Finding the centre of mass for 2D and 3D objects with uniform density

Introduction

The centre of mass is vital in knowing the stability of the structure. If we know the centre of mass of a structure, we can calculate the tipping point of an object. Knowing the centre of mass can help us construct shapes that are harder to knock over, which can then have important uses in automotive construction, aeronautics, and architecture. In a scenario that the centre of mass is too high, skyscrapers can topple over during an earthquake – even if they are resistant enough to withstand it, as seen in **Fig. 1**. The first building has a high centre of mass and thus, when tilted, the centre of mass no longer passes through the base. This would cause it to topple over. On the other hand, the second building's centre of mass is much lower and passes through the base even when tilted, hence it is unlikely to topple over. This is also the reason that the popular Leaning Tower of Pisa does not fall (**Fig. 2**).

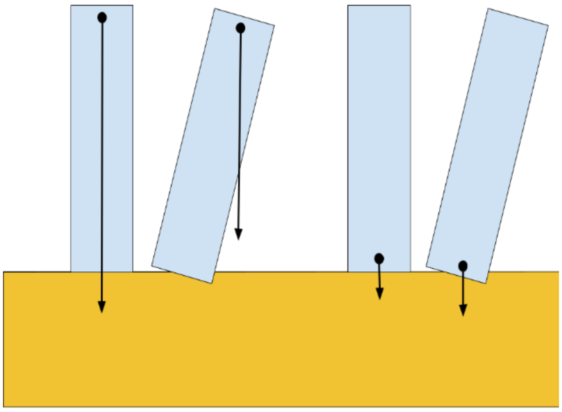

Fig. 1 – Centre of mass passing through a building

Fig. 2 – Centre of mass ensuring that the Leaning Tower of Pisa doesn't tip

As shown, a low centre of mass can lower the risk of tall objects falling over or can even lower the risk of race cars flipping when making sharp turns. Thus, it is of great importance in ensuring our safety, and many objects should be carefully designed to guarantee our safety by confirming the centre of mass mathematically. I look forward to trying to improve the safety of different products across various fields through exploring the topic of centre of mass.

In my individual assessment, I will attempt to create general formulae to find the centre of mass for different shapes. By doing so, I wish to make it so that we can better mathematically determine the improvements that can possibly be made to the aforementioned fields.

Finding the centre of mass in a 2D lamina with uniform density

First, we must find the centre of mass in a 2D lamina[1]. A lamina is a sheet that is thin enough that it can be treated as though it is two-dimensional. We will be assuming that the density of the lamina is uniform, as this can affect the placement of the centre of mass.

A lamina can be represented as a 2D region. The geometric centre of this region is known as the centroid[1]. On either side of the centroid, the area will be equal. On the other hand, on either side of the centre of mass, the mass will be equally distributed. However, as we are assuming that the lamina has uniform density, the geometric centre, the centroid, will also correspond to the centre of mass.

Let's take a look at **Fig. 3** to see the general lamina for which we will be finding the centre of mass.

[1] LibreTexts. (2021, January 2). *6.6: Moments and Centers of Mass*. Mathematics LibreTexts. https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/06%3A_Applications_of_Integration/6.6%3A_Moments_and_Centers_of_Mass

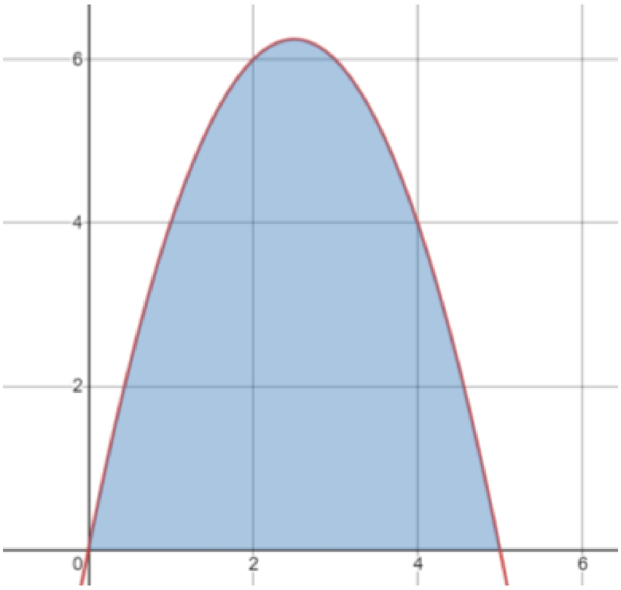

Fig. 3 – Diagram of a lamina modelled by a function f(x)

The lamina is given by the shaded region, bound by a continuous function $f(x)$ and the x-axis. The function used here is $f(x) = x^2 + 5x$, however any function can be used to create a lamina. To find the centre of mass, we need to find the moments of the lamina with respect to the x and y axes as well as the total mass of the lamina. Moment is defined as the product of the mass times the distance from the axis[2]. Then, we can use the following formula[1] to find the coordinates of the centre of mass –

$$x_c = \frac{M_y}{m} \; and \; y_c = \frac{M_c}{m}$$

Where x_c and y_c are the x-coordinate and y-coordinate of the centre of mass, respectively, and where M_x and M_y are the moments with respect to the x and y axes, respectively. m is the total mass of the lamina. To find the horizontal moment, we need to look at the moment with respect to the y-axis. Hence, the lamina's moment with respect to y-axis gives us the x-coordinate of

[2] LibreTexts. (2020, December 21). *3.7: Moments and Centers of Mass*. Mathematics LibreTexts. https://math.libretexts.org/Bookshelves/Calculus/Supplemental_Modules_(Calculus)/Vector_Calculus/3%3A_Multiple_Integrals/3.7%3A_Moments_and_Centers_of_Mass

the centroid. To find the vertical moment, we look at the moment with respect to the x-axis, which is why it is used to find the y-coordinate.

We can now split this curve into several small rectangles by partitioning the interval between the roots 0 and 5 into several n number of rectangles. The m^{th} rectangle would have $x_{m+1} - x_m$ as the interval, where $0 < m < n,\ m \in Z+$. **Fig. 4** shows one such rectangle.

Rectangles always have their centre of mass at the vertical and horizontal centres of the shape when the density is uniform. This is because the horizontal and vertical centres split the rectangle into two symmetrical halves and thus have the same area and mass on both sides, assuming density is constant. This is the symmetry principle[3] which states that if a region, R, is symmetrical about a line, L, then the centroid will lie on that line or the intersection of the lines of symmetry.

We will denote the x-coordinate of the centroid of the m^{th} rectangle as $x_m{}^*$, where $0 < m < n$. The width of one rectangle would just be the change in the x, or $x_{m+1} - x_m$, which is extremely small and can be denoted by Δx. The height of a rectangle can be determined by plugging this point into the function $f(x)$, which is $f(x_m{}^*)$. The height of the centre of mass of the rectangle will be half of the vertical length, that is, the vertical centre. Thus, the centre of mass for rectangle m will be at $\left(x_m{}^*, \frac{f(x_m{}^*)}{2}\right)$, which can be seen in **Fig. 4** below.

[3] *Centre of Mass. (n.d.). Isaac Physics. https://isaacphysics.org/concepts/cp_centre_mass#:~:text=An%20object%20or%20collection%20of,some%20point%20along%20that%20line.*

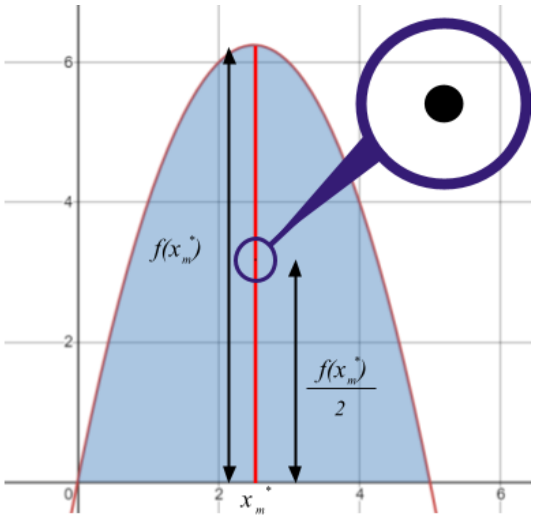

Fig. 4 – Splitting the region into small rectangles. One rectangle (red) shown with the coordinates of its centre of mass

The density of the lamina can be represented by ρ. We are now equipped to find the area and mass of the rectangles. First, the area of the rectangle would be given by $l \times w$, or in our scenario, $f(x_m{}^*) \times \Delta x$. The mass of the rectangle would just be given by multiplying the density, ρ, with our area, thus giving us the formula for finding the mass, which is –

$$\rho f(x_m{}^*)\Delta x$$

To approximate the mass of the whole lamina, we have to add all masses of the rectangles –

$$m = \sum_{m=1}^{n} \rho f(x_m{}^*)\Delta x$$

By taking an infinite number of rectangles, we can approximate the area under the curve with a method in calculus known as the Riemann sum[4]. We can take the limit of n to infinity to estimate the area under the curve as follows –

$$\lim_{n\to\infty} \sum_{m=1}^{n} \rho f(x_m{}^*)\Delta x$$

[4] *Left & right Riemann sums (article)*. (n.d.). Khan Academy. https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/a/left-and-right-riemann-sums

$$= \int_a^b \rho f(x)\,dx$$

$$= \rho \int_a^b f(x)\,dx$$

Where a and b are the limits of the integration which bound the lamina, which in our scenario are the roots of the function, 1 and 5.

Now that we know the total mass of the lamina, we can move on to finding the total moment with respect to the x-axis. We can treat the rectangle as though it were a point mass located at the centre of mass. This can be done as the centre of mass is a theoretical point where all the mass can be considered to be concentrated. Thus, the moment of the lamina with respect to the x-axis is given by mass multiplied by the distance from the x-axis.

We have already derived that the mass of the rectangle is given by $\rho f(x_m{}^*)\Delta x$. The remaining value, the distance of the centre of mass from the x-axis, or the height of the centre of mass, is $\frac{f(x_m{}^*)}{2}$ (**Fig. 4**). Therefore, we get the following formula, which gives us the moment of the rectangle with respect to the x-axis, M_{Rx} –

$$M_{Rx} = \rho f(x_m{}^*)\Delta x \times \frac{f(x_m{}^*)}{2}$$

$$= \rho \frac{\left(f(x_m{}^*)\right)^2}{2} \Delta x$$

However, this is just the moment of one rectangle with respect to the x-axis. We must again use the Riemann Sum, by adding the moments of infinite rectangles and taking the limit to infinity –

$$M_x = \lim_{n\to\infty} \sum_{m=1}^{n} \rho \frac{\left(f(x_m{}^*)\right)^2}{2} \Delta x$$

$$= \int_a^b \rho \frac{\left(f(x)\right)^2}{2} dx$$

$$= \rho \int_a^b \frac{\left(f(x)\right)^2}{2} dx$$

We can follow the same steps to find the moment of the rectangle with respect to the y-axis, M_{Ry}. Looking back at **Fig. 4**, we know the distance from the y-axis to the centroid is ${x_m}^*$, as that is the x-coordinate of the centre of mass. Hence, we can find the moment with respect to the y-axis –

$$M_{Ry} = \rho f({x_m}^*)\Delta x \times {x_m}^*$$

$$= {x_m}^* \rho f({x_m}^*)\Delta x$$

Taking the limit to infinity to find the total moment with respect to the y-axis would give us –

$$M_y = \lim_{n\to\infty} \sum_{m=1}^{n} \rho {x_m}^* f({x_m}^*)\Delta x$$

$$= \int_a^b \rho x f(x)\, dx$$

$$= \rho \int_a^b x f(x)\, dx$$

Plugging the total mass of the lamina and the total moments into our earlier equation which give us the coordinates of the centre of mass, $x_c = \frac{M_y}{m}$ and $y_c = \frac{M_x}{m}$, gives us the final equation–

$$x_c = \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx}$$

$$y_c = \frac{\rho \int_a^b \frac{\left(f(x)\right)^2}{2}\, dx}{\rho \int_a^b f(x)\, dx}$$

$$x_c = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}$$

$$y_c = \frac{\int_a^b \frac{\left(f(x)\right)^2}{2}\, dx}{\int_a^b f(x)\, dx}$$

To prove this, we can try to find the centre of mass of a square lamina and see if it is at the vertical and horizontal centre. We can do this by taking the function $f(x) = 2$, and taking the limits of integration as 2 and 4. This would create the lamina as seen in **Fig. 5**.

The expected point would be (3, 1), as that is the vertical and horizontal geometric centre of the square.

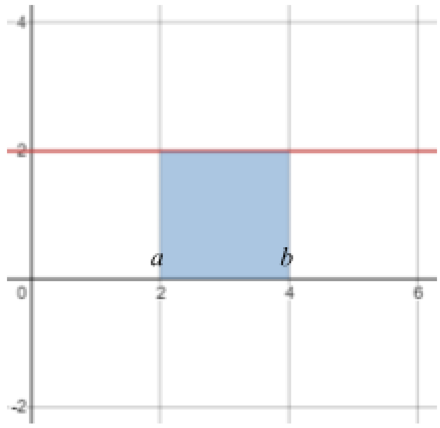

Fig. 5 – Lamina created by f(x)=2 and bound by the region 2<x<4

Now plugging in our values in the formula we derived, we get the following for the x-coordinate –

$$x_c = \frac{\int_2^4 2x\, dx}{\int_2^4 2\, dx}$$

$$= \frac{[x^2]_2^4}{[2x]_2^4}$$

$$= \frac{16-4}{8-4} = \frac{12}{4} = 3$$

Likewise, plugging in our values to find the y-coordinate of the centre of mass gives us–

$$y_c = \frac{\int_2^4 \frac{(2)^2}{2}\, dx}{\int_2^4 2\, dx}$$

$$= \frac{\int_2^4 2\,dx}{\int_2^4 2\,dx} = 1$$

Hence, we get our final coordinates (3,1), which proves that our derived equation was correct.

Combining centre of masses for different laminae to find the centre of mass for an irregular lamina

Now that we know the equation to find the centre of mass of a lamina, we can find the centre of mass of two different 2D laminae and then combine them together to find a new centre of mass for the resulting irregular lamina.

The lamina can be considered as a point mass at the centre of mass as mentioned beforehand – thus, we can find the new centre of mass if we know the coordinates of all the point masses. If we have n number of laminae with the centre of mass at points (x_1, y_1), (x_2, y_2), …, (x_n, y_n), then we can find the total mass of the new lamina created by adding up the individual masses of the component laminae –

$$m = \sum_{\alpha=1}^{n} m_\alpha$$

We can find the moment with respect to the x-axis by the following formula as the moment is the mass multiplied by the distance from the x-axis, or in other words, the y-coordinate of the centre of mass –

$$M_x = \sum_{\alpha=1}^{n} m_\alpha\, y_\alpha$$

Doing the same for the moment with respect to the y-axis we get –

$$M_y = \sum_{\alpha=1}^{n} m_\alpha\, x_\alpha$$

Finally, we can plug in our new mass and moments into our previous formula to get the coordinates of the new combined centre of mass –

$$x_c = \frac{\sum_{\alpha=1}^{n} m_\alpha x_\alpha}{m} \ and \ y_c = \frac{\sum_{\alpha=1}^{n} m_\alpha y_\alpha}{m}$$

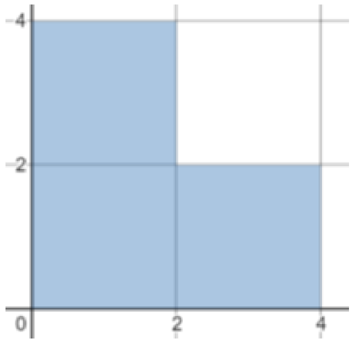

Fig. 6 – Three square laminae

To understand this better, we can look at the three laminae in **Fig. 6**. The three, square laminae have their centre of mass at (1, 1), (3, 1), and (1, 3), respectively.

Let us assume that all have the same unit mass of 4, as they all have the same unit area of 4 and have uniform density. Thus, $m = 4 + 4 + 4 = 12$. And –

$$M_x = 4 \times 1 + 4 \times 1 + 4 \times 3$$

$$= 4 + 4 + 12$$

$$= 20$$

$$y_c = \frac{20}{12} = \frac{5}{3}$$

$$M_y = 4 \times 1 + 4 \times 3 + 4 \times 1$$

$$= 4 + 12 + 4$$

$$= 20$$

$$x_c = \frac{20}{12} = \frac{5}{3}$$

Hence, the final coordinates for the centre of mass of the new lamina would be $(\frac{5}{3}, \frac{5}{3})$.

We can also cut a shape of a lamina out of another lamina through this method. In this case, the mass of the lamina that we are removing would be negative, but the formula remains the same. Looking at **Fig. 7**, we can see a circle lamina cut out of a square lamina and find the centre of mass accordingly.

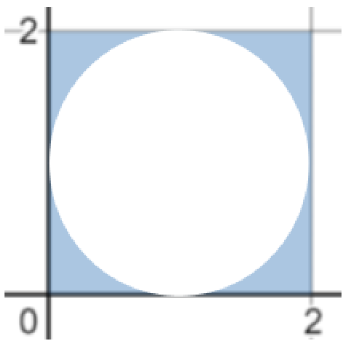

Fig. 7 – A square lamina with a circle lamina cut out from within

The area of the circle is given by πr^2, and as $r = 1$, the area is simply π.

The density is uniform, meaning we can take the mass as π and plug it into the formula. As we are removing the circle, the mass of the circle will be denoted as negative –

$$m = 4 - \pi$$

The centre of mass of the circle is (1,1), just like the square.

Hence, we get the following values for moments –

$$M_x = 4 \times 1 - \pi \times 1 = 4 - \pi$$

$$M_y = 4 \times 1 - \pi \times 1 = 4 - \pi$$

This gives us the final coordinates of the new lamina formed –

$$x_c = \frac{4 - \pi}{4 - \pi} = 1$$

$$y_c = \frac{4 - \pi}{4 - \pi} = 1$$

The centre of mass is remains at (1,1), as at that point, the area of the new lamina is split evenly. This also shows that a centre of mass can be outside a lamina as well.

Finding the centre of mass for 3D shapes using volume of revolution

3D shapes are more complicated than 2D laminae, and thus we can't find a general equation as easily as for 2D shapes. Hence, I will be finding a general equation for 3D cones and segments of cones, as well as for half of an ellipsoid, which forms a turbine. Through the symmetry principle, we already know that shapes such as cubes and cuboids will have their centre of mass at exactly half of their length, width, and height, as that gives the centroid of the regular 3D shape. However, for shapes like cones, truncated cones, and shapes similar to plane turbines, the centre of mass is more complicated to find.

Starting with a cone, we can find the centre of mass through the first moment integral. Similar to the method for finding the centre of mass of a lamina, we can split the cone into several slices of circles starting from the bottom at the largest circle and going up to a point (**Fig. 8**).

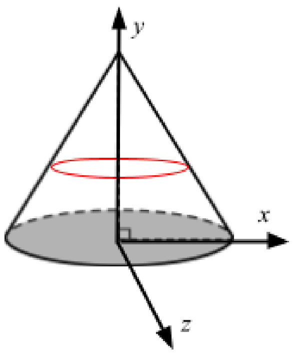

Fig. 8 – A cone with one slice forming a circle (red)

If we imagine the cone to be placed on the $x, y,$ and z axes, we can see that it is symmetrical about two axes, in particular, the x and z axes. Therefore, the centre of mass would lie at a point (0, y_c, 0). To find the y-coordinate of the centre of mass, we have to find the mass of the cone and the moments just like in a lamina. However, in a lamina, the moment is given by mass multiplied by distance. The mass here would be calculated by adding up infinite slices of circles – therefore, we must know the rate of change of volume as we move along the y-axis, that is, dV. The distance of the circle from the origin would be given by the value of y. Instead of integrating along the x-axis as done earlier, to add up the infinite number of circles, we must integrate along the y-axis now. Thus, we can form our equation[5] as follows –

$$\frac{\rho \int_a^b dV \times y \, dy}{m}$$

This can be rewritten as –

$$\frac{\rho \int_a^b dV \times y \, dy}{\rho V}$$

$$= \frac{\int_a^b dV \times y \, dy}{V}$$

[5] Moore, J., Chatsaz, M., d'Entremont, A., Kowalski, J., & Miller, D. (n.d.). *Mechanics Map - The Centroid and Center of Mass in 3D via the First Moment Integral*. Mechanics Map. http://mechanicsmap.psu.edu/websites/A2_moment_intergrals/centroids_3D/centroids3D.html

Here, dV is an equation that describes the cross-sectional area of the shape along the y direction for any given y value.

This equation would differ for cones of different radii and sizes, but we can still formulate a general equation. To do so, we must first take a simple linear equation and use volume of revolution to find the volume of the cone formed (**Fig. 9**).

We will write it in the form of $x = my + c$, as we will be finding the volume of revolution around the y-axis. The gradient will be taken as negative so that we can form a downward sloping line and revolve it around the axis to form a cone. For example, we can take $x = -0.5y + 2$, as seen in **Fig. 9**.

However, the points at which the line cuts the y-axis and x-axis, p and q, can be set as variables to form a large variety of cones. If p is set as a high value, then the cone will be taller and if q is set as a large value then the cone will be wider, with a greater radius.

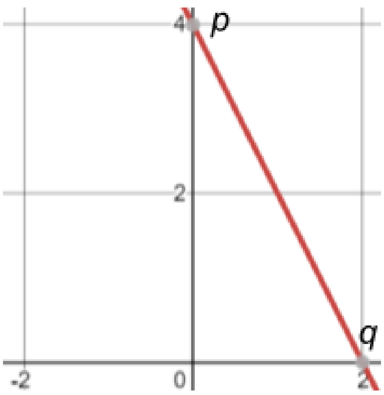

Fig. 9 – A linear equation formed by $x = -0.5y + 2$

We can find the equation of the line by looking at the points it cuts the axes. $x = my + c$, where c is where the line cuts the x-axis, that is, q. At point p, $x = 0$ and hence $0 = mp + q$. This means that $m = -\frac{q}{p}$. We now have the equation of our line –

$$x = -\frac{q}{p}y + q$$

Now that we have the equation of our line, we can plug it into the formula for volume of revolution, $V = \int_a^b \pi x^2 \, dy$. This gives us the following equation –

$$V = \pi \int_a^b \left(-\frac{q}{p}y + q\right)^2 dy$$

$$= \pi \int_a^b \left(\frac{q^2}{p^2} y^2 - \frac{2q^2}{p} y + q^2\right) dy$$

$$= \pi \left[\frac{q^2}{3p^2} y^3 - \frac{2q^2}{2p} y^2 + q^2 y\right]_a^b$$

$$= \pi \left(\left(\frac{q^2}{3p^2} b^3 - \frac{2q^2}{2p} b^2 + q^2 b\right) - \left(\frac{q^2}{3p^2} a^3 - \frac{2q^2}{2p} a^2 + q^2 a\right)\right)$$

This equation then gives us the formula to find the volume of every type of cone that can be created which depend on the variables q and p. To create a cone, the limits of the integration b will be the highest point of the cone and a will be the lowest point, that is, the base of the cone which is equal to 0. Hence, we can rewrite the equation to find the volume as –

$$\pi \left(\frac{q^2}{3p^2} b^3 - \frac{2q^2}{2p} b^2 + q^2 b\right)$$

Now that we know our volume, we need to find an equation for the rate of change of volume at any given y point. As mentioned earlier, this is the same as the cross-sectional area. Thus, we need to find an equation for the cross-sectional area at any given point on the y-axis.

For a cone this is relatively easy, as the circle formed by the cross-sectional area (**Fig. 8**).

decreases linearly according to the equation of the line.

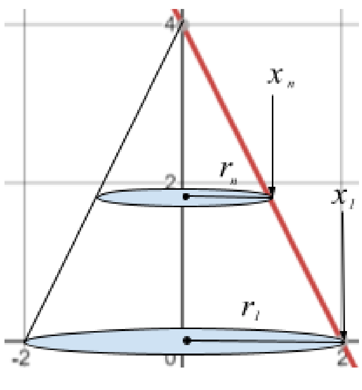

Fig. 10 – Different radii at different y values

The area of the circle is given by πr^2. Looking at **Fig. 10**, we can see that the radius of the circle corresponds with the x value at any given y value. We already have the equation of the line and can replace x with r. This can be done as they both have the same value, given that the centre of the cone is at the origin – which it will be, as we used volume of revolution to create it. Hence, we get our equation for $dV = \pi r^2$, where $r = -\frac{q}{p}y + q$. Therefore –

$$dV = \pi\left(-\frac{q}{p}y + q\right)^2$$

Then, by plugging in our equation for the volume and dV into our original equation of $\frac{\int_a^b dV \times y\ dy}{V}$, we can find our final equation, which is –

$$\frac{\int_a^b \pi\left(-\frac{q}{p}y + q\right)^2 y\, dy}{\pi\left(\frac{q^2}{3p^2}b^3 - \frac{2q^2}{2p}b^2 + q^2 b\right)}$$

$$= \frac{\pi\left[\frac{q^2}{4p^2}y^4 - \frac{2q^2}{3p}y^3 + \frac{q^2y^2}{2}\right]_a^b}{\pi\left(\frac{q^2}{3p^2}b^3 - \frac{q^2}{p}b^2 + q^2 b\right)}$$

$$= \frac{\left(\frac{q^2}{4p^2}b^4 - \frac{2q^2}{3p}b^3 + \frac{q^2b^2}{2}\right) - \left(\frac{q^2}{4p^2}a^4 - \frac{2q^2}{3p}a^3 + \frac{q^2a^2}{2}\right)}{\left(\frac{q^2}{3p^2}b^3 - \frac{q^2}{p}b^2 + q^2 b\right)}$$

However, $a = 0$ as mentioned earlier. Thus, our equation becomes –

$$\frac{\left(\frac{q^2}{4p^2}b^4 - \frac{2q^2}{3p}b^3 + \frac{q^2b^2}{2}\right)}{\left(\frac{q^2}{3p^2}b^3 - \frac{q^2}{p}b^2 + q^2 b\right)}$$

And thus, we can find the final y-coordinate of the centre of the mass of any cone. The centre of mass would be at $(0, \frac{\left(\frac{q^2}{4p^2}b^4 - \frac{2q^2}{3p}b^3 + \frac{q^2b^2}{2}\right)}{\left(\frac{q^2}{3p^2}b^3 - \frac{q^2}{p}b^2 + q^2b\right)}, 0)$. Furthermore, we can also find the centre of mass of any segment of the cone by slicing the cone vertically. We simply have to make the limit of the integrations different. For example, by making the upper limit of the integration, b, to a value which is not at the top of the cone, but rather, in the middle of the cone, and then using this same formula so that the value of p and q remain the same, we can find the centre of mass for an object such as a truncated cone in **Fig. 11**.

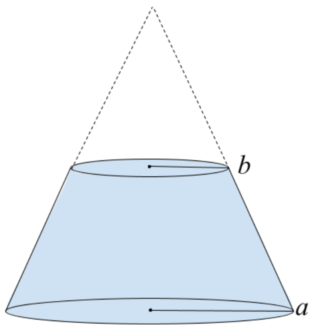

Fig. 11 – Figure formed by setting a smaller value for b

We can find the centre of mass for other 3D shapes as well, using the same method and different equations. For example, using a quadratic equation will give us a smooth aeroplane turbine shaped 3D object. We can also make a variety of prisms, such as a triangular prism, by finding different equations for dV and V. I decided to find the general equation for any cone and truncated cone, as they are one of the most basic and important which are used in a wide variety of products. For example, a truncated cone can be used to resemble a car exhaust or a different type of turbine. We already know the centre of mass of cubes, cuboids, and cylinders as there is so slope, and hence, we can easily find the centroid of the shapes. Furthermore, by combining different 3D shapes together, we can find the centre of mass of more irregular and unique shapes, which can be pieced together to create a new product. For example, placing a cone or triangular prism on a cuboid can give us a simple approximation of a house with a slanted roof. The walls of the house will be mostly made from one material such as brick or concrete and hence will have nearly uniform density, and the roof can also be mainly made from one material and can be said to have uniform density.

Combining 3D shapes to find the centre of mass of an irregular 3D object

Just like with 2D laminae, 3D shapes can also be combined to form an irregular 3D shape[6], and the new centre of mass can be found. This is done through the method of composite parts. For this to work, similar to what we did for the 2D laminae, we must know the x, y, and z coordinates of the 3D objects. We can then treat the centre of mass as a point object and multiply the x, y, and z coordinates of the centre of mass with the mass of the 3D object to find the moment. We can further add all the moments of the 3D objects present in the composite shape to find the total moment in the directions of the x, y, and z axes. To finish, we simply add up the masses of every 3D shape to find the total mass, and then we can find the coordinates of the centre of mass of the composite shape using the formula below. I will use x_k for the coordinates of the new centre of mass to differentiate it from its composite parts' coordinates–

$$x_k = \frac{Moment\ in\ x\ direction}{total\ mass}$$

Hence, we can write the full equation as follows –

$x_k = \frac{\sum_{i=1}^{n} m_i x_{c_i}}{m_{total}}$ for the x-coordinate

$y_k = \frac{\sum_{i=1}^{n} m_i y_{c_i}}{m_{total}}$ for the y-coordinate

$z_k = \frac{\sum_{i=1}^{n} m_i z_{c_i}}{m_{total}}$ for the z-coordinate

[6] Moore, J., Chatsaz, M., d'Entremont, A., Kowalski, J., & Miller, D. (n.d.-a). *Mechanics Map - Centroids via the First Moment Integral*. Mechanics Map. http://adaptivemap.ma.psu.edu/websites/A2_moment_intergrals/method_of_composite_parts/methodofcompositeparts.html#:%7E:text=To%20use%20the%20method%20of,relative%20to%20this%20origin%20point.

Thus, we can find the coordinates for the centre of mass of a new, irregular 3D shape. We can also use the same theory as we did with the laminae to find the centre of mass of irregular objects by removing a portion of the object by taking the mass as negative.

Conclusion

In this exploration, my aim was to investigate centre of masses by creating mathematical models for finding the centre of mass in 2D and 3D shapes. To start with, I found a general equation to find the centre of mass in any 2D sheet or lamina. I then combined various 2D laminae to create a new lamina with a different centre of mass. This method is particularly useful, as it allows us to find the centre of mass for a wide range of laminae with several different combinations.

Next, I attempted to find the centre of mass in 3D shapes. This is significantly more difficult than 2D shapes, as the cross-sectional area differs for each shape. This makes it extremely hard to create a general formula. However, by using volume of revolution, I was able to create a general equation to find the centre of mass in cones and truncated cones. My equation can be modified for other shapes as well, such as a smooth airplane turbine. By considering different cross-sectional areas, it is also possible to create a wide variety of shapes from prisms to cones. Cubes, cuboids and cylinders and other such prism shapes which have no slope have their centre of mass at the centre of the width, height, and length of the shape. Therefore, there is no need to formulate an equation for these shapes.

Once I was able to find the centre of mass of the cone and truncated cone, I moved onto creating a composite 3D shape from other 3D shapes. To do this, we need to know the coordinates of the centre of mass for each shape, which can be calculated, as described earlier. By linking together cones, cubes, cuboids, and cylinders, it is possible to create complex unique 3D objects and find their centre of mass. Moreover, the 3D objects can also be removed to insert a hollow space in the combined 3D object, by taking the mass as negative. This allows for a wide array of objects.

Limitations and Applications:

Although the I was able to create a general formula for 3D cones which can be modified to create other shapes as well, the method to find the centre of mass is still not perfect for 3D objects. Some such calculations are far too complicated, and thus must be calculated either using computer tools or through experimental methods. Further research can be done in creating general formulae for other common 3D shapes to improve upon the method for finding centre of mass in 3D objects.

Overall, the investigation proved to be successful, and general equations were able to be formulated. These may further be used to make basic calculations and approximations when designing a product which requires the centre of mass to be considered, thus potentially improving the safety of the product with regards to the fields mentioned before. The results of this investigation can be effectively applied in the design of a variety of products and show the investigation was successful.

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