Investigating the relationship between radius and pressure difference in a tube

Introduction:

Tubes are an essential device that have thousands of applications in our daily lives. In the form of a blood vessel, tubes allow the transportation of blood cells, nutrients, and oxygen to the tissues of our bodies. However, these blood vessels slowly become narrower through a buildup of plaque, resulting in higher blood pressure and excess strain in order to keep the blood flow rate constant[1]. This can be demonstrated by the following diagram:

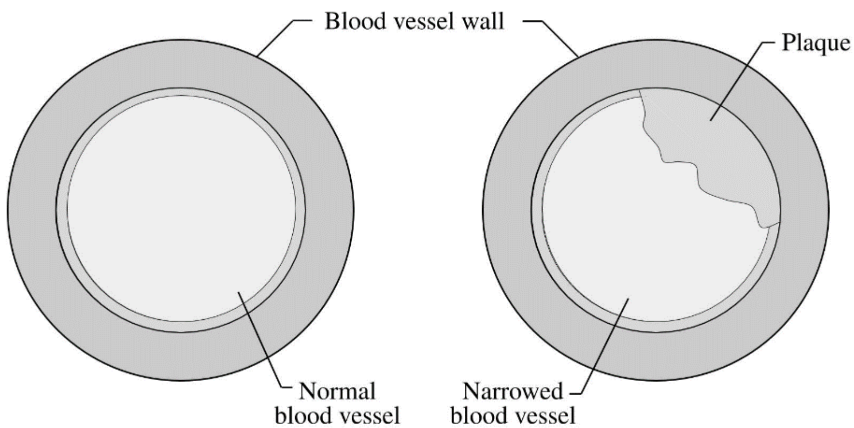

Figure 1. Blood vessel cross-section

As a child, my grandfather had a heart attack[2] – a cardiovascular disease that occurs when blood flow to the heart is severely reduced or cut off completely. Even though I was left traumatized after the experience, it sparked my interest and led me explore the causes and principles behind a heart attack. As I grew up, my curiosity further rose after realizing the physics behind this principle. So, I decided to conduct an experiment exploring the effect of changing the size of the radius of a blood vessel and how much more pressure was required to keep the rate of blood flow constant, which helped me determine the focus of my experiment with the following research question: **To what extent does a change in the radius of a tube affect the pressure difference at both its ends when the volume flow rate is constant?**

Mathematical derivation for the investigation:

The pressure difference between two ends of a tube can be mathematically derived through the Hagen-Poiseuille equation. (Where Q is the volume flow rate, ΔP is the pressure difference, and R is the resistance to the flow.)

$$Q = \frac{\Delta P}{R}$$

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[1] *High Blood Pressure*. (n.d.). Heart. https://www.heart.org/en/health-topics/high-blood-pressure/health-threats-from-high-blood-pressure/how-high-blood-pressure-can-lead-to-a-heart-attack

[2] *What is a Heart Attack?* (n.d.). Heart. https://www.heart.org/en/health-topics/heart-attack/about-heart-attacks

The resistance R can be determined by the following equation. (Where η is the viscosity of the liquid, L is the length of the tube, and r is the radius of the tube.)

$$R = \frac{8\eta L}{\pi r^4}$$

Therefore, the equation can be rewritten as:

$$Q = \frac{\pi \Delta P r^4}{8\eta L} \qquad \text{(Equation 1)}$$

Solving for ΔP: $\left(k = \frac{Q8\eta L}{\pi}\right)$

$$\rightarrow Q = \frac{\pi \Delta P r^4}{8\eta L}$$

$$\rightarrow Q8\eta L = \pi \Delta P r^4$$

$$\rightarrow \Delta P = \frac{Q8\eta L}{\pi r^4} \qquad \text{(Equation 2)}$$

$$\rightarrow \Delta P \propto r^{-4} \qquad \text{(Equation 3)}$$

$$\rightarrow \Delta P = kr^{-4} \qquad \text{(Equation 4)}$$

$$\rightarrow log\, \Delta P = log\, kr^{-4}$$

$$\rightarrow log\, \Delta P = log\, r^{-4} + log\, k$$

$$\rightarrow log\, \Delta P = -4 log\, r + log\, k \qquad \text{(Equation 5)}$$

The relationship between pressure difference ΔP and radius r is determined through equation 3. Since the relationship between them is not linear, I have decided to linearize it with the use of logarithms in equation 5.

The investigation of this relationship is significant as it can assist doctors and biomedical engineers to identify the optimal radii of blood vessels and better understand the excess stress on the heart when a blood vessel is compromised or restricted. In order to investigate this relationship, I have decided to create an experiment where tubes with radii of 1.25mm, 1.50mm, 1.75mm, 2.00mm, 2.25mm and 2.50mm will be used to determine the pressure difference.

Hypothesis:

The pressure difference ΔP will decrease as the radius of tube r is increased. There is a logarithmic relationship between ΔP and r given as: $log\, \Delta P = -4 log\, r + log\, k$, where, given that k remains constant, there will be linear graph with gradient -4 between $log(\Delta P)$ and $log(r)$ passing through the y-intercept at point (0, $log\, k$). Further, the equation $\Delta P = kr^{-4}$, where k remains constant, will give an inverse quartic graph between ΔP and r.

Variables:

A) Independent variable: Radius of the tube (cm). The tubes are kept straight in order to minimize resistance and tubes of the same material is used. The radii used are 1.25mm, 1.50mm, 1.75mm, 2.00mm, 2.25mm and 2.50mm

B) Dependent variable: Pressure difference between the two ends of the tube (Pa)

C) Controlled variables:

1) Viscosity of liquid:
   a) Reason: As observed in equation 2, the viscosity of the liquid is directly related to the dependent variable, ΔP, and hence must remain constant.
   b) Method: The viscosity of the liquid was controlled by only utilizing a single liquid which was relatively pure water at a constant temperature.

2) Length of tube:
   a) Reason: As observed in equation 2, the length of the tube is directly related to the dependent variable, ΔP, and hence must remain constant.
   b) Method: The length of the tube was controlled by cutting all five tubes of different radii to the fixed length of 15cm.

3) Volume flow rate:
   a) Reason: As observed in equation 2, the volume flow rate of the liquid is directly related to the dependent variable, ΔP, and hence must remain constant at all times in the experiment.
   b) Method: According to equation 1, as the radius of the tube is increased, the volume flow rate is also increased. In order to keep the volume flow rate of the liquid constant, first the volume flow rate will be calculated for the radius that is used; This will be done by calculating $\frac{V}{t}$ where V is the volume in the first flask (at $50.0cm^3$) and t is the time taken for volume to reach $0.0cm^3$. Then the pressure difference will be changed using the vacuum system in the second flask in order to make volume flow rate exactly $5.0cm^3s^{-1}$, and that pressure difference will be recorded.

4) Temperature:
   a) Reason: The temperature should be kept constant as the viscosity of a fluid has an inverse relationship with temperature[3]. Also changing the temperature would cause expansion or contraction of the tubes and liquid causing slight variations in the readings.
   b) Method: To keep the value of temperature constant, the entire experiment was conducted in an air-conditioned room with a fixed temperature of 20°C.

[3] Elert, G. (n.d.). *Viscosity*. The Physics Hypertextbook. https://physics.info/viscosity/

Apparatus & Material

Apparatus	Properties	Quantity
Flask	Measurements up to 100.0cm^3 Connector and valve at the bottom Closed flask connected to VS	2
Stopwatch	Least count: 0.1s	1
Digital Manometer	Least count: 1kPa	2
Vacuum System (VS)	-	2

Table 1. Apparatus and properties

Material	Properties	Quantity
Tubes	Radii: (1.25, 1.50, 1.75, 2.00, 2.25, 2.50) mm Length: 15.0cm	5
Water	Relatively pure	-

Table 2. Materials and properties

Method:

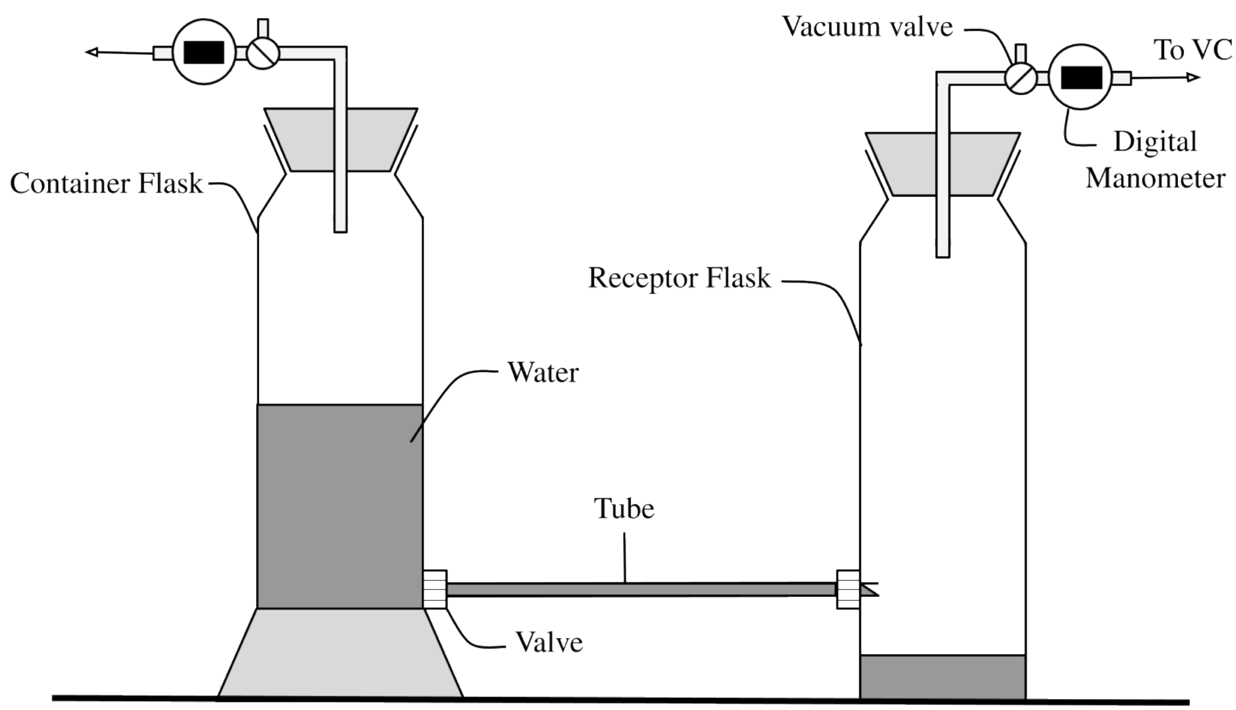

Figure 2. Experiment setup

Procedure:

1) Connect tube of 1.25mm radius between both flasks
2) Enter 50.0cm^3 water in container flask
3) Close container flask lid and set pressure for the left VS to 1000kPa
4) Open the water valve and start timer instantaneously
5) Stop timer when container flask has 0.0cm^3 water left
6) Calculate volume flow rate by $\frac{V}{t}$
7) If volume flow rate is not approximately 5cm^{3}s^{-1} then regulate the right VS (increase or decrease pressure depending on equation 2)
8) Repeat steps 2-6 until volume flow rate is 5cm^{3}s^{-1} approximately
9) Record the reading on manometer and turn off VC
10) Repeat steps 2-8 at least 3 times
11) Repeat steps 2-10 with tube of radius: (1.50, 1.75, 2.00, 2.25, 2.50) mm

Risk assessment

In the experiment there are no major risk assessments; The liquid used is not hazardous and there are no sharp objects to be concerned about. Liquid spills may be dangerous causing one to lose their balance and slip, so a dry cloth should be kept. When preparing for the experiment, tubes must be cut to a certain length, hence scissors must be used carefully to avoid casualties. Finally glass flasks must be handled carefully as if it were to be dropped, it would shatter creating many sharp glass shards. Further, the water will be reused as throughout the experiment with minimal wastage and will be drained into the fields after use. Overall, there are no environmental and ethical concerns in the experiment, thus we can conclude that this is a safe experiment.

Qualitative Assessment

- Throughout the day of data collection, temperature in the room often fluctuated between 19°C and 21°C when measured through a thermometer. The experiment was halted until the temperature remained a steady 20°C.

- The receptor flask lid contraption quickly proved to be insufficient to hold the rarefaction of air as small leakages around the lid kept the flask from lowering its pressure. Blu-Tack (a putty like adhesive) was then applied around the lid every time the vacuum system was in use.

- The volume flow rate was not entirely constant throughout the flow of water from one flask to the other because as the water left the container flask, the pressure of the container flask struggled to stay at a constant of 1000kPa and dipped by 20kPa when the valve was first opened. This dip was negligible as in the first 3 seconds the pressure rose back to 1000kPa as the vacuum system began accounting for the difference.

Data collection & analysis

	Pressure ($P \pm 1$) kPa			
Radius of Tube ($r \pm 0.01$) mm	Trial 1	Trial 2	Trial 3	Mean Pressure P kPa
1.25	221	223	216	220 ± 5
1.50	626	624	620	623 ± 4
1.75	792	796	797	795 ± 4
2.00	878	876	879	878 ± 3
2.25	927	924	923	925 ± 3
2.50	953	950	957	953 ± 4

Table 3. Radius and corresponding pressure (P) at right VS

Calculations for table 3:

The mean pressure was calculated using the formula:

$$\frac{P_1 + P_2 + P_3}{3}$$

The uncertainty of the mean pressure was calculated using the formula

$$\frac{P_{max} - P_{min}}{2} + P_{uncertainity}$$

Sample calculation of mean pressure for r = 1.25mm:

$$\text{Average} = \frac{221+223+216}{3} \approx 220 \text{ kPa (0 decimal places)}$$

$$\text{Uncertainty} = \frac{223-216}{2} + 1 = 4.5 \approx 5 \text{ kPa (0 decimal places)}$$

Here, 1 is added due to the uncertainty in the measurement of pressure (digital manometer).

Radius of Tube ($r \pm 0.01$) mm	Mean Pressure Difference ΔP kPa
1.25	780 ± 6
1.50	377 ± 5
1.75	205 ± 5
2.00	122 ± 4
2.25	75 ± 4
2.50	47 ± 5

Table 4. Radius and corresponding mean pressure difference (ΔP)

Calculations for table 4:

The mean pressure difference was calculated using the formula using values from table 3:

$$(\text{Pressure at left VS}) - P$$

Sample calculation of mean difference pressure for r = 1.25mm:

$$\text{Pressure difference} = 1000 - 220 = 780 \text{ kPa}$$

Uncertainty of mean pressure adds with the uncertainty in measurement of left VC (1kPa).

$$\text{Uncertainty} = 4 + 1 = 5 \text{ kPa}$$

$log(Radius\ of\ Tube)$ $log(r)$ No unit	$log(Mean\ Pressure\ Difference)$ $log(\Delta P)$ No unit
0.0969 ± 0.0035	2.892 ± 0.003
0.1761 ± 0.0029	2.576 ± 0.006
0.2430 ± 0.0025	2.312 ± 0.011
0.3010 ± 0.0022	2.086 ± 0.014
0.3522 ± 0.0019	1.875 ± 0.023
0.3979 ± 0.0017	1.672 ± 0.046

Table 5. Converting the data into logarithmic form

Calculations for table 5:

The $log(Radius\ of\ Tube)$ was calculated from the formula below using table 3:

$$log_{10}(r)$$

Uncertainty for $log(Radius\ of\ Tube)$was calculated from the formula below using table 3:

$$\frac{log_{10}\left(\frac{r + uncertainity\ of\ r}{r - uncertainity\ of\ r}\right)}{2}$$

Sample calculation of $log(Radius\ of\ Tube)$ for r = 1.25mm:

$$log(Radius\ of\ Tube) = log_{10}(1.25) \approx 0.0969 \text{ (4 decimal places)}$$

$$\text{Uncertainty} = \frac{log_{10}\left(\frac{1.26}{1.24}\right)}{2} \approx 0.0035 \text{ (4 decimal places)}$$

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$log(Mean\ Pressure\ Difference)$ was calculated from the formula below using table 3:

$$log_{10}(\Delta P)$$

Uncertainty for $log(Mean\ Pressure\ Difference)$ was calculated from the formula below using table 3:

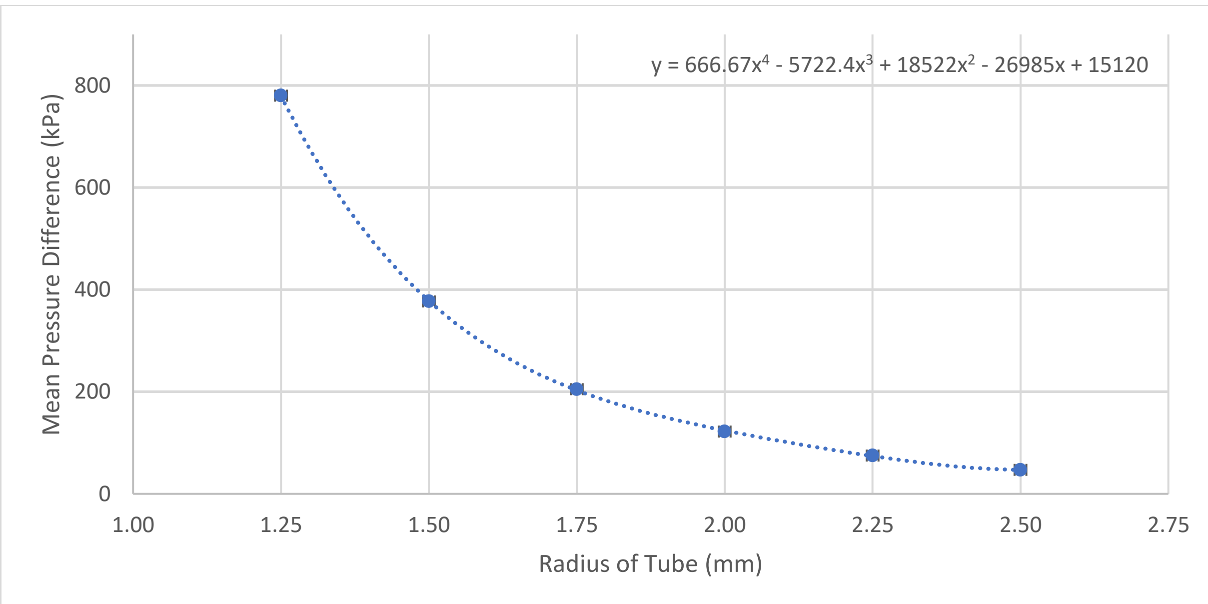
$$\frac{log_{10}\left(\frac{\Delta P + uncertainity\ of\ \Delta P}{\Delta P - uncertainity\ of\ \Delta P}\right)}{2}$$

Sample calculation of $log(Mean\ Pressure\ Difference)$ for r = 1.25mm:

$log(Mean\ Pressure\ Difference) = log_{10}(780) \approx$ 2.892 (3 decimal places)

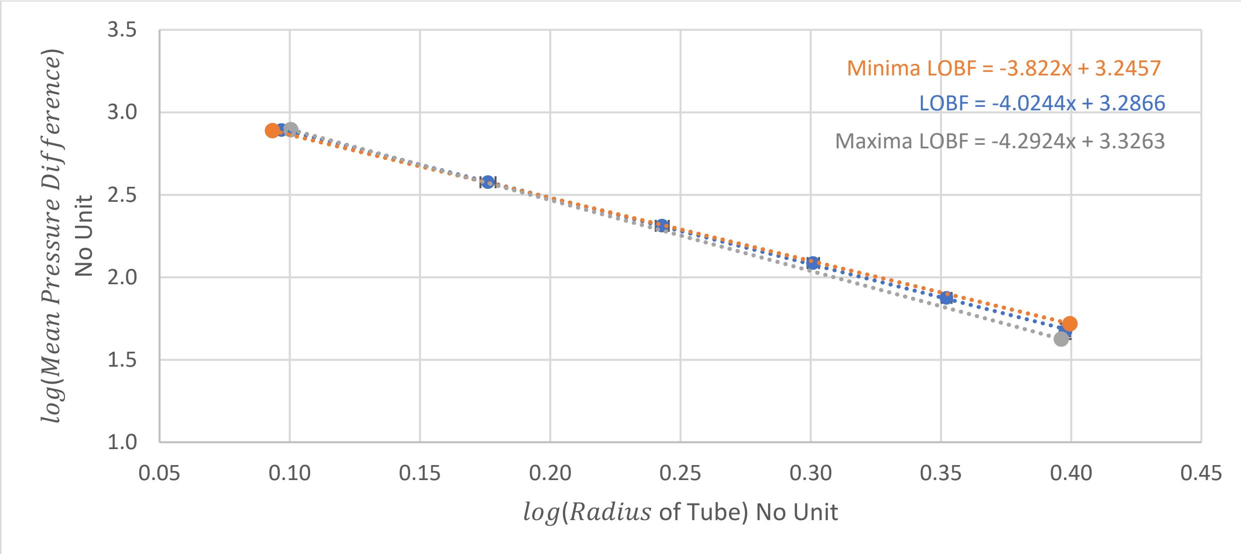
Uncertainty = $\frac{log_{10}\left(\frac{785}{775}\right)}{2} \approx$ 0.003 (3 decimal places)

Graphical analysis



Graph 1. Mean pressure difference(kPa) vs Radius of Tube(mm) with a quartic LOBF

The error bars for the mean pressure difference are very small and thus are not visible. From the data collected and plotted, the graph supports a quartic relationship. This can be realized by seeing that Graph 1, with a quartic line of best fit (LOBF), intersects all 6 plotted error bars. This is consistent with the hypothesis, in that there is an inverse quartic relationship between the radius and pressure difference. However, as Graph 1 suggests, the equation of the line mapped (top right) also includes the coefficients of x^3, x^2, and x, meaning that there are in fact sources of error, and its effects can be visualized.



Graph 2. $log(Mean\ Pressure\ Difference)$ vs $log(Radius)$ with max, min and normal LOBF

Processing of Graph 2 Data	
Gradient = -4.024	y-intercept = 3.287
Minima Gradient = -3.822	Minima y-intercept = 3.246
Maxima Gradient = -4.292	Maxima y-intercept = 3.326
Gradient Uncertainty = $\frac{-4.292+3.822}{2}$ = ± 0.235	y-intercept Uncertainty = $\frac{3.326-3.246}{2}$ = ± 0.080
Gradient Percent Uncertainty = $\frac{-0.235*100}{-4.024}$ = 5.8%	y-intercept Percent Uncertainty = $\frac{0.080*100}{3.287}$ = 2.4%
Gradient Percent Error = $\frac{(-4.024+4)*100}{-4}$ = 0.6%	y-intercept Percent Error = $\frac{(3.287-3.281)*100}{3.281}$ = 0.2%

The LOBF is a negative linear graph and intersects all plotted error bars. The LOBF gradient can be written as -4.024 ± 0.235, and the y-intercept can be written as 3.287 ± 0.080. The minima LOBF and maxima LOBF represent a 5.8% uncertainty in the gradient and 2.4% uncertainty in the y-intercept which can be attributed to random error as precision is relatively low.

The value of R^2, the square of the correlation coefficient, is 0.99965. R^2 evaluates the strength of the relationship between two variables. This value can be assessed using the table below, where we can infer that the experiment provided data has a very strong correlation, thus low systematic error.

Range of R^2	**Strength of Correlation**
Below 0.49	Weak
0.50 to 0.69	Moderate
0.70 to 0.89	Strong
0.90 to 1.00	Very strong

Table 6. R^2 and its corresponding strength of relationship

$$log\ (\Delta P)\ =\ -4log\,r + log\,k \quad \text{(Equation 5)}$$

$$\Delta P\ =\ \frac{Q8\eta L}{\pi r^4} \quad \text{(Equation 2)}$$

Since $k\ =\ \frac{Q8\eta L}{\pi}$, the units for equation 2 can be substituted as (where k's units are unknown)

$$Pa = \frac{k}{mm^4} \approx k = mm^4Pa$$

Thus, calculating k gives $\frac{Q8\eta L}{\pi} = \frac{5000*8*1*150}{\pi} \approx 1.910 * 10^6\ mm^4Pa$. Then, converting the units to mm^4kPa to make the pascal unit the same as the data gathered gives $k = 1910\,mm^4kPa$. Finally logging this value gives $log(k) \approx 3.281$ which is the literature y-intercept. According to the equation 5, the literature gradient is -4. Both calculated gradient and y-intercept in the graph fall within the range of error seen in graph 2 and, hence, the data collected is consistent with the hypothesis. Equation 5 can then be accepted.

Conclusion

After collecting, processing and analyzing the data, the results can confirm the hypothesis. "The pressure difference ΔP will decrease as the radius r is increased" can be supported by looking at graph 1 and 2, where this is evident in the form of a negative correlation. To linearize the equation, both sides were logged, and $log\ (Mean\ pressure\ difference)$ was plotted against $log\ (radius)$ in graph 2. The line of best fit clearly indicated a linear relationship between $log(\Delta P)$ and $log(r)$, which is directly supported by the hypothesis. According to the hypothesis, the gradient of the line of best fit of graph 2 should be -4. This was supported by the data because the gradient fell within range of error of the line of best fit. The hypothesis also indicated that the y-intercept will be at "point (0, $log\,k$)," which also supported as the calculated y-intercept fell within the range of error of the line of best fit. Graph 1 and 2 also revealed that there was an inverse quartic relationship that could be seen when the mean pressure difference was plotted against the radius of the tube. Thus, all points in the hypothesis were confirmed.

On one hand, it can be observed that the experiment had relatively high accuracy, determined by comparing the calculated values to the results. This demonstrates that there was a relatively low percentage systematic error within the experiment, proving that the experiment design was successful. On the other hand, however, we can see that the precision of the experiment was relatively low for table 4, 5 and graph 2, attributed due to the fact that some of the pressure trials for a certain radius had the range of uncertainty up to 12 kPa. This led to the uncertainty in the LOBF in graph 2 to be 5.8% in value. This means there was comparatively higher percent random error than percent systematic error. A few factors may have played into this like the reaction time or not being able to precisely perceive when there is exactly no more water left in the container flask; these conditions will be explored further in the evaluation.

Overall, the research question "**To what extent does a change in the radius of a tube affect the pressure difference at both its ends when the volume flow rate is constant?**" was effectively answered and was worth investigating, where the final equation, that was result of the experiment, can be expressed in the form: $log\,\Delta P\ =\ (-4.024\ \pm\ 0.235)log\,r + (3.287\ \pm\ 0.080)$

Evaluation

Strengths of the experiment

This experiment exhibits very low systematic error, up to only 0.6%, proving that the experiment designed was incredibly accurate and effective. With the use of modulated vacuum systems on both container flasks, the experiment was actively controlled and altered to keep the pressure difference consistent, which wouldn't be the case if one of the flasks was kept open due to air conditioning wind currents or pressure inaccuracies due to temperature. Further, attention to detail to the smallest variables was a priority in my experiment such as taking precautions like regularly checking the temperature and waiting until the temperature stabilizes or using putty to minimize pressure inaccuracies all contributed in substantially lowering systematic errors during the experiment.

Weaknesses of the experiment

Source of error & effects	Significance & evidence	Possible improvements
Systematic errors affecting accuracy		
Volume flow rate: This wasn't consistent as the pressure difference deviated by 20kPa when the valve was first opened.	Low significance: - The vacuum system accounted for the decrease in pressure in the first 3 seconds, so the results were not affected much.	Use a vacuum system which is more responsive to changes in the pressure, thus stabilizing the decrease in pressure more quickly.
Temperature of water in tubes: In smaller radius tubes the pressure and speed of the water is higher, and thus experiences more friction than its larger counterpart.	Low significance: - The temperature change is negligible and thus any systematic error is unnoticeable.	Utilize tubes which are smooth and have a high heat capacity, allowing for heat to get absorbed.
Radius of tube: The radius within the tube may not be of linear radius, shifting results slightly in either direction.	Low significance: - If the radius of the tube was not consistent then the plotted mean pressure difference against radius would be noticeably skewed.	Fill tube with water. Remove water in flask. Measure volume then use the formula $r = \sqrt{\frac{V}{\pi h}}$ to confirm radius is accurate.
Limited amount of data: There were 6 radii used in the experiment for 3 trials. Less trials means lower accuracy given in results	Low significance: - The number of trials done, and radii used proved sufficient to confirm the hypothesis.	Take up to 12 varied radii of tubes and use 9 trials.

Random errors affecting precision		
Reaction time: Led to stopwatch being clicked too late and to the wrong calculation of volume flow, affecting the pressure difference values collected.	High significance: - Some pressure trials in table 4 had range of up to 12kPa leading to high gradient uncertainty in graph 2 LOBF.	Take a video of the water emptying out. Review footage on computer to get exact value of time taken.
Stopwatch precision: The stopwatch had an uncertainty of 0.1s, which gives a inaccurate value for volume flow rate	Low significance: - Uncertainty of 0.1s translates to a less than 1% relative uncertainty.	Put dark food coloring into the water and add a light sensor at the end of the tube, which will give time to the nearest 0.001s.
Barometer precision: The digital barometer had an uncertainty of 1kPa, which gives an inaccurate value for pressure	Moderate significance: - Uncertainty of 1kPa leads to $\frac{1*100}{47}$ = 1.2% percentage uncertainty for pressure difference in 2.5mm radius tube.	Use a new digital barometer of lower uncertainty.
Volume flow rate: The value for volume flow rate was used if it's within $0.1cm^3s^{-1}$, resulting in inaccurate pressure readings	Moderate significance: - Uncertainty of $0.1cm^3s^{-1}$ leads to $\frac{0.1*100}{5.0}$ = 2% percentage uncertainty.	Be more stringent on the volume flow rate, only using them if within a smaller range than $0.1cm^3s^{-1}$, would reduce this error.

Table 6. Weaknesses and limitations effecting results and its improvements

Further research suggestions

The relationship between the pressure difference between both ends of a tube and linear obstruction in the tube can be tested. This would more closely resemble the real-life scenario of narrowed blood vessels by plaque. This can be more clearly seen in the figure 3 where the lighter gray is the flow area.

In this experiment, large tubes of the same radius will be taken, where each tube will contain a circle segment of clay throughout the tube as shown in figure 3. The θ of the segment will be the independent variable and will be increased from 0, the pressure difference between the ends of the tube will be the dependent variable will be measured, and the volume flow rate will be kept constant. The rest of the experiment will be performed the same.

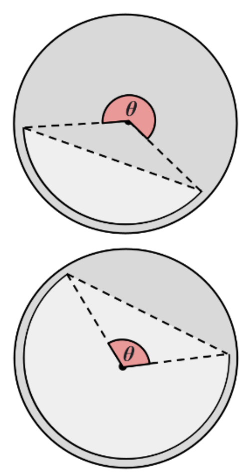


Figure 3. Extension

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