E}(X) \approx 2.75$, $\mathrm{Var}(X) \approx 2.83$

d $\mathrm{E}(Y) \approx 5.40$, $\mathrm{Var}(Y) \approx 16.3$

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